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THE SECOND LARGEST NUMBER OF MAXIMAL INDEPENDENT SETS IN GRAPHS WITH AT MOST k CYCLES

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Abstract. Let G be a simple undirected graph. Denote by $\mathrm{mi}(G)$ (respectively, $\mathrm{xi}(G)$) the number of maximal (respectively, maximum) independent sets in G. In this paper we determine the second largest value of $\mathrm{mi}(G)$ for graphs with at most k cycles. Extremal graphs achieving these values are also determined.

1. INTRODUCTION

Let G be a simple undirected graph. The *neighborhood* $N_G(x)$ of a vertex x in G is the set of vertices adjacent to x, the *closed neighborhood* is the set $N_G[x] = N_G(x) \cup \{x\}$. Denote by $d_G(x) = |N_G(x)|$ the *degree* of x in G. Sometimes, we simply use $N(x)$, $N[x]$ and $d(x)$ for $N_G(x)$, $N_G[x]$ and $d_G(x)$, respectively, if no confusion occurs. Let $\delta(G) = \min\{d(x) \mid x \in V(G)\}$ and $\Delta(G) = \max\{d(x) \mid x \in V(G)\}$. For notation and terminology not defined here, we refer to [1].

An *independent set* is a subset S of $V(G)$ such that no two vertices in S are adjacent in G. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. A *maximum independent set* is an independent set of maximum size. Note that a maximum independent set is maximal but the converse is not always true. Denote by $\mathrm{mi}(G)$ (respectively, $\mathrm{xi}(G)$) the number of maximal (respectively, maximum) independent sets in G.

Erdös and Moser raised the problem of determining the maximum value of $\mathrm{mi}(G)$ for a general graph of order n and the extremal graphs achieving the maximum value. This problem was solved by Moon and Moser [22]. Since then, researchers have

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studied the problem for many special graph classes, see [2, 5, 6, 7, 10, 21, 23, 25]. For other related, including algorithmic, results on $\mathrm{mi}(G)$, see [4, 8, 12, 13, 14, 17, 18]. Compared to $\mathrm{mi}(G)$, there are less results for the parameter $\mathrm{xi}(G)$, see [3, 9, 19]. A survey on counting maximal independent sets in graphs can be found in [15].

In previous results, an interesting problem is to consider the number of the maximal independent set in graphs with restriction on the number of cycles, see [16, 24, 26]. In this paper we determine the second largest value of $\mathrm{mi}(G)$ and $\mathrm{xi}(G)$ for graphs with at most k cycles. Extremal graphs achieving these values are also determined.

The paper is organized as follows. Section 2 presents some preliminaries. We prove the main results in Sections 3 and 4. Finally, we present concluding remarks in the last section.

2. Preliminaries

In this section we present some notation and preliminary results we need in order to prove our main results. Throughout the paper, we use r to denote $\sqrt{2}$. Define

$$g(n) = \begin{cases} r^{n-2}+1, & \text{if } n \equiv 0 \pmod 2; \\ r^{n-1}, & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

$$t(n) = \begin{cases} r^{n}, & \text{if } n \equiv 0 \pmod 2; \\ r^{n-1}, & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

$$f(n) = \begin{cases} 3^{s}, & \text{if } n = 3s; \\ 4 \cdot 3^{s-1}, & \text{if } n = 3s+1; \\ 2 \cdot 3^{s}, & \text{if } n = 3s+2. \end{cases}$$

Lemma 2.1. [10]. *For any vertex x in a graph G, the followings hold.*

(1) $\mathrm{mi}(G) \le \mathrm{mi}(G-x) + \mathrm{mi}(G-N[x])$.

(2) *If x is a leaf adjacent to y, then* $\mathrm{mi}(G) = \mathrm{mi}(G-N[x]) + \mathrm{mi}(G-N[y])$.

Lemma 2.2. [5]. *If $n \ge 6$, then* $\mathrm{mi}(C_n) = \mathrm{mi}(C_{n-2}) + \mathrm{mi}(C_{n-3})$.

Lemma 2.3. [10]. *For any two disjoint graphs G and H,* $\mathrm{mi}(G \cup H) = \mathrm{mi}(G)\mathrm{mi}(H)$.

