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ECCENTRIC SPECTRUM OF A GRAPH

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Dedicated to Ko-Wei Lih on the occasion of his 60th birthday.

Abstract. In the related literatures, the eccentricities of graphs have been studied recently. The main purpose of this paper is to discuss the eccentric spectrum of a graph. For any two vertices u and v in a connected graph G, $d_G(u, v)$ denotes the distance between vertices u and v. The eccentricity $e_G(v)$ of a vertex v in G is the maximum number of $d_G(v, u)$ over all vertex u. A vertex u is an eccentric vertex if there exists a vertex v such that $e_G(v) = d_G(v, u)$. A number k is called an eccentric number of G if, for each vertex v with $e_G(v) = k$, v is an eccentric vertex. The eccentric spectrum S_G of a connected graph G is a set of all eccentric numbers in G. If d is the diameter of G, then $d \in S_G$. In the paper, we show that for positive integers $r \leq d \leq 2r$ and $d \in S \subseteq \{r, r+1, ..., d\}$, there exists a connected graph G with radius r, diameter d and eccentric spectrum S. This result also proves the conjecture of Chartrand, Gu, Schultz, and Winters in [4].

1. Introduction

The discussion about the center and periphery of a connected graph is an important topic in Graph Theory. There are many literatures about studying the graphical eccentricity, center and periphery [1, 2, 5]. Among those studies, to put the emergency plants on the center vertices in a street system (or a network) are well-known. If we can find the farthest vertices(or the eccentric vertices) of a vertex in a graph, then we can determine the center and periphery of a graph.

Here now we set the definitions used in the paper. The graphs considered in the paper are finite, without loops or multiple edges. In a graph $G = (V, E)$, V(or $V(G)$) and E(or $E(G)$) denote the vertex set and the edge set of G, respectively.

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A sequence $(x_1, x_2, ..., x_k)$ of vertices in a graph G is called *a walk*, if x_1x_2, x_2x_3, ...,$x_{k-1}x_k$ are the edges of G. And k is the length of the walk. A walk $(x_1, x_2, ..., x_k)$ is called *a path* in G, if $x_1, x_2, ..., x_k$ are distinct in G. A walk $(x_1, x_2, ..., x_k, k_{k+1})$ is called *a cycle* in G, if $x_1, x_2, ..., x_k$ are the distinct vertices and $x_1 = k_{k+1}$ in G. Suppose u and v are the vertices of G. The *distance* $d_G(u, v)$ of u and v is the length of a shortest path between u and v in G. The *eccentricity* $e_G(v)$ of a vertex v in G is a maximum number of $d_G(v, x)$ over all $x \in V(G)$. Then the *radius* $r(G)$ is $\min\{e_G(v) : v \in V(G)\}$ and the *diameter* $d(G)$ $\max\{e(v) : v \in V(G)\}$ in G. A vertex v is called an *eccentric vertex* of G, if there exists a vertex $u \in V$ such that $e_G(u) = d_G(u, v)$. The *eccentricity* $e(G)$ of G is a minimum number k such that, for each vertex $v \in V(G)$ with $e_G(v) \geq k$, v is an eccentric vertex of G. A number k is an *eccentric number* of G if, for any vertex v with $e_G(v) = k$, v is an eccentric vertex of G. The *eccentric spectrum* S_G of G is a set of all eccentric numbers of G. It is trivial that $d(G) \in S_G$.

A graph is *eccentric* if all vertices of G are eccentric vertices. Chartrand, Gu, Schultz and Winters [4] studied the existence of eccentric graphs and its related relations. They also addressed a conjecture that for positive integers $a \leq b \leq c \leq 2a$, there exists a connected graph G satisfying $r(G) = a$, $e(G) = b$, and $d(G) = c$. Boland and Panrong proved the conjecture in [3].

