TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 4, pp. 1249-1270, August 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

TORSION THEORIES AND ESSENTIAL FLAT ENVELOPES

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Abstract. Let R be a ring with identity and let $\mathcal{M}_R$ be the category of right R-modules. In this article, we study some relations between torsion theories and cotorsion theories in $\mathcal{M}_R$. As applications, we give some new characterizations of IF rings with essential flat envelopes.

1. Notation

In this section, we shall recall some known notions and definitions which we need in the later sections.

Throughout this article, R is an associative ring with identity and all modules are unitary R-modules. We write M_R ($_RM$) to indicate a right (left) R-module. $\mathcal{M}_R$ denotes the category of right R-modules. As usual, $E(M)$ stands for the injective envelope of M. General background material can be found in [1, 15, 28, 31, 32, 33].

A *torsion theory* in $\mathcal{M}_R$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of right R-modules [11, 31] such that

(1) $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$;

(2) If $\mathrm{Hom}(C, F) = 0$ for all $F \in \mathcal{F}$, then $C \in \mathcal{T}$;

(3) If $\mathrm{Hom}(T, C) = 0$ for all $T \in \mathcal{T}$, then $C \in \mathcal{F}$.

For each right R-module M, there exists an exact sequence $0 \to T \to M \to F \to 0$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

A *preradical* τ [7, 31] of $\mathcal{M}_R$ is a subfunctor of the identity functor of $\mathcal{M}_R$, moreover, if τ is left exact, we call τ a *left exact preradical*. To a preradical τ, $\tau(R)$ is a two-sided ideal and $M\tau(R) \subseteq \tau(M)$ for any right R-module M.

If τ_1 and τ_2 are preradicals, one defines preradicals $\tau_1\tau_2$ and $\tau_1 : \tau_2$ as

$$\tau_1\tau_2(M) = \tau_1(\tau_2(M)),$$

Received February 25, 2008, accepted August 11, 2008.
Communicated by Wen-Fong Ke.
2000 *Mathematics Subject Classification*: 16D40, 16S90, 18E40.
Key words and phrases: Torsion theory, Cotorsion theory, Flat envelope, IF ring.
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$$(\tau_1 : \tau_2)(M)/\tau_1(M) = \tau_2(M/\tau_1(M)).$$

A preradical τ is called *idempotent* if $\tau\tau = \tau$ and is called a *radical* if $\tau : \tau = \tau$.

To a preradical τ, one can associate two classes of right R-modules, namely

$$\mathcal{T}_\tau = \{M|\tau(M) = M\},$$
$$\mathcal{F}_\tau = \{M|\tau(M) = 0\}$$

which are called *torsion class* and *torsionfree class* of τ respectively. A right R-module M is called τ-torsion (τ-torsionfree) if $M \in \mathcal{T}_\tau$ ($M \in \mathcal{F}_\tau$). If τ is idempotent, then $\mathcal{T}_\tau = \{\tau(M)|M \in \mathcal{M}_R\}$.

Let $\mathcal{A}$ be a class of right R-modules closed under direct sums and quotients, then there exists only one idempotent preradical τ such that $\mathcal{A} = \mathcal{T}_\tau$, where $\tau(M) = \sum_{M_i \in \mathcal{A}, M_i \subseteq M} M_i$; moreover, if $\mathcal{A}$ is closed under submodules, then the corresponding idempotent preradical τ is left exact. Conversely, let $\mathcal{A}$ be the torsion class of an idempotent preradical τ, then $\tau(M) = \sum_{M_i \in \mathcal{A}, M_i \subseteq M} M_i$.

If τ is an idempotent radical, then $(\mathcal{T}_\tau, \mathcal{F}_\tau)$ is a torsion theory. Conversely, if a pair $(\mathcal{T}, \mathcal{F})$ is a torsion theory, then there exists only one idempotent radical τ such that $(\mathcal{T}, \mathcal{F}) = (\mathcal{T}_\tau, \mathcal{F}_\tau)$.

A torsion theory $(\mathcal{T}, \mathcal{F})$ is called *hereditary* [7, 31] if $\mathcal{T}$ is closed under submodules, in this case, the corresponding idempotent radical τ is left exact; $(\mathcal{T}, \mathcal{F})$ is called *cohereditary* if $\mathcal{F}$ is closed under quotients; $(\mathcal{T}, \mathcal{F})$ is called *cosplitting* if $(\mathcal{T}, \mathcal{F})$ is both hereditary and cohereditary.

Let $\mathcal{C}$ be a class of R-modules and M an R-module. Following [15], we say that a homomorphism $\phi : M \to C$ is a *$\mathcal{C}$-preenvelope* if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}(\phi, C') : \mathrm{Hom}(C, C') \to \mathrm{Hom}(M, C')$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope $\phi : M \to C$ is said to be a *$\mathcal{C}$-envelope* if every endomorphism $g : C \to C$ such that $g\phi = \phi$ is an isomorphism. A monomorphism $\phi : M \to C$ with $C \in \mathcal{C}$ is said to be a *special $\mathcal{C}$-preenvelope* of M if $\mathrm{Ext}^1(C/\phi(M), C') = 0$ for every $C' \in \mathcal{C}$. Dually we have the definitions of a (special) $\mathcal{C}$-precover and a $\mathcal{C}$-cover. $\mathcal{C}$-envelopes ($\mathcal{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

By a *monomorphic* (resp. an *essential*) *$\mathcal{C}$-envelope* of M [24] we shall mean a $\mathcal{C}$-envelope $f : M \to C$ such that f is a monomorphism (resp. an essential monomorphism). By a *ring with monomorphic* (resp. *essential*) *$\mathcal{C}$-envelopes on the right* we shall understand a ring such that all its right modules have monomorphic (resp. essential) $\mathcal{C}$-envelopes.

Let $\mathcal{A}$ be a class of R-modules. $\mathcal{A}$ is called a *resolving class* [16] if $\mathcal{A}$ is closed under extensions, $\mathcal{P}_0 \subseteq \mathcal{A}$ and $A \in \mathcal{A}$, whenever $0 \to A \to B \to C \to 0$ is a short exact sequence such that $B,\ C \in \mathcal{A}$, where $\mathcal{P}_0$ denotes the class of all projective modules; dually, we have the definition of *coresolving classes*.

A *cotorsion theory* [30, 16] for $\mathcal{M}_R$ is a pair $(\mathcal{F}, \mathcal{C})$ of classes of right R-modules such that $\mathcal{F} = {}^\perp\mathcal{C}$ and $\mathcal{C} = \mathcal{F}^\perp$, where ${}^\perp\mathcal{C} = \{F|\mathrm{Ext}_R^1(F,C) = 0 \text{ for all } C \in \mathcal{C}\}$ and $\mathcal{F}^\perp = \{C|\mathrm{Ext}_R^1(F,C) = 0 \text{ for all } F \in \mathcal{F}\}$. Let $\mathcal{A}$ be a class of right R-modules, the cotorsion theory $({}^\perp(\mathcal{A}^\perp), \mathcal{A}^\perp)$ is said to be *generated* by $\mathcal{A}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called *hereditary* if $\mathcal{F}$ is a resolving class or $\mathcal{C}$ is a coresolving class; $(\mathcal{F}, \mathcal{C})$ is called *complete* if every module has a special $\mathcal{C}$-preenvelope or a special $\mathcal{F}$-precover.

Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory, $\Im$ the corresponding Gabriel topology and M a right R-module. Following [31, IX, p.201], an $\Im$-injective envelope of M is an essential monomorphism $M \to E_\Im(M)$ such that $E_\Im(M) \in \mathcal{T}^\perp$ and $E_\Im(M)/M \in \mathcal{T}$. Every right R-module has an $\Im$-injective envelope which is unique up to isomorphism. Let τ be the corresponding radical, and $M_\Im = \varinjlim_{I \in \Im} \mathrm{Hom}(I, M/\tau(M))$, then $M_\Im \cong E_\Im(M/\tau(M))$ [31, IX, p.202]. Moreover, $R_\Im$ is a ring and there exists a functor $q : \mathcal{M}_R \to \mathcal{M}_{R_\Im}$ with $q(M) = M_\Im$ [31, IX, p.197].

A module M is called *cotorsion* [33] if $\mathrm{Ext}\,_R^1(F, M) = 0$ for any flat module F. A ring R is called ***right IF*** [10] if every injective right R-module is flat, an ***IF*** ring is both left and right IF.

2. Introduction

In [13], Enochs proved that for a ring R, every right R-module has a flat preenvelope if and only if R is left coherent, and he gave some characterizations of domains such that every module has a flat envelope. He then asked how rings with every module having flat envelope can be characterized. This problem has been studied by many authors (see, for example, [2, 3, 4, 12, 14, 24, 29]).

If R is a commutative ring with monomorphic flat envelopes on the right, then the flat envelope of every R-module is an essential flat extension [2, Corollary 10]. For the noncommutative situation, Martínez Hernández et al. [24] gave some characterizations of IF rings with essential flat envelopes on the right. A natural question is how the rings with essential flat envelopes on one side can be characterized.