Many researchers have independently considered the problem for trees. Define a *baton* $B(i,j)$ as follows: Start with a basic path P with i vertices and attach j paths of length two to the endpoints of P.

Lemma 2.4. [23]. *If T is a tree of order n, then* $\mathrm{mi}(G) \leq g(n)$. *Furthermore, the equality holds if and only if*

$$T \cong T(n) = \begin{cases} B(2, \dfrac{n-2}{2}) \text{ or } B(4, \dfrac{n-4}{2}), & \text{if } n \equiv 0 \pmod 2; \\ B(1, \dfrac{n-1}{2}), & \text{if } n \equiv 1 \pmod 2). \end{cases}$$

In particular, as a consequence, $\mathrm{mi}(P_n) \leq g(n)$ for any path P_n. Let G and H be two vertex disjoint graphs. Denote by $G \cup H$ the union of G and H. Denote by $G + H$ the graph obtained from $G \cup H$ by adding the edges between all the vertices of G and those of H. When G is a graph each component of which is a complete graph, denote by $K_m * G$ the graph obtained from $K_m \cup G$ by adding an edge between a vertex of K_m and each component G. For forests, Jou [13] obtained the following result.

Theorem 2.5. [13]. *If F is a forest of order $n \geq 1$, then* $\mathrm{mi}(F) \leq t(n)$. *Furthermore, the equality holds if and only if $F \cong F(n)$, where*

$$F(n) = \begin{cases} \dfrac{n}{2} K_2, & \text{if } n \equiv 0 \pmod 2; \\ B(1, \dfrac{n-1-2s}{2}) \cup sK_2, & \text{if } n \equiv 1 \pmod 2). \end{cases}$$

For $n \geq 2$, let

$$G(n) = \begin{cases} sK_3, & \text{if } n = 3s; \\ K_4 \cup (s-1)K_3 \text{ or } 2K_2 \cup (s-1)K_3, & \text{if } n = 3s+1; \\ K_2 \cup sK_3, & \text{if } n = 3s+2. \end{cases}$$

For $n \geq 6$, let

$$H(n) = \begin{cases} (K_3 * K_3) \cup (s-2)K_3, \text{ or } 3K_2 \cup (s-2)K_3, & \\ \text{or } K_4 \cup K_2 \cup (s-2)K_3, & \text{if } n = 3s; \\ (K_4 * K_3) \cup (s-2)K_3, & \text{if } n = 3s+1; \\ (K_3 * K_3) \cup (s-2)K_3 \cup K_2, \text{ or } 4K_2 \cup (s-2)K_3, & \\ \text{or } K_4 \cup 2K_2 \cup (s-2)K_3, \text{ or } 2K_4 \cup (s-2)K_3, & \text{if } n = 3s+2. \end{cases}$$

For general graphs, we have the following result, see [22].

Theorem 2.6. [22]. *If G is a graph of order $n \geq 2$, then* $\mathrm{mi}(G) \leq f(n)$*. Furthermore, the equality holds if and only if $G \cong G(n)$.*

For general graphs, Jin and Li [11] proved the following result.

Theorem 2.7. [11]. *If G is a graph of order $n \geq 3$ and $G \ncong G(n)$, then*

$$\mathrm{mi}(G) \leq \begin{cases} \frac{11}{12} f(n), & \textit{if } n \equiv 1 \ (\textit{mod } 3); \\ \frac{8}{9} f(n), & \textit{otherwise.} \end{cases}$$

Furthermore, the equality holds if and only if $G \cong H(n)$.

For $n \geq 3k-1$ and $k \geq 1$, let

$$G(n,k) = \begin{cases} kK_3 \cup \dfrac{n-3k}{2} K_2, & \text{if } n-k \equiv 0 \pmod 2; \\ (k-1))K_3 \cup \dfrac{n-3k+3}{2} K_2, & \text{if } n-k \equiv 1 \pmod 2. \end{cases}$$

