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THE UNIFORM FORWARD SETS OF $C(X)$ AND UNIFORM CONVERGENCE OF OPERATOR SERIES

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Abstract. In this paper, we introduce the uniform forward sets of $C(X)$ and show that each totally bounded set of $C(X)$ is uniform forward set, moreover, we prove that the uniform forward sets of $C(X)$ are just the largest subset family of $C(X)$ on which each $C(X)$-evaluation convergent operator series is uniformly convergent.

1. INTRODUCTION

Let X and Y be Banach spaces and $C(X) = \big\{(x_j) \in X^{\mathbb{N}} : \lim_j x_j \quad \text{exists}\big\}$, then $C(X)$ is a Banach space with the norm $\|(x_j)\| = \sup_j \|x_j\|$. Denote

$$Y^X = \text{ the family of all } Y\text{-valued mappings on } X,$$

$$C(X)^{\beta Y} = \Big\{(A_j) \subset Y^X : \sum_{j=1}^{\infty} A_j(x_j) \text{ converges, } \forall\, (x_j) \in C(X)\Big\}.$$

As we know studying the classical Banach space $C(X)$ and the evaluation convergent of operator series are very important and interesting topics in Functional Analysis and Summation Theory [1-5], the following investigation determines the largest $\mathcal{M} \subset 2^{C(X)}$ for which $(A_j) \in C(X)^{\beta Y}$ iff $\sum_{j=1}^{\infty} A_j(x_j)$ converges uniformly with respect to (x_j) in any $M \in \mathcal{M}$, that is, in this paper we would like to reveal the strongest intrinsic meaning of $C(X)$-evaluation convergence of mapping series.

A subset B of a topological vector space E is totally bounded if for every neighborhood U of $0 \in E$ there is a finite $F \subset E$ such that $B \subset F + U$.

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2. Main Result and its Proof

First, we introduce the uniform forward set of $C(X)$ as follows:

Definition 1. $M \subset C(X)$ is said to be uniform forward if the following two conditions hold simultaneously,

(a) $\lim_j x_j$ exist uniformly for $(x_j) \in M$.

(b) $S = \{\lim_j x_j : (x_j) \in M\}$ is a totally bounded subset of X.

Next, we show that each totally bounded subset of $C(X)$ is a uniform forward set, that is:

Proposition 1. *Any totally bounded subset of $C(X)$ is a uniform forward set.*

Proof. Assume $M \subset C(X)$ is totally bounded and $\varepsilon > 0$. There is a finite $F = \big\{(z_{ij})_{j=1}^{\infty} : i = 1, 2, \cdots, n\big\} \subseteq C(X)$ such that $M \subseteq F + \big\{(u_j) \in C(X) : \sup_j \|u_j\| < \varepsilon/3\big\}$. Pick a $j_0 \in \mathbb{N}$ for which $\sup_{j \geq j_0} \|z_{ij} - \lim_j z_{ij}\| < \varepsilon/3$, $i = 1, 2, \cdots, n$.

For $(x_j) \in M$, $\sup_j \|x_j - z_{i_0 j}\| < \varepsilon/3$ for some $1 \leq i_0 \leq n$ and so $\sup_j \|\lim_j x_j - \lim_j z_{i_0 j}\| < \varepsilon/3$, thus we have $\sup_{j \geq j_0} \|x_j - \lim_j x_j\| \leq \sup_{j \geq j_0} \|x_j - z_{i_0 j}\| + \sup_{j \geq j_0} \|z_{i_0 j} - \lim_j z_{i_0 j}\| + \sup_j \|\lim_j x_j - \lim_j z_{i_0 j}\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. Hence $\lim_j x_j$ exists uniformly with respect to $(x_j) \in M$.

Furthermore, it follows from the above proof that for each $\varepsilon > 0$, there exists a finite subset $S_0 = \{\lim_j z_{ij} : 1 \leq i \leq n\}$ of X such that $\sup_j \|\lim_j x_j - \lim_j z_{i_0 j}\| < \varepsilon/3$ for some $1 \leq i_0 \leq n$, so $S = \{\lim_j x_j : (x_j) \in M\}$ is a totally bounded subset of X, and hence M is uniform forward subset of $C(X)$. ∎

However, a uniform forward subset of $C(X)$ need not be totally bounded, e.g., for a nonzero $x \in X$, $\{(kx, 0, 0, \cdots) : k \in \mathbb{N}\}$ is uniform forward but it is not totally bounded set in $C(X)$.

Remark. If we denote K_0=$\mathbb{N} \cup \{\infty\}$ as the one point compactification of nature numbers $\mathbb{N}$, then Proposition 1 is similar partly with the vector-valued Ascoli theorem for $C(K, X)$ where K is a compact set.

