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WEAK AND STRONG CONVERGENCE THEOREMS FOR POSITIVELY HOMOGENEOUS NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. Our purpose in this paper is first to prove a weak convergence theorem by Mann's iteration for positively homogeneous nonexpansive mappings in a Banach space. Further, using the shrinking projection method defined by Takahashi, Takeuchi and Kubota, we prove a strong convergence theorem for such mappings. From two results, we obtain weak and strong convergence theorems for linear contractive mappings in a Banach space. These results are new even if the mappings are linear and contractive.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a closed convex subset of H. Let T be a mapping of C into itself. Then we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. From [46] we know a weak convergence theorem by Mann's iteration for nonexpansive mappings in a Hilbert space:

Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and define a sequence $\{x_n\}$ in C by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ such that

$$\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty.$$

Then, $\{x_n\}$ converges weakly to an element z of $F(T)$, where $z = \lim_{n \to \infty} P x_n$ and P is the metric projection of H onto $F(T)$. By Reich [36], such a theorem

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was extended to a uniformly convex Banach space with a Fréchet differentiable norm. However, we do not know whether the fixed point z is characteraized under any projections in a Banach space. On the other hand, Nakajo and Takahashi [33] proved a strong convergence theorem for nonexpansive mappings in a Hilbert space by using the hybrid method in mathematical programming:
Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ such that $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C by $x_1 = x \in C$ and

$$\begin{cases} u_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|u_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$. Then, $\{x_n\}$ converges strongly to an element z of $F(T)$, where $z = P_{F(T)}x$ and $P_{F(T)}$ is the metric projection of H onto $F(T)$. However, we do not know whether such a strong convergence theorem for nonexpansive mappings is extended to a Banach space. Many authors have extended this convergence theorem to a Banach space by using nonlinear mappings which are different from a nonexpansive mapping; see, for instance, [28].

Our purpose in this paper is first to prove a weak convergence theorem by Mann's iteration for positively homogeneous nonexpansive mappings in a Banach space. In the theorem, the limit of weak convergence is characteraized by using a sunny generalized nonexpansive retraction in Ibaraki and Takahashi [14]. Further, using the shrinking projection method defined by Takahashi, Takeuchi and Kubota, we prove a strong convergence theorem for positively homogeneous nonexpansive mappings in a Banach space. From two results, we obtain weak and strong convergence theorems for linear contractive mappings in a Banach space. These results are new even if the mappings are linear and contractive.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to C$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a closed convex subset of E and $T : C \to C$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [17]. From [44] we know the following lemma.

Lemma 2.1. Let E be a uniformly convex Banach space and let δ be the modulus of convexity in E. Let $0 < \epsilon \leq 2r$. Then, $\delta(\frac{\epsilon}{r}) > 0$ and

$$\|\alpha x + (1-\alpha)y\| \leq r\{1 - 2\alpha(1-\alpha)\delta(\frac{\epsilon}{r})\}$$

for all $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x - y\| \geq \epsilon$ and $\alpha \in [0, 1]$.

Further, we know the following result by Browder; see [44].

Lemma 2.2. Let E be a uniformly convex Banach space and let C be a bounded closed convex subset of E. Let $T : C \to C$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$, then u is a fixed point of T.

Let C be a nonempty closed convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C(x)$, we call P_C the metric projection of E onto C. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm to weak* continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm to norm continuous. If E is uniformly smooth, J is uniformly norm to norm continuous on each bounded subset of E. For more details, see [44]. We know the following results; see [44].

Theorem 2.3. Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty closed convex subset of E and let P_C be the metric projection of E onto C. Let $x_0 \in C$ and $x_1 \in E$. Then, $x_0 = P_C(x_1)$ if and only if

$$\langle x_0 - y, J(x_1 - x_0)\rangle \geq 0$$

for all $y \in C$, where J is the duality mapping of E.

Theorem 2.4. Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.

Let E be a reflexive, strictly convex and smooth Banach space. The function ϕ: $E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$, where J is the duality mapping of E; see [1] and [21]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.2}$$

for all $x, y, z \in E$. From $(\|x\|^2 - \|y\|^2) \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Let ϕ_*: $E^* \times E^* \to (-\infty, \infty)$ be the function defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2$$

for $x^*, y^* \in E^*$, where J is the duality mapping of E. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx) \tag{2.3}$$

for $x, y \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{2.4}$$

If C is a nonempty closed convex subset of a smooth, strictly and reflexive Banach space E, then for all $x \in E$ there exists a unique $z \in C$ (denoted by $\Pi_C x$) such that

$$\phi(z, x) = \min_{y \in C} \phi(y, x). \tag{2.5}$$

The mapping Π_C is called the generalized projection from E onto C; see Alber [1], Alber and Reich [2], and Kamimura and Takahashi [21]. The following lemmas are well known; see, for instance, [21].