For $n \geq 3k$, $k \geq 2$ and $(n,k) \neq (7,2)$, let

$$H(n,k) = \begin{cases} (K_3 * K_3) \cup (k-2)K_3 \cup \dfrac{n-3k}{2} K_2, & \\ \text{or } (k-2)K_3 \cup \dfrac{n-3k+6}{2} K_2, & \text{if } n-k \equiv 0 \pmod 2; \\ (K_3 * K_3) \cup (k-3)K_3 \cup \frac{n-3k+3}{2} K_2, & \\ \text{or } (k-3)K_3 \cup \dfrac{n-3k+9}{2} K_2, & \text{if } n-k \equiv 1 \pmod 2. \end{cases}$$

Let

$$f(n,k) = \begin{cases} 3^k r^{n-3k}, & \text{if } n-k \equiv 0 \pmod 2; \\ 3^{k-1} r^{n-3(k-1)}, & \text{if } n-k \equiv 1 \pmod 2. \end{cases}$$

The following lemmas are clear, and we omit the details.

Lemma 2.8. *For any $k \geq \lfloor \frac{n}{2} \rfloor$, $f(n) \leq f(n,k)$.*

Lemma 2.9. *For any $k \geq 0$, $g(n) < f(n,k)$.*

Lemma 2.10. *For any $k' \leq k$ and $n' \leq n$, $f(n',k') \leq f(n,k)$.*

When considering the restriction on the number of cycles in graphs, Ying et al. [26] proved the following result. By Theorem 2.6, the authors [26] only needed to consider the case $n \geq 3k-1$.

Theorem 2.11. [26] *Let G be a graph with n vertices and at most k cycles, $k \geq 1$. If $n \geq 3k-1$, then* $\mathrm{mi}(G) \leq f(n,k)$. *Furthermore, the equality holds if and only if $G \cong G(n,k)$.*

Note that the theorem above also presents an upper bound for trees.

3. The Cases $k = 1$ and $k = 2$, $n \equiv 1 \pmod 2$

In this section we consider the problem for the cases $k = 1$ and $k = 2$, $n \equiv 1 \pmod 2$. First, we present an upper bound for the cycles.

Lemma 3.1. *For $n \geq 4$,*

$$\mathrm{mi}(C_n) \leq \begin{cases} \frac{5}{6} f(n,1), & \textit{if } n \equiv 1 \ (\textit{mod } 2); \\ \frac{3}{4} f(n,1), & \textit{if } n \equiv 0 \ (\textit{mod } 2). \end{cases}$$

Furthermore, the equality holds if and only if $n = 5$.

Proof. Clearly, the equality holds when $n = 5$. By Lemma 2.2, one can easily verify that the result holds for $3 \leq n \leq 8$. We prove the result by the induction hypothesis on n.

Let n be an even integer and $n \geq 9$. By Lemma 2.2, we have

$$\begin{aligned} \mathrm{mi}(C_n) &= \mathrm{mi}(C_{n-2}) + \mathrm{mi}(C_{n-3}) \\ &< \frac{3}{4} f(n-2,1) + \frac{5}{6} f(n-3,1) \\ &= \frac{11}{16} f(n,1) < \frac{3}{4} f(n,1). \end{aligned}$$

So, let n be an odd integer and $n \geq 9$. By Lemma 2.2, we have

$$\begin{aligned} \mathrm{mi}(C_n) &= \mathrm{mi}(C_{n-2}) + \mathrm{mi}(C_{n-3}) \\ &< \frac{5}{6} f(n-2,1) + \frac{3}{4} f(n-3,1) \\ &= \frac{2}{3} f(n,1) < \frac{5}{6} f(n,1). \end{aligned}$$

This completes the proof. ∎

Theorem 3.2. *Let G be a graph of order n ($n \geq 2$) with at most k cycles and $G \neq G(n,k)$.*

(1) *If* $n \equiv 1 \pmod 2$ *and* $k = 1$, *then* $\mathrm{mi}(G) \leq \frac{5}{6}f(n,k)$. *Furthermore, the equality holds if and only if* $G \cong (K_3 * K_2) \cup \frac{n-5}{2}K_2$ *or* $G \cong C_5 \cup \frac{n-5}{2}K_2$.

(2) *If* $n \equiv 1 \pmod 2$ *and* $k = 2$, *then* $\mathrm{mi}(G) \leq \frac{5}{6}f(n,k)$. *Furthermore, the equality holds if and only if* $G \cong (K_3 * K_2) \cup \frac{n-5}{2}K_2$, *or* $G \cong C_5 \cup \frac{n-5}{2}K_2$, *or* $G \cong (K_1 + 2K_2) \cup \frac{n-5}{2}K_2$.

