

RANDOMIZED SERIES AND GEOMETRY OF BANACH SPACES

Han Ju Lee

Abstract. We study some properties of the randomized series and their applications to the geometric structure of Banach spaces. For $n \geq 2$ and $1 < p < \infty$, it is shown that ℓ_∞^n is representable in a Banach space X if and only if it is representable in the Lebesgue-Bochner $L_p(X)$. New criteria for various convexity properties in Banach spaces are also studied. It is proved that a Banach lattice E is uniformly monotone if and only if its p -convexification $E^{(p)}$ is uniformly convex and that a Köthe function space E is upper locally uniformly monotone if and only if its p -convexification $E^{(p)}$ is midpoint locally uniformly convex.

1. RANDOMIZED SERIES

Let $\{r_n\}_{n=1}^\infty$ be a sequence of mutually independent, symmetric and integrable random variables on a probability space (Ω, \mathcal{F}, P) and $\{x_n\}_{n=0}^\infty$ an arbitrary sequence in a Banach space X . A *randomized series* S_n is a vector-valued random variable defined by

$$S_n = x_0 + r_1x_1 + \cdots + r_nx_n, \quad (n = 0, 1, \dots)$$

Let \mathcal{F}_0 be the trivial σ -algebra $\{\emptyset, \Omega\}$ and \mathcal{F}_k , ($k \geq 1$) the σ -algebra generated by random variables $\{r_i\}_{i=1}^k$. It is easy to see that the sequence of randomized series $\{S_n\}_{n=0}^\infty$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$. In this paper, \mathbb{E} stands for the expectation with respect to the probability P .

We begin with the basic properties of the randomized series. For more properties of random series, see [8]. The following lemma is well-known and the proof is omitted.

Received December 4, 2008, accepted February 17, 2009.

Communicated by H. M. Srivastava.

2000 *Mathematics Subject Classification*: 46B20, 46B07, 46B09.

Key words and phrases: Representability, Extreme point, Strict convex, Midpoint locally uniformly convex, Uniformly convex, Upper locally uniformly monotone, Uniformly monotone.

This work was supported by the Dongguk University Research Fund of 2008.

Lemma 1.1. *Let φ be a convex function on \mathbb{R} . Then $\{\varphi(\|S_n\|)\}_{n=0}^\infty$ is a submartingale with respect to $\{\mathcal{F}_n\}_{n=0}^\infty$. In particular,*

$$\mathbb{E}[\varphi(\|S_0\|)] \leq \mathbb{E}[\varphi(\|S_1\|)] \leq \mathbb{E}[\varphi(\|S_2\|)] \leq \dots$$

Proposition 1.2. *Let φ be a convex increasing function on $[0, \infty)$ and $x, y \in X$. Then the function ψ on \mathbb{R} defined by*

$$\psi(\lambda) = \mathbb{E}[\varphi(\|x + \lambda r_1 y\|)]$$

is an increasing convex function on $[0, \infty)$ with $\psi(\lambda) = \psi(|\lambda|)$ for every $\lambda \in \mathbb{R}$.

Proof. By the convexity of φ , we get $\psi(\lambda) = \frac{1}{2}[\psi(\lambda) + \psi(-\lambda)] \geq \psi(0)$ for every real λ , which implies that $\lambda \mapsto \psi(\lambda)$ is increasing on $[0, \infty)$. Clearly $\lambda \mapsto \psi(\lambda)$ is an even function on \mathbb{R} since r_1 is symmetric. ■

Since random variables $\{r_i\}_i$ are independent, Proposition 1.2 shows that for any two real sequences $\{\lambda_i\}_{i=1}^n$ and $\{\xi_i\}_{i=1}^n$ satisfying $|\lambda_i| \leq |\xi_i|$ for $i = 1, \dots, n$, and for any x_0, x_1, \dots, x_n in X , we get

$$\mathbb{E}[\varphi(\|x_0 + \lambda_1 r_1 x_1 + \dots + \lambda_n r_n x_n\|)] \leq \mathbb{E}[\varphi(\|x_0 + \xi_1 r_1 x_1 + \dots + \xi_n r_n x_n\|)].$$

A convex function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is said to be *strictly convex* if

$$\varphi\left(\frac{s+t}{2}\right) < \frac{\varphi(s) + \varphi(t)}{2}$$

holds for all distinct positive numbers s, t . Notice that if φ is strictly convex and increasing on $[0, \infty)$ and a, b are real numbers with $b \neq 0$, then

$$\mathbb{E}[\varphi(|a + r_2 b|)] = \mathbb{E}\left[\frac{\varphi(|a + r_2 b|) + \varphi(|a - r_2 b|)}{2}\right] > \varphi(|a|).$$

Now we state our main theorem concerning the randomized series.