In the paper, we prove that for positive integers r and d with $r \leq d \leq 2r$, and $S \subseteq \{r, r+1, ..., d\}$ with $d \in S$, there exists a connected graph G with the radius r, the diameter d, and the eccentric spectrum S. This result not only gives a another proof of the conjecture proposed in [4] but also gives a profounder result than the conjecture.

2. Eccentric Spectrum

First of all, we construct a special connected graph G with $e(G) = d(G)$. Suppose m and n are positive integers. We define the connected graph $G(m, n)$ for $n \geq 2$ as a graph with the vertex set $\{(i, j) : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ and the edge set $\cup_{j=1}^{n}\{(i, j)(i + 1, j) : 1 \leq i \leq m - 1\} \cup \{(1, j)(1, j + 1) : 1 \leq j \leq n - 1\} \cup \{(1, 1)(1, n)\}$; For example, $G(3, 3)$ is in Figure 1. For $n = 1$, we define $G(m, 1)$ as a connected graph with $V(G(m, 1)) = \{(i, j) : 2 \leq i \leq m \text{ and } j \in \{1, 2\}\} \cup \{(1, 1)\}$ and $E(G(m, 1)) = \{(i, j)(i + 1, j) : 2 \leq i \leq m - 1 \text{ and } j \in \{1, 2\}\} \cup \{(1, 1)(2, 1), (1, 1)(2, 2)\}$. It is clear that $G(m, 1)$ a path of length $2m - 2$.

Lemma 1. *Suppose m and n are positive integers. Let (u, v) and (x, y) be the vertices of $G(m, n)$. Then*

$$d_{G(m,n)}((u, v), (x, y)) = \begin{cases} u + x + w - 2, & \textit{if } v \neq y, \\ |u - x|, & \textit{if } v = y, \end{cases}$$

where w =*min*$\{|v-y|, n-|v-y|\}$.

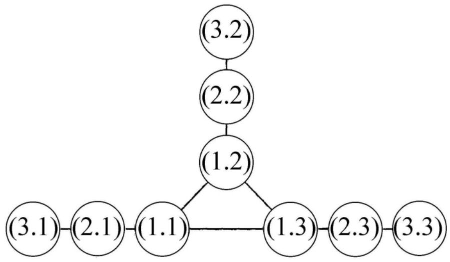

Fig. 1. The graph $G(3,3)$.

In the following, we determine the distances between two vertices of $G(m,n)$.

Proof. If $n=1$ or 2, then $G(m,n)$ is a path with length $2m-2$ or $2m-1$. It is easy to check the distance of each pair of vertices in $G(m,n)$. For $n \geq 3$, by the definition of $G(m,n)$, $G(m,n)$ has a unique cycle. If $v \neq y$, then the shortest path between (u,v) and (x,y) must pass through the cycle and contains the vertices $(1,v)$ and $(1,y)$. Then the distance of (u,v) and (x,y) is $(u-1)+(x-1)+w$ where w =min$\{|v-y|, n-|v-y|\}$. If $v=y$, then there is a unique path(or a shortest path) between (u,v) and (x,y) with length $|u-x|$. ■

Theorem 2. *Suppose* m *and* n *are positive integers and* $G=G(m,n)$*. Then* $r(G)=m+\lfloor n/2\rfloor-1$*,* $d(G)=2m+\lfloor n/2\rfloor-2$*, and* $S_G=\{2m+\lfloor n/2\rfloor-2\}$*.*

Proof. For $n=1$, $G(m,1)$ is a path of length $2m-2$ with radius $m-1$ and diameter $2m-2$. So we assume that $n \geq 2$. By Lemma 1, for each vertex $(i,j) \in V(G)$, the vertex $(m, j+\lfloor n/2\rfloor)$ or $(m, j-\lfloor n/2\rfloor)$ is a farthest vertex from (i,j) with distance $i+m+\lfloor n/2\rfloor-2$. Then the radius of G is min$\{i+m+\lfloor n/2\rfloor-2 : i=1,2,...,m\}$= $m+\lfloor n/2\rfloor-1$, and the diameter of G is max$\{i+m+\lfloor n/2\rfloor-2 : i=1,2,...,m\}$= $2m+\lfloor n/2\rfloor-2$. And, we can observe that only the vertices $(m,1)$, $(m,2)$,...,(m,n) are eccentric vertices in G. Thus, we have $S_G=\{2m+\lfloor n/2\rfloor-2\}$. ■