(3) *If* $n \equiv 0 \pmod 2$ *and* $k = 1$, *then* $\mathrm{mi}(G) \leq \frac{3}{4}f(n,k)$. *Furthermore, the equality holds if and only if* $G \cong (K_1 * (K_3 \cup sK_2)) \cup \frac{n-2s-4}{2}K_2$ *or* $G \cong K_3 \cup B(1,s) \cup \frac{n-2s-4}{2}K_2$ *for some* $0 \leq s \leq \frac{n-4}{2}$

Proof. It is easy to see that the equalities hold for the graphs listed in the theorem. We prove the theorem by the induction hypothesis on n. By a simple computer search, the theorem holds clearly for $n \leq 6$. Now suppose that the graph G is of order $n \geq 7$. If G is disconnected, let $G^{'}$ be a component of order $n^{'} < n$ which contains $k^{'}$ cycles. It is easy to see that at least one of $G^{'} \ncong G(n^{'}, k^{'})$ and $G - G^{'} \ncong G(n - n^{'}, k - k^{'})$ is true.

If $n \equiv 1 \pmod 2$ and $k = 1$, then by the induction hypothesis we have

$$\begin{aligned} \mathrm{mi}(G) &= \mathrm{mi}(G^{'})\mathrm{mi}(G - G^{'}) \\ &\leq \frac{5}{6}f(n^{'}, k^{'})f(n - n^{'}, k - k^{'}) \\ &\leq \frac{5}{6}f(n,k). \end{aligned}$$

Furthermore, by the induction hypothesis, the equality holds if and only if $G^{'} \cong (K_3 * K_2) \cup \frac{n^{'}-5}{2}K_2$ or $C_5 \cup \frac{n^{'}-5}{2}K_2$ and $G - G^{'} \cong G(n - n^{'}, k - k^{'})$, or $G - G^{'} \cong (K_3 * K_2) \cup \frac{n-n^{'}-5}{2}K_2$ or $C_5 \cup \frac{n-n^{'}-5}{2}K_2$ and $G^{'} \cong G(n^{'}, k^{'})$. By construction, that is to say that the equality holds if and only if $G \cong (K_3 * K_2) \cup \frac{n-5}{2}K_2$ or $G \cong C_5 \cup \frac{n-5}{2}K_2$.

The case $n \equiv 1 \pmod 2$ and $k = 2$ can be proved in a similar way. We omit the details.

So, let $n \equiv 0 \pmod 2$ and $k = 1$. Then both $n^{'}$ and $n - n^{'}$ have the same parity. Assume that both $n^{'}$ and $n - n^{'}$ are odd. Since $k = 1$, either $G^{'}$ or $G - G^{'}$ is a forest. Without loss generality, we may assume that $G - G^{'}$ is a forest. By Theorems 2.5 and 2.11, we have

$$\begin{aligned} \mathrm{mi}(G) &= \mathrm{mi}(G^{'})\mathrm{mi}(G - G^{'}) \\ &\leq f(n^{'}, 1)t(n - n^{'}) \\ &\leq \frac{3}{4}f(n,k). \end{aligned}$$

The equality holds if and only if $G' \cong G(n', 1)$ and $G - G' \cong B(1, \frac{n-n'-1-2s'}{2}) \cup s'K_2$ for some $0 \leq s' \leq n - n' - 1$. In fact, this just implies that $G \cong K_3 \cup B(1, s) \cup \frac{n-2s-4}{2} K_2$ for some $0 \leq s \leq \frac{n-4}{2}$. The case both n' and $n - n'$ are even can be proved in similar way.

Hence in the rest of the proof we assume that G is connected. Also, by Lemma 3.1, we may assume that $G \ncong C_n$.

Let $n \equiv 0 \pmod 2$ and $k = 1$. Then we have $\delta(G) = 1$. Let $N(x) = \{y\}$, and then $d(y) \geq 2$. So, both $G - x - y$ and $G - N(y)$ contain at most one cycle.