In this article, we study some relations between torsion theories and cotorsion theories in $\mathcal{M}_R$. We get that, if R is a ring and $\mathcal{F}$ the class of all flat right R-modules, then R is a ring with essential flat envelopes on the right if and only if there exists a torsion theory $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A}^\perp = \mathcal{F}$. As applications, we give some new characterizations of IF rings with essential flat envelopes on the right.

In Section 3, let $\mathcal{A}$ be a class of right R-modules closed under isomorphisms. We prove that if $\mathcal{A}$ is closed under quotients, then $({}^\perp(\mathcal{A}^\perp), \mathcal{A}^\perp)$ is a complete cotorsion theory; moreover, every right R-module M has an essential $\mathcal{A}^\perp$-envelope. Some characterizations of von Neumann regular rings are given.

In Section 4, let R be a ring and $(\mathcal{F}, \mathcal{C})$ be a cotorsion theory in $\mathcal{M}_R$. It is shown that R is a ring with essential $\mathcal{C}$-envelopes on the right if and only if there exists a torsion theory $(\mathcal{A}, \mathcal{B})$ such that $(\mathcal{F}, \mathcal{C})$ is generated by $\mathcal{A}$; moreover, if $(\mathcal{F}, \mathcal{C})$ is a hereditary cotorsion theory, then the above conditions are equivalent to that there exists a hereditary torsion theory $(\mathcal{A}, \mathcal{B})$ such that $(\mathcal{F}, \mathcal{C})$ is generated by $\mathcal{A}$. This result gives some relations between torsion theories and cotorsion theories. As a corollary, we prove that R is a right artinian ring if and only if R is a right noetherian ring with essential cotorsion envelopes on the right.

Section 5 is devoted to rings with essential flat envelopes on the right. Let R be a ring and $\mathcal{F}$ the class of all flat right R-modules, it is proved that R is a ring with essential flat envelopes on the right if and only if there exists a torsion theory $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A}^{\perp} = \mathcal{F}$. This result gives an equivalent condition of rings with essential flat envelopes on the right. As applications, we give some new characterizations of IF rings with essential flat envelopes on the right. It is shown that R is an IF ring with essential flat envelopes on the right if and only if there exists a cosplitting torsion theory $(\mathcal{A}, \mathcal{B})$ with $\mathcal{A}^{\perp} = \mathcal{F}$ if and only if there exists a torsion theory $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A}^{\perp} = \mathcal{F}$, $\tau(R)$ is pure as a right ideal in R and $R/\tau(R)$ is a von Neumann regular ring. Let R be a right IF ring, as a corollary, we have that if there exists a cosplitting torsion theory $(\mathcal{A}, \mathcal{B})$ in $\mathcal{M}_R$ with $\tau(R) = I$ pure as a right ideal in R such that R/I is von Neumann regular, and $E(F)I = FI$ for any flat right R-module F, then R is an IF ring with essential flat envelopes on the right.

3. Cotorsion Theories Generated by a Class of Modules Closed under Quotients

Let $\mathcal{A}$ be a class of right R-modules closed under isomorphisms. For any right R-module M, define $S_{\mathcal{A}}(M) = \sum_{M_i \in \mathcal{A}, M_i \subseteq M} M_i$.

Lemma 3.1. *Let M be a nonzero right R-module.*

(1) *If $M \in \mathcal{A}^{\perp}$, then $S_{\mathcal{A}}(E(M)/M) = 0$. Moreover, if $\mathcal{A}$ is closed under quotients, then the converse is true;*

(2) *If every nonzero cyclic singular module has a nonzero submodule in $\mathcal{A}$, then $M \in \mathcal{A}^{\perp}$ if and only if M is injective;*

(3) *If $\mathcal{A}$ is closed under quotients and every injective right R-module is contained in $\mathcal{A}$, then $M \in \mathcal{A}^{\perp}$ if and only if M is injective.*

Proof.

(1) If $S_{\mathcal{A}}(E(M)/M) \neq 0$, then there exists a nonzero submodule L of $E(M)$ such that M is a proper submodule of L and $L/M \in \mathcal{A}$, so $0 \to M \to$

$L \to L/M \to 0$ is a split exact sequence. Thus M is a direct summand of L. But M is essential in $E(M)$ and hence essential in L. This leads to a contradiction.

Conversely, suppose $\mathcal{A}$ is closed under quotients and $S_{\mathcal{A}}(E(M)/M) = 0$. Note that $M \in \mathcal{A}^{\perp}$ if and only if every A in $\mathcal{A}$ is projective with respect to the exact sequence $0 \to M \to E(M) \xrightarrow{\pi} E(M)/M \to 0$. For any $A \in \mathcal{A}$ and any homomorphism $f : A \to E(M)/M$, $f = 0$ since $S_{\mathcal{A}}(E(M)/M) = 0$. So there exists $g : A \to E(M)$ with $f = \pi g$. Thus $M \in \mathcal{A}^{\perp}$.

(2) By hypothesis, every nonzero singular module has a nonzero submodule in $\mathcal{A}$, that is, for any nonzero singular module N, $S_{\mathcal{A}}(N) \neq 0$. Let $M \in \mathcal{A}^{\perp}$. If M is not injective, then $E(M)/M$ is a nonzero singular module and $S_{\mathcal{A}}(E(M)/M) = 0$ by (1). This is a contradiction. So M is injective.

(3) Let $M \in \mathcal{A}^{\perp}$, then $S_{\mathcal{A}}(E(M)/M) = 0$ by (1). But $\mathcal{A}$ is closed under quotients and contains all injective modules, and so $S_{\mathcal{A}}(E(M)/M) = E(M)/M$. Thus $M = E(M)$ is injective. ■

Let $\mathcal{A}$ be the class of all flat right R-modules in Lemma 3.1 (2), we have

Proposition 3.2. *The following statements are equivalent for a ring R:*

(1) *R is a von Neumann regular ring;*

(2) *every nonzero singular right R-module has a nonzero flat submodule;*

(3) *every nonzero singular right R-module has a nonzero cyclic flat submodule;*

(4) *every nonzero cyclic singular right R-module has a nonzero flat submodule;*

(5) *every nonzero cyclic singular right R-module has a nonzero cyclic flat submodule.*

Proof. $(1) \Longrightarrow (5) \Longrightarrow (4) \Longrightarrow (2)$ and $(1) \Longrightarrow (5) \Longrightarrow (3) \Longrightarrow (2)$ are trivial. $(2) \Longrightarrow (1)$ follows from Lemma 3.1 (2). ■

Recall that R is said to be a *right SF ring* [27] if every simple right R-module is flat.

As applications, we list two corollaries of the above proposition.

Corollary 3.3. [9]. *If R is a right SF ring and every nonzero cyclic singular right R-module has a nonzero socle, then R is a von Neumann regular ring.*

Corollary 3.4. [21, 22]. *The following statements are equivalent for a ring R:*

(1) *R is a von Neumann regular ring;*

(2) *Every nonzero right R-module has a nonzero flat submodule;*

(3) *Every nonzero cyclic singular right R-module is flat.*

Recall that a right R-module M is called *copure injective* [15] if $\mathrm{Ext}\,^1_R(E, M) = 0$ for all injective right R-modules E. Let $\mathcal{A}$ be the class of injective right R-modules. If R is a right hereditary ring, then $\mathcal{A}$ is closed under quotients. So we have the following corollary by Lemma 3.1 (3).

Corollary 3.5. *If R is a right hereditary ring, then every copure injective right R-module is injective.*

Corollary 3.6. [23, Corollary 2.11].

(1) *Every right R-module over a right semihereditary ring R has an $\mathcal{FI}$-injective envelope, where $\mathcal{FI}$ denotes the class of all FP-injective right R-modules.*

(2) *Every right R-module over a right PP-ring R has a $\mathcal{DI}$-injective envelope, where $\mathcal{DI}$ denotes the class of all divisible right R-modules.*

Proof. Note that the class of all $\mathcal{FI}$-injective modules (over a right semihereditary ring) and the class of all $\mathcal{DI}$-injective modules (over a right PP-ring) are closed under quotients and contain all injective modules, so they coincide with the class of injective modules by Lemma 3.1 (3). Hence the above results are trivial. ∎

Following [1], the class of all modules generated by $\mathcal{A}$ is denoted by $\mathrm{Gen}(\mathcal{A})$ and the trace of $\mathcal{A}$ in a right R-module M is defined by $\mathrm{Tr}_M(\mathcal{A}) = \sum\{\mathrm{Im}h | h : A \to M \text{ for some } A \in \mathcal{A}\}$. It is clear that if $\mathcal{A}$ is closed under quotients then $S_{\mathcal{A}}(M) = \mathrm{Tr}_M(\mathcal{A})$ for any right R-module M.