Lemma 2.5. Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.6. Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then, there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq 0\}$.

For a sequence $\{C_n\}$ of nonempty closed convex subsets of a reflexive Banach space E, define s-Li$_n C_n$ and w-Ls$_n C_n$ as follows: $x \in$s-Li$_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and that $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in$w-Ls$_n C_n$ if and only if there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and that $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies that

$$C_0 = \text{s-Li}_n C_n = \text{w-Ls}_n C_n, \tag{2.6}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [30] and we write $C_0 =$M-$\lim_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\cap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [30]. We know the following theorem [12].

Theorem 2.7. Let E be a smooth Banach space and let E^* have a Fréchet differentiable norm. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of E. If $C_0 =$M-$\lim_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\Pi_{C_n} x$ converges strongly to $\Pi_{C_0} x$, where Π_{C_n} and Π_{C_0} are the generalized projections of E onto C_n and C_0, respectively.

Let E be a Banach space and let D be a nonempty closed subset of E. A mapping $R : E \to D$ is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx, \quad \forall x \in E, \ \forall t \geq 0.$$

A mapping $R : E \to D$ is a retraction if $Rx = x$ for all $x \in D$. A nonempty subset of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) of E onto D. From [14], we know the following lemmas.

Lemma 2.8. (Ibaraki and Takahashi [14]). Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty closed subset of E. Then, a sunny generalized nonexpansive retraction of E onto D is uniquely determined.

Lemma 2.9. (Ibaraki and Takahashi [14]). Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty closed subset of E. Suppose that there exists a sunny generalized nonexpansive retraction R of E onto D and let $(x, z) \in E \times D$. Then, the following hold:

(1) $z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0, \ \forall y \in D$;

(2) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

In 2007, Kohsaka and Takahashi [24] proved the following results.

Lemma 2.10. (Kohsaka and Takahashi [24]). Let E be a smooth, strictly convex and reflexive Banach space and let C_* be a nonempty closed convex subset of E^*. Suppose that Π_{C_*} is the generalized projection of E^* onto C_*. Then, R defined by $R = J^{-1}\Pi_{C_*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C_*$.

Lemma 2.11. (Kohsaka and Takahashi [24]). Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty subset of E. Then, the following conditions are equivalent

(1) D is a sunny generalized nonexpansive retract of E;

(2) D is a generalized nonexpansive retract of E;

(3) JD is closed and convex.

In this case, D is closed.

Lemma 2.12. (Kohsaka and Takahashi [24]). Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty closed subset of E. Suppose that there exists a sunny generalized nonexpansive retraction R of E onto D and let $(x, z) \in E \times D$. Then, the following conditions are equivalent

(1) $z = Rx$;
(2) $\phi(x, z) = \min_{y \in D} \phi(x, y)$.

Let E be a smooth Banach space E and let C be a nonempty subset of E. A mapping $T : C \to C$ is generalized nonexpansive [14] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y) \tag{2.7}$$

for all $x \in C$ and $y \in F(T)$. From Ibaraki and Takahashi [15] we know the following lemma.

Lemma 2.13. (Ibaraki and Takahashi [15]). Let E be a smooth, strictly convex and reflexive Banach space and let T be a generalized nonexpansive mapping of E into itself. Then, $F(T)$ is a sunny generalized nonexpansive retract of E.

3. Positively Homogeneous Nonexpansive Mappings

Let E be a Banach space and let C be a closed convex cone of E. Then, a mapping $T : C \to C$ is called positively homogeneous if $T(\alpha x) = \alpha T(x)$ for all $x \in C$ and $\alpha > 0$. In this section, we prove that a nonexpansive mapping $T : C \to C$ under an appropriate condition is generalized nonexpansive. Before proving it, we prove the following lemma.

Lemma 3.1. Let E be a Banach space and let C be a closed convex cone of E. Let $T : C \to C$ be a positively homogeneous nonexpansive mapping. Then, for any $x \in C$ and $m \in F(T)$, there exists $j \in Jm$ such that

$$\langle x - Tx, j \rangle \leq 0,$$

where J is the duality mapping of E into E^*.