By Theorem 2, for any $1 \leq a \leq b \leq 2a$, if $m=b-a+1$, and $n=4a-2b$ or $4a-2b+1$, then $G(m,n)$ is a connected graph with $r(G(m,n))=a$, $d(G(m,n))=b$, and $S_{G(m,n)}=\{b\}$.

Let m and n be positive integers, $d=d(G(m,n))$, $r=r(G(m,n))$, and $S \subseteq \{r, r+1, ..., d\}$ with $d \in S$. Now we definite the graphs $G_i(m,n,S)$ for $i=r, r+1, ..., d$ by the recurrence relation. Define that $G_d(m,n,S)=G(m,n)$. Let f and j be functions defined by $f(a_1,a_2,...,a_k)=a_1$ for any $(a_1,a_2,...,a_k)$,

$j(i) = i$ for $i \in S$, and $j(i) = i - 1$ for $i \notin S$. Take $i \in \{r, r+1, ..., d-1\}$ with $j(i) \geq r$. Define $G_i(m, n, S) = G_i$ as a graph with the vertex set $\{(v, 0) : v \in V(G_{i+1})\} \cup \{(v, 1) : v \in V(G_{i+1})$ and $f(v) \leq m - (d - j(i))\}$ and the edge set $\{(v, 0)(u, 0) : uv \in E(G_{i+1})\} \cup \{(v, 1)(u, 1) : uv \in E(G_{i+1})$ and $f(u), f(v) \leq m-(d-j(i))\} \cup \{(u, 1)(v, 0) : uv \in E(G_{i+1}), f(u) = m-(d-j(i)),$ and $f(v) = m-(d-j(i))+1\} \cup \{(v, 1)(v, 0) : v \in V(G_{i+1})$ and $f(v) \leq m-(d-j(i))\}$ where $((a_1, a_2, ..., a_{k-1}), a_k) = (a_1, a_2, ..., a_{k-1}, a_k)$ for all integers $k \geq 2$. We show two examples $G_3(3, 2, \{5, 4, 3\})$ and $G_3(3, 2, \{5, 3\})$ with radius 3 and diameter 5 in Figures 2 and 3, respectively.

In G_i, if $(x_1, x_2, ..., x_k)(y_1, y_2, ..., y_k) \in E(G_i)$, $x_1 = y_1$ and $x_2 = y_2$, then the edge is called a *vertical edge*; otherwise, the edge is called a *horizontal edge*.

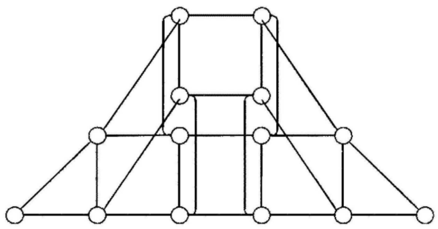

Fig. 2. The graph $G_3(3, 2, \{5, 4, 3\})$.

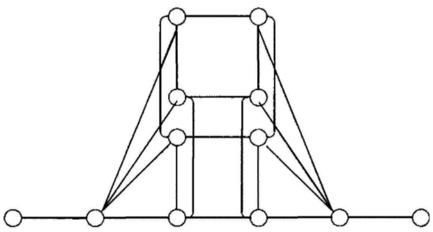

Fig. 3. The graph $G_3(3, 2, \{5, 3\})$.

From above, we have some properties of edges of $G_i(m, n, S)$ in the following two lemmas.