Theorem 1.3. *Let $\{x_i\}_{i=1}^n$ be a finite sequence in a Banach space X , φ a strictly convex, increasing function with $\varphi(0) = 0$ and $\{r_i\}_{i=1}^\infty$ symmetric independent random variables with $\|r_i\|_\infty = 1$, ($i = 1, 2, \dots$).*

Suppose that there is a constant $\rho > 0$ such that the following holds:

$$\sup_{\epsilon_1=\pm 1, \dots, \epsilon_n=\pm 1} \|\epsilon_1 x_1 + \dots + \epsilon_n x_n\| \geq \|x_1\| + \rho.$$

Then there is a constant $\delta = \delta(\rho) > 0$ such that

$$\mathbb{E}[\varphi(\|x_1 + r_2 x_2 + \dots + r_n x_n\|)] \geq \varphi(\|x_1\|) + \delta.$$

In particular, if we take $\rho_1 = \min\{\rho, 1/2\}$, then

$$\delta = \min \left\{ \varphi\left(\frac{\rho_1}{3}\right) \prod_{i=2}^n P\{|r_i - 1| < \frac{\rho_1}{3n}\}, \min_{2 \leq j \leq n} \mathbb{E} \left[\varphi \left(\|x_1\| + r_j \frac{\rho_1}{3n} \right) \right] - \varphi(\|x_1\|) \right\}.$$

Proof. We adapt the argument in the proof of Proposition 2.2 in [3]. We assume that there exist $0 < \rho < 1/2$ and signs $\epsilon_2, \dots, \epsilon_n$ such that

$$\|x_1 + \epsilon_2 x_2 + \dots + \epsilon_n x_n\| \geq \|x_1\| + \rho.$$

Select a unit element x^* in X^* such that $x^*(x_1) = \|x_1\|$ and let $\lambda_i = x^*(x_i)$ for $1 \leq i \leq n$. Now we shall consider two cases according to the size of $|\lambda_i|$. In the first case we suppose that $\max_{2 \leq i \leq n} |\lambda_i| \leq \frac{\rho}{3n}$. If $|\eta_i - \epsilon_i| \leq \frac{\rho}{3n}$, ($2 \leq i \leq n$), then

$$\|x_1 + \eta_2 x_2 + \dots + \eta_n x_n\| \geq \|x_1 + \epsilon_2 x_2 + \dots + \epsilon_n x_n\| - \frac{\rho}{3} \geq \|x_1\| + \frac{2\rho}{3}.$$

Since $|\lambda_1 + \eta_2 \lambda_2 + \dots + \eta_n \lambda_n| \leq |\lambda_1| + \frac{\rho}{3} = \|x_1\| + \frac{\rho}{3}$, we get

$$\|x_1 + \eta_2 x_2 + \dots + \eta_n x_n\| \geq |\lambda_1 + \eta_2 \lambda_2 + \dots + \eta_n \lambda_n| + \frac{\rho}{3},$$

if $|\eta_i - \epsilon_i| \leq \frac{\rho}{3n}$, ($2 \leq i \leq n$). Let

$$F = \bigcap_{j=2}^n \left\{ w \in \Omega : |r_j(w) - \epsilon_j| < \frac{\rho}{3n} \right\}$$

and take $T_n = x_1 + r_2 x_2 + \dots + r_n x_n$. Then we have

$$\mathbb{E}[\varphi(\|T_n\|)] = \mathbb{E}[\varphi(\|T_n\|)\chi_F] + \mathbb{E}[\varphi(\|T_n\|)\chi_{F^c}].$$

Since $\varphi(a + b) \geq \varphi(a) + \varphi(b)$, ($a, b \geq 0$),

$$\begin{aligned} \mathbb{E}[\varphi(\|T_n\|)\chi_F] &\geq \mathbb{E}[\varphi(|\lambda_1 + r_2 \lambda_2 + \dots + r_n \lambda_n| + \rho/3)\chi_F] \\ &\geq \mathbb{E}[\varphi(|\lambda_1 + r_2 \lambda_2 + \dots + r_n \lambda_n|)\chi_F] + \varphi(\rho/3)P(F). \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}[\varphi(\|T_n\|)] &\geq \mathbb{E}[\varphi(|\lambda_1 + r_2 \lambda_2 + \dots + r_n \lambda_n|)] + \varphi(\rho/3)P(F) \\ &\geq \varphi(\lambda_1) + \varphi(\rho/3)P(F) = \varphi(\|x_1\|) + \varphi(\rho/3)P(F). \end{aligned}$$