Proof. Since $0 \in C$ and $\frac{1}{2}T0 = T(\frac{1}{2}0) = T0$, we have $T0 = 0$. So, $F(T) \neq \emptyset$. First, let $x \in C \setminus F(T)$ and $m \in F(T)$. Suppose $m \neq 0$. We have that for $k > 0$,

$$T(km) = kT(m) = km.$$

So, we have that $\frac{1}{k}x - m \neq 0$ for all $k > 0$. We have from the Hahn-Banach theorem that there exists $y_k^* \in E^*$ such that $\langle \frac{1}{k}x - m, y_k^* \rangle = \|\frac{1}{k}x - m\|$ and $\|y_k^*\| = 1$. Then, we have that

$$\begin{aligned}
\langle \frac{1}{k}Tx - m, y_k^* \rangle &\leq \|\frac{1}{k}Tx - m\| = \frac{1}{k}\|Tx - km\| \\
&\leq \frac{1}{k}\|x - km\| = \|\frac{1}{k}x - m\| \\
&= \langle \frac{1}{k}x - m, y_k^* \rangle.
\end{aligned}$$

So, we have $\frac{1}{k}\langle x - Tx, y_k^* \rangle \geq 0$ and hence

$$\langle x - Tx, y_k^* \rangle \geq 0. \tag{3.1}$$

Take a net $\{k > 0\}$ with $k \to \infty$ and put $x_k = \frac{1}{k}x - m$. Then, we have $x_k \to -m$. Further, since $\{y_k^*\}$ is bounded, there exists a subnet $\{y_{k_\alpha}^*\}$ of $\{y_k^*\}$ converging to some $y^* \in E^*$ in the weak* topology. Let us show that $y^* \in E^*$ satisfies $\langle m, -y^* \rangle = \|m\|$ and $\|y^*\| = 1$. Since the norm of E^* is lower semicontinuous in the weak* topology, we have

$$\|y^*\| \leq \liminf_{\alpha \to \infty} \|y_{k_\alpha}^*\| = 1.$$

On the other hand, we have that

$$\begin{aligned} |\langle -m, y^* \rangle - \|x_{k_\alpha}\|| &= |\langle -m, y^* \rangle - \langle x_{k_\alpha}, y_{k_\alpha}^* \rangle| \\ &\leq |\langle -m, y^* - y_{k_\alpha}^* \rangle| + |\langle -m - x_{k_\alpha}, y_{k_\alpha}^* \rangle|. \end{aligned}$$

Since $\langle -m, y^* - y_{k_\alpha}^* \rangle \to 0$ and $\langle -m - x_{k_\alpha}, y_{k_\alpha}^* \rangle \to 0$, we have

$$\|x_{k_\alpha}\| \to -\langle m, y^* \rangle = \langle m, -y^* \rangle.$$

Since $\|x_{k_\alpha}\| \to \|m\|$, we have $\langle m, -y^* \rangle = \|m\|$. So, we have

$$\|m\| = \langle m, -y^* \rangle \leq \|m\| \|y^*\|.$$

From $m \neq 0$, we have $\|y^*\| \geq 1$. Therefore, we have $\|y^*\| = 1$ and $\langle m, -y^* \rangle = \|m\|$. We also have from (3.1) that

$$\langle x - Tx, y^* \rangle \geq 0.$$

Putting $z^* = -y^*$, we have $\|z^*\| = 1$, $\langle m, z^* \rangle = \|m\|$ and

$$\langle x - Tx, z^* \rangle \leq 0. \tag{3.2}$$

So, we have

$$\| \|m\| z^* \|^2 = \|m\|^2 = \|m\| \langle m, z^* \rangle = \langle m, \|m\| z^* \rangle.$$

This implies $\|m\| z^* \in Jm$, where J is the duality mapping of E. From (3.2), we have $\|m\| \langle x - Tx, z^* \rangle \leq 0$ and hence

$$\langle x - Tx, j \rangle \leq 0, \tag{3.3}$$

where $j = \|m\| z^* \in Jm$. In the case of $m = 0$, we have $\{0\} = Jm$. So, we have

$$\langle x - Tx, 0 \rangle = 0. \tag{3.4}$$

From (3.3) and (3.4), we have that for any $x \in C \setminus F(T)$ and $m \in F(T)$, there exists $j \in Jm$ such that

$$\langle x - Tx, j \rangle \leq 0.$$

In the case of $x \in F(T)$, we also have

$$\langle x - Tx, j \rangle = \langle 0, j \rangle = 0,$$

where $j \in Jm$. Therefore, we have that for any $x \in C$ and $m \in F(T)$, there exists $j \in Jm$ such that

$$\langle x - Tx, j \rangle \leq 0. \qquad \blacksquare$$

Using Lemma 3.1, we obtain the following theorem.

