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ON q-HAUSDORFF MATRICES

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Abstract. The q-Hausdorff matrices are defined in terms of symbols from q-mathematics. The matrices become ordinary Hausdorff matrices as $q \to 1$. In this paper, we consider the q-analogues of the Cesàro matrix of order one, both for $0 < q < 1$ and $q > 1$, and obtain the lower bounds for these matrices for any $1 < p < \infty$.

1. INTRODUCTION

Ordinary Hausdorff matrices were introduced by Hurwitz and Silverman [7] to be the class of lower triangular matrix, that commute with C, the Cesàro matrix of order one. Hausdorff [6] reexamined this class, in the process of solving the moment problem over a finite interval, and developed many of the properties of the matrices that now bear his name. The standard reference on Hausdorff means is the book by G. H. Hardy [5].

A Hausdorff matrix H is a lower triangular matrix with entries defined by

$$h_{nk} = \binom{n}{k} \Delta^{n-k}\mu_k \qquad 0 \le k \le n \tag{1.1}$$

where $\binom{n}{k}$ is the ordinary binomial coefficient, $\{\mu_n\}$ is a real or complex sequence, and Δ is the forward difference operator defined by $\Delta\mu_k = \mu_k - \mu_{k+1}$ and $\Delta^{n+1}\mu_k = \Delta(\Delta^n\mu_k)$.

For example, the ordinary Cesàro matrix of order one, $(C, 1)$, has entries

$$c_{nk} = \begin{cases} \dfrac{1}{n+1}, & n \ge k \\ 0, & n < k. \end{cases}$$

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Every Hausdorff matrix also has the representation

$$H = \delta\mu\delta,$$

where μ is the diagonal matrix with entries $\{\mu_n\}$, and δ is the lower triangular matrix defined by

$$\delta_{nk} = (-1)^k \binom{n}{k}.$$

It is easily verified that δ is its own inverse.

We now give a brief introduction to the symbols of q-mathematics and q-Hausdorff matrices. The subject of q-mathematics has many applications in mathematics, and the beginnings of q-mathematics date back to time of Euler. The q-analogue of the integer n, is defined by

$$[n]_q = \frac{1-q^n}{1-q} \quad (q \neq 1).$$

Then one can define the q-analogue of the factorial, the q-factorial, as

$$[n]_q! = \begin{cases} \dfrac{q-1}{q-1}\dfrac{q^2-1}{q-1}\cdots\dfrac{q^n-1}{q-1}, & n = 1, 2, ... \\ 1, & n = 0. \end{cases}$$

and then one can move on to define the q-binomial coefficients, also known Gaussian polynomials,

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q!\,[k]_q!}.$$

Note that, as $q \to 1$, the q-binomial coefficients approach the usual binomial coefficients.

For $q > 0$ (see, e.g., [3]), a q-Hausdorff matrix H_q is defined by

$$h_{nk} = q^{-k(n-k)} \binom{n}{k}_q \Delta_q^{n-k}\mu_k \quad (n, k = 0, 1, ...),$$

where again $\{\mu_k\}$ is any sequence and Δ_q is the q- forward difference operator defined by

$$\left(\Delta_q^n \mu\right)_k = q^{nk} \sum_{i=0}^{n} (-1)^i \binom{n}{i}_q q^{\binom{i}{2}} \mu_{k+i}.$$

A q-Hausdorff matrix H_q has the representation

$$H_q = \delta_q \mu \delta_q^{-1},$$

where , as before, μ is the diagonal matrix with diagonal entries $\{\mu_k\}$ and δ_q is the lower triangular matrix with entries

$$(\delta_q)_{nk} = (-1)^k q^{\binom{k}{2}} \binom{n}{k}_q$$

for $0 \leq k \leq n$. In contrast to ordinary Hausdorff matrices, δ_q is not its own inverse. For $q > 1$, the q-Cesàro matrix, C_q^1 is defined by

$$c_{nk} = \frac{q^k}{1 + q + ... + q^n} \quad 0 \leq k \leq n. \tag{1.2}$$

The corresponding q-Cesàro matrix for $0 < q < 1$ can be obtained by replacing q by $1/q$ in the above definitions. Thus, C_q^1 for $0 < q < 1$ has entries

$$c_{nk} = \frac{q^{n-k}}{1 + q + ... + q^n} \quad 0 \leq k \leq n. \tag{1.3}$$

Bustoz and Gordillo [4], have established a number of results for q-Hausdorff matrices for $0 < q < 1$.