Lemma 3. (Horizontal edges). *Suppose $G_i(m, n, S) = G_i$ is the graph defined by above for $r \leq i \leq d-1$. Let $S' = S \cap \{i, i+1, ..., d\}$, $k = 2 + d - i$, $x = (x_1, x_2, ..., x_k)$, $y = (y_1, y_2, ..., y_k)$ be vertices of G_i with $1 \leq y_1 \leq x_1 \leq m$, $j = d - (m - y_1)$, and $s = 2 + (m - y_1)$. Then xy is a horizontal edge of G_i if and only if $(x_1, x_2)(y_1, y_2) \in E(G(m, n))$, and either*

(a) $x_1 = y_1 + 1$, *and one of the following statements is satisfied.*

(1) $j \in S'$, $j+1 \in S'$, $j \geq i$, *and* $x_h = y_h$ *for* $h \in \{3, 4, ..., k\} - \{s\}$.

(2) $j \notin S'$, $j+1 \in S'$, $j \geq i-1$, *and* $x_h = y_h$ *for* $h \in \{3, 4, ..., k\}$.

(3) $j \in S'$, $j+1 \notin S'$, $j \geq i$, *and* $x_h = y_h$ *for* $h \in \{3, 4, ..., k\} - \{s-1, s\}$.

(4) $j \notin S'$, $j+1 \notin S'$, $j \geq i-1$, *and* $x_h = y_h$ *for* $h \in \{3, 4, ..., k\} - \{s-1\}$.

(5) $i \in S'$, $j \leq i-1$ *and* $x_h = y_h$ *for* $h \in \{3, 4, ..., k\}$.

(6) $i \notin S'$, $j \leq i-2$ *and* $x_h = y_h$ *for* $h \in \{3, 4, ..., k\}$.

or (b) $x_1 = y_1 = 1$ *and* $x_h = y_h$ *for* $h \in \{3, 4, ..., k\}$.

Proof. The proof is by induction. If $i = d-1$ then it is easy to check that the horizontal edges of G_{d-1} satisfy the statement of the lemma. Suppose the statement is true for the horizontal edges of G_{i+1}. Let $r_i = m - (d-i)$ for $i \in S$, $r_i = m - (d-i) - 1$ for $i \notin S$, $V' = \{(x_1, x_2, ..., x_{k-1}) \in V(G_{i+1}) : x_1 \leq r_i\}$ and H be the induced subgraph of V' in G_{i+1}. Then $V(G_i) = \{(x', 0) : x' \in V(G_{i+1})\} \cup \{(y', 1) : y' \in V(H)\}$ and $E(G_i) = \{(v, 0)(u, 0) : uv \in E(G_{i+1})\} \cup \{(v, 1)(u, 1) : uv \in E(H)\} \cup \{(u, 1)(v, 0) : uv \in E(G_{i+1}), f(u) = r_i,$ and $f(v) = r_i + 1\} \cup \{(v, 1)(v, 0) : v \in V(H)\}$. By induction and above structure of G_i, we can get the lemma. ■

Let $a_1, a_2, ..., a_k, b_1, b_2, ..., b_k \in \{0, 1\}$. Define the function hd by $hd((a_1, a_2, ..., a_k), (b_1, b_2, ..., b_k)) = \sum_{i=1}^{k} |a_i - b_i|$.

Lemma 4. (Vertical edges). *Suppose G_i is the graph defined by above with $i \leq d-1$. Let $k = 2 + d - i$ and $x = (x_1, x_2, ..., x_k)$, $y = (y_1, y_2, ..., y_k)$ be vertices of G_i. Then xy is a vertical edge of G_i if and only if $x_1 = y_1$, $x_2 = y_2$, and $hd((x_3, x_4, ..., x_k), (y_3, y_4, ..., y_k)) = 1$.*

Proof. By the definition of G_i. ■

The following two lemmas show the properties of paths in G_i.

