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REGULARIZATION AND ITERATION METHODS FOR A CLASS OF MONOTONE VARIATIONAL INEQUALITIES

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Abstract. We consider the monotone variational inequality of finding $x^* \in C$ such that $\langle (I-T)x^*, x-x^*\rangle \geq 0$ for $x \in C$, where C is a closed convex subset of a real Hilbert space and T is a nonexpansive self-mapping of C. Techniques of nonexpansive mappings are applied to regularize this variational inequality. The regularized solutions and an iteration process are shown to converge in norm to a solution of this variational inequality.

1. Introduction

A *variational inequality* (VI) is formulated as finding a point x^* satisfying the property

$$x^* \in C \quad \text{such that} \quad \langle Fx^*, x-x^*\rangle \geq 0, \quad x \in C \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H and $F : C \to H$ is a (nonlinear) operator. The VI (1.1) is said to be *monotone* if the operator F is monotone. In this paper we are concerned with a special class of monotone variational inequalities where the operator F can be decomposed as $I - T$ with a nonexpansive mapping $T : C \to C$. Namely, the VIs that we will investigate are of the form

$$x^* \in C \quad \text{such that} \quad \langle (I-T)x^*, x-x^*\rangle \geq 0, \quad x \in C. \tag{1.2}$$

It is known that if the operator F is Lipschitz continuous and strongly monotone, then the VI (1.1) is well-defined.

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It is well-known that the VIP (1.1) is equivalent to the fixed point equation

$$x^* = P_C(I - \gamma F)x^* \tag{1.3}$$

where $\gamma > 0$ and P_C is the metric projection of H onto C.

It is also well-known that if F is Lipschitzian and strongly monotone, then for small enough $\gamma > 0$, the mapping $P_C(I - \gamma F)$ is a contraction on C and so the sequence $\{x_n\}$ of Picard iterates, given by $x_n = P_C(I - \gamma F)x_{n-1}$ ($n \geq 1$), converges strongly to the unique solution of the VIP (1.1).

The VI (1.2) is equivalent to the fixed point equation, for any $\gamma > 0$,

$$x^* = P_C(I - \gamma(I - T))x^* = P_C((1 - \gamma)I + \gamma T)x^*.$$

It is observed that the illness of VI (1.2) is because of lack of strong monotonicity of the mapping $I - T$, due the nonexpansivity of T. Regularization by contractions can remove such illness. We therefore replace the nonexpansive mapping T by a family of contractions $tf + (1-t)T$, with $t \in (0,1)$ and $f : C \to C$ a fixed contraction. That is, we consider the regularized problems

$$x_t \in C, \quad \langle (I - [tf + (1-t)T])x_t, x - x_t \rangle \geq 0, \quad x \in C. \tag{1.4}$$

This is equivalent to the fixed point equation, for any $\gamma > 0$,

$$x_t = P_C(I - \gamma(I - [tf + (1-t)T]))x_t. \tag{1.5}$$

The purpose of this paper is to investigate the behavior of the regularized solutions $\{x_t\}$ as $t \downarrow 0$ and also of the following iteration process obtained by discretizing the implicit scheme (1.4):

$$x_{n+1} \in C, \quad \langle x_{n+1} - [t_n f(x_n) + (1 - t_n)Tx_n], x - x_{n+1} \rangle \geq 0, \quad x \in C. \tag{1.6}$$

The paper is organized as follows. The next section introduces some preliminaries. In Sections 3 and 4 we prove convergence of the regularized solutions $\{x_t\}$ and of the iteration process (1.6), respectively. Finally in Section 5 we apply the results obtained in Sections 3 and 4 to a minimization problem.

2. Preliminaries

Throughout this section, H will be a real Hilbert space and C is a nonempty closed convex subset of H. We now recall the following concepts of mappings.

