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A FUNCTIONAL APPROACH TO PROVE COMPLEMENTARITY

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Abstract. Some complementarity among a firm's activities is an important source of its profits. In this paper, we focus on the way to prove complementarity. Though there are many studies on complementarity as supermodularity or the increasing differences of a function, we introduce the notion of self increasing differences with respect to a single activity, which is an essence of convexity from the viewpoint of complementarity, and investigate some interrelations among these three notions of complementarity. Mathematically, we give a sufficient condition for a composite function to have self increasing differences. This proposition is deeply related to Topkis' (1998) Lemma 2.6.4 on sufficient conditions for a composite function to be supermodular. Both propositions are combined and applied to yield and/or strengthen complementarity in an organization, which will also disclose the functional structure of an organization's activities.

1. Introduction

Recently, many authors have developed theories on monotone comparative statics, supermodular games, and so on, that derived distinctive features from supermodularity, a mathematical representation of the Edgeworth = Pareto complementarity defined in Edgeworth [4] in 1925. For example, Topkis [13] considered a firm's profit function:

$$\Pi(x,t) = p\mu(x,t) - c(\mu(x,t), x, t) - k(x,t),$$

where a vector x of decision variables is an element of a constraint set X; a vector t of exogenous parameters is an element of a constraint set T; $\mu(x,t)$ is

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the product demand with all demands satisfied by the current production; p is the constant market price of the product; $c(\mu(x,t),x,t)$ is the production cost, which may depend on the level of production; and $k(x,t)$ is the other cost, which may not depend on the level of production. Topkis [13] assumed in Theorem 3.1 that $\mu(x,t)$ is increasing and supermodular in (x,t), $pz - c(z,x,t)$ is increasing in z for each (x,t), $c(z,x,t)$ is concave in z for each (x,t), $c(z,x,t)$ is submodular in (z,x,t), and $k(x,t)$ is submodular in (x,t). Then, at first, he enduced that (a) $\Pi(x,t)$ is supermodular in (x,t). Fully depending on this result, he also proved that (b) $\pi(t) = \max_{x\in X}\Pi(x,t)$ is supermodular in t, and (c) $\mathrm{argmax}_{x\in X}\Pi(x,t)$ is increasing in (t,X). The supermodularity of $\Pi(x,t)$ comes from [13, Lemma 3.1] with the assumptions for all of the functions that compose $\Pi(x,t)$. This lemma provides a sufficient condition for a composite function to become supermodular, which is generalized in Topkis [14] as follows.

Topkis' Lemma 2.6.4

If X is a lattice, $f_i(x)$ is increasing and supermodular on X for $i = 1,\dots,k$, Z_i is a convex subset of R^1 containing the range of $f_i(x)$ on X for $i = 1,\dots,k$, and $g(z_1,\dots,z_k,x)$ is supermodular in $(z_1,\dots,z_k,x)$ on $(\times_{i=1}^{k} Z_i)\times X$ and increasing and convex in z_i on Z_i for $i = 1,\dots,k$ and for all z'_i in Z'_i for $i' \in \{1,\dots,n\}\setminus\{i\}$ and all x in X, then $g(f_1(x),\dots,f_k(x),x)$ is supermodular in x on X.

On the other hand, though the definition of complementarity usually means a relationship in which two or more different things are connected, what plays another important role in complementarity analysis is a relationship in which one single thing is connected to itself, such as economies of scale or bandwagon effects in large scale coordination games. We define this complementarity as **self complementarity** in this paper. Here, we only remark two things on self complementarity. First, it is a crucial property of the convexity of the function g in Topkis' Lemma, as mentioned in Section 2. Second, it is also implied in the notion of cost complementarity (cf. [14]). Self complementarity is the essence of these points.

In this paper, we prepare some definitions and properties in Section 2. Then, we provide a sufficient condition for a composite function to have self increasing differences as the main result in Section 3. This is a contrastive proposition to Topkis' Lemma. However, both propositions demonstrate the importance of investigating interconnections of (mutual) complementarity and self complementarity, because these relationships give a composite function a complementary property, as in Topkis' Lemma, or a self complementary property, as in Theorem 4. In Section 4, we illustrate some applications of our main results. The first application is on a structural extension of the activities of an organization that maintains or strengthens complementarity. The second application is to build up self complementarity in an organization. This application has two types: (1) self complementarity made from (mutual) complementarity and some appropriate administrative strategies; and (2) self complementarity made from reciprocity in a supermodular game.

2. Preliminaries

First, let a profit function f be a real valued function on R^n. We begin to consider complementarity on f. The Edgeworth = Pareto complementarity is a notion of complementarity between any distinct activities i and j, defined as the additional profit for an arbitrarily fixed increase of the level of activity i is greater when the level of activity j is higher. This definition is straightly generalized to the notion of increasing differences, which is defined later in a more general setting. On the other hand, economies of scale is a notion of self complementarity with respect to a single activity i. That is, f has **self increasing differences** in x_i when the additional profit for an arbitrarily fixed increase of x_i is greater if the initial level x_i is higher. More precisely, we define a real valued function f on R^n to have self increasing differences in x_i if

$$f(x_i+\epsilon, x_{-i}) - f(x_i, x_{-i}) \leq f(x_i'+\epsilon, x_{-i}) - f(x_i', x_{-i}) \tag{2.1}$$

holds for all $x_i < x_i'$, $\epsilon > 0$, and x_{-i}, where x_{-i} denotes $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$, and (y_i, x_{-i}) denotes $(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)$. Furthermore, we say that f has **strictly self increasing differences** in x_i if

$$f(x_i+\epsilon, x_{-i}) - f(x_i, x_{-i}) < f(x_i'+\epsilon, x_{-i}) - f(x_i', x_{-i}) \tag{2.2}$$

holds for all $x_i < x_i'$, $\epsilon > 0$, and x_{-i}. Let X be a convex subset of R^n with the usual ordering. We say that f has self increasing differences in x_i on X if (2.1) holds for all $x_i < x_i'$, $\epsilon > 0$, and x_{-i} such that all the vectors $(x_i, x_{-i}), (x_i+\epsilon, x_{-i}), (x_i', x_{-i})$, and $(x_i'+\epsilon, x_{-i})$ belong to X. If f is continuous, the self increasing differences of f in x_i are equivalent to the convexity of f in x_i. (Cf. [14, Lemma 2.6.2.(c) and Example 2.6.4].) Furthermore, if f is C^2 class, it is also equivalent to $\frac{\partial^2 f}{\partial {x_i}^2}(x) \geq 0$ everywhere. (On the other hand, the definition of (strictly) self increasing differences can be straightly generalized for a discrete function f on Z^n to Z, where Z is the set of integers.)