Lemma 5. *In G_i, let $V_j = \{(x_1, x_2, ..., x_k) \in V(G_i) : x_1 = j\}$ for $1 \leq j \leq m$. Fixed $j \in \{2, 3, ..., m\}$. If $(a_1, a_2, ..., a_l)$ is a path of G_i with $a_1, a_l \in V_{j-1}$ and $a_2, a_3, ..., a_{l-1} \in V_j$, then there exist $b_2, b_3, ..., b_{q-1} \in V_{j-1}$ with $p \leq l$ such that $(a_1, b_2, b_3, ..., b_{q-1}, a_l)$ is a path of G_i.*

Proof. Let $a_h = (a_{h1}, ..., a_{hk})$ for $h = 1, 2, ..., l$, and $s = 2 + (m - a_{11})$. Then $(a_{11}, a_{12}) = (a_{l1}, a_{l2})$, $a_1 a_2$, $a_{l-1} a_l$ are horizontal edges with $a_{11} = a_{l1} = a_{21} - 1 = a_{(l-1)1} - 1$. For the cases of (1), (2), (4), (5), and (6) in Lemma 3 (a), there exists at most one $t \in \{s-1, s\}$ such that $a_{1p} = a_{2p}$ for $3 \leq p \leq k$ except $p = t$. Define $b_h = (b_{h1}, ..., b_{hk})$ by $b_{h1} = a_{11}$, $b_{h2} = a_{12}$, and $b_{hp} = a_{hp}$

for $2 \leq h \leq p-1$ and $3 \leq p \leq k$. Then $b_h \in V_{j-1}$ for all h. By Lemma 4, $(a_1, b_2, b_3, ..., b_{p-1}, a_l)$ is a walk of G_i with $p \leq l$. That is, there is a path from a_1 to a_l with length at most $l-1$ in which all vertices are contained in V_{j-1}.

For the case of (3) in Lemma 3 (a), $a_{1q} = a_{2q}$ for $q \in \{3, 4, ..., k\} - \{s-1, s\}$. By Lemma 4, $hd(a_h, a_{h+1}) = 1$ for $2 \leq h \leq l-2$. Thus there exists $3 \leq t \leq k$ such that $a_{2t} \neq a_{3t}$. Let $b_2 = (b_{21}, ..., b_{2k})$ by $b_{21} = a_{11}$, $b_{22} = a_{12}$, $b_{2p} = a_{1p}$ for $p \in \{3, 4, ..., k\} - \{t\}$ and $b_{2t} = a_{3t}$. Then $b_2 \in V_{j-1}$, $b_2 a_3 \in E(G_i)$ by (3) of Lemma 3, and $a_1 b_2 \in E(G_i)$ by Lemma 4. By the similar way, there exist $b_3, b_4, ..., b_{l-2}$ such that $b_h \in V_{j-1}$ and $b_h b_{h+1}, b_{l-2} a_{l-1} \in E(G_i)$ for $2 \leq h \leq l-3$. By (3) of Lemma 3 (a), $b_{(l-2)h} = a_{(l-1)h} = a_{lh}$ for $h \in \{3, 4, ..., k\} - \{s-1, s\}$. Let $b_{l-1} = (b_{(l-1)1}, b_{(l-1)2}, ..., b_{(l-1)k})$ with $b_{(l-1)h} = b_{(l-2)h}$ for $h \in \{1, 2, ..., k\} - \{s\}$ and $b_{(l-1)s} = a_{ls}$. By Lemma 4, (b_{l-2}, b_{l-1}, a_l) is a walk of G_i. Thus, $(a_1, b_2, b_3, ..., b_{l-2}, b_{l-1}, a_l)$ is a walk of G_i. This implies that there is a path P from a_1 to a_l with length at most $l-1$ in which all vertices of P are contained in V_{j-1}. The proof is complete. ■