(i) A mapping $f : C \to C$ is a ρ-contraction if $\rho \in [0,1)$ and if the following property is satisfied:

$$\|f(x) - f(x')\| \leq \rho\|x - x'\|, \quad x, x' \in C.$$

(ii) A mapping $T : C \to C$ is nonexpansive provided

$$\|Tx - Tx'\| \leq \|x - x'\|, \quad x, x' \in C.$$

(iii) A mapping $F : C \to H$ is

(a) monotone if

$$\langle Fx - Fy, x - y\rangle \geq 0, \quad x, y \in C;$$

(b) strictly monotone if

$$\langle Fx - Fy, x - y\rangle > 0, \quad x, y \in C,\ x \neq y;$$

(c) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \quad x, y \in C.$$

(For more details about the theory of monotone operators, see [1].)

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in C$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

The following properties of projections are useful and often used.

Lemma 2.1. *Given $x \in H$ and $z \in C$.*

(i) *That $z = P_C x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \leq 0 \quad \textit{for all } y \in C.$$

(ii) *There holds the relation*

$$\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2 \quad \textit{for all } x, y \in H.$$

Consequently, P_C is monotone and nonexpansive.

The following lemma is not hard to prove.

Lemma 2.2. (cf. [14]). *Let $f : C \to C$ be a ρ-contraction with $\rho \in [0, 1)$ and $T : C \to C$ be a nonexpansive mapping. Then*

(i) $I - f$ is $(1 - \rho)$-strongly monotone:

$$\langle (I - f)x - (I - f)y, x - y\rangle \geq (1 - \rho)\|x - y\|^2, \quad x, y \in C;$$

(ii) $I - T$ *is monotone:*

$$\langle (I-T)x - (I-T)y, x-y\rangle \geq 0, \quad x, y \in C.$$

Lemma 2.3. (Demiclosedness Principle) (cf. [2]). *Let* $T : C \to C$ *be a nonexpansive mapping with* $Fix(T) \neq \emptyset$*. If* $\{x_n\}$ *is a sequence in* C *weakly converging to* x *and if* $\{(I-T)x_n\}$ *converges strongly to* y*, then* $(I-T)x = y$*.*

Some recent progresses on the investigations on iterative methods for nonexpansive mappings can be found in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15] and the references cited there.

The following plays a key role in proving strong convergence of our algorithms.

Lemma 2.4. [9]. *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n + \beta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ *and* $\{\beta_n\}$ *are sequences in (0,1) and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$*;*

(ii) *either* $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} \gamma_n|\delta_n| < \infty$*;*

(iii) $\sum_{n=1}^{\infty} \beta_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Notation. Let $\{x_n\}$ be a sequence and x be a point in a normed space X.

- $x_n \to x$ means that $\{x_n\}$ converges to x in norm;
- $x_n \rightharpoonup x$ means that $\{x_n\}$ converges to x weakly;
- $\omega_w(x_n)$ is the weak ω-limit set of $\{x_n\}$; that is, $\omega_w(x_n) = \{x : \exists x_{n_i} \rightharpoonup x\}$.

3. A Regulariztion Method

If $F : C \to H$ is L-Lipschitz continuous and η-strongly monotone, then for $0 < \gamma < 2\eta/L^2$, the mapping

$$T_\gamma := P_C(I - \gamma F)$$

is a contraction with contraction coefficient $\sqrt{1-\gamma(2\eta - \gamma L^2)} < 1$. Therefore, for such a γ and for each $x_0 \in C$, the sequence of Picard iterates, $\{T_\gamma^n x_0\}$, converges in norm to the unique solution of VI (1.1) (see [16, 13] for hybrid methods in the case when C is the set of fixed points of another nonexpansive mappings).

However, if either Lipschitz continuity or strong monotonicity of T is violated, the VI (1.1) would be ill-posed; thus regularization is needed. The VI (1.2) falls in this case since the operator $I - T$ is not strongly monotone though it is indeed Lipschitzian. Since contractions can be used to regularize nonexpansive mappings, we can extend this way to regularize the VI (1.2); details are carried out below. Let us copy the VI (1.2) below:

$$x^* \in C, \quad \langle (I-T)x^*, x - x^* \rangle \geq 0, \quad x \in C. \tag{3.1}$$

Let us denote by S the solution set of VI (3.1).