Here, we remark some relationships between complementarity, self complementarity, and convexity. (See also [12, §7].) When a real valued function f on R^n is C^2 class, we know that f is convex if and only if the Hesse matrix is nonnegative definite. From this fact, every convex function $f(x_1, \cdots, x_n)$ of C^2 class has self increasing differences in x_i for each i; however, a convex function, for example $f(x, y) = 2x^2 - 2xy + 2y^2$, may not have increasing differences in (x_i, x_j) for some distinct i and j. Contrarily, having both increasing and self increasing differences is not a sufficient condition for f to be convex. $f(x, y) = ax^2 + 2bxy + ay^2$

is such an example if $0 < a < b$ holds. However, in this example, if $0 < b \leq a$ holds, that is, complementarity is in some sense stronger than self complementarity, f becomes convex.

Next, we recall two definitions of complementarity according to Topkis [14]. Let $f(x)$ be a function from a partially ordered set X to a partially ordered set Y. $f(x)$ is **increasing**, **decreasing**, **strictly increasing**, or **strictly decreasing** if $x' \prec x''$ in X implies, respectively, $f(x') \preceq f(x'')$, $f(x'') \preceq f(x')$, $f(x') \prec f(x'')$, or $f(x'') \prec f(x')$ in Y. Now, let X and T be partially ordered sets and $f(x,t)$ be a real valued function defined on a subset S of $X \times T$. For each $t \in T$, $S_t = \{x \in X : (x,t) \in S\}$ denotes the section of S at t. $f(x,t)$ has **increasing differences**, **decreasing differences**, **strictly increasing differences**, **strictly decreasing differences** in (x,t) on S if $f(x,t'') - f(x,t')$ is, respectively, increasing, decreasing, strictly increasing, or strictly decreasing in x on $S_{t'} \cap S_{t''}$ for any $t' \prec t''$ in T. We generalize the notion of increasing differences. Let A be a set, X_α be a partially ordered set for each α in A, X be a subset of $\times_{\alpha \in A} X_\alpha$, and $f(x)$ be a real valued function on X. $f(x)$ has increasing differences, decreasing differences, strictly increasing differences, or strictly decreasing differences in $x = (x_\alpha : \alpha \in A)$ on X if $f(x)$ has, respectively, increasing differences, decreasing differences, strictly increasing differences, or strictly decreasing differences in $(x_{\alpha'}, x_{\alpha''})$ on the section of X at $(x'_\alpha : \alpha \in A \setminus \{\alpha', \alpha''\})$ for all distinct α' and α'' in A and all x'_α in X_α for all α in $A \setminus \{\alpha', \alpha''\}$. Now, we also clarify the definition of the increasingness of a function in this setting. For each $\alpha' \in A$, $f(x)$ is increasing, decreasing, strictly increasing, or strictly decreasing in $x_{\alpha'}$ if $f(x) = f(x_{\alpha'}, x_{-\alpha'})$ is, respectively, increasing, decreasing, strictly increasing, or strictly decreasing in $x_{\alpha'}$ with $x_{-\alpha'}$ arbitrarily fixed.

Another notion of complementarity is supermodularity. Let $f(x)$ be a real valued function on a lattice X. $f(x)$ is **supermodular** in x on X if

$$f(x') + f(x'') \leq f(x' \vee x'') + f(x' \wedge x'')$$

holds for all x' and x'' in X, and $f(x)$ is **strictly supermodular** in x on X if

$$f(x') + f(x'') < f(x' \vee x'') + f(x' \wedge x'')$$

holds for all unordered x' and x'' in X. We also say that $f(x)$ is **(strictly) submodular** if $-f(x)$ is (strictly) supermodular.

Since every real valued function on a chain is supermodular and submodular, we know that the supermodularity is independent of self complementarity. On the other hand, the following theorems on the relations between two types of (mutual) complementarity are well known.

Theorem 1. ([12, Theorem 3.1]). *Let A be a set, X_α be a lattice for each α in A, X be a sublattice of $\times_{\alpha \in A} X_\alpha$, and $f(x)$ be a real valued function on X.*

If $f(x)$ is (strictly) supermodular in x on X, then $f(x)$ has (strictly) increasing differences in $(x_\alpha : \alpha \in A)$ on X.

Theorem 2. ([14, Theorem 2.6.2]). *Let $X_1, \cdots, X_n$ be lattices and $f(x)$ be a real valued function on $\times_{i=1}^n X_i$. Assume that $f(x)$ has increasing differences in $(x_1, \cdots, x_n)$ on $\times_{i=1}^n X_i$ and that $f(x)$ is supermodular in x_i on X_i for all $x_{i'}$ in $X_{i'}$ for all $i' \neq i$ and for each $i = 1, \ldots, n$. Then, $f(x)$ is supermodular in x on $\times_{i=1}^n X_i$.*

As a special case of Theorem 2, we know the following.

Corollary 1. ([12, Theorem 3.2]). *Let $X_1, \cdots, X_n$ be chains and $f(x)$ be a real valued function on $\times_{i=1}^n X_i$. If $f(x)$ has increasing differences in $(x_1, \cdots, x_n)$ on $\times_{i=1}^n X_i$, then $f(x)$ is supermodular in x on $\times_{i=1}^n X_i$. (Therefore, for an arbitrary real valued function on R^n, as a function of n variables, having increasing differences is equivalent to being supermodular).*

These theorems are reconsidered in Section 4. An objective of this paper is to investigate some further interrelations among self increasing differences, increasing differences, and supermodularity. As the first step along this line, we present a slightly modified version of Topkis' Lemma. We weaken the convexity conditions in the original version to having the self increasing differences. However, we remark that Topkis's proof is still valid for this version.

Theorem 3. (a variation of [14, Lemma 2.6.4]). *Let X be a lattice, a real valued function $g_i(x)$ on X be increasing in x, and supermodular in x for $i = 1, \cdots, m$. Let Z_i be a convex subset of R^1 and contain the range of $g_i(x)$ for $i = 1, \cdots, m$. Assume that $f(z_1, \cdots, z_m, x)$ is a real valued function on $(\times_{i=1}^m Z_i) \times X$ such that it is supermodular in $(z_1, \cdots, z_m, x)$ and increasing in z_i, and has self increasing differences in z_i for $i = 1, \cdots, m$. Then, the composite function $f(g_1(x), \cdots, g_m(x), x)$ is supermodular in x on X.*

3. Main Results

First, we prepare a technical lemma on increasing differences.