If $d(y) \geq 4$, then by the induction hypothesis we have

$$\begin{aligned} \mathrm{mi}(G) &= \mathrm{mi}(G - x - y) + \mathrm{mi}(G - N[y]) \\ &\leq f(n-2, 1) + f(n-5, 1) \\ &= \frac{11}{16} f(n, 1) < \frac{3}{4} f(n, 1). \end{aligned}$$

So, let $d(y) = 3$. If $G - x - y \ncong G(n-2, 1)$ or $G - N(y) \ncong G(n-4, 1)$, then by the induction hypothesis we have

$$\begin{aligned} \mathrm{mi}(G) &= \mathrm{mi}(G - x - y) + \mathrm{mi}(G - N[y]) \\ &\leq \frac{3}{4} f(n-2, 1) + f(n-4, 1) \\ &= \frac{5}{8} f(n, 1) < \frac{3}{4} f(n, 1). \end{aligned}$$

or

$$\begin{aligned} \mathrm{mi}(G) &= \mathrm{mi}(G - x - y) + \mathrm{mi}(G - N[y]) \\ &\leq f(n-2, 1) + \frac{3}{4} f(n-4, 1) \\ &= \frac{11}{16} f(n, 1) < \frac{3}{4} f(n, 1). \end{aligned}$$

So we assume that $G - x - y \cong G(n-2, 1)$ and $G - N(y) \cong G(n-4, 1)$. This implies that $G \cong (K_1 * K_3) \cup \frac{n-4}{2} K_2$.

So let $d(y) = 2$. Suppose that $G - x - y \cong G(n-2, 1)$. By the construction of the graph $G(n-2, 1)$, we have $\mathrm{mi}(G) = \frac{1}{2} f(n, 1) < \frac{3}{4} f(n, 1)$. So we assume that $G - x - y \ncong G(n-2, 1)$. If $G - N(y) \ncong G(n-3, 1)$, then by the induction hypothesis we have

$$\begin{aligned} \mathrm{mi}(G) &= \mathrm{mi}(G - x - y) + \mathrm{mi}(G - N[y]) \\ &\leq \frac{3}{4} f(n-2, 1) + \frac{5}{6} f(n-3, 1) \\ &= \frac{11}{16} f(n, 1) < \frac{3}{4} f(n, 1). \end{aligned}$$

If $G - N(y) \cong G(n-3,1)$, then by the induction hypothesis we have

$$\begin{aligned} \mathrm{mi}(G) &= \mathrm{mi}(G-x-y) + \mathrm{mi}(G-N[y]) \\ &\le \frac{3}{4} f(n-2,1) + f(n-3,1) \\ &= \frac{3}{4} f(n,1). \end{aligned}$$

Furthermore, the equality holds only if $G - x - y \cong (K_1 * (K_3 \cup s)) \cup \frac{n-2s-6}{2} K_2$ or $G - x - y \cong K_3 \cup B(1,s) \cup \frac{n-2s-6}{2} K_2$ for some $0 \le s \le \frac{n-6}{2}$, and $G - N[y] \cong G(n-3,1)$. This implies that the equality holds if and only if $G \cong (K_1 * (K_3 \cup sK_2)) \cup \frac{n-2s-4}{2} K_2$ or $G \cong K_3 \cup B(1,s) \cup \frac{n-2s-4}{2} K_2$ for some $0 \le s \le \frac{n-4}{2}$.

This completes the proof of the case $n \equiv 0 \pmod 2$ and $k = 1$. The case $n \equiv 1 \pmod 2$ and $k = 1$ *or* 2 can be proved in the similar way. For simplicity we omit the details. ∎

4. Remaining Cases

In this section we consider the second largest $\mathrm{mi}(G)$ for the cases other than that in previous section. Since Theorem 2.7 gives a complete answer for $n \le 3k$, we only need to consider the case $n \ge 3k$. We have the following result.

Theorem 4.1. *Let G be a graph of order n ($n \ge 2$) with at most k cycles, and $G \ncong G(n,k)$. If $n \equiv 1$ (mod 2) and $k \ge 3$, or $n \equiv 0$ (mod 2) and $k \ge 2$, then $\mathrm{mi}(G) \le \frac{8}{9} f(n,k)$. Furthermore, the equality holds if and only if $G \cong H(n,k)$.*

Proof. If $n = 3s+1$, since $\frac{11}{12} f(n) < \frac{8}{9} f(n,k)$ for any $k \ge s+6$, by Theorem 2.7, the theorem holds for any $n \le 3k$. So it's left to consider the case $n \ge 3k$ in the following proof.

