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EXISTENCE OF A KIND OF BEST SIMULTANEOUS APPROXIMATIONS IN $L_p(\Omega, \Sigma, X)$

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Abstract. Let X be a Banach space, Y a nonempty locally weakly compact closed convex subset of X, (Ω, Σ, μ) a complete positive σ-finite measure space and Σ_0 a sub-σ-algebra of Σ. This paper gives existence results of best simultaneous approximations to two functions in $L_p(\Omega, \Sigma, X)$ from $L_p(\Omega, \Sigma, Y)/L_p(\Omega, \Sigma_0, Y)$ if $\overline{\operatorname{span} Y}$/and $\overline{\operatorname{span} Y}^*$ has/have the Radon-Nikodym property.

1. Introduction

Throughout this paper, X is a Banach space with norm $\|\cdot\|$, (Ω, Σ, μ) is a complete positive σ-finite measure space, $p \in [1, +\infty)$, and $L_p(\Omega, \Sigma, X)$ denotes the Banach space of all Bochner p-integrable (essentially bounded for $p = \infty$) functions defined on Ω with values in X endowed with the usual norm $\|\cdot\|_p$. Let G be a nonempty subset of $L_p(\Omega, \Sigma, X)$ and let $f \in L_p(\Omega, \Sigma, X)$. Then $g_0 \in G$ is called a best approximation to f from G if

$$\|f - g_0\|_p = \inf\{\|f - g\|_p : \ g \in G\}.$$

The set of all best approximations to f from G is denoted by $P_G(f)$. G is called proximinal in $L_p(\Omega, \Sigma, X)$ if $P_G(f) \neq \emptyset$ for each $f \in L_p(\Omega, \Sigma, X)$.

For a given closed subspace Y of X, many papers have been devoted to studying when the space $L_p(\Omega, \Sigma, Y)$ is proximinal in $L_p(\Omega, \Sigma, X)$ (see the references cited in [3, 7, 8]), and the main problem that these papers address is that if Y is proximinal in X, is $L_p(\Omega, \Sigma, X)$ proximinal $L_p(\Omega, \Sigma, Y)$? Until to 1998, Mendoza [7] solved this problem. He shown that if Y is separable then $L_p(\Omega, \Sigma, Y)$ is proximinal in $L_p(\Omega, \Sigma, X)$ if and only if Y is proximinal in X, and provided an example to shows that the condition that Y is separable can not be dropped.

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In the present paper, we shall study the problem of best simultaneous approximations in $L_p(\Omega, \Sigma, X)$. The setting is as follows. Let

$$U := \{\mathbf{a} \in \mathbb{R}^m : \ \|\mathbf{a}\|_B \leq 1\}, \tag{1.1}$$

where $m \in \mathbb{N}$ and $\|\cdot\|_B$ is a given norm on $\mathbb{R}^m$. Let $G \subset L_p(\Omega, \Sigma, X)$. For any $F = (f_1, \cdots, f_m) \in (L_p(\Omega, \Sigma, X))^m$, define the norm

$$\|F\| := \max_{\mathbf{a} \in U} \left\| \sum_{i=1}^{m} a_i f_i \right\|_p .$$

Then $g_0 \in G$ is called a best simultaneous approximation to F from G if

$$\|F - g_0\| = d(F, G) := \inf\{\|F - g\| : \ g \in G\},$$

here and in the sequel, we adopt the convention that $F - g = (f_1 - g, \cdots, f_m - g)$. The set of all best simultaneous approximations to F from G is denoted by $P_G(F)$. G is called simultaneously proximinal in $L_p(\Omega, \Sigma, X)$ if $P_G(F) \neq \emptyset$ for each $F \in (L_p(\Omega, \Sigma, X))^m$. When $m = 2$ (an extension to any positive integer being straightforward) and Y is a locally weakly compact closed convex subset of X, we shall show in this paper that $L_p(\Omega, \Sigma_0, Y)$ (here, Σ_0 being a sub-σ-algebra of Σ) is simultaneously proximinal in $L_p(\Omega, \Sigma, X)$ for each $1 \leq p < +\infty$ (with the additional assumption that (Ω, Σ, μ) be finite for the case of $p = 1$) if $\overline{\operatorname{span} Y}$ and $\overline{\operatorname{span} Y}^*$ have the Radon-Nikodym property. While for the case when $\Sigma_0 = \Sigma$, we shall show that $L_p(\Omega, \Sigma, Y)$ is simultaneous proximinal in $L_p(\Omega, \Sigma, X)$ for each $1 \leq p < \infty$ if $\overline{\operatorname{span} Y}$ has the Radon-Nikodym property.

