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ON A NONEXPANSIVE RETRACTION RESULT OF R. E. BRUCK IN BANACH SPACES

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Abstract. In this note, we extend and improve the corresponding result of R. E. Bruck [Nonexpansive retracts of Banach spaces, *Bull. Amer. Math. Soc.* **76** (1970), 384-386].

1. Introduction

Let E be a real Banach space and C be a subset of E. We denote by $F(T)$ the set of fixed points of a mapping $T : C \to C$. In this note, a mapping $T : C \to C$ is called (i) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; (ii) *quasi nonexpansive* [3] if $\|Tx - f\| \leq \|x - f\|$ for all $x \in C$ and $f \in F(T)$; (iii) *strongly quasi nonexpansive* [6] if $\|Tx - f\| < \|x - f\|$ for all $x \in C \setminus F(T)$ and $f \in F(T)$; (iv) F-*quasi nonexpansive* (for a subset $F \subseteq F(T)$) if $\|Tx - f\| \leq \|x - f\|$ for all $x \in C$ and $f \in F$; (v) *strongly* F-*quasi nonexpansive* (for a subset $F \subseteq F(T)$) if $\|Tx - f\| < \|x - f\|$ for all $x \in C \setminus F(T)$ and $f \in F$, and (vi) *retraction* if $T^2 = T$.

In [7], a strongly quasi nonexpansive mapping is called *attracting quasi nonexpansive*. From the above definitions, it follows that a nonexpansive mapping must be quasi nonexpansive, a quasi nonexpansive mapping must be F-quasi nonexpansive, and a strongly quasi nonexpansive mapping must be strongly F-quasi nonexpansive. A Banach space E is said to be *strictly convex* if for every $x, y \in E$ satisfying $\|x\|, \|y\| \leq 1$ and $\|x - y\| > 0$, we have $\|(x + y)/2\| < 1$. If C is a convex subset of a strictly convex space and $T : C \to C$ is a quasi nonexpansive mapping then it is easy to verify that the mapping $(T + I)/2$ is a strongly quasi nonexpansive mapping. The existence of the quasi nonexpansive retractions is related to the concept of the *1-local retracts* due to M. A. Khamsi [4]. In [5], it is shown that in a Banach space E, $C_0 \subseteq E$ is a 1-local retract of C if and only if there is a quasi nonexpansive retraction P from C onto C_0.

The following theorem is due to Bruck [1].

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Bruck's Theorem

If C is a closed convex subset of a strictly convex reflexive Banach space, then for any family $\{T_\alpha\}_{\alpha\in I}$ of nonexpansive mappings from C into itself such that $F = \cap_{\alpha\in I} F(T_\alpha) \neq \varnothing$, there exists a nonexpansive retraction R from C onto F.

The existence of such a retraction commuting with the mappings is still unclear. In this note, we give a partial answer to this problem for nonexpansive and quasi nonexpansive mappings. Moreover, we show that if in Bruck's Theorem T_t's are strongly F-quasi nonexpansive then the result holds without the strict convexity assumption on E.

2. Main Theorem

Suppose that C is a nonempty locally weakly compact closed convex subset of a Banach space E and $\{T_\alpha\}_{\alpha\in I}$ is an arbitrary family of (F-quasi) nonexpansive mappings from C into itself, such that $F = \cap_{\alpha\in I} F(T_\alpha) \neq \varnothing$. If either the mappings $\{T_\alpha\}_{\alpha\in I}$ are strongly F-quasi nonexpansive or E is strictly convex then for each $\alpha \in I$ there exists a (quasi) nonexpansive retraction R_α from C onto F such that $R_\alpha T_\alpha = T_\alpha R_\alpha = R_\alpha$, and every closed convex $\{T_\alpha\}_{\alpha\in I}$-invariant subset of C is also R_α-invariant.

This theorem guarantees the existence of retractions that can commute with the mappings. It is worth mentioning that in this result we can replace the nonexpansive mappings with quasi nonexpansive mappings. Closed convex sets in a reflexive Banach space are locally weakly compact.