Lemma 3.7. *For a class $\mathcal{A}$ of right R-modules,* $\mathrm{Tr}\,_M(\mathcal{A}) = \mathrm{Tr}_M(\mathrm{Gen}(\mathcal{A}))$.

Proof. The proof is easy. ∎

Proposition 3.8. *Let R be a ring and $\mathcal{A}$ a class of right R-modules. If $\mathcal{A}$ is closed under quotients, then $\mathcal{A}^{\perp} = (\mathrm{Gen}(\mathcal{A}))^{\perp}$.*

Proof. Note that

$$S_{\mathcal{A}}(M) = \mathrm{Tr}_M(\mathcal{A}) = \mathrm{Tr}_M(\mathrm{Gen}(\mathcal{A})) = S_{\mathrm{Gen}(\mathcal{A})}(M).$$

So $M \in \mathcal{A}^{\perp}$ if and only if $S_{\mathcal{A}}(E(M)/M) = 0$ if and only if $S_{\mathrm{Gen}(\mathcal{A})}(E(M)/M) = 0$ if and only if $M \in (\mathrm{Gen}(\mathcal{A}))^{\perp}$ by Lemma 3.1 (1). ∎

Remark 3.9. In Proposition 3.8, let $\mathcal{A}$ be the class of cyclic modules, then $\mathcal{A}^{\perp} = (\mathrm{Gen}(\mathcal{A}))^{\perp} = (\mathcal{M}_R)^{\perp}$. It is just the Baer Criterion. So Proposition 3.8 may be viewed as a generalization of Baer Criterion.

Definition 3.10. [16, Definition 3.1.1].

(1) *Let μ be an ordinal and $\mathcal{A} = (A_\alpha | \alpha \leq \mu)$ be a sequence of modules. Let $(f_{\beta\alpha} | \alpha \leq \beta \leq \mu)$ be a sequence of monomorphisms (with $f_{\beta\alpha} \in \mathrm{Hom}_R(A_\alpha, A_\beta)$) such that $\mathcal{D} = \{A_\alpha, f_{\beta\alpha} | \alpha \leq \beta \leq \mu\}$ is a direct system of modules. $\mathcal{D}$ is called continuous provided that $A_0 = 0$ and $A_\alpha = \varinjlim_{\beta<\alpha} A_\beta$ for all limit ordinals $\alpha \leq \mu$.*

If all the maps $f_{\beta\alpha}$ are inclusions, then the sequence $\mathcal{A}$ is called a continuous chain of modules. So a continuous chain is just a sequence of modules $\mathcal{A}$ satisfying $A_0 = 0$, $A_\alpha \subseteq A_{\alpha+1}$ for all $\alpha < \mu$ and $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for all limit ordinals $\alpha \leq \mu$.

(2) *Let M be a module and $\mathcal{C}$ be a class of modules. M is $\mathcal{C}$- filtered provided that there exist an ordinal κ and a continuous chain of modules, $(M_\alpha | \alpha \leq \kappa)$, consisting of submodules of M, such that $M = M_\kappa$, and each of the modules $M_{\alpha+1}/M_\alpha$ $(\alpha < \kappa)$ is isomorphic to an element of $\mathcal{C}$. The chain $(M_\alpha | \alpha \leq \kappa)$ is called a $\mathcal{C}$- filtration of M. If κ is finite, then M is said to be finitely $\mathcal{C}$-filtered.*

Lemma 3.11. [16, Lemma 3.1.2]. *Let N be a right R-module and M a ${}^\perp N$-filtered module. Then $M \in {}^\perp N$.*

Recall that for an idempotent preradical τ, there exists a smallest idempotent radical $\overline{\tau}$ larger than τ which is constructed as follows (see [31, VI, p.138]): if α is not a limit ordinal, then τ_α is given by $\tau_\alpha(M)/\tau_{\alpha-1}(M) = \tau(M/\tau_{\alpha-1}(M))$, and for a limit ordinal α, put $\tau_\alpha = \sum_{\beta<\alpha} \tau_\beta$. This gives rise to an increasing sequence of preradicals τ_α, and $\overline{\tau} = \sum_\alpha \tau_\alpha$. It is not difficult to show that (1) if $\tau(M) = M$, then $\overline{\tau}(M) = M$; (2) $\tau(M) = 0$ if and only if $\overline{\tau}(M) = 0$.

Let $\mathcal{A}$ be a class of right R-modules closed under quotients, it is not difficult to show that $S_\mathcal{A} : M \longmapsto S_\mathcal{A}(M) = \sum_{M_i \in \mathcal{A}, M_i \subseteq M} M_i$ is an idempotent preradical of $\mathcal{M}_R$, and the torsion class of $S_\mathcal{A}$ is $\mathrm{Gen}(\mathcal{A})$.

Proposition 3.12. *Let N be a right R-module and $\mathcal{A}$ a class of right R-modules which is closed under quotients, then the following statements are equivalent:*

(1) $N \in \mathcal{A}^\perp$.

(2) $\mathrm{Ext}^1_R(\overline{S_\mathcal{A}}(M), N) = 0$ *for any module M.*

Proof. $(1) \Longrightarrow (2)$. Note that $\mathrm{Gen}(\mathcal{A})$ is closed under direct sums and quotients, by Proposition 3.8, we may assume $\mathcal{A}$ is closed under direct sums and quotients. Since $S_\mathcal{A}$ is an idempotent preradical of $\mathcal{M}_R$ and $S_\mathcal{A}(M) \in \mathcal{A}$, $\mathrm{Ext}^1_R(S_\mathcal{A}(M), N) = 0$ by (1) for any right R-module M.

Let M be a right R-module. Since M is a set, there exists an ordinal κ such that $(S^{\alpha}_{\mathcal{A}}(M)|\alpha \leq \kappa)$ is a continuous chain and $S^{\alpha+1}_{\mathcal{A}}(M)/S^{\alpha}_{\mathcal{A}}(M) = S_{\mathcal{A}}(M/S^{\alpha}_{\mathcal{A}}(M))$ if α is not a limit ordinal. Thus, for any right R-module M, $\overline{S_{\mathcal{A}}}(M)$ is a ${}^{\perp}N$-filtered module, and so $\mathrm{Ext}^1_R(\overline{S_{\mathcal{A}}}(M), N) = 0$.

$(2) \Longrightarrow (1)$. Let $M \in \mathcal{A}$. Then $\overline{S_{\mathcal{A}}}(M) = M$, and hence $\mathrm{Ext}^1_R(M, N) = \mathrm{Ext}^1_R(\overline{S_{\mathcal{A}}}(M), N) = 0$ by (2). So $N \in \mathcal{A}^{\perp}$. ■

Theorem 3.13. *Let R be a ring and $\mathcal{A}$ a class of right R-modules closed under quotients, then $({}^{\perp}(\mathcal{A}^{\perp}), \mathcal{A}^{\perp})$ is a complete cotorsion theory. Moreover, every right R-module M has an essential $\mathcal{A}^{\perp}$-envelope $\iota : M \to E_{\mathcal{A}}(M)$ with $E_{\mathcal{A}}(M)/M = \overline{S_{\mathcal{A}}}(E(M)/M)$.*

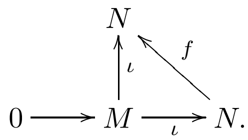
Proof. Let M be a right R-module. If M is injective, then $M \in \mathcal{A}^{\perp}$, we are done.

Next we assume that M is not injective. Let N be the submodule of $E(M)$ such that $M \subseteq N$ and $N/M = \overline{S_{\mathcal{A}}}(E(M)/M)$, then

$$S_{\mathcal{A}}(E(N)/N) = S_{\mathcal{A}}(E(M)/N) = S_{\mathcal{A}}\left(\frac{E(M)/M}{N/M}\right)$$
$$= S_{\mathcal{A}}\left(\frac{E(M)/M}{\overline{S_{\mathcal{A}}}(E(M)/M)}\right) \subseteq \overline{S_{\mathcal{A}}}\left(\frac{E(M)/M}{\overline{S_{\mathcal{A}}}(E(M)/M)}\right) = 0,$$

so $N \in \mathcal{A}^{\perp}$ by Lemma 3.1 (1). Let $\iota : M \to N$ be the canonical injection and X a right R-module in $\mathcal{A}^{\perp}$. Since $N/M = \overline{S_{\mathcal{A}}}(E(M)/M)$, we have $\mathrm{Ext}^1_R(N/M, X) = 0$ by Proposition 3.12. Hence the sequence $\mathrm{Hom}(N, X) \xrightarrow{\iota^*} \mathrm{Hom}(M, X) \to 0$ is exact.