2. A Lower Bound on the q-Cesàro Operator

Let A be a matrix with nonnegative entries, $A \in B(l_p)$ for some $1 < p$ and $\{x_n\}$ a decreasing sequence of nonnegative numbers in l_p. The lower bounds problem is to find the largest number L such that

$$\|Ax\|_p \geq L \|x\|_p .$$

For $p = 2$ and $A = (C, 1)$, the problem was solved by Lyons [8] who found that

$$L^2 = \sum_{k=0}^{\infty} \frac{1}{(1+k)^2}.$$

This result was extended to l_p spaces for $p > 1$ by Bennett [1]. In [1], Bennett established the following result, where $B(l_p)$ denotes the set of bounded linear operators on l_p.

Theorem 2.1. *Let $\{x_n\}$ be a monotone decreasing nonnegative sequence, let $A \in B(l^p)$ with nonnegative entries, and $1 < p < \infty$. Then*

$$\|Ax\|_p \geq L \|x\|_p \tag{2.4}$$

where

$$L^p := \inf_r (r+1)^{-1} \sum_{j=0}^{\infty} \left(\sum_{k=0}^{r} a_{jk} \right)^p = \inf_r f(r). \tag{2.5}$$

For $A = (C, 1)$, the minimum occurs at $f(0)$, which is the sum of the p^{th} power of the first column of $(C, 1)$.

In [9], Rhoades examined the lower bounds questions for Rhaly matrices and obtained some results. In [2], Bennett has shown that $L^p = f(0)$ for each Hausdorff matrix $H \in B(l_p)$ with non-negative entries. Rhoades and Sen ([10, 11]), determined the lower bounds for classes of Rhaly matrices, considered as bounded linear operators on l_p and proved the following Theorem 2.2 and Lemma 2.3 which we will use to make our proofs. A factorable matrix is a lower triangular matrix whose nonzero entries a_{nk} can be written in the form $a_n b_k$, where a_n depends on only n, and b_k depends only on k.

Theorem 2.2. *Let A be factorable matrix with positive entries, row sums t_n, and $\{a_n\}$ monotone decreasing. Then sufficient conditions for $f(0) = L^p$ are that*

$$\Delta y_r^p < 0, \ \ \Delta^2 y_r^p > 0, \tag{2.6}$$

$$\Delta^2 \left(\frac{1}{\Delta y_r^p} \right) \leq 0, \tag{2.7}$$

where $y_r = t_r / a_r$,

$$\lim_{r \to \infty} \frac{a_{r+1}^p \Delta y_{r+1}^p}{\Delta^2 y_r^p} \geq 0, \tag{2.8}$$

$$t_0^p + 2\Delta y_0^p \sum_{j=1}^{\infty} a_j^p \leq 0. \tag{2.9}$$

Lemma 2.3. *Suppose that $v \in C^3[0, \infty)$. If, for all $r > 0$, $p > 1$, one has*

$$\begin{aligned} (a)\ & v' > 0, \\ (b)\ & v'' > 0, \\ (c)\ & 2(v'')^2 - v'v''' > 0, \end{aligned} \tag{2.10}$$

then $\Delta^2(1/\Delta v(r)) \leq 0$.

We shall now determine the lower bounds for the q-Cesàro matrices of order one for $q > 1$ and $0 < q < 1$. First we prove that the q-Cesàro matrices of order one are bounded linear operator on l_p, for $1 < p < \infty$ by making use of the following special case of the Riesz-Thorin Theorem.