Proof. We prove only in the case that T_α's are F-quasi nonexpansive; the proof for the case that T_α's are nonexpansive is similar. Fix an $\alpha_0 \in I$ and consider C^C with the product topology induced by the weak topology on C. Now, define

$$\Re := \{T \in C^C : \|Tx - z\| \leq \|x - z\| \ (\forall x \in C, z \in F),\ TT_{\alpha_0} = T,$$

$$\text{and every closed convex } \{T_\alpha\}_{\alpha\in I}\text{-invariant subset of } C \text{ is also } T\text{-invariant}\}.$$

Since any $z \in F$ is a closed convex $\{T_\alpha\}_{\alpha\in I}$-invariant set we know that $Tz = z$, for all $T \in \Re$. Fix $z_0 \in F$ and for $x \in C$ define $C_x = \{y \in C : \|y - z_0\| \leq \|x - z_0\|\}$. For each $x \in C$ and $T \in \Re$, $T(x) \in C_x$ since $T(C) \subseteq C$ and $\|Tx - z_0\| \leq \|x - z_0\|$. Thus $\Re$ is a subset of the Cartesian product $\Psi = \prod_{x\in C} C_x$. Since C is convex and locally weakly compact, each C_x is convex and weakly compact. If C_x is given the weak topology and Ψ is given the corresponding product topology, then by Tychonoff's theorem Ψ is compact. We show that $\Re$ is closed in Ψ. Suppose that $\{S_\lambda : \lambda \in \Lambda\}$ is a net in $\Re$ which converges to S in Ψ. Then for $z \in F$, $S_\lambda(z) = z$ so $S(z) = w - \lim_\lambda S_\lambda(z) = z$. By the weak lower semi continuity of the norm, for any x in C and $z \in F$, $\|Sx - z\| = \|w - \lim_\lambda S_\lambda(x) - z\| \le \liminf_\lambda \|S_\lambda x - z\| \le \|x - z\|$. On the other hand it easily follows that $ST_{\alpha_0} = S$. So we have shown that $S \in \Re$, hence that $\Re$ is closed in Ψ. Since Ψ is compact, therefore $\Re$ is compact. Furthermore $\Re \neq \varnothing$. Indeed, consider the mapping $S_n = \dfrac{1}{n}\sum_{i=0}^{n-1} T_{\alpha_0}^i$. The sequence $\{S_n\}_{n\ge1}$ satisfies $S_nT_{\alpha_0} - S_n \to 0$ as $n \to \infty$ on C (since C_x is bounded for all $x \in C$) and it has a (pointwise weakly) convergent subnet $\{S_{n(\eta)}\}_\eta$. Define for each $x \in C$, $S(x) = w - \lim_\eta S_{n(\eta)}(x)$. We will check that $S \in \Re$. Fix $x \in C$ and $z \in F(T)$. Since T_{α_0} is quasi nonexpansive $\|Sx - z\| \le \liminf_\eta \|S_{n(\eta)}(x) - z\| \le \|x - z\|$. Moreover, $S(T_{\alpha_0}x) = w - \lim_\eta S_{n(\eta)}(T_{\alpha_0}x) = w - \lim_\eta S_{n(\eta)}(x) = S(x)$, because $S_nT_{\alpha_0} - S_n \to 0$ as $n \to \infty$ on C. Finally, if D is a closed convex $\{T_\alpha\}_{\alpha\in I}$-invariant subset of C, by convexity it is clear that D is S_n-invariant and thus S-invariant. Define a preorder $\preceq$ in $\Re$ by setting $T \prec S$ if $\|Tx - z\| \le \|Sx - z\|$ for all $x \in C$ and $z \in F$, with strict inequality holding for at least one pair of x, z; then set $T \preceq S$ to mean $T \prec S$ or $\|Tx - z\| = \|Sx - z\|$ for all $x \in C$ and $z \in F$ and using the Bruck's method [2] we obtain a minimal element $R \in \Re$. Indeed, by considering Zorn's lemma it suffices to show every linearly ordered subset of $\Re$ has a lower bound in $\Re$. If $\{S_\lambda\}$ is a linearly ordered subset of $\Re$ by $\preceq$, the family of sets $\{T \in \Re : T \preceq S_\lambda\}$ is linearly ordered by inclusion. The proof that $\Re$ is closed in Ψ can be repeated to show that this sets are closed in $\Re$, and hence compact. So there exists $S \in \bigcap_\lambda\{T \in \Re : T \preceq S_\lambda\}$ with $S \preceq S_\lambda$ for each λ. Now, we have shown the existence of a minimal element $R_{\alpha_0} \in \Re$ in the following sense:

$$\text{if } T \in \Re \text{ and } \|T(x) - z\| \le \|R_{\alpha_0}(x) - z\|, \forall x \in C, z \in F,$$

$$\text{then } \|T(x) - z\| = \|R_{\alpha_0}(x) - z\|. \quad (*)$$

We shall prove that R_{α_0} is a quasi nonexpansive retraction from C onto F. Since $R_{\alpha_0} \in \Re$, $T_\alpha R_{\alpha_0} T_{\alpha_0} = T_\alpha R_{\alpha_0}, \forall\alpha$, so it easily follows that $T_\alpha R_{\alpha_0} \in \Re, \forall\alpha$. It is easy to verify that $\Re$ is convex. Therefore using $(*)$ we have