Theorem 3.2. Let E be a smooth Banach space and let C be a closed convex cone of E. Let $T : C \to C$ be a positively homogeneous nonexpansive mapping. Then, T is a generalized nonexpansive mapping.

Proof. Since $0 \in F(T)$, we have that for any $x \in C$,

$$\|Tx\| = \|Tx - 0\| \leq \|x - 0\| = \|x\|.$$

So, we have from Lemma 3.1 that for any $x \in C$ and $m \in F(T)$,

$$\begin{aligned} \phi(Tx, m) &= \|Tx\|^2 - 2\langle Tx, Jm \rangle + \|m\|^2 \\ &\leq \|x\|^2 - 2\langle x, Jm \rangle + \|m\|^2 = \phi(x, m). \end{aligned}$$

Therefore, T is a generalized nonexpansive mapping of C into itself. $\blacksquare$

4. Weak Convergence Theorems

In this section, we prove a weak convergence theorem of Mann's iteration for positively homogeneous nonexpansive mappings in a Banach space. Before proving it, we obtain the following lemma.

Lemma 4.1. Let E be a smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $T : C \to C$ be a generalized nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$ and

$$x_{n+1} = R_C(\alpha_n x_n + (1 - \alpha_n)Tx_n), \quad \forall n \in \mathbb{N},$$

where R_C is a sunny generalized nonexpansive retraction of E onto C. If $R_{F(T)}$ is a sunny generalized nonexpansive retraction of C onto $F(T)$, then $\{R_{F(T)}x_n\}$ converges strongly to an element z of $F(T)$.

Proof. Let $m \in F(T)$. Since R_C and T are generalized nonexpansive,

$$\begin{aligned}\phi(x_{n+1}, m) &= \phi(R_C(\alpha_n x_n + (1-\alpha_n)Tx_n), m) \\ &\leq \phi(\alpha_n x_n + (1-\alpha_n)Tx_n, m) \\ &\leq \alpha_n \phi(x_n, m) + (1-\alpha_n)\phi(Tx_n, m) \\ &\leq \alpha_n \phi(x_n, m) + (1-\alpha_n)\phi(x_n, m) \\ &= \phi(x_n, m).\end{aligned}$$

So, $\lim_{n\to\infty} \phi(x_n, m)$ exists. Since $\{\phi(x_n, m)\}$ is bounded, $\{x_n\}$ and $\{Tx_n\}$ are bounded. Define $y_n = R_{F(T)}x_n$ for all $n \in \mathbb{N}$. Since $\phi(x_{n+1}, m) \leq \phi(x_n, m)$ for all $m \in F(T)$, from $y_n \in F(T)$ we have

$$\phi(x_{n+1}, y_n) \leq \phi(x_n, y_n). \tag{4.1}$$

From Lemma 2.9 and (4.1), we have

$$\begin{aligned}\phi(x_{n+1}, y_{n+1}) &= \phi(x_{n+1}, R_{F(T)}x_{n+1}) \\ &\leq \phi(x_{n+1}, y_n) - \phi(R_{F(T)}x_{n+1}, y_n) \\ &= \phi(x_{n+1}, y_n) - \phi(y_{n+1}, y_n) \\ &\leq \phi(x_{n+1}, y_n) \\ &\leq \phi(x_n, y_n).\end{aligned}$$

So, $\phi(x_n, y_n)$ is a convergent sequence. We also have from (4.1) that for all $m \in \mathbb{N}$,

$$\phi(x_{n+m}, y_n) \leq \phi(x_n, y_n).$$

From $y_{n+m} = R_{F(T)}x_{n+m}$ and Lemma 2.9, we have

$$\phi(y_{n+m}, y_n) + \phi(x_{n+m}, y_{n+m}) \leq \phi(x_{n+m}, y_n) \leq \phi(x_n, y_n)$$

and hence

$$\phi(y_{n+m}, y_n) \leq \phi(x_n, y_n) - \phi(x_{n+m}, y_{n+m}).$$

Using Lemma 2.6, we have that

$$g(\|y_{n+m} - y_n\|) \leq \phi(y_{n+m}, y_n) \leq \phi(x_n, y_n) - \phi(x_{n+m}, y_{n+m}),$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing and convex function such that $g(0) = 0$. Then, the properties of g yield that $R_{F(T)}x_n$ converges strongly to an element z of $F(T)$. ∎

Using Lemma 4.1, we prove the following theorem.