Lemma 6. *Let $V_j = \{(x_1, x_2, ..., x_k) \in V(G_i) : x_1 = j\}$ for $1 \leq j \leq m$. Suppose x and y are vertices of G_i such that $x \in V_s$, $y \in V_t$ for some $s \leq t$. Then there exists a shortest path P from x to y such that $V(P) \subseteq \cup_{j=1}^{t} V_j$.*

Proof. Suppose $P = (x_1, x_2, ..., x_l)$ is a shortest path from x to y with minimum $j \geq t+1$ satisfying $V(P) \cap V_j \neq \emptyset$ and $V(P) \cap V_{j+1} = \emptyset$. Since $V(P) \cap V_j \neq \emptyset$, there is a vertex $x_c \in V(P) \cap V_j$. Then we can find that there exist a, b with $a \leq c \leq b$ such that $x_a, x_b \in V_{j-1}$ and $x_{a+1}, ..., x_{b-1} \in V_j$. By Lemma **??**, there exist $y_{a+1}, ..., y_{q-1} \in V_{j-1}$ such that $a \leq q \leq b$ and $(x_a, y_{a+1}, ..., y_{q-1}, x_b)$ is a path in G_i. To repeat the above step enables us to find a path P' between x and y with length at most $l-1$ and $V(P') \cap V_j = \emptyset$. It contradicts that j is minimum. ■

Let $r_i = m - (d-i)$ for $i \in S$, $r_i = m - (d-i) - 1$ for $i \notin S$, $V'_j = \{(x_1, x_2, ..., x_{k-1}) \in V(G_{i+1}) : x_1 = j\}$ for $1 \leq j \leq m$, and H be the induced subgraph of $V'_1 \cup V'_2 \cup ... \cup V'_{r_i}$ in G_{i+1}. Then $V(G_i) = \{(x', 0) : x' \in V(G_{i+1})\} \cup \{(y', 1) : y' \in V(H)\}$.

From the above lemma of shortest paths of G_i, we deduce the following relation between distances of vertices in G_i and G_{i+1}.

Lemma 7. *Suppose $G_i = G_i(m, n, S)$ is a graph with diameter d and radius r and $i \leq d-1$. Let $x = (x_1, x_2, ..., x_k)$ and $y = (y_1, y_2, ..., y_k)$ be vertices of G_i, $x' = (x_1, x_2, ..., x_{k-1})$, $y' = (y_1, y_2, ..., y_{k-1})$. Then*

$$d_{G_i}(x, y) = \begin{cases} d_{G_{i+1}}(x', y') + 1, & \text{if } x_k \neq y_k \text{ and } x', y' \in V(H), \\ d_{G_{i+1}}(x', y'), & \text{otherwise.} \end{cases}$$

Proof. Let H^* be the induced subgraph of $\{(u, u_k) : u \in V(H)$ and $u_k \in \{0, 1\}\}$ in G_i. Then $E(H^*) = \{(u, j)(v, j) : uv \in E(H)$ and $j \in \{0, 1\}\} \cup \{(u, 0)(u, 1) : u \in V(H)\}$. If $x_k \neq y_k$ and $x', y' \in V(H)$, then by Lemma 6, there is a shortest path between x and y with length $d_{G_i}(x, y)$ in H^*. If $(a_1, a_2, ..., a_l)$ is a shortest path between x and y in H^* where $a_j = (a_{j1}, a_{j2}, ..., a_{jk})$ for $j = 1, 2, ..., l$, then there exists s such that $a_{st} = a_{(s+1)t}$ for $t = 1, 2, ..., k-1$ and $a_{sk} \neq a_{(s+1)k}$ by $x_k \neq y_k$. Let $a'_j = (a_{j1}, a_{j2}, ..., a_{j(k-1)})$ for $j = 1, 2, ..., l$. We have that $(a'_1, a'_2, ..., a'_s, a'_{s+2}, ..., a'_l)$ is a walk between x' and y' with length $l-2$ in H. Then $d_{G_{i+1}}(x', y') + 1 \leq d_{G_i}(x, y)$. And, if $(b_1, b_2, ..., b_q)$ is a shortest path from x' to y' in H, then $((b_1, x_k), (b_2, x_k), ..., (b_q, x_k), (b_q, y_k))$ is a path from x to y in H^*. This implies that $d_{G_i}(x, y) = d_{H^*}(x, y) \leq d_{G_{i+1}}(x', y') + 1$. Therefore $d_{G_i}(x, y) = d_{G_{i+1}}(x', y') + 1$. On the other hand, if $x_k = y_k$ or one of x', y' is not in $V(H)$, then it is easy to see that $d_{G_i}(x, y) = d_{G_{i+1}}(x', y')$. ■

