TAIWANESE JOURNAL OF MATHEMATICS
Vol. 11, No. 3, pp. 807-818, August 2007
This paper is available online at http://www.math.nthu.edu.tw/tjm/

APPROXIMATING ZERO POINTS OF ACCRETIVE OPERATORS BY AN IMPLICIT ITERATIVE SEQUENCES IN BANACH SPACES

Sachiko Atsushiba

Abstract. In this paper, we introduce an implicit iterative sequence to approximate zero points of accretive operators in Banach spaces and then prove weak convergence theorems for resolvents of accretive operators in Banach spaces satisfying Opial's condition. Further, we discuss the strong convergence of the iterative sequences for resolvents of accretive operators with compact domains in general Banach spaces. Using these results, we consider the variational inequality problem of finding a solution of a variational inequality.

1. Introduction

Let E be a real Banach space, let C be a nonempty closed convex subset of E and let T be a nonexpansive mapping of C into itself, that is, $\|Tx-Ty\| \leq \|x-y\|$. Let $A \subset E \times E$ be an accretive operator and let J_r be the resolvent of A for $r > 0$. The problem of finding a solution $u \in E$ such that $0 \in Au$ has been investigated by many authors. One well-known scheme of approximating it is the following: $x_0 = x \in E$ and

$$x_{n+1} = J_{r_n} x_n, \tag{1}$$

for each $n = 0, 1, 2, \dots$, where $\{r_n\}$ is a sequence of positive real numbers. The convergence of (1) has been studied by many authors; see, for example[16, 17, 22]. Motivated by Mann's type [15] and Halpern's type [11], Kamimura and Takahashi [12, 13] proved weak and strong convergence theorems for resolvents of accretive operators.

Received January 8, 2007.
Communicated by J. C. Yao.
2000 *Mathematics Subject Classification*: Primary 47H09, 47H10.
Key words and phrases: Accretive operator, Resolvent, Iteration, Weak convergence, Strong convergence, Solution, Fixed point, Zero point.
This research was supported by Grant-in-Aid for Young Scientists (B), the Ministry of Education, Culture, Sports, Science and Technology, Japan.

Let C be a closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. For each $t \in (0,1)$, the contraction mapping T_t of C into itself defined by

$$T_t x = tu + (1-t)Tx$$

for every $x \in C$, has a unique fixed point x_t, where u is an element of C. Browder [8] proved that $\{x_t\}$ converges strongly to a fixed point of T as $t \to 0$ in a Hilbert space. Motivated by Browder's theorem [8], Takahashi and Ueda [27] proved the strong convergence of the following iterative process in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm (see also [19]):

$$x_k = \frac{1}{k}x + \left(1 - \frac{1}{k}\right)Tx_k \tag{2}$$

for every $k = 1, 2, 3, \ldots,$ where $x \in C$. On the other hand, Xu and Ori [28] studied the following implicit iterative process for finite nonexpansive mappings $T_1, T_2, \ldots, T_r$ in a Hilbert space: $x_0 = x \in C$ and

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n \tag{3}$$

for every $n = 1, 2, 3, \ldots,$ where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $T_n = T_{n+r}$. And they proved a weak convergence of the iterates defined by (3) in a Hilbert space (see also [24]).

In this paper, motivated by [12, 16, 17, 22, 28], we introduce an implicit iterative sequence to approximate zero points of accretive operators in Banach spaces and then prove weak convergence theorems for resolvents of accretive operators in Banach spaces satisfying Opial's condition without strict convexity. Further, we discuss the strong convergence of the iterative sequences for resolvents of accretive operators with compact domains in general Banach spaces (see also [6]). Using these results, we consider the variational inequality problem of finding a solution of a variational inequality.

2. Preliminaries and Notations

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{Z}^+$ the set of all positive integers and the set of all nonnegative integers, respectively. Let E be a real Banach space with norm $\|\cdot\|$. We denote by B_r the set $\{x \in E : \|x\| \le r\}$.