Theorem 4.2. Let E be a smooth and uniformly convex Banach space and let C be a closed convex cone of E such that JC is closed and convex. Let $T : C \to C$ be a positively homogeneous nonexpansive mapping. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(T)$. Further, if E has a Fréchet differentiable norm, then $z = \lim_{n\to\infty} Rx_n$, where R is a sunny generalized nonexpansive retraction of C onto $F(T)$.

Proof. Let $m \in F(T)$. Then, we have

$$\begin{aligned}
\|x_{n+1} - m\| &= \|\alpha_n x_n + (1-\alpha_n)Tx_n - m\| \\
&\leq \alpha_n\|x_n - m\| + (1-\alpha_n)\|Tx_n - m\| \\
&\leq \alpha_n\|x_n - m\| + (1-\alpha_n)\|x_n - m\| \\
&= \|x_n - m\|.
\end{aligned}$$

So, $\lim_{n\to\infty}\|x_n - m\|$ exists. Putting $\lim_{n\to\infty}\|x_n - m\| = c$, without loss of generality, we can assume $c \neq 0$. Using Lemma 2.1, we have that

$$\begin{aligned}
\|x_{n+1} - m\| &= \|\alpha_n x_n + (1-\alpha_n)Tx_n - m\| \\
&\leq \|\alpha_n(x_n - m) + (1-\alpha_n)(Tx_n - m)\| \\
&\leq \|x_n - m\|\{1 - 2\alpha_n(1-\alpha_n)\delta(\frac{\|Tx_n - x_n\|}{\|x_n - m\|})\}.
\end{aligned}$$

Then, we obtain

$$2c\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)\delta(\frac{\|Tx_n - x_n\|}{\|x_n - m\|}) \leq \|x_1 - m\| - c < \infty.$$

From the assumptions of $\{\alpha_n\}$, we have

$$\liminf_{n\to\infty}\delta(\frac{\|Tx_n - x_n\|}{\|x_n - m\|}) = 0.$$

Then, we have

$$\liminf_{n\to\infty}\|Tx_n - x_n\| = 0.$$

On the other hand, we have

$$
\begin{aligned}
&\|Tx_{n+1} - x_{n+1}\| \\
&= \alpha_n \|Tx_{n+1} - x_n\| + (1-\alpha_n)\|Tx_{n+1} - Tx_n\| \\
&\le \alpha_n(\|Tx_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|) + (1-\alpha_n)\|Tx_{n+1} - Tx_n\| \\
&\le \alpha_n \|Tx_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&= \alpha_n \|Tx_{n+1} - x_{n+1}\| + (1-\alpha_n)\|Tx_n - x_n\|.
\end{aligned}
$$

Then, we have $\|Tx_{n+1} - x_{n+1}\| \le \|Tx_n - x_n\|$. So, we obtain that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = \liminf_{n\to\infty} \|Tx_n - x_n\| = 0.$$

Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$ for some $v \in C$. Since E is uniformly convex and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, we have from Lemma 2.2 that v is a fixed point of T. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$. We know that $u, v \in F(T)$. We know from Theorem 3.2 that T is a generalized nonexpansive mapping of C into itself. Then, we have from the convexity of $\|\cdot\|^2$ that for any $m \in F(T)$,

$$
\begin{aligned}
\phi(x_{n+1}, m) &= \phi(\alpha_n x_n + (1-\alpha_n)Tx_n, m) \\
&\le \alpha_n \phi(x_n, m) + (1-\alpha_n)\phi(Tx_n, m) \\
&\le \alpha_n \phi(x_n, m) + (1-\alpha_n)\phi(x_n, m) \\
&= \phi(x_n, m)
\end{aligned}
$$

for all $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} \phi(x_n, m)$ exists. Put