Lemma 1. *Let $N = \{1, \cdots, n\}$, $X_1, \cdots, X_n$ be partially ordered sets, and a real valued function $f(x)$ on $\times_{i \in N} X_i$ have increasing differences in $(x_1, \cdots, x_n)$. Then, for any nonempty proper subset I of N, $f(x)$ has increasing differences in $(x_I, x_{N\setminus I})$. That is,*

$$f(x'_I, x_{N\setminus I}) - f(x_I, x_{N\setminus I}) \leq f(x'_I, x'_{N\setminus I}) - f(x_I, x'_{N\setminus I})$$

holds for all $x_I \preceq x'_I$ *and all* $x_{N\setminus I} \preceq x'_{N\setminus I}$.

Proof. Changing coordinates, if necessary, we can assume that there exists $m < n$ such that $I = \{1, \cdots, m\}$. Since $f(x)$ has increasing differences in (x_i, x_{m+1}) for $i = 1, \cdots, m$, we obtain the following:

$$
\begin{aligned}
& f(x'_1, \cdots, x'_m, x_{m+1}, \cdots, x_n) - f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) \\
= & f(x'_1, \cdots, x'_m, x_{m+1}, \cdots, x_n) - f(x_1, x'_2, \cdots, x'_m, x_{m+1}, \cdots, x_n) \\
& + f(x_1, x'_2, \cdots, x'_m, x_{m+1}, \cdots, x_n) - f(x_1, x_2, x'_3, \cdots, x'_m, x_{m+1}, \cdots, x_n) \\
& \vdots \\
& + f(x_1, \cdots, x_{m-1}, x'_m, x_{m+1}, \cdots, x_n) - f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) \\
\leq & f(x'_1, \cdots, x'_{m+1}, x_{m+2}, \cdots, x_n) - f(x_1, x'_2, \cdots, x'_{m+1}, x_{m+2}, \cdots, x_n) \\
& + f(x_1, x'_2, \cdots, x'_{m+1}, x_{m+2}, \cdots, x_n) - f(x_1, x_2, x'_3, \cdots, x'_{m+1}, x_{m+2}, \cdots, x_n) \\
& \vdots \\
& + f(x_1, \cdots, x_{m-1}, x'_m, x'_{m+1}, x_{m+2}, \cdots, x_n) \\
& - f(x_1, \cdots, x_m, x'_{m+1}, x_{m+2}, \cdots, x_n) \\
= & f(x'_1, \cdots, x'_m, x'_{m+1}, x_{m+2}, \cdots, x_n) - f(x_1, \cdots, x_m, x'_{m+1}, x_{m+2}, \cdots, x_n).
\end{aligned}
$$

Repeating similar estimations for $x_{m+2}, \cdots, x_n$, inductively, we obtain

$$
\begin{aligned}
& f(x'_1, \cdots, x'_m, x_{m+1}, \cdots, x_n) - f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) \\
\leq & f(x'_1, \cdots, x'_m, x'_{m+1}, \cdots, x'_n) - f(x_1, \cdots, x_m, x'_{m+1}, \cdots, x'_n).
\end{aligned}
$$

The proof is completed. ■

Next, we present one of the main theorems on the self complementarity of a composite function.

Theorem 4. *Let* X *be a convex subset of* R^n *with the usual ordering. Fix* ℓ *in* $\{1, \ldots, n\}$. *For* $i = 1, \cdots, m$, *let a real valued function* $g_i(x)$ *on* X *be increasing in* x *and have self increasing differences in* x_ℓ, *and* Z_i *be a convex subset of* R^1 *containing the range of* $g_i(x)$. *Let* $f(z_1, \cdots, z_m, x)$ *be a real valued function on* $(\times_{i=1}^m Z_i) \times X$ *such that (1) it has increasing differences in* $(z_1, \cdots, z_m, x)$, *(2) it is increasing in* z_i *and has self increasing differences in* z_i *for* $i = 1, \cdots, m$, *and (3) it has self increasing differences in* x_ℓ. *Then, the composite function* $f(g_1(x), \cdots, g_m(x), x)$ *has self increasing differences in* x_ℓ.

Proof. Let $x = (x_\ell, x_{-\ell})$ and $x' = (x'_\ell, x_{-\ell})$ be any elements of X with $x_\ell \le x'_\ell$. Fix $\epsilon > 0$ and set $\bar{x} = (x_\ell + \epsilon, x_{-\ell})$ and $\bar{x'} = (x'_\ell + \epsilon, x_{-\ell})$. Assume that $\bar{x}$ and $\bar{x'}$ are in X. Then, we must show that

$$\begin{aligned}
&f(g_1(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) - f(g_1(x), \cdots, g_m(x), x)\\
&\le f(g_1(\bar{x'}), \cdots, g_m(\bar{x'}), \bar{x'}) - f(g_1(x'), \cdots, g_m(x'), x').
\end{aligned}$$

First, since (1) $g_1(x)$ is increasing in x and has self increasing differences in x_ℓ, and (2) $f(z_1, \cdots, z_m, x)$ is increasing in z_1 and has self increasing differences in z_1, we obtain the following:

$$\begin{aligned}
&f(g_1(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) - f(g_1(x), \cdots, g_m(x), x)\\
&= f(g_1(x) + (g_1(\bar{x}) - g_1(x)), \cdots, g_m(\bar{x}), \bar{x}) - f(g_1(x), g_2(\bar{x}), \cdots, g_m(\bar{x}), \bar{x})\\
&+ f(g_1(x), g_2(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) - f(g_1(x), \cdots, g_m(x), x)\\
&\le f(g_1(x') + (g_1(\bar{x}) - g_1(x)), g_2(\bar{x}), \cdots, g_m(\bar{x}), \bar{x})\\
&\quad - f(g_1(x'), g_2(\bar{x}), \cdots, g_m(\bar{x}), \bar{x})\\
&+ f(g_1(x), g_2(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) - f(g_1(x), \cdots, g_m(x), x)\\
&\le f(g_1(\bar{x'}), g_2(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) - f(g_1(x'), g_2(x), \cdots, g_m(x), x)\\
&+ f(g_1(x'), g_2(x), \cdots, g_m(x), x) - f(g_1(x'), g_2(\bar{x}), \cdots, g_m(\bar{x}), \bar{x})\\
&+ f(g_1(x), g_2(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) - f(g_1(x), \cdots, g_m(x), x).
\end{aligned}$$

