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RICCI OPERATORS AND STRUCTURAL JACOBI OPERATORS ON REAL HYPERSURFACES IN A COMPLEX SPACE FORM

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Abstract. We give a classification of real hypersurfaces in a non-flat complex space form, whose almost contact structure operator, induced from the complex structure of the complex space form, commutes with the Ricci operator and at the same time commutes with the structural Jacobi operator. In particular, we classify real hypersurfaces in 2-dimensional complex projective and hyperbolic spaces satisfying the first commutativity condition.

1. Introduction

Let M be an oriented real hypersurface in a non-flat complex space form $\widetilde{M}_n(c)$, $c \neq 0$. Let $(\tilde{g}, J)$ be a Hermitian structure of $\widetilde{M}_n(c)$ and N be a unit normal vector field on M in $\widetilde{M}_n(c)$. A real hypersurface M is called a *Hopf hypersurface* if the Reeb vector field $\xi = -JN$ is a principal vector field (with respect to N), that is, $A\xi = \alpha_1 \xi$, where A denotes the shape operator. Hopf hypersurfaces in P_nC are realized as tubes over certain Kähler submanifolds with constant rank of the focal maps $\varphi_r : N_1M \to P_nC$, which are defined by $\varphi_r(N) = F(rN)$, where $F : NM \to P_nC$ is the normal exponential map and NM (N_1M, respectively) denotes the normal bundle (the unit normal bundle, respectively) of M (cf. [3]). R. Takagi [17], [18] classified homogeneous real hypersurfaces of P_nC as six model spaces. By making use of those model spaces and the tube construction in [3], M. Kimura [8] proved the following

Theorem 1. *Let M be a Hopf hypersurface of P_nC. Then M has constant principal curvatures if and only if M is locally congruent to one of the following ones:*

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(A_1) *a geodesic hypersphere of radius* r*, where* $0 < r < \pi/2$*,*

(A_2) *a tube of radius* r *over a totally geodesic* P_kC*, where* $0 < r < \pi/2$ *and* $1 \leq k \leq n-2$*,*

(B) *a tube of radius* r *over a complex quadric* Q_{n-1}*, where* $0 < r < \pi/4$*,*

(C) *a tube of radius* r *over* $P_1C \times P_{(n-1)/2}C$*, where* $0 < r < \pi/4$ *and* $n(\geq 5)$ *is odd,*

(D) *a tube of radius* r *over a complex Grassmann* $G_{2,5}C$*, where* $0 < r < \pi/4$ *and* $n = 9$*,*

(E) *a tube of radius* r *over a Hermitian symmetric space* $SO(10)/U(5)$*, where* $0 < r < \pi/4$ *and* $n = 15$*.*

On the other hand, real hypersurfaces of a complex hyperbolic space H_nC have also been studied by S. Montiel [13], S. Montiel and A. Romero [14], J. Berndt [2] and so on. J. Berndt [2] classified Hopf hypersurfaces with constant principal curvatures of H_nC. Namely, he proved the following

Theorem 2. *Let* M *be a Hopf hypersurface of* H_nC*. Then* M *has constant principal curvatures if and only if* M *is locally congruent to one of the following ones:*

(A_0) *a horosphere,*

(A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane* $H_{n-1}C$*,*

(A_2) *a tube over a totally geodesic* H_kC*, where* $1 \leq k \leq n-2$*,*

(B) *a tube over a totally real hyperbolic space* H_nR*.*

A real hypersurface of type (A_1), (A_2) in Theorem 1 and of type (A_0), (A_1), (A_2) in Theorem 2 is simply called a real hyperspace of type (A). There are many characterizations of real hypersurfaces of type (A) (cf. [15]). In particular, M. Okumura ([16]) (resp. Montiel and A. Romero ([14])) proved that $\phi A = A\phi$ if and only if M is locally congruent to one of type (A) in P_nC (resp. H_nC). Actually, he obtained the result by showing the commutativity of ϕ and A is equivalent to the parallelism of the second fundamental form of the hypersurface of the sphere S^{2n+1} which is a S^1-bundle on M via the restriction of the Hopf fibration $\pi : S^{2n+1} \to P_nC$. It is easily seen that the condition $\phi A = A\phi$ implies that $\phi S = S\phi$ in a real hypersurface of P_nC or H_nC, where S denotes the Ricci operator. Motivated by the Okumura's work one may try to find a geometric property for the hypersurface $\pi^{-1}M$ in S^{2n+1} when M satisfies $\phi S = S\phi$ using the lifts by the Hopf fibration or the corresponding principal S^1-bundle over H_nC. (See Proposition 1 in Section 2). In this context, it is interesting to ask that ϕ commute only with S (that is, $\phi S = S\phi$