$$\|T_\alpha R_{\alpha_0}x - z\| = \|R_{\alpha_0}x - z\| = \|\frac{1}{2}(T_\alpha R_{\alpha_0}x + R_{\alpha_0}x) - z\|, \quad (**)$$

for all $x \in C, z \in F(T)$ and $\alpha \in I$. Now, if $\{T_\alpha\}_{\alpha\in I}$ are strongly F-quasi nonexpansive, using the first equality in $(**)$ we have $R_{\alpha_0}x \in F(T_\alpha)$ ($\forall x \in C, \alpha \in I$) and it means that $R_{\alpha_0}x \in F = \cap_{\alpha\in I}F(T_\alpha)$. Therefore, R_{α_0} is the desired quasi nonexpansive retraction from C onto F and the proof is completed by considering the condition that the mappings $\{T_\alpha\}_{\alpha\in I}$ are strongly F-quasi nonexpansive. To prove in the case that E is strictly convex, it suffices to consider the second equality in $(**)$. ∎

Corollary 1. *Let E, C, $\{T_\alpha\}_{\alpha\in I}$, F and $\{R_\alpha\}_{\alpha\in I}$ be as above. Then, for all $x \in C$*

$$\overline{\{T^n_\alpha x : \alpha \in I,\ n \in \mathbb{N}\}} \cap F \subseteq \overline{\{R_\alpha x : \alpha \in I\}}.$$

Proof. Assume that $f \in \overline{\{T^n_\alpha x : \alpha \in I,\ n \in \mathbb{N}\}} \cap F$ and consider an $\varepsilon > 0$. Then there exist $\alpha \in I$ and $n \in \mathbb{N}$ such that $\|T^n_\alpha x - f\| < \varepsilon$. On the other hand

$$\|R_\alpha x - f\| = \|R_\alpha T^n_\alpha x - f\| \leq \|T^n_\alpha x - f\| < \varepsilon,$$

which implies $f \in \overline{\{R_\alpha x : \alpha \in I\}}$ and the proof is complete. ■

In what follows, by embedding E in $l^\infty(E)$, we study the existence of a retraction that satisfy a kind of commutativity for all mappings. In order to state this result we need the following notations:

$$l^\infty(E) = \{(x_\alpha) : \ \ x_\alpha \in E \text{ for all } \alpha \in I \text{ and } \|(x_\alpha)\| = \sup_\alpha \|x_\alpha\|\};$$

$$\widetilde{C} = \{(x_\alpha) \in l^\infty(E) : \ \ x_\alpha \in C \text{ for all } \alpha \in I\}.$$

For a family $\{T_\alpha\}_{\alpha\in I}$ of nonexpansive mappings from C into C, the mapping $(T_\alpha) : \widetilde{C} \to \widetilde{C}$ will be defined as $(T_\alpha)(x_\alpha) = (T_\alpha x_\alpha)$, for all $(x_\alpha) \in \widetilde{C}$. For a subset K in E, when we consider K in $l^\infty(E)$ we mean that K is embedded in $l^\infty(E)$ by the inclusion mapping $x \hookrightarrow (x_\alpha)$, where $x_\alpha = x$ for all $\alpha \in I$.

Corollary 2. *Suppose that C is a nonempty locally weakly compact closed convex subset of a Banach space E and $\{T_\alpha\}_{\alpha\in I}$ is an arbitrary family of (F-quasi) nonexpansive mappings from C into itself, such that $F = \cap_{\alpha\in I} F(T_\alpha) \neq \varnothing$. If either the mappings $\{T_\alpha\}_{\alpha\in I}$ are strongly F-quasi nonexpansive or E is strictly convex then there exists a (quasi) nonexpansive retraction R from $\widetilde{C}$ onto $F \subset l^\infty(E)$ such that $Ro(T_\alpha) = (T_\alpha)oR = R$ and every closed convex φ-invariant subset of C is R-invariant in $l^\infty(E)$.*

Proof. Let $\mathfrak{U}$ be an ultrafilter defined on a set I, and $\{R_\alpha\}_{\alpha\in I}$ be retractions as in the above theorem. Now, define

$$R(x_\alpha) = w - \lim_{\mathfrak{U}} R_\alpha x_\alpha \in F.$$

Then for all $(x_\alpha) \in \widetilde{C}$ we have that

$$Ro(T_\alpha)(x_\alpha) = R(T_\alpha x_\alpha) = w - \lim_{\mathfrak{U}} R_\alpha T_\alpha x_\alpha = w - \lim_{\mathfrak{U}} R_\alpha x_\alpha = R(x_\alpha) = (T_\alpha)R(x_\alpha),$$

where the last equality follows from $R(x_\alpha) \in F$. Therefore, $Ro(T_\alpha) = (T_\alpha)oR = R$. ■

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