It is easy to see that $\mathrm{mi}(H(n,k)) = \frac{8}{9} f(n,k)$. We prove the theorem by the induction hypothesis on n. By simple computer search, the theorem holds for $3 \le n \le 6$. Now we consider the graph G of order n, $n \ge 7$, which contains k cycles.

If G is disconnected, let G' be a component of order $n' < n$ which contains k' cycles. Since $G \ncong G(n,k)$, it is easy to see that at least one of $G' \ncong G(n',k')$ and $G - G' \ncong G(n-n', k-k')$ is true. Then

$$\begin{aligned} \mathrm{mi}(G) &= \mathrm{mi}(G')\mathrm{mi}(G-G') \\ &\le \frac{8}{9} f(n',k') f(n-n', k-k') \\ &\le \frac{8}{9} f(n,k). \end{aligned}$$

Furthermore, by the construction of $H(n,k)$ and $G(n,k)$, the equality holds if and only if $G \cong H(n,k)$. Hence we may assume that G is connected. We distinguish the following cases.

Case 1. $\delta(G) = 1$. Let $N(x) = \{y\}$. Since $G \ncong K_2$, $d(y) \geq 2$. So, both $G - x - y$ and $G - N(y)$ contain at most k cycles. By the induction hypothesis we have

$$\begin{aligned}
\text{mi}(G) &= \text{mi}(G - x - y) + \text{mi}(G - N[y]) \\
&\leq f(n-2,k) + f(n-3,k) \\
&= \begin{cases} 3^k r^{n-2-3k} + 3^{k-1} r^{n-3-3(k-1)}, \text{ if } & n-k \equiv 0 \pmod 2; \\ 3^{k-1} r^{n-2-3(k-1)} + 3^k r^{n-3-3k}, \text{ if } & n-k \equiv 1 \pmod 2; \end{cases} \\
&= \begin{cases} \frac{5}{6} f(n,k), \text{ if } & n-k \equiv 0 \pmod 2; \\ \frac{7}{8} f(n,k), \text{ if } & n-k \equiv 1 \pmod 2; \end{cases} \\
&< \frac{8}{9} f(n,k).
\end{aligned}$$

Case 2. $\delta(G) = \triangle(G) = 2$. Then $G \cong C_n$, and Lemma 3.1 implies that the theorem is true.

Case 3. $\delta(G) \geq 2$, $\triangle(G) \geq 3$, and there are two cycles sharing the same vertex y.

Then both $G - y$ and $G - N[y]$ contain at most $k-2$ cycles. Without loss of generality, we can choose the vertex y with $d(y) \geq 3$. By the induction hypothesis we have

$$\begin{aligned}
\text{mi}(G) &\leq \text{mi}(G - y) + \text{mi}(G - N[y]) \\
&\leq f(n-1,k-2) + f(n-4,k-2) \\
&= \begin{cases} 3^{k-3} r^{n-1-3(k-3)} + 3^{k-2} r^{n-4-3(k-2)}, & \text{if } n-k \equiv 0 \pmod 2; \\ 3^{k-2} r^{n-1-3(k-2)} + 3^{k-3} r^{n-4-3(k-3)}, & \text{if } n-k \equiv 1 \pmod 2; \end{cases} \\
&= \begin{cases} \frac{22}{27} f(n,k), \text{ if } n-k \equiv 0 \pmod 2; \\ \frac{8}{9} f(n,k), \quad \text{if } n-k \equiv 1 \pmod 2; \end{cases} \\
&\leq \frac{8}{9} f(n,k).
\end{aligned}$$

Furthermore, the equality holds if and only if $n - k \equiv 1 \pmod 2$, $G - y \cong G(n-1,k-2)$ and $G - N[y] \cong G(n-4,k-2)$. From $G - N[y] \cong G(n-4,k-2)$

we have $d(y) = 3$; and from $G-y \cong G(n-1, k-2) = (k-2)K_3 \cup \frac{n-1-3(k-2)}{2}K_2$, we have that there are exactly two cycles passing through y. This is a contradiction.