and $\phi A \neq A\phi$). This problem was studied by M. Kimura ([9]), U-H. Ki and Y. J. Suh ([7]). (We may also refer to [15]). Indeed, in the range of Hopf hypersurfaces of P_nC, $n \geq 3$, all types $(A) \sim (E)$ with some restrictions to the radii and a non-homogeneous real hypersurface satisfy $\phi S = S\phi$. Among Hopf hypersurfaces of H_nC, $n \geq 3$, only the type (A) satisfy the commutation condition. However, we do not know so far a non-Hopf hypersurface in a non-flat complex space form which satisfies $\phi S = S\phi$. In these situations, we may raise the following question:

> *Is a real hypersurface in P_nC or H_nC which satisfies $\phi S = S\phi$ always a Hopf hypersurface? In particular, classify such a real hypersurface in P_2C or H_2C.*

The structural Jacobi operator $R_\xi = R(\cdot, \xi)\xi$, which is a self-adjoint operator along the Reeb flow ξ, has a fundamental role in (almost) contact geometry. Recently, it is investigated actively for real hypersurfaces in a non-flat complex space form (cf. [4, 5, 6]).

Concerning the above question, we consider an additional condition, namely that the induced complex operator ϕ commutes also with the structural Jacobi operator $R_\xi = R(\cdot, \xi)\xi$. Here, it is notable that the condition $\phi S = S\phi$ already implies $\phi R_\xi = R_\xi \phi$ when $n = 2$. Then we prove the following two theorems.

Theorem 3. *Let M be a real hypersurface of P_nC. If M satisfies $\phi S = S\phi$ and $\phi R_\xi = R_\xi \phi$ at the same time, then M is locally congruent to one of the following:*

(1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$,*

(2) *a tube of radius r over a totally geodesic P_kC, where $0 < r < \pi/2$ and $1 \leq k \leq n-2$,*

(3) *a non-homogeneous real hypersurface in P_nC which lies on a tube of radius $\pi/4$ over a $(n/2)$-dimensional (n: even) Kähler submanifold $\tilde{N}$ with the rank of each shape operator is not greater than 2 and with non-zero principal curvatures $\neq \pm 1$, where the rank of the corresponding focal map $\varphi_{\pi/4}$ is constant.*

Theorem 4. *Let M be a real hypersurface of H_nC. If M satisfies $\phi S = S\phi$ and $\phi R_\xi = R_\xi \phi$ at the same time, then M is locally congruent of the following:*

(1) *a horosphere,*

(2) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}C$,*

(3) *a tube over a totally geodesic H_kC, where $1 \leq k \leq n-2$.*

In the process of proving the above theorems, we find the following interesting results:

- A real hypersurface of $\widetilde{M}_n(c)$ $(c \neq 0)$ satisfies $\phi \cdot R = 0$ if and only if M is locally congruent to homogeneous real hypersurface of type (A), where $\cdot$ means that the $(1,1)$-tensor field ϕ operates on the tensor R as a derivation. (Corollary 3.)

  This is a development of the result of Y. Maeda (Theorem 5.4 in [11]). In reality, he proved the above fact under the assumption that ξ is a principal curvature vector field and $n \geq 3$.
- A three-dimensional real hypersurface of $\widetilde{M}_2(c)$ $(c \neq 0)$ is η-Einstein or satisfies $\phi S = S\phi$ if and only if it is locally congruent to type (A_1), a non-homogeneous real hypersurface in P_2C which lies on a tube of radius $\pi/4$ over a complex curve Σ_1 with non-zero principal curvature $\neq \pm 1$, where the rank of the corresponding focal map $\varphi_{\pi/4}$ is constant; (A_0) or (A_1) in H_2C. (Corollary 4).

  This fact gives an answer to the question mentioned above for the three-dimensional case and an answer to the open problems (Question 9.5 and Question 9.10) in [15].