Now, we show that the uniform forward sets of $C(X)$ are just the largest subset family of $C(X)$ on which each $C(X)$-evaluation convergent operator series is uniformly convergent.

Theorem 1. *For $M \subset C(X)$, the following (1) and (2) are equivalent.*

(1) *M is uniform forward.*

(2) *For every Fréchet space E and $(A_j) \in C(X)^{\beta E}$, $\sum_{j=1}^{\infty} A_j(x_j)$ converges uniformly for $(x_j) \in M$.*

Proof. $(1) \Longrightarrow (2)$: Assume that M is uniform forward but (2) fails to hold for M and so there exists a Fréchet space E with the paranorm $\|\cdot\|$ and $(A_j) \in C(X)^{\beta E}$ such that the convergence of $\sum_{j=1}^{\infty} A_j(x_j)$ is not uniform for $(x_j) \in M$. Then there is an $\varepsilon > 0$ such that for every $m_0 \in \mathbb{N}$ we have an $m > m_0$ and a $(x_j) \in M$ for which $\|\sum_{j=m}^{\infty} A_j(x_j)\| \geq \varepsilon$ and, moreover, $\|\sum_{j=m}^{n} A_j(x_j)\| > \varepsilon/2$ for some $n > m$.

Since M is uniform forward, there is a $j_0 \in \mathbb{N}$ for which $\|x_j - \lim_j x_j\| < \varepsilon/2$, $\forall\, j > j_0$, $\forall\, (x_j) \in M$. Then there exist integers $n_1 > m_1 > j_0$ and $(x_{1j}) \in M$ such that $\|\sum_{j=m_1}^{n_1} A_j(x_{1j})\| > \varepsilon/2$. For n_1, there exist integers $n_2 > m_2 > n_1$ and $(x_{2j}) \in M$ such that $\|\sum_{j=m_2}^{n_2} A_j(x_{2j})\| > \varepsilon/2$. Continuing this construction produces an integer sequence $j_0 < m_1 < n_1 < m_2 < n_2 < \cdots$ and $\{(x_{kj})_{j=1}^{\infty} : k \in \mathbb{N}\} \subset M$ such that

$$\Big\| \sum_{j=m_k}^{n_k} A_j(x_{kj}) \Big\| > \varepsilon/2, k = 1, 2, 3, \cdots.$$

For $k \in \mathbb{N}$ define $u_k = \lim_j x_{kj}$, then there exists $\{u_{k_h}\}_{h=1}^{\infty} \subset \{u_k\}$ such that $\lim_h u_{k_h} = u_0$ for some $u_0 \in S$ since X is a Banach space and S is a totally bounded subset of X ([6], $p.102$). Without loss of generality assume $\lim_k u_k = u_0$, then for the above ε, there exists an $k_0 \in \mathbb{N}$ for which $\|u_k - u_0\| < \varepsilon/2$, $\forall\, k > k_0$. So for $j \geq j_0, k \geq k_0$ we have

$$\|x_{kj} - u_0\| \leq \|x_{kj} - u_k\| + \|u_k - u_0\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Let
$$x_j = \begin{cases} x_{kj}, & m_k \leq j \leq n_k, k \geq k_0, \\ u_0, & \text{otherwise.} \end{cases}$$
Then, $\|x_j - u_0\| = \|x_{kj} - u_0\| < \varepsilon$, $\forall\, j \geq j_0$, thus $(x_j) \in C(X)$ but

$$\Big\| \sum_{j=m_k}^{n_k} A_j(x_j) \Big\| = \Big\| \sum_{j=m_k}^{n_k} A_j(x_{kj}) \Big\| > \varepsilon/2, k \geq k_0.$$

This contradicts that $(A_j) \in C(X)^{\beta E}$ and so $(1) \Longrightarrow (2)$ holds.

$(2) \Longrightarrow (1)$: Assume that $M \subset C(X)$ is not uniform forward, that is, either (a) or (b) in Definition 1 does not hold.

Firstly, if (a) fails, i.e., $\lim_j x_j$ does not exist uniformly for $(x_j) \in M$, then there exist $\varepsilon > 0$, integer sequence $j_1 < j_2 < \cdots$ and $\big\{(x_{kj})_{j=1}^{\infty} : k \in \mathbb{N}\big\} \subset M$ such that $\|x_{kj_k} - \lim_k x_{kj_k}\| > \varepsilon$, $k = 1, 2, 3, \cdots$.