For $X = N$, consider the commutative diagram:



Since ι is an essential monomorphism, it follows that f is an essential monomorphism. So $f(N) \in \mathcal{A}^{\perp}$. Hence $S_{\mathcal{A}}(E(f(N))/f(N)) = 0$ by Lemma 3.1 (1). But $S_{\mathcal{A}}(N/f(N)) \subseteq S_{\mathcal{A}}(E(f(N))/f(N))$, so $S_{\mathcal{A}}((N/f(N)) = 0$. Hence $\overline{S_{\mathcal{A}}}(N/f(N)) = 0$. Since $\overline{S_{\mathcal{A}}}$ is an idempotent radical, we have

$$\overline{S_{\mathcal{A}}}(N/M) = \overline{S_{\mathcal{A}}}(\overline{S_{\mathcal{A}}}(E(M)/M)) = \overline{S_{\mathcal{A}}}(E(M)/M) = N/M.$$

Note that the torsion class of $\overline{S_{\mathcal{A}}}$ is closed under quotients and $N/f(N)$ is a quotient module of N/M, and so $\overline{S_{\mathcal{A}}}(N/f(N)) = N/f(N)$. Thus $N = f(N)$, and hence f is an isomorphism. Therefore $\iota : M \to N$ is an essential monomorphic $\mathcal{A}^{\perp}$-envelope of M. ■

Remark 3.14. Let $(\mathcal{A}, \mathcal{B})$ be a hereditary torsion theory and $\Im$ the corresponding Gabriel topology, then $E_{\Im}(M) \cong E_{\mathcal{A}}(M)$ for every right R-module M.

4. Some Relations between Torsion Theories and Cotorsion Theories

Let $\mathcal{C}$ be a class of right R-modules closed under isomorphisms. If a right R-module M has a $\mathcal{C}$-envelope, we always let $\varepsilon_M : M \to C(M)$ denote the $\mathcal{C}$-envelope of M. Let $\mathcal{X} = \{C(M)/\varepsilon_M(M) | M \text{ has a } \mathcal{C} - \text{envelope } \varepsilon_M : M \to C(M)\}$.

Lemma 4.1. *Let $\mathcal{C}$ be a class of right R-modules closed under isomorphisms, direct summands and extensions, then*

(1) $\mathrm{Ext}^1_R(\mathcal{X}, \mathcal{C}) = 0$*;*

(2) *If every right R-module has a monomorphic $\mathcal{C}$-envelope, then $\mathcal{X}^{\perp} = \mathcal{C}$.*

Proof. (1) follows from Wakamatsu's Lemma [16, Lemma 2.1.13].

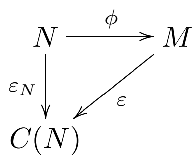
(2). Let $M \in \mathcal{X}^{\perp}$, there exists a monomorphic $\mathcal{C}$-envelope $\varepsilon_M : M \to C(M)$. We have the split exact sequence $0 \to M \to C(M) \to C(M)/\varepsilon_M(M) \to 0$ by hypothesis. Thus M is a direct summand of $C(M)$, and so $M \in \mathcal{C}$. Hence $\mathcal{X}^{\perp} \subseteq \mathcal{C}$. But $\mathcal{C} \subseteq \mathcal{X}^{\perp}$ by (1). So $\mathcal{X}^{\perp} = \mathcal{C}$. ■

Corollary 4.2. [14, Theorem 2.5 (1)]. *Let R be a ring and $\mathcal{F}$ the class of all flat right R-modules. If every right R-module has a monomorphic flat envelope, then $({}^{\perp}\mathcal{F}, \mathcal{F})$ is a cotorsion theory.*

Proposition 4.3. *Let $\mathcal{C}$ be a class of right R-modules closed under isomorphisms, direct summands and extensions. Then the following statements are equivalent:*

(1) *$\mathcal{X}$ is closed under quotients;*

(2) *If a right R-module N has a $\mathcal{C}$-envelope $\varepsilon_N : N \to C(N)$ and there exists a commutative diagram*

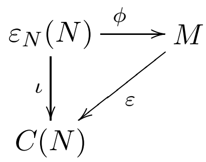


with ϕ and ε monomorphisms, then $\varepsilon : M \to C(N)$ is a $\mathcal{C}$-envelope of M.

Proof. $(1) \Longrightarrow (2)$. Let N be a right R-module satisfying the condition of (2). Then $C(N)/\varepsilon(M) \cong \frac{C(N)/\varepsilon_N(N)}{\varepsilon(M)/\varepsilon_N(N)} \in \mathcal{X}$ by (1), and hence $\mathrm{Ext}^1_R(C(N)/\varepsilon(M), C) = 0$ for any $C \in \mathcal{C}$ by Lemma 4.1(1). So $\varepsilon : M \to C(N)$ is a $\mathcal{C}$-preenvelope of M.

Note that for any $f : C(N) \to C(N)$ with $f\varepsilon = \varepsilon$, we have $f\varepsilon_N = f\varepsilon\phi = \varepsilon\phi = \varepsilon_N$. Thus f is an automorphism of $C(N)$ and so $\varepsilon : M \to C(N)$ is a $\mathcal{C}$-envelope of M.

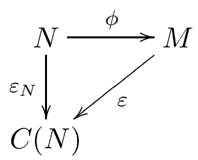
$(2) \Longrightarrow (1)$. Let X be a quotient of $C(N)/\varepsilon_N(N) \in \mathcal{X}$, where $\varepsilon_N : N \to C(N)$ is a $\mathcal{C}$-envelope of N. We may assume $X = C(N)/M$ with $\varepsilon_N(N) \subseteq M \subseteq C(N)$. Let $\iota : \varepsilon_N(N) \to C(N)$, $\phi : \varepsilon_N(N) \to M$ and $\varepsilon : M \to C(N)$ be the inclusions. Then we have a commutative diagram:



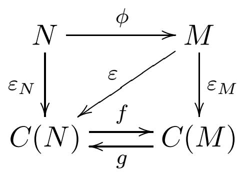
Clearly, $\iota : \varepsilon_N(N) \to C(N)$ is a $\mathcal{C}$-envelope of $\varepsilon_N(N)$. By (2), $\varepsilon : M \to C(N)$ is a $\mathcal{C}$-envelope of M, and so $X = C(N)/M \in \mathcal{X}$ by definition. ∎

Proposition 4.4. *Let $\mathcal{C}$ be a class of right R-modules closed under isomorphisms, direct summands and extensions. If every right R-module has an essential $\mathcal{C}$-envelope, then $\mathcal{X}$ is closed under quotients.*

Proof. Let N be a right R-module with a $\mathcal{C}$-envelope $\varepsilon_N : N \to C(N)$. Consider the commutative diagram



with ϕ and ε monomorphisms. For a $\mathcal{C}$-envelope $\varepsilon_M : M \to C(M)$, we have the diagram:



with $f\varepsilon_N = \varepsilon_M\phi$ and $\varepsilon = g\varepsilon_M$. Thus $gf\varepsilon_N = g\varepsilon_M\phi = \varepsilon\phi = \varepsilon_N$, and hence gf is an automorphism of $C(N)$ and g is an epimorphism. Note that ε and ε_M are essential monomorphisms, and so g is an essential monomorphism. It follows that g is an isomorphism, and so $\varepsilon : M \to C(N)$ is a $\mathcal{C}$-envelope of M. Therefore $\mathcal{X}$ is closed under quotients by Proposition 4.3. ∎

Theorem 4.5. *Let R be a ring and $(\mathcal{F}, \mathcal{C})$ be a cotorsion theory in $\mathcal{M}_R$. Then the following statements are equivalent:*

(1) *R is a ring with essential $\mathcal{C}$-envelopes;*

(2) *$(\mathcal{F}, \mathcal{C})$ is generated by a class $\mathcal{A}$ of right R-modules closed under quotients;*

(3) *There exists a torsion theory $(\mathcal{A}, \mathcal{B})$ such that $(\mathcal{F}, \mathcal{C})$ is generated by $\mathcal{A}$.*

Moreover, if $(\mathcal{F}, \mathcal{C})$ is a hereditary cotorsion theory, then the above statements are equivalent to:

(4) *There exists a hereditary torsion theory $(\mathcal{A}, \mathcal{B})$ such that $(\mathcal{F}, \mathcal{C})$ is generated by $\mathcal{A}$.*

Proof. $(3) \Longrightarrow (2)$ follows since $\mathcal{A}$ is closed under quotients.

$(2) \Longrightarrow (1)$ follows from Theorem 3.13.

$(1) \Longrightarrow (3)$. By Lemma 4.1 (2) and Proposition 4.4, $(\mathcal{F}, \mathcal{C})$ is generated by $\mathcal{X}$ and $\mathcal{X}$ is closed under quotients. Hence $\overline{S_{\mathcal{X}}}$ is an idempotent radical, and so $(\mathcal{T}_{\overline{S_{\mathcal{X}}}}, \mathcal{F}_{\overline{S_{\mathcal{X}}}})$ is a torsion theory. Let $(\mathcal{A}, \mathcal{B}) = (\mathcal{T}_{\overline{S_{\mathcal{X}}}}, \mathcal{F}_{\overline{S_{\mathcal{X}}}})$. By Proposition 3.12, $(\mathcal{F}, \mathcal{C})$ is generated by $\mathcal{A}$.