2. Preliminaries

All manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented. At first, we review the fundamental facts on a real hypersurface of a n-dimensional complex space form $\widetilde{M}_n(c)$ with constant holomorphic sectional curvature c. Let M be an orientable real hypersurface of $\widetilde{M}_n(c)$ and let N be a unit normal vector on M. We denote by $\tilde{g}$ and J a Kähler metric tensor and its Hermitian structure tensor, respectively. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi, \tag{1}$$

where ϕ is a (1,1)-type tensor field, η is a 1-form and ξ is a unit vector field on M, which is called ***Reeb vector field***. The induced Riemannian metric on M is denoted by g. Then by properties of $(\tilde{g}, J)$ we see that the structure (ϕ, ξ, η, g) is an almost contact metric structure on M, that is, from (1) it follows that

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1 \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \end{aligned} \tag{2}$$

for any vector fields X and Y tangent to M. From (2), we have

$$\phi\xi = 0, \ \eta \circ \phi = 0, \ \eta(X) = g(X, \xi)$$

The Gauss and Weingarten formula for M are given as

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,$$

$$\tilde{\nabla}_X N = -AX$$

for any tangent vector fields X, Y, where $\tilde{\nabla}$ and ∇ denote the Levi-Civita connections of $(\widetilde{M}_n(c), \tilde{g})$ and (M, g), respectively, A is the shape operator. From (1) and $\tilde{\nabla} J = 0$, we then obtain

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX. \tag{3}$$

Then we have the following Gauss equation:

$$\begin{aligned} R(X,Y)Z = (c/4)\{&g(Y,Z)X - g(X,Z)Y \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &+ g(AY, Z)AX - g(AX, Z)AY. \end{aligned} \tag{4}$$

From (4) together with (2) the Ricci operator S is given by

$$SX = (c/4)\{(2n+1)X - 3\eta(X)\xi\} + HAX - A^2X, \tag{5}$$

where H =trace A. Also, from (4) the structural Jacobi operator $R_\xi = R(\cdot, \xi)\xi$, which is self-adjoint, is given by

$$R_\xi X = (c/4)\{X - \eta(X)\xi\} + g(A\xi, \xi)AX - \eta(AX)A\xi. \tag{6}$$

Recall ([16]) that the commutativity between ϕ and the shape operator A for a real hypersurface M of P_nC interpreted in terms of the parallelism of the shape operator of the hypersurface of the sphere S^{2n+1} which is a S^1-bundle on M by the restriction of the principal fiber bundle, so-called the Hopf fibration

$$\pi : S^{2n+1} \to P_nC.$$

Analogously, the ***anti-de Sitter*** H_1^{2n+1} is considered as a principal fiber bundle over H_nC with the structure group S^1 and the projection π. We adapt the the terminology according to those of [16], and we review the contents in brief. We let again $\widetilde{M}$ represent P_nC or H_nC and $\widehat{M}$ represent S^{2n+1} or H_1^{2n+1}, respectively, with the canonical projection $\pi : \widehat{M} \to \widetilde{M}$. Then $\bar{M} = \pi^{-1}M$ is an S^1-invariant hypersurface in $\widehat{M}$, and the horizontal lift $\bar{N} = N^L$ of a unit normal vector field N is a unit normal for $\bar{M}$. Since π is a Riemannain submersion, there are vertical vector field $\bar{V}$ and a Riemannian metric $\bar{g}$ of $\bar{M}$ such that $\bar{V}$ is a unit Killing vector field for $\bar{g}$. The induced connection $\bar{\nabla}$ and the shape operator $\bar{A}$ for $\bar{M}$ satisfy

$$\tilde{\nabla}_{\bar{X}} \bar{Y} = \bar{\nabla}_{\bar{X}} \bar{Y} + \bar{g}(\bar{A}\bar{X}, \bar{Y})\bar{N}, \quad \tilde{\nabla}_{\bar{X}} \bar{N} = -\bar{A}\bar{X},$$

and the well-known form of the Gauss equation is derived by

$$\bar{R}(\bar{X},\bar{Y})\bar{Z}=\bar{g}(\bar{Y},\bar{Z})\bar{X}-\bar{g}(\bar{X},\bar{Z})\bar{Y}+\bar{g}(\bar{A}\bar{Y},\bar{Z})\bar{A}\bar{X}-\bar{g}(\bar{A}\bar{X},\bar{Z})\bar{A}\bar{Y}.$$