For each $j \in \mathbb{N}$ define $A_j : C(X) \longrightarrow C(X)$ by

$$A_j(x) = (0, \cdots, 0, \overset{(j)}{x_j - \lim_j x_j}, 0, 0, \cdots), \forall\, (x_j) \in C(X).$$

Each A_j is continuous linear, and $\sum_{j=1}^{\infty} A_j(x_j) = (x_j - \lim_j x_j)$ in $(C(X), \|\cdot\|_\infty)$ for each $(x_j) \in C(X)$ so $(A_j) \in C(X)^{\beta C(X)}$. However,

$$\Big\| \sum_{j=j_k}^{\infty} A_j(x_{kj}) - \sum_{j=j_k+1}^{\infty} A_j(x_{kj}) \Big\|_\infty = \|x_{kj_k} - \lim_k x_{kj_k}\| > \varepsilon, \quad k = 1, 2, 3, \cdots.$$

This contradicts (2) and so $(2) \Longrightarrow (1)$ holds.

Next, if (b) fails, i.e., $S = \{\lim_j x_j : (x_j) \in M\}$ is not a totally bounded subset of X, then it follows from [6] again that there exists a sequence $\{u_j\}_{j=1}^{\infty} \subset S$ which has no convergent subsequence, where $u_k = \lim_j x_{kj}, (x_{kj}) \in M$. Obviously, there exist $\varepsilon > 0$ and $\{u_{k_h}\} \subset \{u_k\}$ such that $\sup_h \|u_{k_h}\| > \varepsilon$, otherwise, $u_k \to 0$, which contradicts the hypothesis.

For each $j \in \mathbb{N}$ define $A_j : C(X) \longrightarrow C(X)$ by

$$A_j(x) = (0, \cdots, 0, \overset{(j)}{\lim_j x_j}, 0, 0, \cdots), \, \forall \, (x_j) \in C(X).$$

Then $\sum_{j=1}^{\infty} A_j(x_j) = (\lim_j x_j)$ in $(C(X), \|\cdot\|_\infty)$ for each $(x_j) \in C(X)$, so $(A_j) \in C(X)^{\beta C(X)}$. However,

$$\Big\| \sum_{j=m}^{\infty} A_j(x_{k_h j}) - \sum_{j=m+1}^{\infty} A_j(x_{k_h j}) \Big\|_\infty = \| \lim_j x_{k_h j}\| = \|u_{k_h}\| > \varepsilon, \, \forall \, m \in \mathbb{N},$$
$$h = 1, 2, 3, \cdots.$$

This contradicts (2) and so $(2) \Longrightarrow (1)$ holds. ■

Let $f, f_n \in Y^{C(X)}, \forall \, n \in \mathbb{N}$. Let $f_n \overset{ufC(X)}{\longrightarrow} f$ denote that $\lim_n f_n[(x_j)] = f[(x_j)]$ uniformly for (x_j) in any uniform forward subset of $C(X)$.

We say that $\{f_n\} \subset Y^{C(X)}$ is $(C(X), Y)$-convergent or, simply, $C(X)$-convergent if there exist an $\mathcal{M} \subset 2^{C(X)}$ and $f \in Y^{C(X)}$ such that $\lim_n f_n[(x_j)] = f[(x_j)]$ uniformly for (x_j) in any $M \in \mathcal{M}$. Obviously, $f_n \overset{ufC(X)}{\longrightarrow} f$ is $C(X)$-convergence.

Corollary. *For every Fréchet space E and $(A_j) \in C(X)^{\beta E}$ define $f_{(A_j),n} : C(X) \longrightarrow E \, (n \in \mathbb{N})$ and $f_{(A_j)} : C(X) \longrightarrow E$ by*

$$f_{(A_j),n}[(x_j)] = \sum_{j=1}^{n} A_j(x_j), \; f_{(A_j)}[(x_j)] = \sum_{j=1}^{\infty} A_j(x_j), \; (x_j) \in C(X).$$

Then $f_{(A_j),n} \overset{ufC(X)}{\longrightarrow} f_{(A_j)}$. Moreover, this convergence is the strongest $C(X)$-convergence for $\{f_{(A_j),n}\}$ and the family of uniform forward subsets of $C(X)$ is just the largest subfamily of $2^{C(X)}$ inducing $f_{(A_j),n} \overset{ufC(X)}{\longrightarrow} f_{(A_j)}$.

Proof. It is obvious that $f_{(A_j),n}$ converges to $f_{(A_j)}$ at each $(x_j) \in C(X)$ since $(A_j) \in C(X)^{\beta E}$. By Theorem 1, we can easily get that $\lim_n f_n[(x_j)] = f[(x_j)]$ uniformly for (x_j) in any uniform forward subset of $C(X)$, i.e., $f_{(A_j),n} \xrightarrow{ufC(X)} f_{(A_j)}$. Also, this convergence is the strongest $C(X)$-convergence for $\{f_{(A_j),n}\}$, since if there exist an $\mathcal{M} \subset 2^{C(X)}$ and $f \in Y^{C(X)}$ such that $\lim_n f_n[(x_j)] = f[(x_j)]$ uniformly for (x_j) in any $M \in \mathcal{M}$, then M must be uniform forward by Theorem 1.

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