We note that the results of the present paper are corresponding to those given in [3], in which another kind of best simultaneous approximation problem in $L_p(\Omega, \Sigma, X)$ based to a so-called monotonic norm in $\mathbb{R}^m$ is considered. Also, it should be pointed that the study of this paper is motivated by works in [4, 5, 6], in which the problems of best simultaneous approximation in normed spaces under a monotonic norm in $\mathbb{R}^m$ are investigated.

2. Auxiliary Lemmas

Let $(X, \|\cdot\|)$, (Ω, Σ, μ), p and $L_p(\Omega, \Sigma, X)$ be explained as in the beginning of Section 1. Let Y be a subset of X and Σ_0 be a sub-σ-algebra of Σ. By $L_p(\Omega, \Sigma_0, Y)$ we mean the subset of $L_p(S, \Sigma, X)$ defined by

$$L_p(\Omega, \Sigma_0, Y) := \{g \in L_p(\Omega, \Sigma_0, X) : \ g(s) \in Y \text{ for a.e. } s \in \Omega\}.$$

For a set $E \in \Sigma$, χ_E denotes the characteristic function of E, i.e., $\chi_E(s) = 1$ if $s \in E$ and $\chi_E(s) = 0$ otherwise. Recall that $Y \subset X$ is called locally weakly compact if for each point $y \in Y$ there exists $\delta > 0$ such that $\mathbf{B}(y, \delta) \cap Y$ is weakly compact, where $\mathbf{B}(y, \delta)$ stands for the closed ball with center δ and radius r. In what follows, we always assume that $m = 2$ and the closed unit ball of $\mathbb{R}^2$ is defined by (1.1). Furthermore, we assume that Y is a locally weakly compact closed convex subset of X such that $L_p(\Omega, \Sigma_0, Y)$ is nonempty. Without loss of generality, we may assume that $0 \in Y$ as pointed in [3].

The following Lemmas 1, 2 and Lemma 3, which will be used in the next section, are corresponding to [8, Lemma 1 and 2] and [3, Lemma 2.2], respectively,

Lemma 1. *Let $G \subset X$ be a weakly closed subset of X and $F = (x_1, x_2) \in X^2$. If $\{g_n\} \subset G$ is a minimizing sequence for best simultaneous approximation to F from G and $\{g_n\}$ converges weakly to g_0, then $g_0 \in P_G(F)$.*

Proof. Let $(a_1, a_2) \in U$. Then, since $\lim_{n\to\infty} g_n = g_0 \in G$ weakly, one has that

$$\lim_{n\to\infty} (a_1(x_1 - g_n) + a_2(x_2 - g_n)) = a_1(x_1 - g_0) + a_2(x_2 - g_0) \quad \text{weakly.}$$

By the weak lower semicontinuity of the norm, we obtain that

$$\begin{aligned} \|a_1(x_1 - g_0) + a_2(x_2 - g_0)\| &\le \liminf_{n\to\infty} \|a_1(x_1 - g_n) + a_2(x_2 - g_n)\| \\ &\le \liminf_{n\to\infty} \|F - g_n\| = d(F, G) \end{aligned}$$

because $\{g_n\} \subset G$ is a minimizing sequence for best simultaneous approximation to F form G. Consequently, $\|F - g_0\| \le d(F, G)$ because $(a_1, a_2) \in U$ is arbitrary and $g_0 \in P_G(F)$, which completes the proof. ∎

Lemma 2. *Let $f^1, f^2 \in L_p(\Omega, \Sigma, X)$ and let $\{g_n\} \subset L_p(\Omega, \Sigma_0, Y)$ be a minimizing sequence for best simultaneous approximation to f^1, f^2 from $L_p(\Omega, \Sigma_0, Y)$. If $\{A_n\} \subset \Sigma_0$ satisfies that $\lim_{n\to\infty} \mu(A_n) = 0$, then $\{g_n \chi_{A_n^c}\}$ is a minimizing sequence for best simultaneous approximation to f^1, f^2 from $L_p(\Omega, \Sigma_0, Y)$.*

Proof. Let $(a_1, a_2) \in U$. It then follows from Minkowski Inequality that

$$\begin{aligned} &\|a_1(f_1 - g_n\chi_{A_n^c}) + a_2(f_2 - g_n\chi_{A_n^c})\|_p \\ &= \|[a_1(f_1 - g_n) + a_2(f_2 - g_n)]\chi_{A_n^c} + (a_1 f_1 + a_2 f_2)\chi_{A_n}\|_p \\ &\le \|a_1(f_1 - g_n) + a_2(f_2 - g_n)\|_p + |a_1|\|f_1\chi_{A_n}\|_p + |a_2|\|f_2\chi_{A_n}\|_p \\ &\le \|F - g_n\| + M\left(\|f_1\chi_{A_n}\|_p + \|f_2\chi_{A_n}\|_p\right), \end{aligned}$$

where $M := \max_{\mathbf{a}\in U}(|a_1| + |a_2|)$ is some positive number. This implies that

$$d(F, L_p(\Omega, \Sigma_0, Y)) \le \|F - g_n\chi_{A_n^c}\| \le \|F - g_n\| + M\left(\|f_1\chi_{A_n}\|_p + \|f_2\chi_{A_n}\|_p\right).$$