$(4) \Longrightarrow (2)$ is trivial.

$(1) \Longrightarrow (4)$. Let $\widetilde{\mathcal{X}} = \{L | L \subseteq X \in \mathcal{X}\}$ and $\mathcal{Y} = \{M | M$ is singular and $M \in \mathcal{F}\}$.

Let $L \in \widetilde{\mathcal{X}}$, then there exists some right R-module M such that $L = N/\varepsilon_M(M) \subseteq C(M)/\varepsilon_M(M)$.

Let $H/\varepsilon_M(M)$ be a submodule of $N/\varepsilon_M(M)$. The canonical injection $\iota_H : H \to C(M)$ is a $\mathcal{C}$-envelope of H by Proposition 4.3. Hence the quotient module $\frac{N/\varepsilon_M(M)}{H/\varepsilon_M(M)} \cong N/H$ is a submodule of $C(M)/H \in \mathcal{X}$. Thus $\frac{N/\varepsilon_M(M)}{H/\varepsilon_M(M)} \in \widetilde{\mathcal{X}}$, and so $\widetilde{\mathcal{X}}$ is closed under quotients.

Since $\varepsilon_M : M \to C(M)$ is essential, $C(M)/\varepsilon_M(M)$ is singular. Hence $N/\varepsilon_M(M)$ is singular. By Lemma 4.1 (1), $C(M)/\varepsilon_M(M) \in \mathcal{F}$. Note that the canonical injection $\iota_N : N \to C(M)$ is a $\mathcal{C}$-envelope of N by Proposition 4.3, and so $C(M)/N \in \mathcal{F}$ by Lemma 4.1 (1). Since $(\mathcal{F}, \mathcal{C})$ is a hereditary cotorsion theory, the exact sequence $0 \to N/\varepsilon_M(M) \to C(M)/\varepsilon_M(M) \to C(M)/N \to 0$ shows that $N/\varepsilon_M(M) \in \mathcal{F}$. So $\widetilde{\mathcal{X}} \subseteq \mathcal{Y}$.

Now let $M' \in \mathcal{Y}$. Since M' is singular, there exists an exact sequence $0 \to K \xrightarrow{f} N \to M' \to 0$ with f an essential monomorphism. Note that $M' \in \mathcal{F}$, we have the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{f} & N & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & K & \xrightarrow{\varepsilon_K} & C(K) & \longrightarrow & C(K)/K & \longrightarrow & 0.
\end{array}$$

Since f is an essential monomorphism, h is monic. So g is monic by the Five Lemma. Hence $M' \cong g(M') \in \widetilde{\mathcal{X}}$ and so $\mathcal{Y} \subseteq \widetilde{\mathcal{X}}$. Thus $\widetilde{\mathcal{X}} = \mathcal{Y}$. Since $\mathcal{F}$ and the class of all singular right R-modules are both closed under direct sums, $\mathcal{Y}$ is closed under direct sums. So $\widetilde{\mathcal{X}}$ is closed under direct sums.

It is clear that $\widetilde{\mathcal{X}}$ is closed under submodules, the above proof shows that $\widetilde{\mathcal{X}}$ is closed under quotients and direct sums. Hence $S_{\widetilde{\mathcal{X}}}$ is a left exact preradcial. So $\overline{S_{\widetilde{\mathcal{X}}}}$ is a left exact radical. Note that $\mathcal{X} \subseteq \widetilde{\mathcal{X}} \subseteq \mathcal{F}$, we have $\mathcal{C} = \mathcal{F}^{\perp} \subseteq \widetilde{\mathcal{X}}^{\perp} \subseteq \mathcal{X}^{\perp} = \mathcal{C}$. So $(\mathcal{A}, \mathcal{B}) = (\mathcal{T}_{\overline{S_{\widetilde{\mathcal{X}}}}}, \mathcal{F}_{\overline{S_{\widetilde{\mathcal{X}}}}})$ is a hereditary torsion theory, and $(\mathcal{F}, \mathcal{C})$ is generated by $\mathcal{A}$ by Proposition 3.12. ∎

Proposition 4.6. *Let R be a right noetherian ring and $(\mathcal{F}, \mathcal{C})$ a cotorsion theory in $\mathcal{M}_R$. If every right R-module has an essential $\mathcal{C}$-envelope, then $\mathcal{C}$ is closed under direct sums.*

Proof. By Theorem 4.5, there exists a torsion theory $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{C} = \mathcal{A}^{\perp}$. Let τ be the corresponding idempotent radical of $(\mathcal{A}, \mathcal{B})$. It is clear that $\tau = S_{\mathcal{A}}$. Let $\{C_i\}_{i \in I}$ be a family of right R-modules in $\mathcal{C}$. Since R is a right noetherian ring, $E(\bigoplus_{i \in I} C_i) = \bigoplus_{i \in I} E(C_i)$. Thus

$$\tau(E(\bigoplus_{i \in I} C_i)/\bigoplus_{i \in I} C_i) = \tau(\bigoplus_{i \in I} E(C_i)/\bigoplus_{i \in I} C_i) = \bigoplus_{i \in I} \tau(E(C_i)/C_i) = 0,$$

and hence $\bigoplus_{i \in I} C_i \in \mathcal{C} = \mathcal{A}^{\perp}$ by Lemma 3.1 (1). So $\mathcal{C}$ is closed under direct sums. ∎

It is well known that if the class of cotorsion right R-modules is closed under direct sums, then R is right perfect [19]. Hence we have

Corollary 4.7. *Let R be a ring, then the following statements are equivalent:*

(1) *R is a right artinian ring;*

(2) *R is a right noetherian ring with essential cotorsion envelopes on the right.*

Proof. $(1) \Longrightarrow (2)$. Since R is right artinian, R is right perfect. Thus every right R-module is cotorsion [33], and so (2) follows.

$(2) \Longrightarrow (1)$ Since the class of cotorsion modules is closed under direct sums by Proposition 4.6, R is right perfect [19]. But a right noetherian and right perfect ring is right artinian by [18, Theorem 5.9]. ∎

Remark 4.8. It is well known that every right R-module over any ring R has a cotorsion envelope [6]. But, in general, not every cotorsion envelope is an essential monomorphism. For example, let $R = \mathbf{Z}$, the ring of integers, $\mathbf{P} = \{p|p$ is a prime number$\}$, $\mathbf{Z}_{(p)} = \{\frac{a}{b} : b \notin \mathbf{Z}p, (a,b) = 1\}$, where $p \in \mathbf{P}$. Then

$$\phi : \mathbf{Z} \longrightarrow \prod_{p\in\mathbf{P}} \mathbf{Z}_{(p)}$$

$$x \mapsto (x/1)$$

is a cotorsion envelope of $\mathbf{Z}$. But ϕ is not essential. In fact, it is easy to check that $\prod_{p\in\mathbf{P}}(\frac{p}{p+1}) \neq 0$, but $Im(\phi) \cap \prod_{p\in\mathbf{P}}(\frac{p}{p+1}) = 0$.

5. IF Rings with Essential Flat Envelopes on the Right

In this section, we always let $\mathcal{F}$ denote the class of all flat right R-modules, and for a torsion theory $(\mathcal{A}, \mathcal{B})$, we always let τ be the corresponding idempotent radical.

If R is a commutative ring with monomorphic flat envelopes on the right, then the flat envelope of every R-module is an essential flat extension [2, Corollary 10]. In [29], Saorín described the structure of commutative rings with essential flat envelopes. For the noncommutative situation, Martínez Hernández et al. [24] gave some characterizations of IF rings with essential flat envelopes on the right. A natural question is how the rings with essential flat envelopes on one side can be characterized. The following proposition gives an equivalent condition of rings with essential flat envelopes on the right.

Proposition 5.1. *If R is a ring and $\mathcal{F}$ the class of all flat right R-modules, then R is a ring with essential flat envelopes on the right if and only if there exists a torsion theory $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A}^{\perp} = \mathcal{F}$.*

Proof. This result follows from Corollary 4.2 and Theorem 4.5 immediately. ■

Proposition 5.2. *Let R be a ring and $(\mathcal{A}, \mathcal{B})$ a torsion theory with $\mathcal{A}^{\perp} = \mathcal{F}$.*

(1) *If $\tau(R) = R$, then R is a QF-ring;*

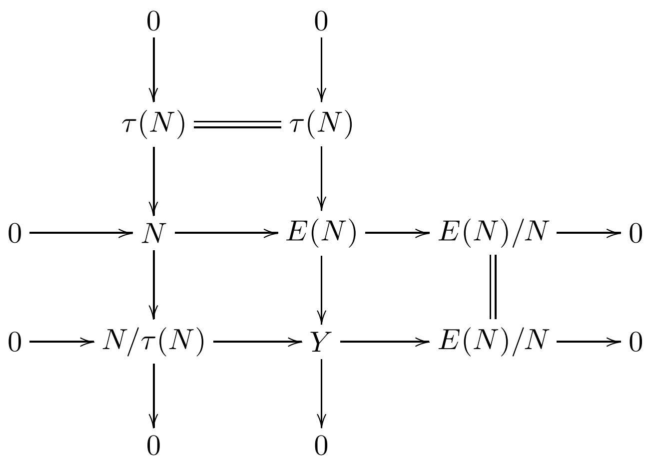
(2) *If $\tau(R) = 0$ and $\mathcal{B}$ is closed under direct limits, then R is a von Neumann regular ring.*

Proof. (1). If $\tau(R) = R$, then $M = M\tau(R) \subseteq \tau(M) \subseteq M$ for any right R-module M, and hence $(\mathcal{A}, \mathcal{B}) = (\mathcal{M}_R, 0)$. Thus $\mathcal{F}$ is just the class of injective right R-modules and so R is a QF-ring.