Since $x' \ge x$ and each $g_i(x)$ is increasing in x for $i = 1, \cdots, m$, $g_1(x') \ge g_1(x)$ and $(g_2(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) \ge (g_2(x), \cdots, g_m(x), x)$. Then, applying Lemma 1 for f as $I = \{1\}$, we obtain

$$\begin{aligned}
&- \Big(\big(f(g_1(x'), g_2(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) - f(g_1(x), g_2(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) \big)\\
&- \big(f(g_1(x'), g_2(x), \cdots, g_m(x), x) - f(g_1(x), g_2(x), \cdots, g_m(x), x) \big) \Big)\\
&\le 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&f(g_1(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) - f(g_1(x), \cdots, g_m(x), x)\\
&\le f(g_1(\bar{x'}), g_2(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) - f(g_1(x'), g_2(x), \cdots, g_m(x), x).
\end{aligned}$$

Next, assume that

$$
\begin{aligned}
& f(g_1(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) - f(g_1(x), \cdots, g_m(x), x) \\
\leq\ & f(g_1(\bar{x'}), \cdots, g_i(\bar{x'}), g_{i+1}(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) \\
& - f(g_1(x'), \cdots, g_i(x'), g_{i+1}(x), \cdots, g_m(x), x)
\end{aligned}
$$

for some i satisfying $1 \leq i \leq m-1$. Then, since (1) $g_{i+1}(x)$ is increasing in x and has self increasing differences in x_ℓ, (2) $f(z_1, \cdots, z_m, x)$ has increasing differences in $(z_1, \cdots, z_m, x)$, and (3) $f(z_1, \cdots, z_m, x)$ is increasing in z_i and has self increasing differences in z_i, using Lemma 1 as $I = \{i+1\}$, we similarly obtain

$$
\begin{aligned}
& f(g_1(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) - f(g_1(x), \cdots, g_m(x), x) \\
\leq\ & f(g_1(\bar{x'}), \cdots, g_{i+1}(\bar{x'}), g_{i+2}(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) \\
& - f(g_1(x'), \cdots, g_{i+1}(x'), g_{i+2}(x), \cdots, g_m(x), x).
\end{aligned}
$$

Inductively, we obtain

$$
\begin{aligned}
& f(g_1(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) - f(g_1(x), \cdots, g_m(x), x) \\
\leq\ & f(g_1(\bar{x'}), \cdots, g_m(\bar{x'}), \bar{x}) - f(g_1(x'), \cdots, g_m(x'), x).
\end{aligned}
$$

Since $f(z_1, \cdots, z_m, x)$ has self increasing differences in x_ℓ, the right side is estimated as follows.

$$
\begin{aligned}
& f(g_1(\bar{x'}), \cdots, g_m(\bar{x'}), \bar{x}) - f(g_1(x'), \cdots, g_m(x'), x) \\
\leq\ & f(g_1(\bar{x'}), \cdots, g_m(\bar{x'}), \bar{x'}) - f(g_1(\bar{x'}), \cdots, g_m(\bar{x'}), x') \\
& + f(g_1(\bar{x'}), \cdots, g_m(\bar{x'}), x) - f(g_1(x'), \cdots, g_m(x'), x) \\
=\ & f(g_1(\bar{x'}), \cdots, g_m(\bar{x'}), \bar{x'}) - f(g_1(x'), \cdots, g_m(x'), x') \\
& + f(g_1(\bar{x'}), \cdots, g_m(\bar{x'}), x) + f(g_1(x'), \cdots, g_m(x'), x') \\
& - f(g_1(\bar{x'}), \cdots, g_m(\bar{x'}), x') - f(g_1(x'), \cdots, g_m(x'), x).
\end{aligned}
$$

Again, applying Lemma 1 as $I = \{m+1\}$, we know that the sum of the last 4 terms of the right side of the above inequality is nonpositive. Therefore,

$$
\begin{aligned}
& f(g_1(\bar{x}), \cdots, g_m(\bar{x}), \bar{x}) - f(g_1(x), \cdots, g_m(x), x) \\
\leq\ & f(g_1(\bar{x'}), \cdots, g_m(\bar{x'}), \bar{x'}) - f(g_1(x'), \cdots, g_m(x'), x').
\end{aligned}
$$

The proof is completed. ∎

Corollary 2. *Let X be a convex subset of R^n, ℓ be in $\{1,\ldots,n\}$, a real valued function $g_i(x)$ on X be increasing in x and convex in x_ℓ for $i=1,\cdots,m$, Z_i be a convex subset of R^1 containing the range of $g_i(x)$ for $i=1,\cdots,m$, and $f(z_1,\cdots,z_m,x)$ be a real valued function on $(\times_{i=1}^m Z_i)\times X$ such that (1) it has increasing differences in $(z_1,\cdots,z_m,x)$, (2) it is increasing and convex in z_i for $i=1,\cdots,m$, and (3) it is convex in x_ℓ. Then, the composite function $f(g_1(x),\cdots,g_m(x),x)$ is convex in x_ℓ.*

Proof. (If $X=R^n$ and each $Z_i=R^1$, then the convexity of $f(z_1,\cdots,z_m,x)$ and each $g_i(x)$ implies continuity. Therefore, since any composite function of finite numbers of continuous functions is continuous, the proof is a direct result of Theorem 4. However, in general, a convex function is only continuous relative to the relative interior of its domain. (Cf. Theorem 10.1 of Rockafellar [10].) Thus, we need another way.) Let x and x' be the same as those in the proof of Theorem 4. Fix $\alpha: 0\le\alpha\le 1$ arbitrarily and set $\bar\alpha=1-\alpha$. Then, from assumptions, we obtain the following estimation.