Recall that a necessary and sufficient condition for $x^* \in C$ to solve (3.1) is that $x^* \in C$ solves the fixed point equaiton

$$x^* = P_C(I - \gamma(I-T))x^* = P_C((1-\gamma)I + \gamma T)x^* \tag{3.2}$$

where $\gamma > 0$ is any fixed constant. The focus of our technique lies in the fact that the operator $P_C(I - \gamma(I-T))$ is nonexpansive $0 < \gamma \leq 1$. This fact is summarized below.

Proposition 3.1. *The operator $P_C(I-\gamma(I-T))$ is nonexpansive if $0 < \gamma \leq 1$ and $\frac{1+\gamma}{2}$-averaged if $0 < \gamma < 1$.*

Our idea is to turn the VI (3.1) into its equivalent fixed point formulation (3.2) and then regularize the nonexpansive mapping T by contractions. Details are carried out below.

We fix $\gamma \in (0, 1]$. Now for each $t \in (0, 1)$ and a ρ-contraction $f : C \to C$, we introduce a contraction $V_t : C \to C$ by

$$V_t = (1-\gamma)I + \gamma[tf + (1-t)T].$$

It is easily found that V_t is a contraction with coefficient $1 - (1-\rho)\gamma t$, so is $P_C V_t$. Hence $P_C V_t$ has a unique fixed point in C which is denoted by x_t. That is, x_t is the unique solution in C of the following fixed point equation

$$x_t = P_C V_t x_t = P_C((1-\gamma)I + \gamma[tf + (1-t)T])x_t. \tag{3.3}$$

Equivalently, x_t is the unique solution of the following VI

$$x_t \in C, \quad \langle x_t - V_t x_t, x - x_t \rangle \geq 0, \quad x \in C. \tag{3.4}$$

Before stating the main result of this section, we need the equivalence of monotone VIs with their dual counterparts.

Proposition 3.2. *Assume that $F : C \to H$ is monotone and weakly continuous along segments (i.e., $F(x+ty) \rightharpoonup Fx$ as $t \to 0$). Then the VI (1.1) is equivalent to its dual variational inequality*

$$x^* \in C, \quad \langle Fx, x - x^* \rangle \geq 0, \quad x \in C.$$

In particular, the VI (3.1) is equivalent to its dual

$$x^* \in C, \quad \langle (I-T)x, x-x^* \rangle \geq 0, \quad x \in C.$$

Theorem 3.3. *Fix* $\gamma \in (0,1]$. *The solution set* S *of the VI (3.1) is nonempty if and only of the net* $\{x_t\}$ *of the solutions of the regularized fixed point problem (3.3) remains bounded as* $t \downarrow 0$. *Moreover, if* $S \neq \emptyset$, *then* $\{x_t\}$ *converges in norm as* $t \downarrow 0$ *to a solution* $x^* \in S$ *which also solves the following VI*

$$x^* \in S, \quad \langle (I-f)x^*, x-x^* \rangle \geq 0, \quad x \in S. \tag{3.5}$$

Proof. Write $T_\gamma = I - \gamma(I-T) = (1-\gamma)I + \gamma T$. Notice that $S = Fix(P_C T_\gamma)$.
First assume $S \neq \emptyset$. Then we can take $p \in S$ to deduce that

$$\begin{aligned}\|x_t - p\| &= \|(P_C V_t)x_t - (P_C T_\gamma)p\| \\ &\leq \|V_t x_t - T_\gamma p\| \\ &\leq \|V_t x_t - V_t p\| + \|V_t p - T_\gamma p\| \\ &\leq [1-(1-\rho)\gamma t]\|x_t - p\| + \gamma t \|f(p) - Tp\|.\end{aligned}$$

It follows that

$$\|x_t - p\| \leq \frac{1}{1-\rho}\|f(p) - Tp\|$$

hence $\{x_t\}$ is bounded.