Theorem 2.4. *[12]. If A is an infinite matrix for which $A \in B(l_\infty)$ and $A \in B(l_1)$, then $A \in B(l_p)$ for every $1 < p < \infty$.*

It is easily shown that each q-Cesàro matrix of order one for $q > 1$ is a bounded linear operator from l_1 to l_1 and from l_∞ to l_∞.

$$\begin{aligned}\left\|C_q^1\right\|_1 &= \sup_k \sum_{n=k}^{\infty} \left| \frac{q^k}{1+q+...+q^n} \right| \\ &= \sup_k q^k \sum_{n=k}^{\infty} \frac{q-1}{q^{n+1}-1} \\ &= \sup_k q^k (q-1) \sum_{n=k}^{\infty} \frac{1}{q^{n+1}} \frac{q^{n+1}}{q^{n+1}-1}.\end{aligned}$$

Note that

$$\left\{ \frac{q^{n+1}}{q^{n+1}-1} \right\}$$

is a convergent monotone decreasing sequence. Therefore

$$\begin{aligned}\left\|C_q^1\right\|_1 &\leq \sup_k q^k (q-1) \left(\frac{q}{q-1} \right) \sum_{n=k}^{\infty} \frac{1}{q^{n+1}} \\ &= \sup_k q^{k+1} \sum_{n=k}^{\infty} \frac{1}{q^{n+1}} \\ &= \sup_k \sum_{j=0}^{\infty} \frac{1}{q^j} \\ &= \frac{1}{1-1/q} = \frac{q}{q-1}.\end{aligned}$$

Similarly we can show that an upper bound for $||C_q^1||_1$ $(0 < q < 1)$ is $1/(1-q)$.

In [3] it has been shown that each row sum of each q-Hausdorff matrix is μ_0. Since $\mu_0 = 1$ for C_q^1 and the entries of each C_q^1 are positive, $\left\|C_q^1\right\|_\infty = 1$, and $C_q^1 \in B(l^p)$ for $1 < p < \infty$, and $0 < q < 1$.

Theorem 2.5. *For the q-Cesàro matrix of order one* $(q > 1)$, $L^p = f(0)$.

Proof. From (1.2) it is clear that $C_q^1, (1 < q < \infty)$ is a factorable matrix. To prove our result we will use Theorem 2.2. To show that the sufficient conditions in Theorem 2.2 are satisfied by C_q^1, we shall use Lemma 2.3 with

$$t_n = a_n \sum_{k=0}^{n} b_k = 1, \quad a_n = \frac{1}{[n+1]_q}, \quad y_n = \frac{t_n}{a_n} = [n+1]_q\,.$$

Define

$$v(r) = \left(\frac{q^{r+1}-1}{q-1}\right)^p.$$

Then,

$$\begin{aligned} v'(r) &= \frac{pq^{r+1}(\ln q)(q^{r+1}-1)^{p-1}}{(q-1)^p} > 0, \\ v''(r) &= \frac{p(\ln q)^2 q^{r+1}(q^{r+1}-1)^{p-2}\left(pq^{r+1}-1\right)}{(q-1)^p} > 0, \end{aligned}$$

and

$$\begin{aligned} v'''(r) &= \frac{p(\ln q)^3(q^{r+1}-1)^{p-3}q^{r+1}[p^2q^{2(r+1)}+q^{r+1}(1-3p)+1]}{(q-1)^p} \\ 2(v'')^2 - v'v''' &= \frac{p^2(\ln q)^4 q^{2(r+1)}(q^{r+1}-1)^{2p-4}}{(q-1)^{2p}} \\ &\quad \times[2(pq^{r+1}-1)^2 - (p^2q^{2(r+1)}+q^{r+1}(1-3p)+1)] > 0. \end{aligned}$$

Hence by using Lemma 2.3, conditions (2.6) and (2.7) in Theorem 2.2 are satisfied. Also

$$\lim_{r\to\infty}\left(\frac{1-q}{1-q^{r+2}}\right)^p \times \left(\frac{\left(1-q^{-(r+2)}\right)^p - \left(q-q^{-(r+2)}\right)^p}{\left(q^{-1}-q^{-(r+2)}\right)^p - 2\left(1-q^{-(r+2)}\right)^p + (q-q^{-(r+2)})^p}\right) = 0.$$