$$
\begin{aligned}
&f(g_1(\alpha x+\bar\alpha x'),g_2(\alpha x+\bar\alpha x'),\cdots)\\
\le\ & f(\alpha g_1(x)+\bar\alpha g_1(x'),g_2(\alpha x+\bar\alpha x'),\cdots)\\
\le\ & \alpha f(g_1(x),g_2(\alpha x+\bar\alpha x'),\cdots)+\bar\alpha f(g_1(x'),g_2(\alpha x+\bar\alpha x'),\cdots)\\
\le\ & \alpha\{\alpha f(g_1(x),g_2(x),\cdots)+\bar\alpha f(g_1(x),g_2(x'),\cdots)\}\\
&+\bar\alpha\{\alpha f(g_1(x'),g_2(x),\cdots)+\bar\alpha f(g_1(x'),g_2(x'),\cdots)\}\\
=\ & \alpha f(g_1(x),g_2(x),\cdots)+\bar\alpha f(g_1(x'),g_2(x'),\cdots)\\
&+\alpha\bar\alpha\{f(g_1(x),g_2(x'),\cdots)-f(g_1(x),g_2(x),\cdots)\\
&+f(g_1(x'),g_2(x),\cdots)-f(g_1(x'),g_2(x'),\cdots)\}\\
\le\ & \alpha f(g_1(x),g_2(x),\cdots)+\bar\alpha f(g_1(x'),g_2(x'),\cdots).
\end{aligned}
$$

The rest of the proof is quite similar to it of Theorem 4 and is omitted. ■

We remark that Corollary 2 is a generalization of Theorem 5.1 of Rockafellar [10] to a multi variables version. The next corollary is obtained by carefully investigating the proof of Theorem 4. The increasingness of $f(z_1,\cdots,z_m,x)$ in each z_i is not necessary if each $g_i(x)$ is affine (defined to be both convex and concave) in x.

Corollary 3. *Let X be a convex subset of R^n, ℓ be in $\{1,\ldots,n\}$, a real valued function $g_i(x)$ on X be increasing and affine in x for $i=1,\cdots,m$, Z_i be a convex* *subset of* R^1 *containing the range of* $g_i(x)$ *for* $i = 1, \cdots, m$, $f(z_1, \cdots, z_m, x)$ *be a real valued function on* $(\times_{i=1}^m Z_i) \times X$ *such that (1) it has increasing differences in* $(z_1, \cdots, z_m, x)$ *and (2) it has self increasing differences in* z_i *for* $i = 1, \cdots, m$ *and in* x_ℓ. *Then, the composite function* $f(g_1(x), \cdots, g_m(x), x)$ *has self increasing differences in* x_ℓ.

Combining Theorems 3 and 4 yields the following.

Theorem 5. *Let* X *be a convex sublattice of* R^n *with the usual ordering. Fix* ℓ *in* $\{1, \ldots, n\}$. *For* $i = 1, \cdots, m$, *let a real valued function* $g_i(x)$ *on* X *be increasing and supermodular in* x *and have self increasing differences in* x_ℓ, *and* Z_i *be a convex subset of* R^1 *containing the range of* $g_i(x)$. *Let* $f(z_1, \cdots, z_m, x)$ *be a real valued function on* $(\times_{i=1}^m Z_i) \times X$ *such that (1) it is supermodular in* $(z_1, \cdots, z_m, x)$, *(2) it is increasing in* z_i *and has self increasing differences in* z_i *for* $i = 1, \cdots, m$, *and (3) it has self increasing differences in* x_ℓ. *Then, the composite function* $f(g_1(x), \cdots, g_m(x), x)$ *on* X *is supermodular in* x *and has self increasing differences in* x_ℓ.

4. Applications

Let us reconsider the difference between the two notions of complementarity in a firm with n departments. For $i = 1, \cdots, n$, let the department i have several activities, and x_i be a vector of activity levels in the department. Assume that the board can select vectors $x_1, \cdots, x_n$ independently. Let X_i be a lattice of feasible vectors of the activity levels in the department i for $i = 1, \cdots, n$, $X = \times_{i=1}^n X_i$, and $f(x_1, \cdots, x_n)$ be a profit function of the firm on X. Then, if $f(x_1, \cdots, x_n)$ has increasing differences in $(x_1, \cdots, x_n)$, it means that an activity in a department is complementary to any activity in another department, and if $f(x_1, \cdots, x_n)$ is supermodular in x_i for $i = 1, \cdots, n$, it means that an activity in an arbitrarily fixed department is complementary to another activity in the same department. From Theorem 2, we obtain that these two types of complementarity, say inter-department complementarity and in-department complementarity, together, both let the firm have complementarity as the whole. This is the basic model of a functional decomposition of an organization from the viewpoint of complementarity.

We study some types of complementarity extension based on the functional structure of an organization as applications of the main results in Section 3. The first application is a complementarity extension theorem based on the consistency (or coherency) of activities.

Theorem 6. *Let* Z_i *be* R^1 *for* $i = 1, \cdots, m$, X_i *be a convex sublattice of* R^{q_i} *for* $i = m + 1, \cdots, n$, *and* $f(z_1, \cdots, z_m, x_{m+1}, \cdots, x_n)$ *be a real valued function* *on* $(\times_{i=1}^{m} Z_i) \times (\times_{i=m+1}^{n} X_i)$*. Assume that (1)* $f(z_1, \cdots, z_m, x_{m+1}, \cdots, x_n)$ *is supermodular in* $(z_1, \cdots, z_m, x_{m+1}, \cdots, x_n)$ *and (2) it is increasing in* z_i *and has self increasing differences in* z_i *for* $i = 1, \cdots, m$*. Let* A *be a convex sublattice of* R^p *with the usual ordering, and* ℓ *be in* $\{1, \cdots, p\}$*. (Assume that* $a_1, \cdots, a_p$ *are independent variables of* $x_{m+1}, \cdots, x_n$*.) For* $i = 1, \cdots, m$*, let* $\mu_i(a)$ *be a real valued function on* A *such that it is increasing and supermodular in* a*, and has self increasing differences in* a_ℓ*. Then,* $f(\mu_1(a), \cdots, \mu_m(a),\ x_{m+1}, \cdots, x_n)$ *is supermodular in* $(a, x_{m+1}, \cdots, x_n)$ *and has self increasing differences in* a_ℓ *on* $A \times (\times_{i=m+1}^{n} X_i)$*.*