Conversely, assume that $\{x_t\}$ remains bounded as $t \downarrow 0$. Observe that VI (3.4) can be rewritten as (using the definition of V_t)

$$t\langle x_t - f(x_t), x - x_t \rangle + (1-t)\langle x_t - Tx_t, x - x_t \rangle \geq 0, \quad x \in C. \tag{3.6}$$

Now since $I - T$ is monotone, we get from (3.6), for $x \in C$,

$$\begin{aligned}\langle x - Tx, x_t - x \rangle &\leq \langle x_t - Tx_t, x_t - x \rangle \\ &\leq \frac{t}{1-t}\langle x_t - f(x_t), x - x_t \rangle.\end{aligned} \tag{3.7}$$

The boundedness of $\{x_t\}$ and (3.7) imply that

$$\limsup_{t \downarrow 0} \langle x - Tx, x_t - x \rangle \leq 0. \tag{3.8}$$

It follows that if a sequence $\{t_n\}$ in $(0,1)$ is such that $t_n \to 0$ and $x_{t_n} \rightharpoonup \tilde{x}$, then

$$\langle x - Tx, \tilde{x} - x \rangle \leq 0, \quad x \in C.$$

By Proposition 3.2, we conclude that $\tilde{x}$ solves the VI (3.2); hence $S \neq \emptyset$ and $\omega_w(x_t) \subset S$.

Finally we prove that the entire net $\{x_t\}$ indeed converges in norm to the unique solution x^* of the VI (3.5) provided we assume $S \neq \emptyset$. To see this, take $\bar{x} \in S$; hence, by Proposition 3.2, $\langle (I-T)x, x-\bar{x}\rangle \geq 0$ for $x \in C$. This together with (3.6) results in

$$\langle x_t - f(x_t), \bar{x} - x_t\rangle \geq \frac{t}{1-t}\langle (I-T)x_t, x_t - \bar{x}\rangle \geq 0. \tag{3.9}$$

Next since $I-f$ is $(1-\rho)$-strongly monotone, we get

$$\begin{aligned}(1-\rho)\|x_t - \bar{x}\|^2 &\leq \langle (I-f)x_t - (I-f)\bar{x}, x_t - \bar{x}\rangle \\ &= \langle (I-f)x_t, x_t - \bar{x}\rangle - \langle (I-f)\bar{x}, x_t - \bar{x}\rangle \\ &\leq -\langle (I-f)\bar{x}, x_t - \bar{x}\rangle.\end{aligned}$$

Therefore, for $\bar{x} \in S$,

$$\|x_t - \bar{x}\|^2 \leq -\frac{1}{1-\rho}\langle (I-f)\bar{x}, x_t - \bar{x}\rangle. \tag{3.10}$$

It turns out that if $\{x_{t_n}\}$ is a subsequence of $\{x_t\}$ such that $x_{t_n} \rightharpoonup \bar{x}$, then since $\bar{x} \in S$, we get from (3.10) that $x_{t_n} \to \bar{x}$. This proves that $\{x_t\}$ is actually relatively compact in the norm topology (at $t \downarrow 0$). To see that the whole net $\{x_t\}$ is strongly convergent, we assume that $x_{t'_n} \to \tilde{x}$, where $t'_n \to 0$. Taking limits in (3.9) as $t = t_n \to 0$ and $t = t'_n \to 0$, respectively, we get

$$\langle (I-f)\bar{x}, \tilde{x} - \bar{x}\rangle \geq 0 \quad \text{and} \quad \langle (I-f)\tilde{x}, \bar{x} - \tilde{x}\rangle \geq 0.$$

Adding up these two inequalities and using the strong monotonicity of $I-f$ yield

$$(1-\rho)\|\tilde{x} - \bar{x}\|^2 \leq \langle (I-f)\tilde{x} - (I-f)\bar{x}, \tilde{x} - \bar{x}\rangle \leq 0.$$

Hence $\tilde{x} = \bar{x}$ and $\{x_t\}$ converges in norm to some point (say) $x^* \in S$. It remains to prove that x^* solves the VI (3.5). As a matter of fact, taking the limit as $t \to 0$ in (3.9) gives that $\langle (I-f)x^*, \bar{x} - x^*\rangle \geq 0$ for all $\bar{x} \in S$. This is (3.5). ∎

Corollary 3.4. *If the solution set S of the VI (3.1) is empty, then* $\lim_{t\downarrow 0} \|x_t\| = \infty$.