$$a = \lim_{n\to\infty} (\phi(x_n, u) - \phi(x_n, v)).$$

Since $\phi(x_n, u) - \phi(x_n, v) = 2\langle x_n, Jv - Ju\rangle + \|u\|^2 - \|v\|^2$, we have

$$a = 2\langle u, Jv - Ju\rangle + \|u\|^2 - \|v\|^2$$

and

$$a = 2\langle v, Jv - Ju\rangle + \|u\|^2 - \|v\|^2.$$

From these equalities, we obtain

$$\langle u - v, Ju - Jv\rangle = 0.$$

Since J is strictly monotone, it follows that $u = v$; see [44]. Therefore, $\{x_n\}$ converges weakly to an element u of $F(T)$. On the other hand, we know from Lemma 4.1 that $\{R_{F(T)}x_n\}$ converges strongly to an element z of $F(T)$. From Lemma 2.9, we also have

$$\langle x_n - R_{F(T)}x_n, JR_{F(T)}x_n - Ju\rangle \geq 0.$$

Since E has a Fréchet differentiable norm, the duality mapping J is norm-to-norm continuous. So, we have $\langle u - z, Jz - Ju\rangle \geq 0$. Since J is monotone, we also have $\langle u - z, Jz - Ju\rangle \leq 0$. So, we have $\langle u - z, Jz - Ju\rangle = 0$. Since E is strictly convex, we have $z = u$. This completes the proof. ∎

5. Strong Convergence Theorems

In this section, we prove a strong convergence theorem by a hybrid method called the shrinking projection method for positively homogeneous nonexpansive mappings in a Banach space.

Theorem 5.1. Let E be a uniformly convex Banach space which has a Fréchet differentiable norm. Let $T : E \to E$ be a positively homogeneous nonexpansive mapping. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq a < 1$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$, $C_1 = E$ and

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_{n+1}}$ is the sunny generalized nonexpansive retraction of E onto C_{n+1}. Then, $\{x_n\}$ converges strongly to $z = R_{F(T)}x$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction of E onto $F(T)$.

Proof. We know that T is a generalized nonexpansive mapping of E into itself. So, we have from Lemma 2.13 that $F(T)$ is a sunny generalized nonexpansive retract of E. We shall show that JC_n are closed and convex, and $F(T) \subset C_n$ for all $n \in \mathbb{N}$. It is obvious from the assumption that $JC_1 = JE = E^*$ is closed and convex, and $F(T) \subset C_1$. Suppose that JC_k is closed and convex, and $F(T) \subset C_k$ for some $k \in \mathbb{N}$. From the definition of ϕ, we know that for $z \in C_k$,

$$\begin{aligned} &\phi(u_k, z) \leq \phi(x_k, z) \\ \Longleftrightarrow &\|u_k\|^2 - \|x_k\|^2 - 2\langle u_k - x_k, Jz\rangle \leq 0. \end{aligned}$$

So, JC_{k+1} is closed and convex. If $z \in F(T) \subset C_k$, then we have

$$\begin{aligned}\phi(u_n, z) &= \phi(\alpha_n x_n + (1-\alpha_n)Tx_n, z)\\ &\leq \alpha_n \phi(x_n, z) + (1-\alpha_n)\phi(Tx_n, z)\\ &\leq \alpha_n \phi(x_n, z) + (1-\alpha_n)\phi(x_n, z)\\ &= \phi(x_n, z).\end{aligned}$$

Hence, we have $z \in C_{k+1}$. By induction, we have that JC_n are closed and convex, and $F(T) \subset C_n$ for all $n \in \mathbb{N}$. Since JC_n is closed and convex, from Lemma 2.8 there exists a unique sunny generalized nonexpansive retraction R_{C_n} of E onto C_n. We also know from Lemma 2.10 that such R_{C_n} is denoted by $J^{-1}\Pi_{JC_n}J$, where J is the duality mapping of E and Π_{JC_n} is the generalized projection of E onto JC_n. Thus, $\{x_n\}$ is well-defined.

Since $\{JC_n\}$ is a nonincreasing sequence of nonempty closed convex subsets of E^* with respect to inclusion, it follows that

$$\emptyset \neq JF(T) \subset \text{M-}\lim_{n\to\infty} JC_n = \cap_{n=1}^{\infty} JC_n. \tag{5.1}$$

Put $C_0^* = \cap_{n=1}^{\infty} JC_n$. Then, by Theorem 2.7 we have that $\{\Pi_{JC_{n+1}}Jx\}$ converges strongly to $x_0^* = \Pi_{C_0^*}Jx$. Since E^* has a Fréchet differencial norm, J^{-1} is continuous. So, we have

$$x_{n+1} = R_{n+1}x = J^{-1}\Pi_{JC_{n+1}}Jx \to J^{-1}x_0^*.$$

To complete the proof, it is sufficient to show that $J^{-1}x_0^* = R_{F(T)}x$.