Proof. We use the ˆ mark to define elements denoted in Theorem 5. Let $\hat{n} = p + q_{m+1} + \cdots + q_n$, $\hat{\ell} = \ell$, $\hat{m} = m$, $\hat{X} = A \times (\times_{i=m+1}^{n} X_i)$, and $\hat{x} = (a, x_{m+1}, \cdots, x_n)$. Let $\hat{g}_i(\hat{x}) = \mu_i(a)$, $\hat{Z}_i = R^1$, and $\hat{z}_i = z_i$ for $i = 1, \cdots, \hat{m}$. Let $\hat{f}(\hat{z}_1, \cdots, \hat{z}_{\hat{m}}, \hat{x}) = f(z_1, \cdots, z_m, x_{m+1}, \cdots, x_n)$. Then, since $\hat{X}$ is a finite product of the convex sublattices of Euclidean spaces, it is a convex sublattice of $R^{\hat{n}}$. Next, since $\hat{x} = (a, x_{m+1}, \cdots, x_n)$, for $i = 1, \cdots, \hat{m}$, $\hat{g}_i(\hat{x})$ is increasing and supermodular in $\hat{x}$ because $\mu_i(a)$ is increasing and supermodular in a, and $\hat{g}_i(\hat{x})$ has self increasing differences in $\hat{x}_{\hat{\ell}}$ because μ_i has self increasing differences in a_ℓ. Similarly, the supermodularity of f in $(z_1, \cdots, z_m, x_{m+1}, \cdots, x_n)$ implies the supermodularity of $\hat{f}$ in $(\hat{z}_1, \cdots, \hat{z}_{\hat{m}}, \hat{x})$. $\hat{f}$ is increasing in $\hat{z}_i$ and has self increasing differences in $\hat{z}_i$ because f is increasing in z_i and has self increasing differences in z_i. Finally, $\hat{f}$ has self increasing differences in $\hat{x}_{\hat{\ell}}$ because $\hat{\ell} = \ell \leq p$ and f does not have a_ℓ as a variable. (Condition (2.1) always holds as an equation.) Then, from Theorem 5, we know that $\hat{f}(\hat{g}_1(\hat{x}), \cdots, \hat{g}_{\hat{m}}(\hat{x}), \hat{x})$ is supermodular in $\hat{x}$ and has self increasing differences in $\hat{x}_{\hat{\ell}}$ on $\hat{X}$. This means that $f(\mu_1(a), \cdots, \mu_m(a), x_{m+1}, \cdots, x_n)$ is supermodular in $(a, x_{m+1}, \cdots, x_n)$ and has self increasing differences in a_ℓ. ■

We present an interpretation of Theorem 6. As the first model of a firm, let $x_1, \cdots, x_n$ be real valued levels of core activities, and $f(x_1, \cdots, x_n)$ be a profit function of the firm. Assume that (1) $f(x_1, \cdots, x_n)$ is supermodular in $(x_1, \cdots, x_n)$ and (2) it is increasing in x_i and has self increasing differences in x_i for each i in $\{1, \cdots, n\}$. Let $a_1, \cdots, a_q$ be real valued levels of the ancillary activities of the firm. Suppose that a subset of ancillary activities $A_i = \{i_1, \cdots, i_{k_i}\} \subset \{1, \cdots, q\}$ reinforces the activity i for each i in $\{1, \cdots, n\}$. Here, let us consider the second model of the firm as an improvement of the first model. In addition, we assume that for $i = 1, \cdots, n$, there exists a real valued estimation measure $z_i = \mu_i(x_i, a_{i_1}, \cdots, a_{i_{k_i}})$ such that it is increasing and supermodular in $(x_i, a_{i_1}, \cdots, a_{i_{k_i}})$ and has self increasing differences in x_i. Roughly, this means that μ_i gives a real valued level of an abstract activity defined through the abstraction of activity i with the possibility of a slight influence by ancillary activities $i_1, \cdots, i_{k_i}$, and that all variables of μ_i are consistent. (The idea of mapping from activity i to the real valued function μ_i for each i is analogous to it of the duality mapping in mathematics by Beurling and Livingston [2].) In this setting, since the other core activities $\{1, \cdots, n\} \setminus \{i\}$ are thought not to be significant variables of μ_i, μ_i has self increasing differences in x_j for each j in $\{1, \cdots, n\} \setminus \{i\}$. (Condition (2.1) always holds as an equation.) Furthermore, in the second model, we redefine the profit of the firm as $f(z_1, \cdots, z_n)$ and assume on that (1) it is supermodular in $(z_1, \cdots, z_n)$, and (2) it is increasing in z_i and has self increasing differences in z_i for each i in $\{1, \cdots, n\}$. Then, from Theorem 6, we know that the profit of the firm is a supermodular function of the levels of core activities $1, \cdots, n$, and ancillary activities $1, \cdots, q$. This means that the core and ancillary activities have complementarity as a whole with respect to the profit of the firm. Furthermore, from Theorem 6, the profit of the firm still has self increasing differences in each core activity i in $\{1, \cdots, n\}$. Therefore, this process can be repeated as necessary.

Note that Theorem 6 does not require $A_i \cap A_j = \emptyset$ for any distinct i and j in $\{1, \cdots, n\}$. Therefore, it is applicable to a case where an ancillary activity significantly supports several core activities. (For example, assume that a new support center offers services to several departments.) This point differs greatly from Theorem 2 (which requires an exclusive decomposition of all activities).

As the second application of our main results, we investigate a functional mechanism to yield self complementarity based on (mutual) complementarity. A feature of this mechanism is an administrative strategy in a firm that appropriately influences some capabilities of the firm with the result that some complementary activities are simultaneously activated so highly as to bring the firm more and more profits. When the constraint set of feasible activities is determined by a wide range of capabilities, including hardware resources, technologies, information, know-how, routines, etc., an aim of the firm's administrative strategies is to keep, heighten, or restruct the capabilities so as to obtain more profits in the current external environment including customers and rivals. To formalize this dependency, we let the constraint set be parametrized by the vector of levels of administrative strategies. Let M be a lattice of all vectors of the levels of some administrative strategies, m be an element of M, and S_m be the constraint set that depends on m. Here, we assume that if we execute the considered administrative strategies, it increases some capabilities of the firm, and then, we have room to activate some activities to higher levels.

Now, we formally state the first mechanism.