Proof. If $S = \emptyset$, then no subsequences of $\{x_t\}$ can be bounded. Therefore, $\|x_t\| \to \infty$ as $t \downarrow 0$. ∎

4. An Iteration Method

In this section, we discretize the implicit scheme (3.3) to get an iterative algorithm which will be proved to converge in norm. Again fix $\gamma \in (0,1]$. Now define V_n by

$$V_n = (1-\gamma)I + \gamma[t_n f + (1-t_n)T],$$

where $\{t_n\}$ is a sequence in $(0,)]$ and f is a ρ-contraction on C. It is seen that V_n is a contraction with coefficient $1-(1-\rho)\gamma t_n$. Initializing $x_0 \in C$, we define a sequence $\{x_n\}$ in C in the manner: x_{n+1} is the solution of the VI

$$x_{n+1} \in C, \quad \langle x_{n+1} - V_n x_n, x - x_{n+1}\rangle \ge 0, \quad x \in C. \tag{4.1}$$

Remark. Since V_n is a contraction on C, VI (4.1) has a unique solution. Hence the sequence $\{x_n\}$ is well-defined. Indeed, x_{n+1} is alternatively given by

$$x_{n+1} = (P_C V_n)x_n. \tag{4.2}$$

Theorem 4.1. *Assume the solution set S of the VI (3.1) is nonempty. Assume the sequence $\{t_n\}$ of parameters satisfies the conditions*

(i) $t_n \to 0$;

(ii) $\sum_{n=1}^{\infty} t_n = \infty$;

(iii) *either* $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$ *or* $\lim_{n\to\infty} t_n/t_{n+1} = 1$.

Then the sequence $\{x_n\}$ generated by the algorithm (4.1) converges in norm to the unique solution x^* of the VI

$$x^* \in S, \quad \langle (I-f)x^*, x - x^*\rangle \ge 0, \quad x \in S. \tag{4.3}$$

Equivalently, x^* is the unique fixed point of the contraction $P_C f$; i.e., $x^* = (P_C f)x^*$.

Proof. (1°) The sequence $\{x_n\}$ is bounded. As a matter of fact, it follows from (4.1) that, for $x' \in S$,

$$\begin{aligned}
0 \ge\ & \langle (1-\gamma)x_n + \gamma[t_n f(x_n) + (1-t_n)Tx_n] - x_{n+1}, x' - x_{n+1}\rangle \\
=\ & (1-\gamma)\langle x_n - x_{n+1}, x' - x_{n+1}\rangle \\
& +\gamma[t_n\langle f(x_n) - x_{n+1}, x' - x_{n+1}\rangle + (1-t_n)\langle Tx_n - x_{n+1}, x' - x_{n+1}\rangle] \\
=\ & (1-\gamma)(\langle x_n - x', x' - x_{n+1}\rangle + \|x' - x_{n+1}\|^2) \\
& +\gamma[t_n(\langle f(x_n) - f(x'), x' - x_{n+1}\rangle + \langle f(x') - x', x' - x_{n+1}\rangle + \|x' - x_{n+1}\|^2) \\
& +(1-t_n)(\langle Tx_n - Tx', x' - x_{n+1}\rangle + \langle Tx' - x', x' - x_{n+1}\rangle + \|x' - x_{n+1}\|^2)] \\
=\ & \|x' - x_{n+1}\|^2 + (1-\gamma)\langle x_n - x', x' - x_{n+1}\rangle \\
& +\gamma[t_n(\langle f(x_n) - f(x'), x' - x_{n+1}\rangle + \langle f(x') - x', x' - x_{n+1}\rangle) \\
& +(1-t_n)(\langle Tx_n - Tx', x' - x_{n+1}\rangle + \langle Tx' - x', x' - x_{n+1}\rangle)].
\end{aligned}$$