(2). Since $\tau(R) = 0$, R is τ-torsionfree. But $\mathcal{B}$ is closed under submodules and direct products, and $R^{(I)}$ is a submodule of R^I, $\tau(R^{(I)}) = 0$. Hence for any projective right R-module P, $\tau(P) = 0$. Let F be a flat right R-module. Note that F is a direct limit of projective modules and $\mathcal{B}$ is closed under direct limits, and so $\tau(F) = 0$. Let M be any right R-module. Since $\mathcal{A}^{\perp} = \mathcal{F}$, M has an essential flat envelope $M \to E_{\mathcal{A}}(M)$ by Proposition 5.1. Note that $E_{\mathcal{A}}(M) \in \mathcal{B}$ by the foregoing proof, and so $M \in \mathcal{B}$. It follows that $(\mathcal{A}, \mathcal{B}) = (0, \mathcal{M}_R)$ and $\mathcal{F} = \mathcal{M}_R$. Thus R is a von Neumann regular ring. ■

Proposition 5.3. *Let R be a ring with essential flat envelopes on the right and $(\mathcal{A}, \mathcal{B})$ a torsion theory with $\mathcal{A}^{\perp} = \mathcal{F}$. Then $\tau(N) \cong \tau(E(N)) \in \mathcal{F}$ for any flat right R-module N.*

Proof. Let N be a flat right R-module and consider the commutative diagram:



Note that $\tau = S_{\mathcal{A}}$ and $N \in \mathcal{A}^{\perp} = \mathcal{F}$, and so $E(N)/N \in \mathcal{B}$ by Lemma 3.1 (1). But $N/\tau(N) \in \mathcal{B}$, so $Y \in \mathcal{B}$. Since $\tau(N) \in \mathcal{A}$ and $Y \in \mathcal{B}$, the exact sequence $0 \to \tau(N) \to E(N) \to Y \to 0$ shows that $\tau(N) \cong \tau(E(N))$ by [26, Lemma 1.1]. It is clear that the injective envelope $E(\tau(N))$ of $\tau(N)$ is a submodule of $E(N)$, and hence $\tau(E(\tau(N))/\tau(N)) \subseteq \tau(E(N)/\tau(N)) = \tau(Y) = 0$. So $\tau(N) \in \mathcal{A}^{\perp} = \mathcal{F}$ by Lemma 3.1 (1). ■

Lemma 5.4. [31, Proposition XI.3.13] *The following statements are equivalent for a two-sided ideal I of R:*

(1) *R/I is flat as a right R-module;*

(2) *I is pure as a right ideal in R;*

(3) *For any $a \in I$, there exists $c \in I$ such that $a = ca$;*

(4) *Every injective left R/I-module is injective as an R-module;*

(5) *Every flat right R/I-module is flat as an R-module.*

The following lemma is well known, so we omit its proof.

Lemma 5.5. *Let R be a ring, I a two-sided ideal of R and F a right R-module.*

(1) *If F is a flat right R-module and $FI = 0$, then $F_{R/I}$ is a flat right R/I-module;*

(2) *If I is pure as a right ideal in R, then $IN \cong I \otimes N$ for any left R-module N;*

(3) *If F is a flat right R-module and I is pure as a right ideal in R, then F/FI is a flat right R/I-module.*

Lemma 5.6. *Let R be a ring, I a two-sided ideal of R pure as a right ideal and M a right R/I-module. If $M \to F(M)$ is a flat preenvelope of M_R, then $M \to \frac{F(M)}{F(M)I}$ is a flat preenvelope of $M_{R/I}$.*

Proof. Straightforward. ■

We call a two-sided ideal I pure in R if I is pure as a left and right ideal in R.

Proposition 5.7. *Let R be a ring with essential flat envelopes on the right and I a pure ideal in R, then R/I is a ring with essential flat envelopes on the right.*

Proof. Since every right R-module has an essential flat envelope, R is a right IF ring. Let $E_{R/I}$ be an injective right R/I-module. Since I is pure as a left ideal in R, E_R is an injective right R-module by Lemma 5.4. But R is right IF, so E_R is a flat right R-module. By Lemma 5.5 (1), $E_{R/I}$ is a flat right R/I-module. Hence R/I is a right IF ring.

Let N be a right R/I-module. N_R has an essential flat envelope $\varepsilon_N : N \to E_{\mathcal{A}}(N)$. Let $\pi : E_{\mathcal{A}}(N) \to \frac{E_{\mathcal{A}}(N)}{E_{\mathcal{A}}(N)I}$ be the quotient map and $\iota : N_{R/I} \to E(N_{R/I})$ be the inclusion with $E(N_{R/I})$ an injective envelope of $N_{R/I}$. Note that $\pi\varepsilon_N : N \to \frac{E_{\mathcal{A}}(N)}{E_{\mathcal{A}}(N)I}$ is a flat preenvelope of $N_{R/I}$ by Lemma 5.6 and R/I is right IF, and so there exists $g : \frac{E_{\mathcal{A}}(N)}{E_{\mathcal{A}}(N)I} \to E(N_{R/I})$ with $g\pi\varepsilon = \iota$. Since ι is monic, $\pi\varepsilon$ is monic. Hence $\mathrm{Ker}(\pi\varepsilon_N) = E_{\mathcal{A}}(N)I \cap N = 0$. But N is essential in $E_{\mathcal{A}}(N)$, so $E_{\mathcal{A}}(N)I = 0$. Thus $E_{\mathcal{A}}(N)$ may be viewed as a right R/I-module and $\varepsilon_N : N \to E_{\mathcal{A}}(N)$ is a flat preenvelope of $N_{R/I}$. It is not difficult to show that $\varepsilon_N : N \to E_{\mathcal{A}}(N)$ is an essential flat envelope of $N_{R/I}$. ■

Lemma 5.8. *Let R be a ring and $(\mathcal{A}, \mathcal{B})$ a torsion theory in $\mathcal{M}_R$, then*

(1) *$(\mathcal{A}, \mathcal{B})$ is cohereditary if and only if $\tau(M) = M\tau(R)$ for any right R-module M;*

(2) $(\mathcal{A}, \mathcal{B})$ *is cosplitting if and only if* $\operatorname{Ext}_R^1(A, B) = 0$ *for any* $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$;

(3) *If* $(\mathcal{A}, \mathcal{B})$ *is cohereditary, then* $(\mathcal{A}, \mathcal{B})$ *is cosplitting if and only if* $\tau(R)$ *is pure as a left ideal in* R.

Proof. (1) follows from [7, I.2.10].

(2). If $(\mathcal{A}, \mathcal{B})$ is a cosplitting torsion theory, then $\mathcal{B}$ is closed under quotients by definition and injective envelopes by [31, Proposition VI.3.2]. Hence for any $B \in \mathcal{B}$, $E(B)/B \in \mathcal{B}$. Note that $\tau = S_{\mathcal{A}}$, and so $B \in \mathcal{A}^{\perp}$ by Lemma 3.1 (1).

Conversely, let $B \in \mathcal{B}$. Since $B \in \mathcal{A}^{\perp}$ and $\tau = S_{\mathcal{A}}$, $E(B)/B \in \mathcal{B}$ by Lemma 3.1 (1). Thus $E(B) \in \mathcal{B}$, and so $\mathcal{B}$ is closed under injective envelopes. Hence $(\mathcal{A}, \mathcal{B})$ is a hereditary torsion theory. On the other hand, let $0 \to B' \to B \to B'' \to 0$ be an exact sequence. Since $\mathcal{B}$ is closed under submodules and $B \in \mathcal{B}$, $B' \in \mathcal{B}$. For any $A \in \mathcal{A}$, applying the functor $\operatorname{Hom}(A, -)$ to this sequence, we get the exact sequence $0 = \operatorname{Hom}(A, B) \to \operatorname{Hom}(A, B'') \to \operatorname{Ext}_R^1(A, B')$. Since $\operatorname{Ext}_R^1(A, B') = 0$ by hypothesis, $\operatorname{Hom}(A, B'') = 0$. Thus $B'' \in \mathcal{B}$, and so $(\mathcal{A}, \mathcal{B})$ is a cohereditary torsion theory.