Then the Ricci tensor $\bar{S}$ is given by

$$\bar{S}X=(2n-1)\bar{X}+\bar{H}\bar{A}\bar{X}-\bar{A}^2\bar{X}, \tag{7}$$

where $\bar{H}$ =trace $\bar{A}$. Since $\xi^L=\bar{W}=-J^L\bar{N}$, we see that the Jacobi operator $\bar{R}_W=\bar{R}(\cdot,\bar{W})\bar{W}$ is given by

$$\bar{R}_{\bar{W}}\bar{X}=\bar{X}-\bar{g}(\bar{X},\bar{W})\bar{W}+\bar{g}(\bar{A}\bar{W},\bar{W})\bar{A}\bar{X}-\bar{g}(\bar{A}\bar{X},\bar{W})\bar{A}\bar{W}. \tag{8}$$

Moreover, we have (cf. [16])

$$\bar{g}(X^L,Y^L)=g(X,Y)^L,\quad \bar{\nabla}_{\bar{V}}\bar{X}=-J^LX^L, \tag{9}$$

$$\bar{g}(\bar{A}X^L,Y^L)=g(AX,Y)^L,\quad \bar{H}=H^L. \tag{10}$$

Together with (9) and (10), we further find that

$$\bar{g}(\bar{S}X^L,Y^L)=g(SX,Y)^L,\quad \bar{g}(\bar{R}_{\bar{W}}X^L,Y^L)=g(R_\xi X,Y)^L.$$

Thus, differentiate the above equations covariantly in the direction $\bar{V}$ and make use of (9) and (10) again to have

Proposition 1. *Let M^{2n-1} be a real hypersurface in a complex space form $\widetilde{M}^n(c)$, $c\neq 0$. If the Ricci operator $\bar{S}$ (the Jacobi operator $\bar{R}_W$, respectively) of $\bar{M}(=\pi^{-1}M)$ is parallel, then ϕ and S (ϕ and R_ξ, respectively) commutes.*

The converse problem of the above proposition seems to be closely related with the classification of real hypersurfaces satisfying each commutativity condition. Now we return to a real hypersurface M in a non-flat complex space form $\widetilde{M}_n(c)$. Consider the vector field $U=\nabla_\xi\xi$ and denote $\alpha_m=\eta(A^m\xi)$. Then from (2) and (3) we easily observe that

$$g(U,\xi)=0,\ \ g(U,A\xi)=0,$$

$$\|U\|^2=g(U,U)=\alpha_2-\alpha_1^2.$$

From (2) and (3) we see at once that ξ is a principal curvature vector field if and only if $\|U\|^2=\alpha_2-\alpha_1^2=0$.

3. Proofs of Theorems 3 and 4

Let M be a real hypersurface of a complex space form $\widetilde{M}_n(c)$, $c \neq 0$. Suppose that

$$\phi S = S\phi \tag{11}$$

and

$$\phi R_\xi = R_\xi \phi. \tag{12}$$

We first prove that ξ is principal. From (5) and (11), we have

$$(A^2\phi - \phi A^2) = H(A\phi - \phi A). \tag{13}$$

Also, from (6) and (12), it is obtained that

$$\alpha_1(A\phi - \phi A)X = -g(U, X)A\xi - \eta(AX)U. \tag{14}$$

We get at once a relation:

$$\alpha_1(A\phi^2 - \phi^2 A) = \alpha_1\Big((A\phi - \phi A)\phi + \phi(A\phi - \phi A)\Big). \tag{15}$$

But, from (14) it follows that

$$\alpha_1\Big(\phi(A\phi - \phi A)X + (A\phi - \phi A)\phi X\Big) = -\eta(AX)\phi U - g(U, \phi X)A\xi. \tag{16}$$

From (15) and (16) it follows that

$$\alpha_1(A\phi^2 - \phi^2 A) = -\eta(AX)\phi U - g(U, \phi X)A\xi.$$

Use (13) to obtain

$$\alpha_1 H(A\phi - \phi A)X = -\eta(AX)\phi U - g(U, \phi X)A\xi. \tag{17}$$

On the other hand, from (14) we get

$$\alpha_1 H(A\phi - \phi A)X = -Hg(U, X)A\xi - H\eta(AX)U. \tag{18}$$

The above two equations (17) and (18) yield that

$$-\eta(AX)\phi U - \big(g(U, \phi X) - Hg(U, X)\big)A\xi + H\eta(AX)U = 0.$$

We put $X = \xi$, then we get

$$-\alpha_1\phi U + H\alpha_1 U = 0.$$

From this we can see that $\alpha_1 = 0$ or $\|U\| = 0$. But, from (14) we can easily see that ξ is principal where $\alpha_1 = 0$. We eventually have shown that ξ is principal on all M. Namely, we have