Notice that $P(F) = \prod_{i=2}^n P\{w \in \Omega : |r_i(w) - 1| < \frac{\rho}{3n}\} > 0$ for r_i 's are independent symmetric random variables with $\|r_i\|_\infty = 1$, ($i = 1, 2, \dots$). In the second

case we suppose that there exists i_0 ($2 \leq i_0 \leq n$) such that $|\lambda_{i_0}| \geq \frac{\rho}{3n}$. It follows from Lemma 1.1, 1.2 and strict convexity of φ that

$$\begin{aligned} \mathbb{E}[\varphi(\|T_n\|)] &\geq \mathbb{E}[\varphi(|\lambda_1 + r_2\lambda_2 + \cdots + r_n\lambda_n|)] \\ &\geq \mathbb{E}[\varphi(|\lambda_1 + r_{i_0}\lambda_{i_0}|)] \\ &\geq \mathbb{E}\left[\varphi\left(\left|\lambda_1 + r_{i_0}\frac{\rho}{3n}\right|\right)\right] \\ &> \varphi(\lambda_1) = \varphi(\|x_1\|). \end{aligned}$$

The proof is complete. ■

2. REPRESENTABILITY OF ℓ_∞^n

A Banach space Y is said to be *representable* in X if, for each $\lambda > 1$, there is a bounded linear map $T : Y \rightarrow X$ such that $\|x\| \leq \|Tx\| \leq \lambda\|x\|$ for every $x \in Y$. A Banach space Y is said to be *finitely representable* in X if every finite dimensional subspace of Y is representable in X .

It is well-known due to the work of B. Maurey and G. Pisier [12] that c_0 is finitely representable in X if and only if c_0 is finitely representable in $L_p(X)$ for all $1 \leq p < \infty$. S. J. Dilworth considered the quantitative version of this theorem in [3], where he showed that if X is a complex Banach space, $n \geq 2$ and $0 < p < \infty$, then $\ell_\infty^n(\mathbb{C})$ is representable in X if and only if it is representable in $L_p(X)$. As we see in the next example it is not true for real Banach spaces.

Example 2.1. Let X be a nontrivial real Banach space. Then ℓ_∞^2 is representable in $L_1([0, 1]; X)$. Indeed, if we choose the Rademacher sequence $\{r_n = \text{sign}(\sin 2^n \pi t)\}_{n=1}^\infty$ in $L_1[0, 1]$ and $x_0 \in S_X$, then $x = r_1 x_0$ and $y = r_2 x_0$ are the elements of unit sphere of $L_1(X)$ and they satisfy

$$\|x + y\|_{L_1(X)} = \|x - y\|_{L_1(X)} = 1,$$

which means that ℓ_∞^2 is representable in $L_1(X)$.

The subharmonicity of absolute value of holomorphic functions on \mathbb{C} plays the crucial role in the proof in [3]. In this paper, the strict convexity of $\varphi(t) = |t|^p$ ($1 < p < \infty$) on \mathbb{R} plays the analogous role, and thus it is shown here that ℓ_∞^n is representable in X if and only if it is representable in $L_p(X)$ for every $1 < p < \infty$.

The following proposition is a real version of Proposition 2.2 in [3].

Proposition 2.2. *Let $\{r_i\}_{i=1}^\infty$ be symmetric independent random variables with $\|r_i\|_\infty = 1$, ($i = 1, 2, \dots$). Suppose that X is a real Banach space and that $n \geq 2$. The following properties are equivalent:*

- (1) ℓ_∞^n is not representable in X .
- (2) There exists $\rho > 0$ such that whenever x_1, \dots, x_n are unit vectors in X , then there exist signs $\epsilon_1, \dots, \epsilon_n$ such that

$$\|\epsilon_1 x_1 + \dots + \epsilon_n x_n\| \geq 1 + \rho.$$

- (3) There exist strictly convex, increasing function φ on $[0, \infty)$ with $\varphi(0) = 0$ and $\rho > 0$ such that whenever x_1, \dots, x_n are unit vectors in X then

$$\mathbb{E}[\varphi(\|x_1 + r_2 x_2 \dots + r_n x_n\|)] \geq \varphi(1) + \rho.$$