Since $x_n = R_{C_n}x$ and $x_{n+1} = R_{C_{n+1}}x \in C_{n+1} \subset C_n$, we have from Lemma 2.9 and (2.2) that

$$\begin{aligned}0 &\leq 2\langle x - x_n, Jx_n - Jx_{n+1}\rangle\\ &= \phi(x, x_{n+1}) - \phi(x, x_n) - \phi(x_n, x_{n+1})\\ &\leq \phi(x, x_{n+1}) - \phi(x, x_n).\end{aligned}$$

So, we get that

$$\phi(x, x_n) \leq \phi(x, x_{n+1}). \tag{5.2}$$

Further, since $x_n = R_{C_n}x$ and $z \in F(T) \subset C_n$, from Lemma 2.12 we have

$$\phi(x, x_n) \leq \phi(x, z). \tag{5.3}$$

So, we have that $\lim_{n\to\infty}\phi(x,x_n)$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{u_n\}$ and $\{Tx_n\}$ are also bounded. From

$$\begin{aligned}\phi(x_n,x_{n+1}) &= \phi(R_{C_n}x,x_{n+1})\\ &= \phi(x,x_{n+1})-\phi(x,R_{C_n}x)\\ &= \phi(x,x_{n+1})-\phi(x,x_n)\to 0,\end{aligned}$$

we have that

$$\phi(x_n,x_{n+1})\to 0. \tag{5.4}$$

From $x_{n+1}\in C_{n+1}$, we have that $\phi(u_n,x_{n+1})\le\phi(x_n,x_{n+1})$. So, we get that $\phi(u_n,x_{n+1})\to 0$. Using Lemma 2.5, we have

$$\lim_{n\to\infty}\|u_n-x_{n+1}\|=\lim_{n\to\infty}\|x_n-x_{n+1}\|=0.$$

So, we have

$$\|u_n-x_n\|\le\|u_n-x_{n+1}\|+\|x_{n+1}-x_n\|\to 0. \tag{5.5}$$

Since $\|x_n-u_n\|=\|x_n-\alpha_n x_n-(1-\alpha_n)Tx_n\|=(1-\alpha_n)\|x_n-Tx_n\|$ and $0\le\alpha_n\le a<1$, we have that

$$\|Tx_n-x_n\|\to 0. \tag{5.6}$$

Since $x_{n+1}\to J^{-1}x_0^*$ and T is continuous, we have $J^{-1}x_0^*\in F(T)$.

Put $z_0=R_{F(T)}x$. Since $z_0=R_{F(T)}x\subset C_{n+1}$ and $x_{n+1}=R_{C_{n+1}}x$, we have that

$$\phi(x,x_{n+1})\le\phi(x,z_0). \tag{5.7}$$

So, we have that

$$\begin{aligned}\phi(x,J^{-1}x_0^*) &= \|x\|^2-2\langle x,x_0^*\rangle+\|J^{-1}x_0^*\|^2\\ &= \lim_{n\to\infty}(\|x\|^2-2\langle x,Jx_n\rangle+\|x_n\|^2)\\ &= \lim_{n\to\infty}\phi(x,x_n)\\ &\le \phi(x,z_0).\end{aligned}$$

So, we get $z_0=J^{-1}x_0^*$. Hence, $\{x_n\}$ converges strongly to z_0. This completes the proof. ∎

Using Theorem 5.1, we prove a strong convergence theorem for linear contractive mappings in a Banach space.

Theorem 5.2. Let E be a uniformly convex Banach space which has a Fréchet differentiable norm. Let $T : E \to E$ be a linear contractive mapping. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq a < 1$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$, $C_1 = E$ and

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_{n+1}}$ is the sunny generalized nonexpansive retraction of E onto C_{n+1}. Then, $\{x_n\}$ converges strongly to $z = R_{F(T)}x$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction of E onto $F(T)$.

Proof. A linear contractive mapping $T : E \to E$ is positively homogeneous and nonexpansive. So, using Theorem 5.1, we obtain the desired result. ∎

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