Theorem 7. *Let M be a convex sublattice of R^n with the usual ordering, S be a sublattice of $M \times R^p$ such that a section S_m of S at m is nonempty for each m in M, ℓ be in $\{1, \cdots, n\}$, $f(x_1, \cdots, x_p, y)$ be a real valued function on $R^p \times R^q$ such that* (1) *it is supermodular in $(x_1, \cdots, x_p, y)$,* (2) *it is increasing in x_i for $i = 1, \cdots, p$,* (3) *it has self increasing differences in x_i for $i = 1, \cdots, p$,*

(4) $\operatorname{argmax}_{x \in S_m} f(x,y)$ *is nonempty for each* m *in* M *and each* y *in* R^q, *and* (5) *there exists an increasing optimal selection* $\bar{x}(m,y)$ *from* $\operatorname{argmax}_{x \in S_m} f(x,y)$ *with parameter* (m,y) *in* $M \times R^q$ *such that* $\bar{x}_i(m,y)$ *has self increasing differences in* m_ℓ *for* $i = 1, \cdots, p$. *Then,* $\max_{x \in S_m} f(x,y)$ *is supermodular in* (m,y) *and increasing in* m_ℓ*, and has self increasing differences in* m_ℓ*.*

Proof. First, we show the supermodularity. The idea to prove the supermodularity of $\bar{f}(m,y) = \max_{x \in S_m} f(x,y)$ in (m,y) on $M \times R^q$ is similar to the proof of [14, Theorem 2.7.6]. Since f is supermodular,

$$f(x',y') + f(x'',y'') \le f(x' \vee x'', y' \vee y'') + f(x' \wedge x'', y' \wedge y'')$$

holds for all $m', m'' \in M$, all $x' \in S_{m'}$, all $x'' \in S_{m''}$, and all $y', y'' \in R^q$. Since (m', x') and (m'', x'') are elements of S and S is a sublattice, $(m' \wedge m'', x' \wedge x'')$ and $(m' \vee m'', x' \vee x'')$ are in S; that is, $x' \wedge x'' \in S_{m' \wedge m''}$ and $x' \vee x'' \in S_{m' \vee m''}$. Therefore, we obtain

$$\begin{aligned}
&\bar{f}(m',y') + \bar{f}(m'',y'') \\
&\le \max\{f(x' \vee x'', y' \vee y'') + f(x' \wedge x'', y' \wedge y'') : x' \in S_{m'}, x'' \in S_{m''}\} \\
&\le \max_{x \in S_{m' \vee m''}} f(x, y' \vee y'') + \max_{x \in S_{m' \wedge m''}} f(x, y' \wedge y'') \\
&= \bar{f}(m' \vee m'', y' \vee y'') + \bar{f}(m' \wedge m'', y' \wedge y'').
\end{aligned}$$

Thus, the supermodularity is proved.

To prove the self increasing differences in m_ℓ, we use Theorem 4. We employ the ˆ mark to define elements denoted in Theorem 4. Let $\hat{n} = n + q$, $\hat{\ell} = \ell$, $\hat{m} = p$, $\hat{X} = M \times R^q$, and $\hat{x} = (m,y)$. Let $\hat{g}_i(\hat{x}) = \bar{x}_i(m,y)$, $\hat{Z}_i = R^1$, $\hat{z}_i = x_i$, for $i = 1, \cdots, \hat{m}$, and $\hat{f}(\hat{z}_1, \cdots, \hat{z}_{\hat{m}}, \hat{x}) = f(x_1, \cdots, x_p, y)$. Then, $\hat{X}$ is a convex sublattice of $R^{\hat{n}}$. For $\hat{g}_i(\hat{x})$, (1) it is increasing in $\hat{x}$ directly by assumptions and (2) it has self increasing differences in $\hat{x}_{\hat{\ell}} = m_\ell$ because $\bar{x}(m,y)$ has self increasing differences in m_ℓ. $\hat{f}$ is supermodular in $(\hat{z}_1, \cdots, \hat{z}_{\hat{m}}, \hat{x})$ because f is supermodular in $(x_1, \cdots, x_n, y)$. Since f is increasing in x_i and has self increasing differences in x_i, $\hat{f}$ is increasing in $\hat{z}_i$ and has self increasing differences in $\hat{z}_i$. Finally, $\hat{f}$ is increasing in $\hat{x}_{\hat{\ell}}$ and has self increasing differences in $\hat{x}_{\hat{\ell}}$ because f does not have m_ℓ as a variable. Then, from Theorem 4, we obtain that $\hat{f}(\hat{g}_1(\hat{x}), \cdots, \hat{g}_{\hat{m}}(\hat{x}), \hat{x}) = f(\bar{x}(m,y), y)$ has self increasing differences in $\hat{x}_{\hat{\ell}} = m_\ell$. $f(\bar{x}(m,y), y)$ is also increasing in m_ℓ from the assumptions on increasingness. ■

Remark 1. In Theorem 7, if there exists some j in $\{1, \cdots, q\}$ such that (1) f is increasing in y_j and has self increasing differences in y_j and (2) $\bar{x}_i(m,y)$ has self increasing differences in y_j for $i = 1, \cdots, p$, then, $\max_{x \in S_m} f(x,y)$ is increasing in y_j and has self increasing differences in y_j. This is because the proof of Theorem 7 is valid for $\hat{x}_\ell$ for any ℓ in $\{n+1, \cdots, n+q\}$ with a slight modification. Thus, self complementarity in y_j is also preserved.

Remark 2. In the assumptions of Theorem 7, a sufficient condition of the existence of an increasing optimal selection $\bar{x}(m, y)$ is given in [14, Theorem 2.8.3(a)].

The last application is a functional mechanism to yield self complementarity based on reciprocity in a supermodular game. We start by recalling some more definitions. Now suppose that each player determines the levels of his activities exclusively and independently of the other players. In this case, first, we must clarify complementarity with respect to whose profit function. Next, we must divide all of the activities to all of the players without duplication. Let n be the number of players; x_i be a decision variable, that is a vector of activity levels, of player i; and $f_i(x_1, \cdots, x_n)$ be a profit function of player i for $i = 1, \cdots, n$. We assume that $f_i(x_1, \cdots, x_n)$ has increasing differences in $(x_1, \cdots, x_n)$ for $i = 1, \cdots, n$. This complementarity is related to plural players and is called **strategic complementarity** (Cf. Bulow et al. [3]). Additionally, if $f_i(x_1, \cdots, x_n)$ is supermodular in x_i for $i = 1, \cdots, n$, these conditions define a **supermodular game**. We investigate self complementarity in this situation.