- (4) For each strictly convex, increasing function φ on $[0, \infty)$ with $\varphi(0) = 0$, there is $\rho > 0$ such that whenever x_1, \dots, x_n are unit vectors in X then

$$\mathbb{E}[\varphi(\|x_1 + r_2 x_2 \dots + r_n x_n\|)] \geq \varphi(1) + \rho.$$

Proof. The implications (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) are clear. To show (1) \Rightarrow (2), suppose that (2) fails, so for any $\rho > 0$ there exist unit vectors x_1, \dots, x_n in X such that for all signs $\epsilon_1, \dots, \epsilon_n$, we have

$$\|\epsilon_1 x_1 + \dots + \epsilon_n x_n\| < 1 + \rho.$$

It follows that for all $\lambda_1, \dots, \lambda_n$ with $1 = |\lambda_{i_0}| = \max_{1 \leq i \leq n} |\lambda_i|$, we have

$$\|\lambda_1 x_1 + \dots + \lambda_n x_n\| < 1 + \rho.$$

Hence

$$\begin{aligned} 1 = \|\lambda_{i_0} x_{i_0}\| &\leq \frac{1}{2} \|\lambda_1 x_1 + \dots + \lambda_n x_n\| + \frac{1}{2} \|\lambda_1 x_1 + \dots + \lambda_n x_n - 2\lambda_{i_0} x_{i_0}\| \\ &\leq \frac{1}{2} \|\lambda_1 x_1 + \dots + \lambda_n x_n\| + \frac{1}{2} (1 + \rho). \end{aligned}$$

Thus, $\|\lambda_1 x_1 + \dots + \lambda_n x_n\| \geq 1 - \rho$. Since ρ is arbitrary, it follows that ℓ_∞^n is representable in X . Now we have only to show that (2) \Rightarrow (4). Suppose that (2) holds and that φ is a strictly convex, increasing function on $[0, \infty)$ with $\varphi(0) = 0$. There is $0 < \rho < 1/2$ such that whenever x_1, \dots, x_n are unit vectors in X , there exist signs $\epsilon_2, \dots, \epsilon_n$ such that

$$\|x_1 + \epsilon_2 x_2 + \dots + \epsilon_n x_n\| \geq 1 + \rho.$$

Then Theorem 1.3 shows that (2) implies (4). \blacksquare

Notice that in the case of the Rademacher sequence $\{r_n\}$, for every x_1, \dots, x_n in X ,

$$\mathbb{E}[\varphi(\|r_1x_1 + \dots + r_nx_n\|)] = \mathbb{E}[\varphi(\|x_1 + r_2x_2 + \dots + r_nx_n\|)].$$

The following theorem shows the lifting property of representability of ℓ_∞^n .

Theorem 2.3. *Suppose X is a Banach space, (M, \mathfrak{M}, μ) is a measure space with a measurable subset A satisfying $0 < \mu(A) < \infty$, and $1 < p < \infty$, $n \geq 2$. Then ℓ_∞^n is representable in X if and only if it is representable in $L_p(M, \mathfrak{M}, \mu; X)$.*

Proof. One implication is clear. To prove the other implication, suppose that ℓ_∞^n is not representable in X and let $\{r_n\}$ be the Rademacher sequence. By Proposition 2.2, there exists $0 < \rho < 1/2$ such that whenever x_1, \dots, x_n are unit vectors in X , we have

$$\mathbb{E}\|r_1x + r_2x_2 + \dots + r_nx_n\|^p \geq (1 + \rho)^p.$$

Suppose that f_1, \dots, f_n are unit vectors in $L_p(X)$. We define the following functions on M . For $w \in M$, let

$$q(w)^p = \mathbb{E}\|r_1f_1(w) + \dots + r_nf_n(w)\|^p,$$

$$M(w) = \max\{\|f_i(w)\| : 1 \leq i \leq n\},$$

$$m(w) = \min\{\|f_i(w)\| : 1 \leq i \leq n\}.$$

By Lemma 1.1, $q(w) \geq M(w)$ for all $w \in M$. Now the argument divides into two cases according to the relative sizes of $M(w)$ and $m(w)$. In the first case we suppose that $(1 - \rho/3)M(w) \geq m(w)$. Then