It follows that (noticing $\langle Tx' - x', x' - x_{n+1}\rangle \geq 0$ as $x' \in S$)

$$\begin{aligned}
\|x_{n+1} - x'\|^2 \leq\ & (1-\gamma)\langle x_n - x', x_{n+1} - x'\rangle \\
& +\gamma[t_n(\langle f(x_n) - f(x'), x_{n+1} - x'\rangle + \langle f(x') - x', x_{n+1} - x'\rangle) \\
& +(1-t_n)\langle Tx_n - Tx', x_{n+1} - x'\rangle] \\
\leq\ & (1-\gamma)\|x_n - x'\|\|x_{n+1} - x'\| \\
& +\gamma[t_n(\|f(x_n) - f(x')\|\|x_{n+1} - x'\| + \langle f(x') - x', x_{n+1} - x'\rangle) \\
& +(1-t_n)\|Tx_n - Tx'\|\|x_{n+1} - x'\|] \\
=\ & (1-(1-\rho)\gamma t_n)\|x_n - x'\|\|x_{n+1} - x'\| \\
& +\gamma t_n\langle f(x') - x', x_{n+1} - x'\rangle \qquad (4.4) \\
\leq\ & (1-(1-\rho)\gamma t_n)\|x_n - x'\|\|x_{n+1} - x'\| \\
& +\gamma t_n\|f(x') - x'\|\|x_{n+1} - x'\|.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_{n+1} - x'\| \leq\ & (1-(1-\rho)\gamma t_n)\|x_n - x'\| + \gamma t_n\|f(x') - x'\| \\
\leq\ & \max\left\{\|x_n - x'\|, \frac{1}{1-\rho}\|f(x') - x'\|\right\}. \qquad (4.5)
\end{aligned}$$

By induction, we get

$$\|x_n - x'\| \leq \max\left\{\|x_0 - x'\|, \frac{1}{1-\rho}\|f(x') - x'\|\right\}$$

for all $n \geq 0$; in particular, $\{x_n\}$ is bounded.

(2°) We show $\|x_{n+1} - x_n\| \to 0$. Indeed, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| =\ & \|P_C V_n x_n - P_C V_{n-1} x_{n-1}\| \\
\leq\ & \|V_n x_n - V_{n-1} x_{n-1}\| \\
=\ & \|(1-\gamma)x_n + \gamma[t_n f(x_n) + (1-t_n)Tx_n] \\
& -(1-\gamma)x_{n-1} - \gamma[t_{n-1}f(x_{n-1}) + (1-t_{n-1})Tx_{n-1}]\| \\
=\ & \|(1-\gamma)(x_n - x_{n-1}) \qquad (4.6) \\
& +\gamma(t_n[f(x_n) - f(x_{n-1})] + (1-t_n)(Tx_n - Tx_{n-1})) \\
& +\gamma(t_n - t_{n-1})(f(x_{n-1}) - Tx_{n-1})\| \\
\leq\ & (1-(1-\rho)\gamma t_n)\|x_n - x_{n-1}\| + M|t_n - t_{n-1}|
\end{aligned}$$

where $M > 0$ is a constant big enough so that $M > \|f(x_n) - Tx_n\|$ for all n. Noticing conditions (ii) and (iii), we can apply Lemma 2.4 to () to assert that

$\|x_{n+1} - x_n\| \to 0$.

(3°) The weak ω-limit set $\omega_w(x_n)$ of $\{x_n\}$ satisfies the property: $\omega_w(x_n) \subset S$. To see this, we again use (4.1) to get

$$\langle (1-\gamma)x_n + \gamma[t_n(f(x_n) - Tx_n) + Tx_n] - x_{n+1}, x - x_{n+1}\rangle \le 0, \quad x \in C.$$

This results in

$$\gamma\langle Tx_n - x_{n+1}, x - x_{n+1}\rangle \le (1-\gamma)\langle x_{n+1} - x_n, x - x_{n+1}\rangle + \gamma t_n\langle Tx_n - f(x_n), x - x_{n+1}\rangle.$$