By the absolute continuity of a calculus, one has that $\lim_{n\to\infty}\|f_1\chi_{A_n}\|_p = \lim_{n\to\infty}\|f_2\chi_{A_n}\|_p = 0$ as $\lim_{n\to\infty}\mu(A_n) = 0$. Thus, letting $n \to \infty$ in above inequality yields

$$\lim_{n\to\infty}\|F - g_n\chi_{A_n^c}\| = d(F, L_p(\Omega, \Sigma_0, Y)).$$

This means that $\{g_n\chi_{A_n^c}\}$ is a minimizing sequence for best simultaneous approximation to f^1, f^2 from $L_p(\Omega, \Sigma_0, Y)$. The proof is complete. ∎

Lemma 3. *Let $f^1,\ f^2 \in L_p(\Omega, \Sigma, X)$ be a pair of countable valued functions. Then $(f^1,\ f^2)$ admits a best simultaneous approximation from $L_p(\Omega, \Sigma, Y)$.*

Proof. Let $k = 1, 2$ and assume that $f^k = \sum_{i=1}^{\infty} x_i^k\chi_{A_i}$ for some sequence $\{A_i\}$ of disjoint measurable sets in Ω and some sequence $\{x_i^k\} \subset X$. Then $\mu(A_i) < \infty$ whenever $x_i^k \neq 0$ because

$$\|f^k\|_p^p = \sum_{i=1}^{\infty}\|x_i^k\|^p\mu(A_i) < \infty.$$

Thus, we may assume that $0 < \mu(A_i) < \infty$ for each $i \in \mathbb{N}$. Set

$$G := \left\{ g = \sum_{i=1}^{\infty} y_i\chi_{A_i} :\ g \in L_p(\Omega, \Sigma, Y) \right\}$$

and

$$\phi(f^1, f^2; g) := \|F - g\| \quad \text{for each } g \in G.$$

We first show that there exists $g_0 \in G$ such that

$$\phi(f^1, f^2; g_0) = \phi(f^1, f^2) := \inf\{\phi(f^1, f^2; g) :\ g \in G\}. \tag{2.1}$$

To this end, let $\{g^n\} \subset G$ be a sequence such that $\phi(f^1, f^2; g^n) \to \phi(f^1, f^2)$. Then there exists some positive number M_1 such that $\phi(f^1, f^2; g^n) \leq M_1$ for all n. Let $n \in \mathbb{N}$ and assume that $g^n = \sum_{i=1}^{\infty} y_i^n\chi_{A_i}$. Then

$$\begin{aligned}
&\|g^n\|_p \max_{\mathbf{a}\in U}|a_1 + a_2| \\
&= \left(\sum_{i=1}^{\infty}\|y_i^n\|^p\mu(A_i)\right)^{\frac{1}{p}} \max_{\mathbf{a}\in U}|a_1 + a_2| \\
&= \max_{\mathbf{a}\in U}\|a_1 g^n + a_2 g^n\|_p \\
&\leq \max_{\mathbf{a}\in U}\|a_1(f^1 - g^n) + a_2(f^2 - g^n)\|_p + \max_{\mathbf{a}\in U}\|a_1 f^1 + a_2 f^2\|_p \\
&\leq M_1 + \phi(f^1, f^2; 0).
\end{aligned}$$

Noting that $\max_{\mathbf{a}\in U} |a_1 + a_2| > 0$, we have that $\{g^n\}$ is bounded. Furthermore, for each i, $\{y_i^n\}_{n=1}^{\infty}$ is also bounded in Y because $\|y_i^n\| \leq \frac{M_1+\phi(f^1,f^2;0)}{(\mu(A_i))^{1/p}}$ for each $n \in \mathbb{N}$. Since Y is locally weakly compact, it follows that $\{y_1^n\}$ has a weakly convergent subsequence, say $\{y_1^{n,1}\}$, with the weak limit y_1. Then $y_1 \in Y$ because Y is a closed convex subset of X. Similarly, noting that $\{y_2^{n,1}\}$ is a subsequence of $\{y_2^n\}$, there exists a subsequence $\{y_2^{n,2}\}$ of $\{y_2^{n,1}\}$ such that $\lim_{n\to\infty} y_2^{n,2} = y_2$ weakly for some $y_2 \in Y$. Keeping on going, one has that, for each i, there exists a subsequence $\{y_{i+1}^{n,i+1}\}$ of $\{y_{i+1}^{n,i}\}$ such that $\{y_{i+1}^{n,i+1}\}$ weakly converges to some element $y_{i+1} \in Y$. Since, for each fixed natural number m and each $i = 1, \cdots, m$, $\{y_i^{n,m}\}$ is a subsequence of $\{y_i^{n,i}\}$, we have that $\lim_n y_i^{n,m} = y_i$ weakly. Let $(a_1, a_2) \in U$. By the weak lower semicontinuity of the norm in X, one has that