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \|f_i(w)\|^p &\leq \frac{n-1}{n} M(w)^p + \frac{1}{n} m(w)^p \\ &\leq \left(1 - \frac{\rho}{3n}\right) M(w)^p \\ &\leq q(w)^p - \frac{\rho}{3n} \left(\frac{1}{n} \sum_{i=1}^n \|f_i(w)\|^p\right) \end{aligned}$$

and so

$$q(w)^p \geq \left(1 + \frac{\rho}{3n}\right) \frac{1}{n} \sum_{i=1}^n \|f_i(w)\|^p.$$

In the second case, we suppose that $(1 - \rho/3)M(w) < m(w)$. Then

$$\begin{aligned} q(w)^p &\geq (1 + \rho)^p m^p(w) \\ &\geq (1 + \rho)^p \left(1 - \frac{\rho}{3}\right)^p \frac{1}{n} \sum_{i=1}^n \|f_i(w)\|^p \\ &\geq \left(1 + \frac{\rho}{2}\right)^p \frac{1}{n} \sum_{i=1}^n \|f_i(w)\|^p. \end{aligned}$$

Hence by the Fubini theorem,

$$\mathbb{E}\|rf_1 + \dots + r_n f_n\|_{L^p(X)}^p = \int_M q(w)^p d\mu \geq \min \left\{ \left(1 + \frac{\rho}{2}\right)^p, \left(1 + \frac{\rho}{3n}\right)^p \right\},$$

which shows that ℓ_∞^n is not representable in $L_p(X)$ by Proposition 2.2. The proof is completed. ■

3. APPLICATIONS TO THE CONVEXITY OF BANACH SPACES

Recall that a point x in S_X is an *extreme point* of B_X if $\max\{\|x+y\|, \|x-y\|\} = 1$ for some $y \in X$ implies $y = 0$. A point $x \in S_X$ is called a *strongly extreme point* of B_X if, given $\epsilon > 0$, there is a $\delta = \delta(x, \epsilon) > 0$ such that

$$\inf\{\max\{\|x + y\|, \|x - y\|\} : \|y\| \geq \epsilon\} \geq 1 + \delta.$$

A Banach space is said to be *strictly convex* (resp. *midpoint locally uniformly convex*) if every point of S_X is (resp. strongly) extreme point of B_X . A Banach space is called *uniformly convex* if, given $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that

$$\inf\{\max\{\|x + y\|, \|x - y\|\} : \|y\| \geq \epsilon, \|x\| = 1\} \geq 1 + \delta.$$

Theorem 1.3 gives the following criteria for the various convexity properties.

Theorem 3.1. *Let X be a real Banach space and φ , a strictly convex increasing function on $[0, \infty)$ with $\varphi(0) = 0$ and r , a symmetric random variable with $\|r\|_\infty = 1$. Then*

- (1) *A point x in S_X is an extreme point of B_X if and only if $\mathbb{E}[\varphi(\|x + ry\|)] = \varphi(1)$ for $y \in X$ implies $y = 0$.*
- (2) *A point x in S_X is a strongly extreme point of B_X if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $\|y\| \geq \epsilon$, we get*

$$\mathbb{E}[\varphi(\|x + ry\|)] \geq \varphi(1) + \delta.$$

- (3) X is uniformly convex if and only if the modulus $\delta_\varphi(\epsilon) > 0$ for every $\epsilon > 0$, where $\delta_\varphi(\epsilon) = \inf\{\mathbb{E}[\varphi(\|x + ry\|)] - \varphi(1) : x \in S_X, \|y\| \geq \epsilon\}$.

Proof. We prove only (3) because the proofs of the others are similar. Suppose that X is uniformly convex. Given $\epsilon > 0$, there is a $\rho > 0$ such that for any $x \in S_X$ and $y \in X$ with $\|y\| \geq \epsilon$, we have

$$\max\{\|x + y\|, \|x - y\|\} \geq 1 + \rho.$$

Then Theorem 1.3 shows that there is $\delta > 0$ such that for any $x \in S_X$ and y with $\|y\| \geq \epsilon$,

$$\mathbb{E}[\varphi(\|x + ry\|)] \geq \varphi(1) + \delta.$$

Conversely, suppose that $\delta_\varphi(\epsilon) > 0$ for every $\epsilon > 0$. Then given $\epsilon > 0$ for any $x \in S_X$ and $y \in X$ with $\|y\| \geq \epsilon$, we have

$$\max\{\varphi(\|x + y\|), \varphi(\|x - y\|)\} \geq \max_{-1 \leq t \leq 1} \{\varphi(\|x + ty\|)\} \geq \mathbb{E}[\varphi(\|x + ry\|)].$$