Then we can determine the eccentricities of vertices in G_i.

Theorem 8. *Let $x = (x_1, x_2, ..., x_k) \in V(G_i)$, then $e_{G_i}(x) = e_{G_d}(x_1, x_2)$.*

Proof. In $G_d = G(m, n)$, there exists $s \in \{1, 2, ..., n\}$ such that $d_{G_d}((x_1, x_2), (m, s)) = e_{G_d}(x_1, x_2)$. By Lemma 7, $d_{G_i}((x_1, x_2, ..., x_k), (m, s, 0, ..., 0)) = d_{G_d}((x_1, x_2), (m, s)) = e_{G_d}(x_1, x_2)$. Thus, $e_{G_i}(x) \geq e_{G_d}(x_1, x_2)$.

Claim that $e_{G_i}(x) \leq e_{G_d}(x_1, x_2)$. The proof is made by induction. If $i = d$, then the statement is true. Suppose the statement is true in G_{i+1}. Let $r_i = m - (d - i)$ for $i \in S$, $r_i = m - (d-i) - 1$ for $i \notin S$, $x = (x_1, x_2, ..., x_k), y = (y_1, y_2, ..., y_k) \in V(G_i)$, $x' = (x_1, x_2, ..., x_{k-1})$ and $y' = (y_1, y_2, ..., y_{k-1})$. If $x_1 \geq r_i + 1$, then $x \notin V(H)$, by Lemma 7, $d_{G_i}(x, y) = d_{G_{i+1}}(x', y') \leq e_{G_{i+1}}(x') \leq e_{G_d}(x_1, x_2)$. If $y_1 \leq x_1 \leq r_i$, then by Lemma 6 and 7, $d_{G_i}(x, y) = d_{G_d}((x_1, x_2), (y_1, y_2)) + hd((x_3, x_4, ..., x_k), (y_3, y_4, ..., y_k))$. Then $d_{G_i}(x, y) \leq x_1 + y_1 + \lfloor n/2 \rfloor - 2 + d - i \leq x_1 + r_i + \lfloor n/2 \rfloor - 2 + d - i \leq x_1 + \lfloor n/2 \rfloor + m - 2 = e_{G_d}(x_1, x_2)$. The proof is complete. ■

Finally, we prove the main theorem.

Theorem 9. *For every two positive integers r, d with $r \leq d \leq 2r$ and $S \subseteq \{r, r+1, ..., d\}$ with $d \in S$, there exists a connected graph G with radius r, diameter d and eccentric spectrum S.*

Proof. Suppose r, d with $r \leq d \leq 2r$ and $S \subseteq \{r, r+1, ..., d\}$ with $d \in S$. Let $m = d - r + 1$ and $n = 4r - 2d + 1$. According to Theorem 2, $G(m, n)$ is a connected graph with radius r, diameter d and eccentric spectrum $\{d\}$. By Theorem 8, G_i is a graph with the radius r and the diameter d.

Claim that $x = (x_1, x_2, ..., x_k)$ is an eccentric vertex of G_i if and only if $x_1 \geq m - (d - i)$ and $d - (m - x_1) \in S$. We now proceed by induction. If $i = d$, then the statement is true by $G_d = G(m, n)$ and Theorem 2. Suppose the statement is true in G_{i+1}. Let $x = (x_1, x_2, ..., x_k), y = (y_1, y_2, ..., y_k) \in V(G_i)$, $x' = (x_1, x_2, ..., x_{k-1})$ and $y' = (y_1, y_2, ..., y_{k-1})$.