Lemma 2. *Let M be a real hypersurface of $\widetilde{M}_n(c)$, $c \neq 0$. If M satisfies $\phi S = S\phi$ and $\phi R_\xi = R_\xi \phi$ at the same time, then M is a Hopf hypersurface.*

Then, due to the results in [11, 12] and [7], we already know that α_1 is constant along M. From (14) we have

$$\alpha_1(\phi A - A\phi) = 0.$$

Now we divide our arguments into two cases: (i) $n \geq 3$ and (ii) $n = 2$. First, we treat the case (i). For $\widetilde{M}_n(c) = P_nC$ we consider the above equation together with the classification theorem for Hopf hypersurfaces in P_nC satisfying $\phi S = S\phi$ ([9]). Then, we can find other than real hypersurfaces of type (A) in P_nC a non-homogeneous real hypersurface in P_nC which lies on a tube of radius $\pi/4$ over a $(n/2)$-dimensional (n: even) Kähler submanifold $\tilde{N}$ with the rank of each shape operator is not greater than 2 and with non-zero principal curvatures $\neq \pm 1$, where the rank of the corresponding focal map $\varphi_{\pi/4}$ is constant, holds our conditions. In case that $\widetilde{M}_n(c) = H_nC$, since α_1 can not be zero (cf. [2]), we have $\phi A = A\phi$, and hence by the result of S. Montiel and A. Romero [14], we see that M is locally congruent to a real hypersurface of type (A) in H_nC.

Next, we look at the case $n = 2$. It is well-known that the curvature tensor R of a three-dimensional Riemannian manifold is written as :

$$\begin{aligned} R(X,Y)Z =& \{g(Y,Z)SX - g(X,Z)SY + g(SY,Z)X - g(SX,Z)Y\} \\ & - (r/2)\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned}$$

where r denotes the scalar curvature. We easily see that $\phi S = S\phi$ implies $S\xi = \sigma\xi$ and further implies that the relation $\phi R_\xi = R_\xi\phi$ in a 3-dimensional real hypersurface M. By Lemma 2 we may write $A\xi = \alpha_1\xi$ and may put

$$AV = \beta V, \quad A\phi V = \gamma\phi V$$

for a unit vector V orthogonal to ξ. From (5) it follows that

$$S\xi = p\xi, \ SV = qV, \ S\phi V = d\phi V,$$

where we have put $p = c/2 + H\alpha_1 - \alpha_1^2,\ q = 5c/4 + H\beta - \beta^2,\ d = 5c/4 + H\gamma - \gamma^2$. The assumption $\phi S = S\phi$ gives $q = d$. So, together with the above relations we easily get

$$\alpha_1(\beta - \gamma) = 0,$$

where we have used $H = \alpha_1 + \beta + \gamma$. So, we see that $\alpha_1 = 0$ or M is totally η-umbilical, that is $A = aI + b\eta \otimes \xi$ for smooth functions a, b on M. It is known that these two functions a and b are already constant. As mentioned in the argument of (i), there can not occur $\alpha_1 = 0$ in the case H_nC. Hence, with the classification of totally η-umbilical hypersurfaces (cf. [3, 13, 18]) we can see that M is locally congruent to type (A_1) or a non-homogeneous real hypersurface in P_2C which lies on a tube of radius $\pi/4$ over a complex curve Σ_1 with non-zero principal curvatures $\neq \pm 1$, where the rank of the corresponding focal map $\varphi_{\pi/4}$ is constant; (A_0) or (A_1) in H_2C.

Summing up all the arguments so far, we have completed Theorems 3 and 4.