(3). If $(\mathcal{A}, \mathcal{B})$ is a cosplitting torsion theory, then $(\mathcal{A}, \mathcal{B})$ is hereditary. Hence for any right ideal J, $J \cap \tau(R) = \tau(J)$. But $(\mathcal{A}, \mathcal{B})$ is cohereditary, and so $J \cap \tau(R) = \tau(J) = J\tau(R)$ by (1). Thus $R/\tau(R)$ is a flat left R-module by [28, Theorem 3.55], and so $\tau(R)$ is pure as a left ideal in R.

Conversely, let $0 \to N \to M \to M/N \to 0$ be an exact sequence with $M \in \mathcal{A}$. Applying the functor $- \otimes \tau(R)$ to this sequence gives the exactness of $0 \to N\tau(R) \to M\tau(R) \to (M/N)\tau(R) \to 0$ by Lemma 5.5 (2). Since $(\mathcal{A}, \mathcal{B})$ is a cohereditary torsion theory, $M\tau(R) = M$ and $(M/N)\tau(R) = M/N$ by (1). Hence $N\tau(R) = N$. Thus $\mathcal{A}$ is closed under submodules, and so $(\mathcal{A}, \mathcal{B})$ is a hereditary torsion theory. ■

Let M be a right R-module. Following [32], $\sigma[M]$ denotes the full subcategory of $\mathcal{M}_R$ consisting of the M-subgenerated R-modules. The injective objects in $\sigma[M]$ are precisely the M-injective M-generated modules. M is said to be a self-generator if $\operatorname{Tr}_M(N) = N$ for every submodule N of M. Let $M = P$ be a projective module, a flat object in $\sigma[P]$ is a direct limit of modules in $\operatorname{add}(P) = \{X | X$ is a direct summand of finite direct sums of copies of $P\}$. Since the inclusion functor $\sigma[P] \hookrightarrow \mathcal{M}_R$ preserves direct limits in $\sigma[P]$, it is clear that each flat object in $\sigma[P]$ is also flat in $\mathcal{M}_R$; conversely, any P-subgenerated flat right R-module is a flat object in $\sigma[P]$, as may be seen by adapting the proof of Lazard in [20, Theorem 1.2]. Moreover, P_R is a self-generator if and only if $\operatorname{Gen}(P) = \sigma[P]$. We call a quotient of a submodule of a right R-module M a subfactor of M.

Theorem 5.9. *The following statements are equivalent for a ring* R:

(1) *R is an IF ring with essential flat envelopes on the right;*

(2) *There exists a cosplitting torsion theory $(\mathcal{A}, \mathcal{B})$ with $\mathcal{A}^{\perp} = \mathcal{F}$;*

(3) *There exists a cohereditary torsion theory $(\mathcal{A}, \mathcal{B})$ with $\mathcal{A}^{\perp} = \mathcal{F}$ and $\tau(R)$ pure as a two-sided ideal in R;*

(4) *There exists a hereditary torsion theory $(\mathcal{A}, \mathcal{B})$ with $\mathcal{A}^{\perp} = \mathcal{F}$, $\mathcal{B}$ closed under direct limits and $\tau(R)$ pure as a two-sided ideal in R;*

(5) *There exists a hereditary torsion theory $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A}^{\perp} = \mathcal{F}$, $\tau(R)$ is pure as a right ideal in R and the functor $q : \mathcal{M}_R \to \mathcal{M}_{R_{\Im}}$ is exact, where $\Im$ is the corresponding Gabriel topology of $(\mathcal{A}, \mathcal{B})$;*

(6) *There exists a torsion theory $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A}^{\perp} = \mathcal{F}$, $\tau(R)$ is pure as a right ideal in R and $R/\tau(R)$ is a von Neumann regular ring;*

(7) *R is a ring with essential flat envelopes on the right and $\mathcal{F}$ is a coresolving class.*

Proof. $(1) \Longrightarrow (2)$. Let

$$\mathcal{A} = \{M \in \mathcal{M}_R | M \text{ has no flat simple subfactor}\}$$

and

$$\mathcal{B} = \{M \in \mathcal{M}_R | \text{all simple subfactors of } M \text{ are flat}\}.$$

In [24, Lemma 10], Mart´nez Hernández et al. proved that $(\mathcal{A}, \mathcal{B})$ is a cosplitting torsion theory with $\mathcal{B} \subseteq \mathcal{F}$. Moreover, there exists a projective module P such that $\mathcal{A} = \sigma[P] = \mathrm{Gen}(P)$ by the proof of $(a) \Longrightarrow (b)$ in [24, Theorem 7].

Let $M \in \mathcal{A}^{\perp}$. Since R is IF, $E(M)$ is flat. Note that $\tau = S_{\mathcal{A}}$ and $\mathcal{B} \subseteq \mathcal{F}$, and so $E(M)/M \in \mathcal{B}$ by Lemma 3.1 (1). Thus $E(M)/M$ is flat, and hence M is flat. So $\mathcal{A}^{\perp} \subseteq \mathcal{F}$.

Conversely, let F be a flat right R-module. Since $F/\tau(F) \in \mathcal{B} \subseteq \mathcal{F}$, $\tau(F)$ is a flat right R-module. Note that $\mathcal{A} = \mathrm{Gen}(P) = \sigma[P]$ and P is a projective generator of $\sigma[P]$, by adapting the proof of Lazard in [20, Theorem 1.2], we may show that there exists a direct system $\{P_j | j \in J \text{ with } P_j \in \mathrm{add}(P)\}$ such that $\tau(F) = \varinjlim P_j$. Hence $\tau(F)$ is a flat object in $\mathcal{A}$. But the class of injective objects and the class of flat objects in $\mathcal{A}$ coincide by [24, Lemma 11], so $\tau(F) \in \mathcal{A}^{\perp}$. Note that $(\mathcal{A}, \mathcal{B})$ is cosplitting and $F/\tau(F) \in \mathcal{B} \subseteq \mathcal{A}^{\perp}$ by Lemma 5.8 (2), and so $F \in \mathcal{A}^{\perp}$. Thus $\mathcal{F} \subseteq \mathcal{A}^{\perp}$.

$(1) \Longrightarrow (5)$. By $(1) \Longrightarrow (2)$, there exists a cosplitting torsion theory $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A}^{\perp} = \mathcal{F}$, and hence every right R-module M has an essential flat envelope $\varepsilon_M : M \to E_{\mathcal{A}}(M)$ by Proposition 5.1. Let $\tau(R) = I$. For any finitely generated right subideal H of I, take a flat envelope $H \to F_X$. Since I is flat by Proposition 5.3, F_X is contained in I. So F_X is a direct summand of R by [3, Corollary 3].

Thus I is a direct union of direct summands of R, and so it is pure as a right ideal in R. Note that $(\mathcal{A}, \mathcal{B})$ is a cosplitting torsion theory, and so $\mathcal{B} \subseteq \mathcal{F}$ by Lemma 5.8 (2). Hence $q(M) = M_{\Im} \cong E_{\mathcal{A}}(M/\tau(M)) = M/\tau(M) \cong M \otimes R/I$. Since I is pure as a left ideal in R by Lemma 5.8 (3), R/I is a flat left R-module, and so q is exact.

$(5) \Longrightarrow (6)$. Since $\mathcal{A}^{\perp} = \mathcal{F}$, every right R-module M has an essential flat envelope $\varepsilon_M : M \to E_{\mathcal{A}}(M)$ by Proposition 5.1. Let $\tau(R) = I$. Note that I is pure as a right ideal in R, $E_{\mathcal{A}}(R/I) = R/I$. Let $\{M_i\}_I$ be a family of right R-modules. Since $\mathcal{F}$ is closed under direct sums, we have

$$q(\bigoplus_I M_i) \cong E_{\mathcal{A}}((\bigoplus_I M_i)/\tau(\bigoplus_I M_i)) = E_{\mathcal{A}}(\bigoplus_I (M_i/\tau(M_i)))$$
$$= \bigoplus_I E_{\mathcal{A}}(M_i/\tau(M_i)),$$

i.e., q preserves direct sums. Thus $q(M) \cong M \otimes_R R/I \cong M/MI$ for any right R-module M [31, XI, p.231]. Let N be a right R/I-module. Since $NI = 0$, $N \cong q(N)$ is flat. By Lemma 5.5 (1), $N_{R/I}$ is a flat right R/I-module. So R/I is a von Neumann regular ring.