This further results in

$$\begin{aligned}\gamma\langle Tx_n - x_n, x - x_n\rangle &\le \langle x_{n+1} - x_n, x - x_n\rangle + \gamma\langle Tx_n - x_n, x_{n+1} - x_n\rangle \\ &\quad + \gamma t_n\langle Tx_n - f(x_n), x - x_{n+1}\rangle \\ &\le d(t_n + \|x_n - x_{n+1}\|),\end{aligned} \tag{4.7}$$

where d is a constant (which may depend on x) such that

$$d > \sup_n\{\|x - x_n\| + \gamma(\|x_n - Tx_n\| + \|x - x_{n+1}\|\|f(x_n) - Tx_n\|\}.$$

Next use the monotonicity of $I - T$ to obtain from (4.7)

$$\begin{aligned}\langle (I-T)x, x_n - x\rangle &\le \langle (I-T)x_n, x_n - x\rangle \\ &\le (d/\gamma)(t_n + \|x_n - x_{n+1}\|).\end{aligned}$$

By Step (2°), we get

$$\limsup_{n\to\infty}\langle (I-T)x, x_n - x\rangle \le 0, \quad x \in C. \tag{4.8}$$

This immediately implies that if $\tilde{x} \in \omega_w(x_n)$, then $\langle (I-T)x, \tilde{x} - x\rangle \le 0$ for $x \in C$; hence, $\tilde{x} \in S$ by Proposition 3.2. Therefore, $\omega_w(x_n) \subset S$.

(4°) We claim that

$$\limsup_{n\to\infty}\langle (I-f)x^*, x^* - x_n\rangle \le 0. \tag{4.9}$$

where x^* is the unique solution of the VI (4.3). To see this, we take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (I-f)x^*, x^* - x_n\rangle = \limsup_{i\to\infty}\langle (I-f)x^*, x^* - x_{n_i}\rangle.$$

With no loss of generality, we may assume $x_{n_i} \rightharpoonup \hat{x}$. By Step (3°), $\hat{x} \in S$. Hence the VI (4.3) ensures that

$$\limsup_{n\to\infty}\langle (I-f)x^*, x^* - x_n\rangle = (I-f)x^*, x^* - \hat{x}\rangle \le 0.$$

(5°) Finally we prove that $x_n \to x^*$, where $x^* = (P_C f)x^*$ is the unique solution of VI (4.3). To see this, we substitute x^* for x' in the relation (4.4) to get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \le\ & (1-(1-\rho)\gamma t_n)\|x_n - x^*\|\|x_{n+1} - x^*\| \\ & +\gamma t_n \langle f(x^*) - x^*, x_{n+1} - x^*\rangle) \\ \le\ & \frac{1}{2}(1-(1-\rho)\gamma t_n)(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\ & +\gamma t_n \langle f(x^*) - x^*, x_{n+1} - x^*\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \le\ & \frac{1-(1-\rho)\gamma t_n}{1+(1-\rho)\gamma t_n}\|x_n - x^*\|^2 \\ & + \frac{2\gamma t_n}{1+(1-\rho)\gamma t_n}\langle (I-f)x^*, x^* - x_{n+1}\rangle. \end{aligned} \tag{4.10}$$

Setting

$$\gamma_n = \frac{2(1-\rho)\gamma t_n}{1+(1-\rho)\gamma t_n}, \quad \delta_n = \frac{1}{1-\rho}\langle (I-f)x^*, x^* - x_{n+1}\rangle,$$

we can rewrite (4.10) as

$$\|x_{n+1} - x^*\|^2 \le (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n \delta_n. \tag{4.11}$$

Observing that $\gamma_n \to 0$, $\sum_{n=1}^{\infty}\gamma_n = \infty$, and $\limsup_{n\to\infty}\delta_n \le 0$, we can apply Lemma 2.4 to (4.11) to conclude that $\|x_n - x^*\| \to 0$. ■

5. Application in Minimization

Let H be a Hilbert space and C a closed convex subset of H. Let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous convex function. Consider the following minimization problem

$$\min_{x\in C}\varphi(x). \tag{5.1}$$

Let S denote the solution set of (5.1) and assume $S \ne \emptyset$.

Assume φ is differentiable and its gradient is Lipschitz continuous:

$$\|\nabla\varphi(x) - \nabla\varphi(y)\| \le L\|x - y\|$$

for $x, y \in C$ and $L > 0$ is a constant.