A Banach space E is said to be *strictly convex* if $\|x+y\|/2 < 1$ for each $x, y \in B_1$ with $x \neq y$, and it is said to be *uniformly convex* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x+y\|/2 \le 1 - \delta$ for each $x, y \in B_1$ with $\|x-y\| \ge \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex (see [26]). We also know that if C is a closed convex subset of a uniformly convex

Banach space E, then for each $x \in E$, there exists a unique element $u = Px \in C$ with

$$\|x-u\| = \inf\{\|x-y\| : y \in C\}.$$

Such a P is called the metric projection of E onto C.

Let E^* be the dual space of a Banach space E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. We say that a Banach space E satisfies *Opial's condition* [18] if for each sequence $\{x_n\}$ in E which converges weakly to x,

$$\varliminf_{n\to\infty} \|x_n - x\| < \varliminf_{n\to\infty} \|x_n - y\|$$

for each $y \in E$ with $y \neq x$. Since if the duality mapping $x \mapsto \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$ from E into E^* is single-valued and weakly sequentially continuous, then E satisfies Opial's condition. Each Hilbert space and the sequence spaces ℓ^p with $1 < p < \infty$ satisfy Opial's condition (see [14, 18]). Though an L^p-space with $p \neq 2$ does not usually satisfy Opial's condition, each separable Banach space can be equivalently renormed so that it satisfies Opial's condition (see [9, 18]). The following plays an important role in the proofs of our results (see [18]).

Proposition 2.1. [18]. *Let C be a nonempty weakly compact convex subset of a Banach space which satisfies Opial's condition and let T be a nonexpansive mapping of C into itself. Let $\{x_n\}$ be a sequence in C such that it converges weakly to an element x of C and $\{x_n - Tx_n\}$ converges strongly to* 0. *Then x is a fixed point of T.*

Let C be a closed subset of a Banach space and let T be a mapping of C into itself. We denote by $F(T)$ the set $\{x \in C : x = Tx\}$. We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors converges strongly to x. Similarly, we write $x_n \rightharpoonup x$ (or w-$\lim_{n\to\infty} x_n = x$) will symbolize weak convergence. Let I denote the identity operator on E. The duality mapping J from E into 2^{E^*} is defined by

$$J(x) = \{y^* \in E^* : \langle x, y^* \rangle = \|x\|^2 = \|y^*\|^2\}, x \in E.$$

Let $A \subset E \times E$ be a malutivalued opeartor. We denote by $D(A)$ and $A^{-1}0$ the effective domain of A, that is $D(A) = \{z \in E : Az \neq \emptyset\}$ and the set of zero points of A, that is, $A^{-1}0 = \{x \in E : 0 \in Ax\}$, respectively. We also denote by $R(A)$ the range of A, that is, $R(A) = \bigcup\{Az : z \in D(A)\}$. An operator A is said to be accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$. If A is accretive, then we have

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$$

for all $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$ and $r > 0$. An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$. If A is accretive, we can define, for each $r > 0$, a nonexpansive single valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$. It is called the resolvent of A. We also define the Yosida approximation A_r by $(I - J_r)/r$. We know that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$ and $\|A_r x\| \le \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. We also know that for an m-accretive operator A, we have $A^{-1}0 = F(J_r)$ for all $r > 0$. In a Hilbert space, an operator A is m-accretive if and only if A is maximal monotone.

In the sequel, unless otherwise stated, we assume that $A \subset E \times E$ is an m-accretive operator and J_r is the resolvent of A for all $r > 0$.

3. Weak Convergence Theorem

In the section, we study the weak convergence of the following implicit iterative sequences in a Banach space satisfying Opial's condition (see also [28]): $x_0 = x \in E$ and

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) J_{r_n} x_n \tag{4}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{r_n\}$ is a sequence in $(0, \infty)$. Before proving convergence theorems, we need some lemmas.