Case 4. $\delta(G) \geq 2$, $\triangle(G) \geq 3$, and the cycles of G are vertex-disjoint.

Then, since $G \ncong C_n$ and $\delta(G) \geq 2$, there is at least one cycle, denoted by C_l, such that there is a unique cut-vertex x of G on C_l, i.e., x is the unique vertex of C_l adjacent to vertices not on C_l. It is easy to see that $d(x) \geq 3$ and $G-x$ contains exactly $k-1$ cycles. We distinguish the following subcases.

Subcase 4.1. $n-k \equiv 1 \pmod 2$ and $l = 3$.

If $d(x) \geq 4$, by the induction hypothesis we have

$$\begin{aligned} \text{mi}(G) &\leq \text{mi}(G-x) + \text{mi}(G-N[x]) \\ &= \text{mi}(G-C_l)\text{mi}(P_{l-1}) + \text{mi}(G-N[x]) \\ &\leq f(n-3, k-1)\text{mi}(P_2) + f(n-5, k-1) \\ &= 2 \cdot 3^{k-2} r^{n-3-3(k-2)} + 3^{k-2} r^{n-5-3(k-2)} \\ &= \frac{5}{6} f(n,k) < \frac{8}{9} f(n,k). \end{aligned}$$

So, let $d(x) = 3$. We claim that $G - C_l \ncong G(n-3, k-1)$. Otherwise, from the construction of the graph $G(n-3, k-1)$, $G - C_l$ contains exactly $k-2$ cycles, which implies that G contains exactly $k-1$ cycles, a contradiction. Then, by the induction hypothesis we have

$$\begin{aligned} \text{mi}(G) &\leq \text{mi}(G-x) + \text{mi}(G-N[x]) \\ &= \text{mi}(G-C_l)\text{mi}(P_{l-1}) + \text{mi}(G-N[x]) \\ &\leq \frac{8}{9} f(n-3, k-1)\text{mi}(P_2) + f(n-4, k-1) \\ &= \frac{16}{9} \cdot 3^{k-2} r^{n-3-3(k-2)} + 3^{k-1} r^{n-4-3(k-1)} \\ &= \frac{91}{108} f(n,k) < \frac{8}{9} f(n,k). \end{aligned}$$

Thus, if $n-k \equiv 1 \pmod 2$ and $l = 3$, we have $\text{mi}(G) < \frac{8}{9} f(n,k)$.

Subcase 4.2. $n-k \equiv 0 \pmod 2$ and $l = 3$.

If $d(x) \geq 4$, by the induction hypothesis we have

$$\begin{aligned} \text{mi}(G) &\leq \text{mi}(G-x) + \text{mi}(G-N[x]) \\ &= \text{mi}(G-C_l)\text{mi}(P_{l-1}) + \text{mi}(G-N[x]) \\ &\leq f(n-3, k-1)\text{mi}(P_2) + f(n-5, k-1) \\ &= 2 \cdot 3^{k-1} r^{n-3-3(k-1)} + 3^{k-1} r^{n-5-3(k-1)} \\ &= \frac{5}{6} f(n,k) < \frac{8}{9} f(n,k). \end{aligned}$$

So, let $d(x) = 3$. Then, by the induction hypothesis we have

$$\begin{aligned}\mathrm{mi}(G) &\le \mathrm{mi}(G-x) + \mathrm{mi}(G-N[x]) \\ &= \mathrm{mi}(G-C_l)\mathrm{mi}(P_{l-1}) + \mathrm{mi}(G-N[x]) \\ &\le f(n-3,k-1)\mathrm{mi}(P_2) + f(n-4,k-1) \\ &= 2\cdot 3^{k-2}r^{n-3-3(k-2)} + 3^{k-1}r^{n-4-3(k-1)} \\ &= \frac{8}{9}f(n,k).\end{aligned}$$

Furthermore, the equality holds if and only if $G - C_l \cong G(n-3, k-1)$ and $G - N[x] \cong G(n-4, k-1)$. Since $\delta(G) \ge 2$ and $G - C_l$ is connected, we have $G - C_l \cong K_3$, and $G - N[x] \cong K_2$. That is to say, $G \cong H(6, 2)$. This contradicts to the assumption $n \ge 7$.