$$\begin{aligned}&\|a_1(x_i^1 - y_i) + a_2(x_i^2 - y_i)\|\\ \leq\ &\liminf_n \|a_1(x_i^1 - y_i^{n,m}) + a_2(x_i^2 - y_i^{n,m})\| \text{ for each } i = 1, \cdots, m.\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{i=1}^{m} \|a_1(x_i^1 - y_i) + a_2(x_i^2 - y_i)\|^p \mu(A_i)\\ \leq\ &\sum_{i=1}^{m} [\liminf_n \|a_1(x_i^1 - y_i^{n,m}) + a_2(x_i^2 - y_i^{n,m})\|]^p \mu(A_i)\\ \leq\ &\liminf_n \sum_{i=1}^{m} \|a_1(x_i^1 - y_i^{n,m}) + a_2(x_i^2 - y_i^{n,m})\|^p \mu(A_i)\\ \leq\ &\liminf_n \sum_{i=1}^{\infty} \|a_1(x_i^1 - y_i^{n,m}) + a_2(x_i^2 - y_i^{n,m})\|^p \mu(A_i)\\ =\ &\liminf_n \|a_1(f^1 - y^{n,m}) + a_2(f^2 - y_{n,m})\|^p\\ \leq\ &\liminf_n [\phi(f^1, f^2; g^{n,m})]^p = [\phi(f^1, f^2)]^p,\end{aligned}$$

where the last equality holds because $\{g^{n,m}\}_{n=1}^{\infty}$ is a subsequence of $\{g^n\}$. Passing onto limit and noting that $(a_1, a_2) \in U$ is arbitrary, one has that

$$\max_{\mathbf{a}\in U} \left(\sum_{i=1}^{\infty} \|a_1(x_i^1 - y_i) + a_2(x_i^2 - y_i)\|^p \mu(A_i) \right)^{\frac{1}{p}} \leq \phi(f^1, f^2). \tag{2.2}$$

This implies that $\sum_{i=1}^{\infty} \|y_i\|^p \mu(A_i) < \infty$. Define $g_0 = \sum_{i=1}^{\infty} y_i \chi_{A_i}$. Then $g_0 \in G$ and (2.1) is seen to holds thanks to (2.2).

We then verify that

$$\phi(f^1, f^2) \leq \|F - w\| \quad \text{for each } w \in L_p(\Omega, \Sigma, Y). \tag{2.3}$$

Granting this, one has that g_0 is a best simultaneous approximation to $(f_1,\ f_2)$ from $L_p(\Omega,\Sigma,Y)$ and completes the proof.

To show (2.3), let $w\in L_p(\Omega,\Sigma,Y)$ be a countably valued function that has the expression $w=\sum_{j=1}^{\infty}w_j\chi_{B_j}$ for some sequence $\{B_j\}$ of disjoint measurable sets in Ω and some sequence $\{w_j\}\subset Y$. Then f^k and w can be respectively rewritten as

$$f^k=\sum_{i,j=1}^{\infty}x_{ij}^k\chi_{A_i\cap B_j}\quad\text{and}\quad w=\sum_{i,j=1}^{\infty}w_{ij}\chi_{A_i\cap B_j},$$

where

$$x_{ij}^k=x_i^k\quad\text{and}\quad w_{ij}=w_j\quad\text{for each } i,\ j=1,2,\cdots.$$

We claim that

$$\sum_{j=1}^{\infty}\mu(A_i\cap B_j)\|w_j\|\le\|w\|_p(\mu(A_i))^{\frac{1}{q}}\quad\text{for each } i\in\mathbb{N}. \tag{2.4}$$

In fact, (2.4) is clear in the case of $p=1$. While in the case of $1<p<\infty$, we obtain from Hölder Inequality that

$$\begin{aligned}\sum_{j=1}^{\infty}\mu(A_i\cap B_j)\|w_j\| &\le \left(\sum_{j=1}^{\infty}\left(\frac{\mu(A_i\cap B_j)}{\mu(B_j)^{\frac{1}{p}}}\right)^q\right)^{\frac{1}{q}}\left(\sum_{j=1}^{\infty}\mu(B_j)\|w_j\|^p\right)^{\frac{1}{p}}\\ &= \|w\|_p\left(\sum_{j=1}^{\infty}\left(\frac{\mu(A_i\cap B_j)}{\mu(B_j)}\right)^q\mu(B_j)\right)^{\frac{1}{q}}\\ &\le \|w\|_p\left(\sum_{j=1}^{\infty}\frac{\mu(A_i\cap B_j)}{\mu(B_j)}\mu(B_j)\right)^{\frac{1}{q}}\\ &= \|w\|_p(\mu(A_i))^{\frac{1}{q}}.\end{aligned}$$