If $i \in S$, then let $r_i = m-(d-i)$. If $x_1 \geq r_i+1$, then by the proof of Theorem 8, $d_{G_i}(x, y) = d_{G_{i+1}}(x', y')$. By induction, for $x_1 \geq r_i+1$, x' is an eccentric vertex in G_{i+1} if and only if x is an eccentric vertex of G_i. Thus, G_i satisfies the claim for $x_1 \geq r_i+1$. If $x_1, y_1 \leq r_i$, then by the proof of Theorem 8, $d_{G_i}(x, y) \leq x_1 + y_1 + \lfloor n/2 \rfloor - 2 + d - i \leq e_{G_i}(y) = y_1 + \lfloor n/2 \rfloor + m - 2$. We have that for $x_1, y_1 \leq r_1$, $d_{G_i}(x, y) = e_{G_i}(y)$ if and only if $x_1 = r_1$, $d_{G_d}((x_1, x_2), (y_1, y_2)) = 2r_i + \lfloor n/2 \rfloor - 2$, and $hd((x_3, x_4, ..., x_k), (y_3, y_4, ..., y_k)) = d - i$. By $i \in S$, for each vertex $x = (x_1, x_2, ..., x_k)$ with $x_1 = r_i$, there exists a vertex $y = (y_1, y_2, ..., y_k)$ with $y_1 = r_1$, $d_{G_d}((x_1, x_2), (y_1, y_2)) = 2r_i + \lfloor n/2 \rfloor - 2$, and $hd((x_3, x_4, ..., x_k), (y_3, y_4, ..., y_k)) = d - i$. This implies that vertices $(x_1, x_2, ..., x_k)$ with $x_1 = r_i$ are eccentric vertices in G_i. By above, we have that for $i \in S$, $x = (x_1, x_2, ..., x_k)$ is a vertex with $x_1 \geq m - (d - i)$ and $d - (m - x_1) \in S$ if and only if x is an eccentric vertex of $V(G_i)$.

If $i \notin S$, then let $r_i = m - (d - i) - 1$. If $x_1 \geq r_i + 1$, then by the proof of Theorem 8, $d_{G_i}(x, y) = d_{G_{i+1}}(x', y')$. By induction, for $x_1 \geq r_i + 1$, x' is an eccentric vertex in G_{i+1} if and only if x is an eccentric vertex of G_i. Thus, G_i satisfies the claim for $x_1 \geq r_i + 1$. If $x_1, y_1 \leq r_i$, then by the proof of Theorem 8, $d_{G_i}(x, y) \leq x_1 + y_1 + \lfloor n/2 \rfloor - 2 + d - i \leq e_{G_i}(y) = y_1 + \lfloor n/2 \rfloor + m - 2$. We have that, by $x_1 < r_i + 1 = m - (d - i)$, $d_{G_i}(x, y) < e_{G_i}(y)$; that is, if $x = (x_1, x_2, ..., x_k)$ is a vertex of G_i with $x_1 \leq r_1$, then x is not an eccentric vertex. By above, we have that for $i \notin S$, $x = (x_1, x_2, ..., x_k)$ is a vertex with $x_1 \geq m - (d - i)$ and $d - (m - x_1) \in S$ if and only if x is an eccentric vertex of $V(G_i)$. The proof is complete. ∎

As a consequence, we have

Corollary 10. *For positive integers a, b, and c with $a \leq b \leq c \leq 2a$, there exists a connected graph satisfying $r(G) = a$, $e(G) = b$, and $d(G) = c$.*

Corollary 11. *For positive integers a and c with $a \leq c \leq 2a$, there exists an eccentric graph satisfying $r(G) = a$ and $d(G) = c$.*

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