$(6) \Longrightarrow (7)$. Let $\tau(R) = I$ and F a flat right R-module. Since $\mathcal{A}^{\perp} = \mathcal{F}$ and $\tau = S_{\mathcal{A}}$, $E(F)/F \in \mathcal{B}$ by Lemma 3.1 (1). Hence $(E(F)/F)I \subseteq \tau(E(F)/F) = 0$. Since R/I is von Neumann regular, $(E(F)/F)_{R/I}$ is a flat right R/I-module. But I is pure as a right ideal in R, so $(E(F)/F)_R$ is a flat right R-module by Lemma 5.4. Thus $\mathrm{Ext}^1_R(A, E(F)/F) = 0$, and hence $\mathrm{Ext}^2_R(A, F) = 0$. Now let $0 \to F_1 \to F_2 \to F_3 \to 0$ be an exact sequence with $F_1,\ F_2 \in \mathcal{F}$. For any $A \in \mathcal{A}$, applying the functor $\mathrm{Hom}_R(A, -)$ to this sequence, we get the exact sequence $0 = \mathrm{Ext}^1_R(A, F_2) \to \mathrm{Ext}^1_R(A, F_3) \to \mathrm{Ext}^2_R(A, F_1) = 0$. Hence $\mathrm{Ext}^1_R(A, F_3) = 0$, i.e., $F_3 \in \mathcal{A}^{\perp} = \mathcal{F}$. So $\mathcal{F}$ is a coresolving class.

$(2) \Longrightarrow (7)$. The proof is similar to that of $(6) \Longrightarrow (7)$.

$(7) \Longrightarrow (1)$. Since every right R-module has an essential flat envelope, R is a right IF ring. So every FP-injective right R-module is flat. For any flat right R-module F, since R is a right IF ring and $\mathcal{F}$ is a coresolving class, $E(F)$ and $E(F)/F$ are flat. Hence F is pure in $E(F)$ and so F is FP-injective. Thus the class of flat right R-modules and the class of FP-injective right R-modules coincide. So R is an IF ring by [17, Proposition 2.3].

$(1) \Longrightarrow (3)$ follows from $(1) \Longrightarrow (5)$ and Lemma 5.8 (3).

$(3) \Longrightarrow (4)$. Let $\tau(R) = I$. By Lemma 5.8 (3), $(\mathcal{A}, \mathcal{B})$ is a cosplitting torsion theory. Hence $(\mathcal{A}, \mathcal{B})$ is a hereditary torsion theory.

Let $\{B_i, \varphi^i_j\}$ be a direct system with each $B_i \in \mathcal{B}$. Applying the functor $- \otimes I$ to the pure exact sequence $0 \to K \to \prod_{i \in I} B_i \to \varinjlim_{i \in I} B_i \to 0$, we get the exact sequence $0 \to KI \to (\prod_{i\in I} B_i)I \to (\varinjlim_{i\in I} B_i)I \to 0$ by Lemma 5.5 (2). Since $\mathcal{B}$ is closed under direct products and submodules, $\prod_{i\in I} B_i \in \mathcal{B}$ and $K \in \mathcal{B}$. So $(\prod_{i\in I} B_i)I = \tau(\prod_{i\in I} B_i) = 0$ and $KI = \tau(K) = 0$ by Lemma 5.8 (1). Hence $\tau(\varinjlim_{i\in I} B_i) = (\varinjlim_{i\in I} B_i)I \cong (\prod_{i\in I} B_i)I/KI = 0$. Thus $\varinjlim_{i\in I} B_i \in \mathcal{B}$, and so $\mathcal{B}$ is closed under direct limits.

$(4) \Longrightarrow (2)$. Let $\tau(R) = I$. It is clear that $\tau(R/I) = 0$. But $\mathcal{B}$ is closed under direct products and submodules, and $(R/I)^{(I)}$ is a submodule of $(R/I)^I$, so $\tau((R/I)^{(I)}) = 0$. Hence, for any projective right R/I-module P, $\tau(P_R) = 0$. Let F be a flat right R/I-module. Note that F is a direct limit of projective R/I-modules and $\mathcal{B}$ is closed under direct limits, and so $\tau(F_R) = 0$.

Note that $\mathcal{A}^{\perp} = \mathcal{F}$, and so every right R-module has an essential flat envelope by Proposition 5.1. Let M be a right R-module and $\varepsilon_{M/MI} : M/MI \to E_{\mathcal{A}}(M/MI)$ be an essential flat envelope of $(M/MI)_R$. By the proof of Proposition 5.7, $\varepsilon_{M/MI} : M/MI \to E_{\mathcal{A}}(M/MI)$ is an essential flat envelope of $(M/MI)_{R/I}$. Since $E_{\mathcal{A}}(M/MI) \in \mathcal{B}$ by the foregoing proof, $M/MI \in \mathcal{B}$. On the other hand, since $MI \subseteq \tau(M) \in \mathcal{A}$ and $(\mathcal{A}, \mathcal{B})$ is a hereditary torsion theory, $MI \in \mathcal{A}$. Thus $MI = \tau(M)$ by the exact sequence $0 \to MI \to M \to M/MI \to 0$ and [26, Lemma 1.1]. Thus $(\mathcal{A}, \mathcal{B})$ is a cohereditary torsion theory by Lemma 5.8 (1), and so $(\mathcal{A}, \mathcal{B})$ is a cospitting torsion theory. ∎

Corollary 5.10. *Let R be a right IF ring. If there exists a cosplitting torsion theory $(\mathcal{A}, \mathcal{B})$ in $\mathcal{M}_R$ with $\tau(R) = I$ pure as a right ideal in R such that R/I is von Neumann regular, and $E(F)I = FI$ for any flat right R-module F, then R is an IF ring with essential flat envelopes on the right.*

Proof. Take $N \in \mathcal{A}^{\perp}$ and consider the exact sequence $0 \to N \to E(N) \to E(N)/N \to 0$. By Lemma 3.1 (1), $E(N)/N \in \mathcal{B}$. So $(E(N)/N)I = 0$ and $E(N)/N$ is a right R/I-module. Since R/I is von Neumann regular and I is pure as a right ideal in R, $E(N)/N$ is a flat right R-module by Lemma 5.4. Note that R is right IF, and so $E(N)$ is flat. Hence N is flat. On the other hand, take $F \in \mathcal{F}$ and consider the exact sequence $0 \to F \to E(F) \to E(F)/F \to 0$. Since I is pure as a left ideal in R by Lemma 5.8 (3), we have the exact sequence $0 \to FI \to E(F)I \to (E(F)/F))I \to 0$ by Lemma 5.5 (2). But $FI = E(F)I$, and so $(E(F)/F))I = 0$. Since $S_{\mathcal{A}} = \tau$, $F \in \mathcal{A}^{\perp}$ by Lemma 3.1 (1). Thus $\mathcal{A}^{\perp} = \mathcal{F}$, and so R is an IF ring with essential flat envelopes on the right by Theorem 5.9. ∎

We conclude this paper with the following example as an application of Corollary 5.10.

Example 5.11. Let R be the K-algebra with basis $\{1\} \cup \{e_n | n \in \mathbf{Z}\} \cup \{x_n | n \in \mathbf{Z}\}$, where 1 is an identity for R, and for all $i, j \in \mathbf{Z}$, we have

$$e_i e_j = \delta_{i,j} e_i,\ e_i x_j = \delta_{i-1,j} x_j,\ x_i e_j = \delta_{i,j} x_i,\ x_i x_j = 0.$$

Then R is an IF ring with essential flat envelopes on both sides by [24, Example 3]. Next we show that this fact can be obtained from Corollary 5.10. In fact, it is easily seen that R is a right IF ring by the argument described in [10, Example 2]. Let $I = \oplus_{n\in\mathbf{Z}} e_n R$. It is easy to check that I is pure as a right ideal and it is a self-generator as a right R-module, and that $R/I \cong K$ is von Neumann regular. After a routine inspection we can show that $(\mathrm{Gen}(I), \{M_R | MI = 0\})$ is a cosplitting torsion theory. Note that $e_n R$ is $e_m R$-injective for all $n, m \in \mathbf{Z}$ by [24, Example 3]. Adapting the proof of [25, Proposition 1.5 and Theorem 1.11], we get that every flat object in $\mathrm{Gen}(I)$ is an injective object. Now, let F be a flat right R-module, since I is pure as a right ideal in R and K is von Neumann regular, F/FI is a flat right R-module by Lemma 5.4. But I is a projective generator of $\mathrm{Gen}(I)$, and so FI is a flat object in $\mathrm{Gen}(I)$. Hence FI coincides with its injective envelope E in $\mathrm{Gen}(I)$. Furthermore, $E(F)$ is a flat right R-module (for R is a right IF ring), and so $E(F)I \in \mathrm{Gen}(I)$. It is clear that $E(F)I \subseteq E$, then $FI \subseteq E(F)I \subseteq E = FI$, i.e., $E(F)I = FI$. It follows that R is an IF ring with essential flat envelopes on the right by Corollary 5.10. Similarly, one sees that $I' = \oplus_{n\in\mathbf{Z}} Re_n$ satisfies the left version of the conditions in Corollary 5.10. So R is an IF ring with essential flat envelopes on both sides.

Acknowledgments

This research was partially supported by NSFC (No. 10771096) and Natural Science Foundation of Jiangsi Province of China (No. 2008365). The authors would like to thank the referee for the helpful comments and suggestions.

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