Hence (2.4) holds and the claim is proved. Write

$$\bar{y}_i=\frac{\sum_{j=1}^{\infty}\mu(A_i\cap B_j)w_j}{\mu(A_i)}\quad\text{for each } i\in\mathbb{N}.$$

Then $\bar{y}_i\in Y$ because $\sum_{j=1}^{\infty}[\mu(A_i\cap B_j)]/\mu(A_i)=1$ and Y is a closed convex set. Define $\bar{g}=\sum_{i=1}^{\infty}\bar{y}_i\chi_{A_i}$. Then $\bar{g}\in G$. Furthermore, let $(a_1,a_2)\in U$. Then

$$\begin{aligned}&\|a_1(f^1-w)+a_2(f_2-w)\|_p^p\\ &=\sum_{i,j=1}^{\infty}\|a_1(x_{ij}^1-w_{ij})+a_2(x_{ij}^2-w_{ij})\|^p\mu(A_i\cap B_j)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \mu(A_i) \sum_{j=1}^{\infty} \frac{\mu(A_i \cap B_j)}{\mu(A_i)} \|a_1(x_i^1 - w_j) + a_2(x_i^2 - w_j)\|^p \\
&\geq \sum_{i=1}^{\infty} \mu(A_i) \left(\sum_{j=1}^{\infty} \frac{\mu(A_i \cap B_j)}{\mu(A_i)} \|a_1(x_i^1 - w_j) + a_2(x_i^2 - w_j)\| \right)^p \\
&\geq \sum_{i=1}^{\infty} \mu(A_i) \|a_1(x_i^1 - \bar{y}_i) + a_2(x_i^2 - \bar{y}_i)\|^p \\
&= \|a_1(f^1 - \bar{g}) + a_2(f^2 - \bar{g})\|_p^p,
\end{aligned}
$$

where we use the fact that the function $t \mapsto t^p$ is convex on $[0, +\infty)$ and the definition of norm in X. Since $(a_1, a_2) \in U$ is arbitrary and $\bar{g} \in G$, it follows from (2.1) that $\|F - w\| \geq \|F - \bar{g}\| \geq \phi(f^1, f^2)$; hence (2.3) holds and the proof is complete. ■

Investigating the proof of Lemma 2.1, we obtain the following result.

Lemma 4. *Let $f^1,\ f^2 \in L_p(\Omega, \Sigma, X)$ be a pair of countable valued functions. Then there exists a best simultaneous approximation g_0 to $(f^1,\ f^2)$ from $L_p(\Omega, \Sigma, Y)$ such that for each $E \in \Sigma$, so is $g_0|_E$ to $(f^1|_E,\ f^2|_E)$ from $L_p(E, \Sigma|_E, Y)$.*

Recall that a Banach space X is said to have the Radon-Nikodym property if, for each finite measure space (Ω, Σ, μ) and each μ-continuous vector measure $G : \Sigma \to X$ of bounded variation, there exists $g \in L_1(\Omega, \Sigma, \mu)$ such that $G(E) = \int_E g d\mu$ for all $E \in \Sigma$.

The following lemma (see, [3, Lemma 3.1]), which is an extension of Dunford Theorem ([2, Theorem IV.2.1])), plays an important role in establishing main results of this paper.

Lemma 5. *Let (Ω, Σ_1, μ) be a σ-finite measure space with Σ_1 generated by a countable field. Suppose that X has the Radon-Nikodym property. Let $1 \leq p < \infty$ and let $\{g_n\}$ be a sequence of $L_p(\Omega, \Sigma_1, X)$ satisfying the following conditions.*

(*i*) *$\{g_n\}$ is bounded in $L_p(\Omega, \Sigma_1, X)$.*

(*ii*) *$\{g_n\}$ is uniformly integrable.*

(*iii*) *For each $E \in \Sigma_1$ with $\mu(E) < \infty$, $\{\int_E g_n d\mu\}$ is relatively weakly compact in X.*

Then there exist a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and $g_0 \in L_p(\Omega, \Sigma_1, X)$ such that for each $E \in \Sigma_1$ with $\mu(E) < \infty$,

$$\lim_k \langle g_{n_k} - g_0, h^* \chi_E \rangle = 0 \quad \text{for each } h^* \in L_q(\Omega, \Sigma_1, X^*). \tag{2.5}$$