Since φ is strictly increasing we get, for any $x \in S_X$ and $y \in X$ with $\|y\| \geq \epsilon$,

$$\max\{\|x + y\|, \|x - y\|\} \geq \varphi^{-1}(1 + \delta_\varphi(\epsilon)) > 1.$$

Therefore X is uniformly convex and this completes the proof. \blacksquare

It is worthwhile to notice that Theorem 3.1 does not hold if we consider the general increasing convex function φ with $\varphi(0) = 0$. Indeed, it is easily checked that if $\varphi(t) = |t|$ and $\{r_n\}_n$ is the Rademacher sequence then for any nontrivial Banach space X ,

$$\mathbb{E}\|x + r_1x\| = 1 \quad (x \in S_X).$$

Consequently, we cannot characterize the extreme point of B_X with $\varphi(t) = t$.

We shall discuss the uniform convexity of p -convexification $E^{(p)}$ for uniformly monotone Banach lattice E . For more details on Banach lattices, order continuity and Köthe function spaces, see [11]. For the definition of p -convexification $E^{(p)}$ of E and the addition \oplus and multiplication \odot there, see [9, 11]. A Banach lattice is said to be *uniformly monotone* (resp. *upper locally uniformly monotone*) if given $\epsilon > 0$

$$M_p(\epsilon) = \inf\{\|(|x|^p + |y|^p)^{1/p}\| - 1 : \|y\| \geq \epsilon, \|x\| = 1\} > 0$$

$$\text{(resp. } N_p(\epsilon; x) = \inf\{\|(|x|^p + |y|^p)^{1/p}\| - 1 : \|y\| \geq \epsilon\} > 0)$$

for some $1 \leq p < \infty$. It is shown in [9, 10] that, given $\epsilon > 0$ and $1 \leq p < \infty$ there is a $C_p > 0$ such that for every $\epsilon > 0$,

$$(3.1) \quad C_p^{-1} M_1(C_p^{-1} \epsilon^p) \leq M_p(\epsilon) \leq M_1(\epsilon).$$

In the case when E is an order continuous Banach lattice or a Köthe function space, we also get the following relations by Lemma 2.3 in [10]: There is a $C_p > 0$ such that every $x \in S_X$ and $\epsilon > 0$,

$$(3.2) \quad C_p^{-1} N_1(C_p^{-1} \epsilon^p; x) \leq N_p(\epsilon; x) \leq N_1(\epsilon; x).$$

Notice that relations (3.1), (3.2) show that if a Banach lattice E is uniformly monotone then $E^{(1/p)}$ is uniformly monotone quasi-Banach lattice for $1 < p < \infty$. Similarly, if E is upper locally uniformly monotone order continuous Banach lattice or Köthe function space, then $E^{(1/p)}$ is also upper locally uniformly monotone quasi-Banach lattice for $1 < p < \infty$ (cf. [9]). The characterizations of local uniform monotonicity of various function spaces have been discussed in [6].

In [7], H. Hudzik, A. Kamińska and M. Mastyło showed that if a Köthe function space E is uniformly monotone then its p -convexification $E^{(p)}$ is uniformly convex for $1 < p < \infty$. A partial generalization of this result has been studied by the author in [9], where it was shown that if a Banach lattice is uniformly monotone then $E^{(p)}$ is uniformly convex for all $2 \leq p < \infty$. In the next theorem, the gap is completed.

Theorem 3.2. *Let E be a Banach lattice. The following statements are equivalent.*

- (1) E is uniformly monotone.
- (2) $E^{(p)}$ is uniformly convex for all $1 < p < \infty$.
- (3) $E^{(p)}$ is uniformly convex for some $1 < p < \infty$.

Proof. Proposition 4.4 in [9] shows that uniformly convex Banach lattice is uniformly monotone. So if we assume (3), then $E^{(p)}$ is uniformly monotone and E is uniformly monotone. Hence (3) \Rightarrow (1) is proved. The implication (2) \Rightarrow (3) is clear. So we have only to show that (1) implies (2).