Recall that the optimality condition for $x^* \in C$ to be a solution of (5.1) is the following VI

$$x^* \in C, \quad \langle \nabla\varphi(x^*), x - x^* \rangle \ge 0, \quad x \in C. \tag{5.2}$$

Let

$$T = I - \lambda\nabla\varphi,$$

where $\gamma > 0$ is a parameter. Then VI (5.2) is rewritten as

$$x^* \in C, \quad \langle (I - T)x^*, x - x^* \rangle \ge 0, \quad x \in C. \tag{5.3}$$

Since for $0 < \lambda < 2/L$, the mapping T is nonexpansive, Theorems 3.3 and 4.1 are applicable. In particular, taking $\gamma = 1$ and $f = 0$ in both theorems, we arrive at the following theorem.

Theorem 5.1. *Assume the solution set S of the minimization (5.1) is nonempty and fix $0 < \lambda < 2/L$. (I) For each $t \in (0, 1)$, let x_t be the unique solution to the fixed point equation*

$$x_t = P_C((1 - t)I - \lambda(1 - t)\nabla\varphi)x_t. \tag{5.4}$$

Then $\lim_{t\downarrow 0} x_t$ exists in the norm topology and is the minimum-norm solution of the minimization (5.1).

(II) Define a sequence $\{x_n\}$ via the recursive algorithm:

$$x_{n+1} = P_C((1 - t_n)(I - \lambda\nabla\varphi)x_n), \tag{5.5}$$

where the sequence $\{t_n\}$ satisfy the conditions (i)-(iii) of Theorem 4.1. Then $\{x_n\}$ converges in norm to the minimum-norm solution of the minimization (5.1).

Proof. If we choose $\gamma = 1$, $f = 0$ and $T = I - \lambda\nabla\varphi$, then it is easily found that the fixed point equation (3.3) is reduced to the fixed point equation (5.4) and that the algorithm (4.1) (or its equivalence (4.2)) is reduced to the algorithm (5.5). Also notice that when $f = 0$, the solution x^* of the VI (3.5) is precisely the minimum-norm element of the solution set S. ∎

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REFERENCES

1. H. Brezis, *Operateurs Maximaux Monotones et Semi-Groups de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, 1973.

2. K. Geobel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics, Vol. 28, Cambridge University Press, 1990.

3. B. Halpern, Fixed points of nonexpanding maps, *Bull. Amer. Math. Soc.*, **73** (1967), 957-961.

4. P. L. Lions, Approximation de points fixes de contractions, *C.R. Acad. Sci. Sèr. A-B Paris*, **284** (1977), 1357-1359.

5. G. Marino and H. K. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.*, **318** (2006), 43-52.

6. G. Marino and H. K. Xu, *Explicit hierarchical fixed point approach to variational inequalities*, Preprint.

7. N. Shioji and W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, *Proc. Amer. Math. Soc.*, **125** (1997), 3641-3645.

8. R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.*, **58** (1992), 486-491.

9. H. K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.*, **66** (2002), 240-256.

10. H. K. Xu, Another control condition in an iterative method for nonexpansive mappings, *Bull. Austral. Math. Soc.*, **65** (2002), 109-113.

11. H. K. Xu, Remarks on an iterative method for nonexpansive mappings, *Comm. Applied Nonlinear Anal.*, **10(1)** (2003), 67-75.

12. H. K. Xu, An iterative approach to quadratic optimization, *J. Optimiz. Theory Appl.*, **116** (2003), 659-678.

13. H. K. Xu and T. H. Kim, Convergence of hybrid steepest-descent methods for variational inequalities, *J. Optimiz. Theory Appl.*, **119** (2003), 185-201.

14. H. K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.*, **298** (2004), 279-291.

15. H. K. Xu, Viscosity method for hierarchical fixed point approach to variational inequalities, *Taiwanese J. Math.*, (To appear in 2009).

16. I. Yamada, The hybrid steepest descent for the variational inequality problems over the intersection of fixed point sets of nonexpansive mappings, in: *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, Elservier, New York, 2001, pp. 473-504.

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