3. Main Result

Theorem 1. *Let Y be a locally weakly compact closed convex subset of X such that $\overline{\operatorname{span} Y}$ and $\overline{\operatorname{span} Y}^*$ have the Radon-Nikodym property. Suppose that $p > 1$ or $p = 1$ and (Ω, Σ, μ) is finite. Then $L_p(\Omega, \Sigma_0, Y)$ is simultaneously proximinal in $L_p(\Omega, \Sigma, X)$.*

Proof. Let $f^1,\ f^2 \in L_p(\Omega, \Sigma, X)$ and let $\{g_n\} \subset L_p(\Omega, \Sigma_0, Y)$ be a minimizing sequence for best simultaneous approximation to $f^1,\ f^2$ from $L_p(\Omega, \Sigma_0, Y)$. Then $\{\|F - g_n\|\}$ is bounded. Note that

$$
\begin{aligned}
&\max_{\mathbf{a} \in U} |a_1 + a_2| \|g_n\|_p \\
&= \max_{\mathbf{a} \in U} \|a_1(f_1 - g_n) + a_2(f_2 - g_n) - a_1 f_1 - a_2 f_2\|_p \\
&\leq \max_{\mathbf{a} \in U} \|a_1(f_1 - g_n) + a_2(f_2 - g_n)\|_p + \max_{\mathbf{a} \in U} \|a_1 f_1 + a_2 f_2\|_p \\
&= \|F - g_n\| + \max_{\mathbf{a} \in U} \|a_1 f_1 + a_2 f_2\|_p.
\end{aligned}
$$

One has that $\{g_n\}$ is bounded.

Let $\Sigma_1 \subset \Sigma_0$ be a σ-algebra generated by a countable algebra such that each g_n is measurable with respect to (Ω, Σ_1, μ). Then $\{g_n\} \subset L_p(\Omega, \Sigma_1, Y)$. By [1, Lemma 2.1.3], there exist a subsequence of $\{g_n\}$, denoted by $\{g_n\}$, and a sequence $\{E_n\}$ of pairwise disjoint measurable sets in Σ_1 such that $\{g_n \chi_{E_n^c}\}$ is uniformly integrable in $L_1(\Omega, \Sigma_1, \overline{\operatorname{span} Y})$. Define

$$
\bar{g}_n = \begin{cases} g_n \chi_{E_n^c}, & p = 1, \\ g_n, & 1 < p < +\infty. \end{cases}
$$

Then, for each $1 \leq p < \infty$, it follows from Lemma 2 that $\{\bar{g}_n\}$ is a minimizing sequence for a best simultaneous approximation to $f^1,\ f^2$ from $L_p(\Omega, \Sigma_0, Y)$. By the same proof as that given to [3, Theorem 3.1], we result that there exist a subsequence of $\{\bar{g}_n\}$, again denoted as $\{\bar{g}_n\}$, and $g_0 \in L_p(\Omega, \Sigma_0, Y)$ such that $\{\bar{g}_n\}$ converges weakly to g_0 in $L_p(\Omega, \Sigma, X)$. Therefore, g_0 is a best simultaneous approximation to f_1, f_2 from $L_p(\Omega, \Sigma_0, Y)$ thanks to Lemma 1, which completes the proof. ■

Theorem 2. *Let $1 \leq p < \infty$ and let Y be a locally weakly compact closed convex subset of X. If $\overline{\operatorname{span} Y}$ has the Radon-Nikodym property, then $L_p(\Omega, \Sigma, Y)$ is best simultaneously proximinal in $L_p(\Omega, \Sigma, X)$.*

Proof. Let $F = (f^1,\ f^2) \in (L_p(\Omega, \Sigma, X))^2$. We shall show that there exists $g_0 \in L_p(\Omega, \Sigma, Y)$ such that

$$
\|F - g_0\| \leq \|F - g\| \quad \text{for each } g \in L_p(\Omega, \Sigma, Y). \tag{3.1}
$$

For each $k = 1, 2$, let $\{f_n^k\}$ be a sequence of countably valued measurable functions in $L_p(\Omega, \Sigma, X)$ such that

$$\lim_n \|f_n^k - f^k\|_p = 0 \quad \text{and} \quad \lim_n \|f_n^k(s) - f^k(s)\| = 0 \quad \text{for a.e. } s \in \Omega. \tag{3.2}$$