Thus, if $n - k \equiv 0 \pmod 2$ and $l = 3$, we have $\mathrm{mi}(G) < \frac{8}{9}f(n,k)$.

Subcase 4.3. Otherwise, by the induction hypothesis we have

$$\begin{aligned}\mathrm{mi}(G) &\le \mathrm{mi}(G-x) + \mathrm{mi}(G-N[x]) \\ &= \mathrm{mi}(G-C_l)\mathrm{mi}(P_{l-1}) + \mathrm{mi}(G-C_l-N(x))\mathrm{mi}(P_{l-3}) \\ &\le f(n-l,k-1)\mathrm{mi}(P_{l-1}) + f(n-l-1,k-1)\mathrm{mi}(P_{l-3}) \\ &\le \begin{cases} 3^{k-2}r^{n-l-3(k-2)}r^{l-2} & \\ +3^{k-1}r^{n-l-1-3(k-1)}r^{l-4}, & \text{if } n-k\equiv 0 \pmod 2, \\ & l\equiv 0 \pmod 2; \\ 3^{k-1}r^{n-l-3(k-1)}(r^{l-3}+1) & \\ +3^{k-2}r^{n-l-1-3(k-2)}(r^{l-5}+1), & \text{if } n-k\equiv 0 \pmod 2, \\ & l\equiv 1 \pmod 2, \\ & \text{and } l\ge 5; \\ 3^{k-1}r^{n-l-3(k-1)}r^{l-2} & \\ +3^{k-2}r^{n-l-1-3(k-2)}r^{l-4}, & \text{if } n-k\equiv 1 \pmod 2, \\ & l\equiv 0 \pmod 2; \\ 3^{k-2}r^{n-l-3(k-2)}(r^{l-3}+1) & \\ +3^{k-1}r^{n-l-1-3(k-1)}(r^{l-5}+1), & \text{if } n-k\equiv 1 \pmod 2, \\ & l\equiv 1 \pmod 2, \\ & \text{and } l\ge 5; \end{cases}\end{aligned}$$

$$\leq \begin{cases} \dfrac{11}{18}f(n,k), & \text{if } n-k \equiv 0 \pmod 2, l \equiv 0 \pmod 2; \\ \dfrac{13}{18}f(n,k), & \text{if } n-k \equiv 0 \pmod 2, l \equiv 1 \pmod 2, \\ & \text{and } l \geq 5; \\ \dfrac{2}{3}f(n,k), & \text{if } n-k \equiv 1 \pmod 2, l \equiv 0 \pmod 2; \\ \dfrac{3}{4}f(n,k), & \text{if } n-k \equiv 1 \pmod 2, l \equiv 1 \pmod 2, \\ & \text{and } l \geq 5; \end{cases}$$

$$< \frac{8}{9}f(n,k).$$

This completes the proof. ∎

5. Concluding Remarks

Note that, if $G \cong (K_3 * K_2) \cup \frac{n-5}{2}K_2$, or $G \cong C_5 \cup \frac{n-5}{2}K_2$, or $G \cong H(n,k)$, an independent set of G is maximal if and only if it is maximum, i.e., $\mathrm{xi}(G) = \mathrm{mi}(G)$. From Theorems 3.2 and 4.1, we have the following result for maximum independent sets.

Theorem 1.1. *Let G be a graph of order n with at most k cycles, $n \geq 3k$, and $G \not\cong G(n,k)$.*

(1) *If $n \equiv 1$ (mod 2) and $k = 1$ or 2, then $\mathrm{xi}(G) \leq \frac{5}{6}f(n,k)$. Furthermore, the equality holds if and only if $G \cong (K_3 * K_2) \cup \frac{n-5}{2}K_2$ or $G \cong C_5 \cup \frac{n-5}{2}K_2$.*

(2) *If $n \equiv 1$ (mod 2) and $k \geq 3$, or $n \equiv 0$ (mod 2) and $k \geq 2$, then $\mathrm{xi}(G) \leq \frac{8}{9}f(n,k)$. Furthermore, the equality holds if and only if $G \cong H(n,k)$.*

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