We shall use Theorem 3.1 (3) with the Rademacher function $|r| = 1$. Let $\epsilon > 0$ and let $f, g \in E^{(p)}$ with $\|f\|_{E^{(p)}} = \|f\|_E^{1/p} = 1$ and $\|g\|_{E^{(p)}} = \|g\|_E^{1/p} \geq \epsilon$. Recall the following well-known inequality (cf. Lemma 4.1 [9]) : for any $1 < p < \infty$ there is $C = C(p)$ such that for any reals s, t ,

$$\left(\left| \frac{s-t}{C} \right|^2 + \left| \frac{s+t}{2} \right|^2 \right)^{\frac{1}{2}} \leq \left(\frac{|s|^p + |t|^p}{2} \right)^{\frac{1}{p}}.$$

Then applying the Krivine functional calculus to the inequality above, we get

$$\begin{aligned}
\mathbb{E} \left[\|f \oplus (r \odot g)\|_{E^{(p)}}^p \right] &= \mathbb{E}[\| |f^{1/p} + rg^{1/p}|^p \|_E] \\
&= \frac{\mathbb{E}[\| |f^{1/p} + rg^{1/p}|^p \|_E] + \mathbb{E}[\| |f^{1/p} - rg^{1/p}|^p \|_E]}{2} \\
(3.3) \quad &\geq \mathbb{E} \left[\left\| \frac{|f^{1/p} + rg^{1/p}|^p + |f^{1/p} - rg^{1/p}|^p}{2} \right\|_E \right] \\
&\geq \mathbb{E} \left\| \left(|f|^{2/p} + \frac{|2g|^{2/p}}{C^{2/p}} \right)^{p/2} \right\|_E \\
&= \left\| \left(|f|^{2/p} + \frac{|2g|^{2/p}}{C^{2/p}} \right)^{p/2} \right\|_E.
\end{aligned}$$

By (3.3), if $1 < p \leq 2$, then $\mathbb{E} \left[\|f \oplus (r \odot g)\|_{E^{(p)}}^p \right] \geq 1 + M_{2/p}(2\epsilon^p/C)$. In the case of $2 < p < \infty$, (3.3) shows that

$$\begin{aligned}
\mathbb{E} \left[\|f \oplus (r \odot g)\|_{E^{(p)}}^p \right] &\geq \left\| \left(|f|^{2/p} + \frac{|2g|^{2/p}}{C^{2/p}} \right)^{p/2} \right\|_E \\
&\geq \left\| \left(|f| + \frac{|2g|}{C} \right) \right\|_E \geq 1 + M_1(2\epsilon^p/C).
\end{aligned}$$

Hence

$$\mathbb{E}[\|f \oplus (r \odot g)\|_{E^{(p)}}^p] \geq 1 + M_{\max\{1, 2/p\}}(2\epsilon^p/C)$$

completes the proof. \blacksquare

Now we discuss the the local version of Theorem 3.2. A point $x \in S_X$ in a complex Banach space X is said to be a *complex strongly extreme point* if there is $0 < p < \infty$ such that given $\epsilon > 0$,

$$H_p(\epsilon; x) = \inf \left\{ \left(\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta}y\|^p d\theta \right)^{1/p} - 1 : \|y\| \geq \epsilon \right\} > 0.$$

It is known in [4] that $x \in S_X$ is a complex strongly extreme point if and only if for every $\epsilon > 0$,

$$H_\infty(\epsilon; x) = \inf \left\{ \max_{0 \leq \theta \leq 2\pi} \|x + e^{i\theta}y\| - 1 : \|y\| \geq \epsilon \right\} > 0.$$

For more details about these moduli, see [2, 3, 4]. A complex Banach space X is said to be *locally uniformly complex convex* if every point of S_X is a complex

strongly extreme point. The *complexification* $E^{\mathbb{C}}$ of a real Banach lattice E consists of all elements $x + iy$ for $x, y \in E$ with norm $\|x + iy\| = \left(|x|^2 + |y|^2 \right)^{\frac{1}{2}} \|E$. Then $E^{\mathbb{C}}$ is a complex Banach space (see [11]).

Theorem 3.3. *Let E be an order continuous Banach lattice or a Köthe function space. Then the following are equivalent:*

- (1) E is upper locally uniformly monotone.
- (2) $E^{\mathbb{C}}$ is locally uniformly complex convex.
- (3) $E^{(p)}$ is midpoint locally uniformly convex for all $1 < p < \infty$.
- (4) $E^{(p)}$ is midpoint locally uniformly convex for some $1 < p < \infty$.