By Lemma 4, for each n, there exists a best simultaneous approximation g_n to $(f_n^1,\ f_n^2)$ from $L_p(S, \Sigma, Y)$ such that for each $E \in \Sigma$, so is $g_n|_E$ to $f^1|_E,\ f^2|_E$ from $L_p(E, \Sigma|_E, Y)$. Let Σ_1 be a σ-algebra generated by a countable algebra such that each f_n^k and g_n are measurable with respect to (Ω, Σ_1, μ). Thus, f^1 and f^2 are measurable with respect to (Ω, Σ_1, μ). Consequently, $\{f^1, f^2\},\ \{f_n^k\},\ \{g_n\} \subset L_p(\Omega, \Sigma_1, X)$. We assert that there exist a subsequence of $\{g_n\}$, denoted by itself, and $g_0 \in L_p(\Omega, \Sigma_1, \overline{\text{span}Y})$ such that, for each $E \in \Sigma_1$ with $\mu(E) < \infty$,

$$\lim_n \langle g_n - g_0, h^* \chi_E \rangle = 0 \quad \text{for each } h^* \in L_q(\Omega, \Sigma_1, X^*). \tag{3.3}$$

By Lemma 3, it suffices to verify that $\{g_n\}$ satisfies the following conditions:

(i) $\{g_n\}$ is bounded in $L_p(\Omega, \Sigma_1, \overline{\text{span}Y})$;

(ii) $\{g_n\}$ is uniformly integrable in $(\Omega, \Sigma_1, \overline{\text{span}Y})$;

(iii) for each $E \in \Sigma_1$ with $\mu(E) < \infty$, $\{\int_E g_n(s) d\mu\}$ is relatively weakly compact in $\overline{\text{span}Y}$.

Since, for each n, g_n is a best simultaneous approximation to $f^1,\ f^2$ from $L_p(\Omega, \Sigma, Y)$ and $0 \in L_p(\Omega, \Sigma, Y)$, we have that

$$\begin{aligned}
&\|g_n\|_p \max_{\mathbf{a} \in U} |a_1 + a_2| \\
&= \max_{\mathbf{a} \in U} \|a_1 g_n + a_2 g_n\|_p \\
&\le \max_{\mathbf{a} \in U} \|a_1(f_n^1 - g_n) + a_2(f^2 - g_n)\|_p + \max_{\mathbf{a} \in U} \|a_1 f_n^1 + a_2 f_n^2\|_p \\
&\le 2 \max_{\mathbf{a} \in U} \|a_1 f_n^1 + a_2 f_n^2\|_p \\
&\le 2 \max_{\mathbf{a} \in U} (|a_1| + |a_2|) \max\{\|f_n^1\|_p, \|f_n^2\|_p\}.
\end{aligned}$$

Thus g_n is bounded since $\{\|f_n^1\|_p\}$ and $\|f_n^2\|_p$ are bounded by (3.2) and (i) is proved. To prove (ii), we first consider the case of $p = 1$. Since $\lim_n \|f_n^k - f^k\|_1 = 0$ by (3.2), $\{f_n^k\}$ is uniformly integrable for each $k = 1, 2$. On the other hand, for each $E \in \Sigma$, since $g_n|_E$ is best simultaneous approximation to $(f_n^1|_E, f_n^2|_E)$ from $L_p(E, \Sigma|_E, Y)$, we have that

$$\|g_n|_E\|_p \max_{\mathbf{a} \in U} |a_1 + a_2| \le 2 \max_{\mathbf{a} \in U} (|a_1| + |a_2|) \max\{\|f_n^1|_E\|_p, \|f_n^2|_E\|_p\}.$$

Thus, $\{g_n\}$ is uniformly integrable. For the case of $1 < p < \infty$, let $E \in \Sigma$ with $\mu(E) < \infty$. Then, by Hölder Inequality, we get that

$$\int_E \|g_n(s)\| d\mu \le \left(\int_E \|g_n(s)\|^p d\mu\right)^{\frac{1}{p}} \left(\int_E d\mu\right)^{\frac{1}{q}} \le M_2(\mu(E))^{\frac{1}{q}},$$

where $M_2 = \sup_{n\ge 1} \|g_n\|_p$, which implies that (ii) holds. Finally, let $E \in \Sigma_1$ with $0 < \mu(E) < \infty$. Note that $\{\int_E g_n(s)d\mu\}$ is bounded by (ii), and

$$\frac{1}{\mu(E)} \int_E g_n(s) d\mu \in \overline{\mathrm{co}(g_n(E))} \subset Y \quad \text{for each } n \in \mathbb{N}$$

thanks to [2, Corollary II.2.8]. Hence (iii) follows and the assertion holds.

Next we assert that $g_0 \in L_p(\Omega, \Sigma, Y)$, the proof of which can be completed by same technique as that given in proving [3, Theorem 3.2].