Theorem 8. *Let S_1 be a convex sublattice of R^p with the usual ordering, S_2 be a rectangular subset of R^q with the usual ordering, and $\{\{1, 2\}, \{S_1, S_2\}, \{f_1, f_2\}\}$ be a supermodular game. We denote the coordinates of vectors x_1 and x_2 by ℓ in $\{1, \cdots, p\}$ and i in $\{1, \cdots, q\}$, respectively. Fix coordinate ℓ in $\{1, \cdots, p\}$ arbitrarily. Assume that (1) $f_1(x_1, x_2)$ is supermodular in (x_1, x_2), (2) $f_1(x_1, x_2)$ is increasing in $x_{1\ell}$ and has self increasing differences in $x_{1\ell}$, (3) $f_1(x_1, x_2)$ is increasing in x_{2i} and has self increasing differences in x_{2i} for $i = 1, \cdots, q$, and (4) there exists an increasing selection $\mu_2(x_1)$ from the optimal responses of player 2 for each strategy x_1 of S_1 such that $\mu_{2i}(x_1)$ has self increasing differences in $x_{1\ell}$ for coordinate $i = 1, \cdots, q$. Then, $f_1(x_1, \mu_2(x_1))$ is increasing and has self increasing differences in $x_{1\ell}$ on S_1.*

Proof. We apply Theorem 4 to prove self increasing differences. We use the ˆ mark to define the elements denoted in Theorem 4. Let $\hat{n} = p$, $\hat{\ell} = \ell$, $\hat{m} = q$, $\hat{X} = S_1$, and $\hat{x} = x_1$. Let $\hat{g}_i(\hat{x}) = \mu_{2i}(x_1)$, $\hat{Z}_i$ be the projection of S_2 onto the ith coordinate, $\hat{z}_i = x_{2i}$, for $i = 1, \cdots, \hat{n}$. Let $\hat{f}(\hat{z}_1, \cdots, \hat{z}_{\hat{m}}, \hat{x}) = f_1(x_1, x_2)$. Then, $\hat{X}$ is a convex subset of $R^{\hat{n}}$ by the assumption on S_1. Next, $\hat{g}_i(\hat{x})$ is increasing in $\hat{x}$ by the assumptions on μ_2. $\hat{g}_i(\hat{x})$ also has self increasing differences in $\hat{x}_{\hat{\ell}} = x_{1\ell}$ by the assumption on μ_{2i}. $\hat{f}$ has increasing differences in $(\hat{z}_1, \cdots, \hat{z}_{\hat{m}}, \hat{x})$ because $f_1(x_1, x_2)$ is supermodular in (x_1, x_2). Since f_1 is increasing in x_{2i} and has self increasing differences in x_{2i} for $i = 1, \cdots, q$, $\hat{f}$ is increasing in $\hat{z}_i$ and has self increasing differences in $\hat{z}_i$ for $i = 1, \cdots, \hat{m}$. Finally, $\hat{f}$ is increasing and has self increasing differences in $\hat{x}_\ell$ because f_1 is increasing and has self increasing differences in $x_{1\ell}$. Then, from Theorem 4, we obtain $\hat{f}(\hat{g}_1(\hat{x}), \cdots, \hat{g}_{\hat{m}}(\hat{x}), \hat{x}) = f_1(x_1, \mu_2(x_1))$ as having self increasing differences in $\hat{x}_{\hat{\ell}} = x_{1\ell}$. It is also increasing in $x_{1\ell}$ from the assumptions on f_1 and μ_2. ∎

Remark 3. For Theorem 8, a sufficient condition of the existence of an increasing optimal selection $\mu_2(x_1)$ is given in [14, Theorem 2.8.3(a)].

Since any affine function has increasing differences in any component of its variable, we summarize a special case as a corollary of Theorem 8.

Corollary 4. *Let S_1 be a convex sublattice of R^p with the usual ordering, S_2 be a rectangular subset of R^q with the usual ordering, and $\{\{1, 2\}, \{S_1, S_2\}, \{f_1, f_2\}\}$ be a supermodular game. We denote the coordinate of vector x_1 by ℓ in $\{1, \cdots, p\}$. Fix coordinate ℓ in $\{1, \cdots, p\}$ arbitrarily. Assume that (1) $f_1(x_1, x_2)$ is increasing and affine in $x_{1\ell}$, (2) $f_1(x_1, x_2)$ is increasing and affine in x_2, and (3) there exists an increasing selection $\mu_2(x_1)$ from the optimal responses of player 2 for each strategy x_1 of S_1 such that $\mu_2(x_1)$ is affine in x_1. Then, $f_1(x_1, \mu_2(x_1))$ is increasing and has self increasing differences in $x_{1\ell}$. Furthermore, $f_1(x_1, \mu_2(x_1))$ is supermodular in x_1.*

Proof. From assumptions, since $f_1(x_1, x_2)$ is affine in x_2, it is supermodular in x_2 by [14, Theorem 2.6.4]. Therefore, from Theorem 2, we know that $f_1(x_1, x_2)$ is supermodular in (x_1, x_2). Therefore, Theorem 8 is applicable. Furthermore, since $\mu_{2i}(x_1)$ is also supermodular in x_1 for $i = 1, \cdots, q$, using Theorem 3, we obtain that $f_1(x_1, \mu_2(x_1))$ is supermodular in x_1. ∎

Corollary 4 can be interpreted as follows. Assume that a firm and its customer keep a reciprocal good relationship on a good (or service); that is, we assume that there exists a supermodular game. Let x_1 be the p-vector of the activity levels of the firm that especially influence the customer's satisfaction through the good, and x_2 be the q-vector of the activity levels of the customer on the concerned good that especially influence the firm's profit, $f_1(x_1, x_2)$ be the profit of the firm on this good, and $f_2(x_1, x_2)$ be the customer's level of satisfaction. Assume that $f_1(x_1, x_2)$ is increasing and linear in (x_1, x_2), and has increasing differences in (x_1, x_2). In addition, assume that there exists a unique optimal response $x_2 = \mu_2(x_1)$ of the customer with respect to f_2 for each x_1, and that it is linear in x_1. Under the optimal response of the customer, if the other assumptions of Corollary 4 are satisfied, the profit function of the firm has self increasing differences in each concerned activity x_{1i} for i in $\{1, \cdots, p\}$, and all activities of the firm are complementary. This indicates that this is a straight way to obtain strong complementarity for a firm to build up a reciprocal relationship with its customer.

5. Conclusion

Suppose the repeated application of our results to the complementarity analysis of a firm. Then, we must know the functional structures, typically hierarchical structures, of activities, capabilities, administrative strategies, and so on of the firm. Thus, this way is a functional approach to complementarity analysis. This approach will be useful both for designing a new organization and redesigning an organization that has lost fitness for the current environment.

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