4. Corollaries and Remarks

The condition $\phi \cdot R = 0$ means ϕ operates R as a derivation. Namely,

(19) $$(\phi \cdot R)(X,Y)Z = \phi R(X,Y)Z - R(\phi X,Y)Z - R(X,\phi Y)Z - R(X,Y)\phi Z$$

for any vector fields X, Y, Z on M. For an adapted orthonormal basis $\{e_I\} = \{e_i, \phi e_i,\ e_{2n-1} = \xi\}_{i=1,2,\cdots,n-1}$ if we put $Y = Z = e_I$, and summing for I, then we can deduce that $\phi S = S\phi$. Also, if we put $Y = Z = \xi$, then we get also easily $\phi R_\xi = R_\xi \phi$. But, we note that under the condition $\phi \cdot R = 0$ and $n \geq 3$ the case $A\xi = 0$ only contributes to a real hypersurface of type (A) in P_nC, $n \geq 3$ (see, Lemmas 5.2 and 5.3 in [11]). Next, in order to treat the case $n = 2$ and $\alpha_1 = 0$ we derive the following relation which express the equation (19) with ϕ, A and g :

(20) $$\begin{aligned} &g(AY,Z)(A\phi - \phi A)X + g((A\phi - \phi A)Y,Z)AX \\ &- g((A\phi - \phi A)X,Z)AY - g(AX,Z)(A\phi - \phi A)Y = 0 \end{aligned}$$

for any tangent vector fields X, Y, Z on M. Since $A\xi = 0$ we may assume that $AV = \lambda V$ and $A\phi V = (1/\lambda)\phi V$, $V \perp \xi$, $\|V\| = 1$ (cf. Lemma 2.2 in [11]). If we put $X = V$ and $Y = Z = \phi V$ in (20), then we get $\lambda = 1/\lambda$, which says that $A\phi = \phi A$. Thus we have

Corollary 3. *Let M be a real hypersurface of $\widetilde{M}_n(c)$, $c \neq 0$. Then M satisfies $\phi \cdot R = 0$ if and only if M is locally congruent to one of homogeneous real hypersurface of type (A).*

Remark 1. The above corollary is a development of the result of Y. Maeda ([11]). Actually, he determined a Hopf hypersurface in P_nC, $n \geq 3$ which satisfies $\phi \cdot R = 0$.

We can easily show that in 3-dimensional M the η-Einstein condition (i.e., $S = \lambda I + \mu\eta \otimes \xi$) is equivalent to the condition $\phi S = S\phi$, where λ, μ are functions in M. Thus we have

Corollary 4. *Let M be a three-dimensional real hypersurface of $\widetilde{M}_2(c)$, $c \neq 0$. Then M is η-Einstein or satisfies $\phi S = S\phi$ if and only if M is locally congruent to type (A_1) or a non-homogeneous real hypersurface in P_2C which lies on a tube of radius $\pi/4$ over a complex curve Σ_1 with non-zero principal curvatures $\neq \pm 1$, where the rank of the corresponding focal map $\varphi_{\pi/4}$ is constant; (A_0) or (A_1) in H_2C.*

Remark 2. Corollary 4 gives the answers to Question 9.5 and Question 9.10 in [15].

Remark 3. Ruled real hypersufaces in P_nC and H_nC given in [10] and [1], respectively. Let $\gamma : I \to \widetilde{M}_n(c)$ be a regular curve in $\widetilde{M}_n(c)$ $(P_nC$ or $H_nC)$. For each $t \in I$, let $M^{(t)}_{n-1}(c)$ be a totally geodesic complex hypersurfaces which is orthogonal to holomorphic plane Span$\{\dot\gamma, J\dot\gamma\}$. Then we have a ruled real hypersurface $M = \bigcup_{t\in I} M^{(t)}_{n-1}(c)$. Ruled real hypersurfaces are non-Hopf, but fail to satisfy $\phi S = S\phi$. In fact, the shape operator A of M is written as :

$$\begin{aligned} &A\xi = \alpha_1\xi + \nu V \ (\nu \neq 0), \\ &AV = \nu\xi, \\ &AX = 0 \text{ for any } X \perp \xi, V, \end{aligned}$$

where V is a unit vector field orthogonal to ξ, α_1, ν are differentiable functions on M. From (5), we have

$$\begin{aligned} &S\xi = f\xi, \\ &SV = gV, \\ &SX = \frac{c}{4}(2n+1)X \quad \text{for any } X \perp \xi, V, \end{aligned}$$

where $f = \frac{c}{2}(n-1) - \nu^2$ and $g = \frac{c}{4}(2n+1) - \nu^2$. From $\phi SV = S\phi V$, we get $\nu = 0$, impossible.

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