Finally, we show that g_0 is a best simultaneous approximation to f^1, f^2 from $L_p(\Omega, \Sigma, Y)$. To do this, let $\epsilon > 0$ and $k = 1, 2$. Then there exists $f_\epsilon^k \in L_p(\Omega, \Sigma_1, X)$ with countable values such that

$$\|f_\epsilon^k - (f^k - g_0)\|_p < \epsilon. \tag{3.4}$$

Let $(a_1, a_2) \in U$. Then, by (3.4), we have that

$$\begin{aligned} &\|a_1(f^1 - g_0) + a_2(f^2 - g_0)\|_p \\ &\le \|a_1(f^1 - g_0 - f_\epsilon^1) + a_2(f^2 - g_0 - f_\epsilon^1)\|_p + \|a_1 f_\epsilon^1 + a_2 f_\epsilon^2\|_p \\ &\le (|a_1| + |a_2|)\epsilon + \|a_1 f_\epsilon^1 + a_2 f_\epsilon^2\|_p. \end{aligned} \tag{3.5}$$

Since $a_1 f_\epsilon^1 + a_2 f_\epsilon^2$ is countably valued, by [3, Lemma 2.3], there is $h_\epsilon^* \in L_q(\Omega, \Sigma_1, X^*)$ such that $\|h_\epsilon^*\|_q \le 1$ and

$$\langle a_1 f_\epsilon^1 + a_2 f_\epsilon^2, h_\epsilon^* \rangle = \|a_1 f_\epsilon^1 + a_2 f_\epsilon^2\|_p. \tag{3.6}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} \|a_1 f_\epsilon^1 + a_2 f_\epsilon^2\|_p \le\ & |\langle a_1(f_\epsilon^1 - (f^1 - g_0)) + a_2(f_\epsilon^2 - (f^1 - g_0)), h_\epsilon^* \rangle| \\ & + |\langle a_1(f^1 - g_0) + a_2(f^1 - g_0), h_\epsilon^* \rangle| \\ \le\ & (|a_1| + |a_2|)\epsilon + |\langle a_1(f^1 - g_0) + a_2(f^1 - g_0), h_\epsilon^* \rangle| \end{aligned} \tag{3.7}$$

On the other hand, there exists $E \in \Sigma_1$ with $\mu(E) < +\infty$ such that $\|(f^k - g_0)\chi_{\Omega\setminus E}\|_p < \epsilon$ for each $k = 1, 2$. Thus,

$$\begin{aligned} &|\langle [a_1(f^1 - g_0) + a_2(f^1 - g_0)], h_\epsilon^* \rangle| \\ &\le |\langle [a_1(f^1 - g_0) + a_2(f^1 - g_0)]\chi_{\Omega\setminus E}, h_\epsilon^* \rangle| \\ &\quad + |\langle a_1(f^1 - g_0) + a_2(f^1 - g_0), h_\epsilon^* \chi_E \rangle| \\ &\le (|a_1| + |a_2|)\epsilon + |\langle a_1(f^1 - g_0) + a_2(f^1 - g_0), h_\epsilon^* \chi_E \rangle|. \end{aligned} \tag{3.8}$$

Let us estimate $|\langle a_1(f^1 - g_0) + a_2(f^1 - g_0), h^*_\epsilon \chi_E\rangle|$. By (3.2) and (3.3) , we have that

$$\lim_{n\to\infty} \langle f^k_n - g_n, h^*_\epsilon \chi_E\rangle = \langle f^k - g_0, h^*_\epsilon \chi_E\rangle \quad \text{for each } k = 1, 2.$$

Thus,

$$\begin{aligned} & |\langle a_1(f^1 - g_0) + a_2(f^1 - g_0), h^*_\epsilon \chi_E\rangle| \\ = & \lim_n |\langle a_1(f^1_n - g_n) + a_2(f^2_n - g_n), h^*_\epsilon \chi_E\rangle| \\ \le & \liminf_n \|\langle a_1(f^1_n - g_n) + a_2(f^2_n - g_n)\|_p \\ \le & \liminf_n \max_{\mathbf{a}\in U} \|\langle a_1(f^1_n - g_n) + a_2(f^2_n - g_n)\|_p \\ \le & \liminf_n \max_{\mathbf{a}\in U} \|\langle a_1(f^1_n - g) + a_2(f^2_n - g)\|_p \\ = & \max_{\mathbf{a}\in U} \|\langle a_1(f^1 - g) + a_2(f^2 - g)\|_p = \|F - g\|. \end{aligned}$$

This together with (3.5), (3.7) and (3.8) implies that

$$\|a_1(f^1 - g_0) + a_2(f^2 - g_0)\|_p \le 3(|a_1| + |a_2|)\epsilon + \|F - g\|.$$

Since $\epsilon > 0$ and $(a_1, a_2) \in U$ are arbitrary, we have that (3.1) holds. The proof is complete. ∎

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