Hence condition (2.8) in Theorem 2.2 is satisfied. Finally,

$$\begin{aligned} t_0^p + 2\Delta y_0^p \sum_{j=1}^{\infty} a_j^p &= 1 + 2(1-(q+1)^p)\sum_{j=1}^{\infty}\frac{(q-1)^p}{(q^{j+1}-1)^p} \\ &= 1 + \frac{2(1-(q+1)^p)(q-1)^p}{(q^2-1)^p} + 2(1-(q+1)^p)\sum_{j=2}^{\infty}\frac{(q-1)^p}{(q^{j+1}-1)^p}. \end{aligned}$$

Since $q > 1$, the sum is less than

$$1 + \frac{2(1-(q+1)^p)}{(q+1)^p} = 1 + \frac{2}{(q+1)^p} - 2 = \frac{2}{(q+1)^p} - 1 < 0,$$

so that

$$t_0^p + 2\Delta y_0^2 \sum_{j=1}^{\infty} a_j^2 < 0.$$

and condition (2.9) in Theorem 2.2 is satisfied. ■

The following theorem describes the lower bound condition for C_q^1 $(0 < q < 1)$.

Theorem 2.6. *For the q-Cesàro matrix of order one* $(0 < q < 1)$, $L^p = f(0)$.

Proof. From (1.3) it is clear that C_q^1 $(0<q<1)$ is factorable matrix. Again we use Lemma 2.3 to prove that the conditions in Theorem 2.2 are satisfied by C_q^1. Using Lemma 2.3 with

$$t_n = a_n \sum_{k=0}^{n} b_k = 1, \quad a_n = \frac{q^n}{[n+1]_q}, \quad y_n = \frac{t_n}{a_n} = \frac{[n+1]_q}{q^n},$$

define

$$v(r) = \frac{(q^{-r}-q)^p}{(1-q)^p},$$

then

$$\begin{aligned} v'(r) &= \frac{p(\ln(1/q))(1-q^{r+1})^{p-1}}{q^{pr}(1-q)^p} > 0, \\ v''(r) &= \frac{p(\ln(1/q))^2(1-q^{r+1})^{p-2}}{q^{pr}(1-q)^p}[p-q^{r+1}] > 0. \end{aligned}$$

and

$$v'''(r) = \frac{p(\ln(1/q))^3(1-q^{r+1})^{p-3}}{q^{pr}(1-q)^p}[q^{2(r+1)}-(3p-1)q^{r+1}+p^2].$$

Thus

$$2(v'')^2 - v'v''' = \frac{p^2(\ln(1/q))^4(1-q^{r+1})^{2p-4}}{q^{2pr}(1-q)^{2p}}[q^{2(r+1)}-(p+1)q^{r+1}+p^2] > 0,$$

From Lemma 2.3, conditions (2.6) and (2.7) in Theorem 2.2 are satisfied, and

$$\begin{aligned} \lim_{r\to\infty} \frac{a_{r+1}^p \Delta y_{r+1}^p}{\Delta^2 y_r^p} &= \lim_{r\to\infty} q^{p(r+1)} \left(\frac{1-q}{1-q^{r+2}}\right)^p \\ &\times \left[\frac{q^p(1-q^{r+2})^p-(1-q^{r+3})^p}{q^{2p}(1-q^{r+1})^p-2q^p(1-q^{r+2})^p+(1-q^{r+3})^p}\right] = 0. \end{aligned}$$

and the condition (2.8) in Theorem 2.2 is satisfied. Since $0<q<1$,

$$\begin{aligned} t_0^p + 2\Delta y_0^p \sum_{j=1}^{\infty} a_j^p &= 1+2\left(1-\left(\frac{1+q}{q}\right)^p\right) \times \left[q^p\left(\frac{1}{q+1}\right)^p + \sum_{j=2}^{\infty}\left(\frac{q^j(1-q)}{1-q^{j+1}}\right)^p\right] \\ &< 1+2\left(\left(\frac{q}{q+1}\right)^p - 1\right) \\ &= 2\left(\frac{q}{q+1}\right)^p - 1 < 0 \end{aligned}$$

Hence condition (2.9) in Theorem 2.2 is satisfied. ∎

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