Proof. First we prove the equivalence of (1) and (2). We shall use a similar argument as in the proof of [1, Proposition 3.7] in the sequence space. Suppose that E is locally uniformly complex convex. Then for each $x \in S_E$ and $\epsilon > 0$ there is $\delta = \delta(x, \epsilon) > 0$ such that for all $y \in X$ with $\|y\| \geq \epsilon$

$$\| |x| + |y| \| \geq \frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta} y\| \, d\theta \geq 1 + \delta.$$

So X is upper locally uniformly monotone.

For the converse, suppose that E is upper locally uniformly monotone. Now, if we use Theorem 7.1 in [2], then we have for every pair x, y in $E^{\mathbb{C}}$,

$$\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta} y\| \, d\theta \geq \left\| \left(|x|^2 + \frac{1}{2}|y|^2 \right)^{1/2} \right\|.$$

Hence for every $x \in S_X$ and $\epsilon > 0$, we get

$$\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta} y\| \, d\theta \geq 1 + N_2(\epsilon/\sqrt{2}; x).$$

Therefore, the upper local uniform monotonicity of E implies the local uniform complex convexity of E .

For (1) \Rightarrow (3), fix f with $\|f\|_{E^{(p)}} = \|f\|^{1/p} = 1$ and for any $g \in E^{(p)}$ with $\|g\|_{E^{(p)}} = \|g\|^{1/p} \geq \epsilon$, (3.3) holds. Hence

$$\mathbb{E}[\|f \oplus (r \odot g)\|_{E^{(p)}}^p] \geq 1 + N_{\max\{1, 2/p\}}(2\epsilon^p/C; f),$$

which shows that (1) \Rightarrow (3) holds.

The implication (3) \Rightarrow (4) is clear. Finally assume that (4) holds. Note that every midpoint locally uniformly convex Banach lattice is upper locally uniformly monotone. Indeed, if $x \in S_X$ and $\epsilon > 0$, there is $\delta > 0$ such that

$$1 + \delta \leq \max\{\|x + y\|, \|x - y\|\} \leq \| |x| + |y| \|.$$

Since the midpoint local uniform convexity of $E^{(p)}$ implies the upper local uniform monotonicity of $E^{(p)}$, E is upper locally uniformly monotone. This completes the proof. ■

ACKNOWLEDGMENT

The author thanks A. Kamińska for useful comments.

REFERENCES

1. Y. S. Choi, K. H. Han and H. J. Lee, Boundaries for algebras of holomorphic functions on Banach spaces, *Illinois J. Math.*, **51(3)** (2007), 883-896.
2. W. Davis, D. J. H. Garling and N. Tomczak-Jaegermann, The complex convexity of quasi-normed linear spaces, *J. Funct. Anal.*, **55** (1984), 110-150.
3. S. J. Dilworth, Complex convexity and the geometry of Banach spaces, *Math. Proc. Cambridge Philos. Soc.*, **99** (1986), 495-506.
4. P. N. Dowling, Z. Hu and D. Mupasiri, Complex convexity in Lebesgue-Bochner function spaces, *Trans. Amer. Math. Soc.*, **348** (1996), 127-139.
5. P. Dowling, Z. Hu and M. A. Smith, Geometry of spaces of vector-valued harmonic functions, *Canad. J. Math.*, **46(2)** (1994), 274-283.
6. P. Foralewski and P. Kolwicz, Local uniform rotundity in Calderón-Lozanovskii spaces, *J. Convex Anal.*, **14(2)** (2007), 395-412.
7. H. Hudzik, A. Kamińska and M. Mastyło, Geometric properties of some Calderon-Lozanovskii spaces and Orlicz-Lorentz spaces, *Houston J. Math.*, **22(3)** (1996), 639-663.
8. J. P. Kahane, *Some random series of functions*, Second ed., Cambridge Studies in Advanced Mathematics, 5. Cambridge University Press, Cambridge, 1985.
9. H. J. Lee, Complex convexity and monotonicity in quasi-Banach lattices, *Israel J. Math.*, **159** (2007), 57-91.
10. H. J. Lee, Monotonicity and complex convexity in Banach lattices, *J. Math. Anal. Appl.*, **307** (2005), 86-101.
11. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II*, Springer-Verlag, 1979.
12. B. Maurey and G. Pisier, Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach, *Studia Math.*, **58(1)** (1976), 45-90.

Han Ju Lee
Department of Mathematics Education,
Dongguk University,
Seoul, Korea
E-mail: hanjulee@dongguk.edu