Lemma 3.1. *Let E be a Banach space. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$ for every $n \in \mathbb{N}$ and let $\{r_n\}$ be a sequence of positive real numbers. Let $x \in E$ and let $\{x_n\}$ be the sequence defined by $x_0 = x$ and*

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) J_{r_n} x_n$$

for every $n \in \mathbb{N}$. If $A^{-1}0 \neq \emptyset$, then $\|x_{n+1} - w\| \le \|x_n - w\|$ and $\lim_{n\to\infty} \|x_n - w\|$ exists for each $w \in A^{-1}0$.

Proof. We know that J_r is a nonexpansive mapping and $A^{-1}0 = F(J_r)$ for each $r > 0$. Let $w \in A^{-1}0$. For $x \in E$, put $R_0 = \|x - w\|$ and set $D = \{u \in E : \|u - w\| \le R_0\}$. Then, D is a nonempty bounded closed convex subset of E which is J_r-invariant for each $r > 0$ and contains $x_0 = x$. So, without loss of generality, we may assume that J_r is a nonexpansive mapping of a bounded closed convex subset D into itself for each $r > 0$. Define a mapping T_1 of D into itself by $T_1 y = \alpha_1 x_0 + (1 - \alpha_1) J_{r_1} y$ for all $y \in D$. Then, we have, for $y_1, y_2 \in D$,

$$\begin{aligned}\|T_1 y_1 - T_1 y_2\| &= (1 - \alpha_1)\|J_{r_1} y_1 - J_{r_1} y_2\| \\ &\le (1 - \alpha_1)\|y_1 - y_2\|.\end{aligned}$$

So, we obtain that T_1 is a contraction mapping of D into itself. By the Banach contraction principle, there exists a unique point $x_1 \in D$ such that $x_1 = T_1 x_1$. Similarly, for each $n \in \mathbb{N}$, we define a mapping T_n of D into itself by $T_n y = \alpha_n x_{n-1} + (1-\alpha_n) J_{r_n} y$ for all $y \in D$ and obtain a unique point $x_n \in D$ such that $x_n = T_n x_n$. By $w \in A^{-1}0$ and the definition of $\{x_n\}$, we obtain that

$$\begin{aligned} \|x_n - w\| &= \|\alpha_n(x_{n-1} - w) + (1-\alpha_n)(J_{r_n}x_n - w)\| \\ &\le \alpha_n \|x_{n-1} - w\| + (1-\alpha_n)\|J_{r_n}x_n - w\| \\ &\le \alpha_n \|x_{n-1} - w\| + (1-\alpha_n)\|x_n - w\| \end{aligned}$$

and hence $\alpha_n \|x_n - w\| \le \alpha_n \|x_{n-1} - w\|$. It follows from $\alpha_n > 0$ that

$$\|x_n - w\| \le \|x_{n-1} - w\|.$$

Hence, it follows that $\lim_{n\to\infty} \|x_n - w\|$ exists. ∎

The following lemma is crucial in the proof of the main result (Theorem 3.3).

Lemma 3.2. *Let C be a weakly compact convex subset of a Banach space E which satisfies Opial's condition. Let $A \subset E \times E$ be an m-accretive operator such that $D(A) \subset C$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \alpha_n = 0$ and let $\{r_n\}$ be a sequence of positive real numbers such that $\varliminf_{n\to\infty} r_n > 0$. Let $x \in C$ and let $\{x_n\}$ be the sequence defined by $x_0 = x$ and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) J_{r_n} x_n$$

for every $n \in \mathbb{N}$. Then, weak subsequential limit of $\{x_n\}$ is an element of $A^{-1}0$.

Proof. As in the proof of Lemma 3.1, without loss of generality, we may assume that J_r is a nonexpansive mapping of a bounded closed convex subset C into itself for each $r > 0$. We see that $\{x_n\} \subset C$ is bounded.

By the definition of $\{x_n\}$, we have

$$(x_n - J_{r_n}x_n) = \alpha_n (x_{n-1} - J_{r_n} x_n).$$

So, it follows that

$$\|x_n - J_{r_n}x_n\| = \alpha_n \|x_{n-1} - J_{r_n}x_n\| \le 2\alpha_n M,$$

where $M = \sup_{z\in C} \|z\|$. So, by $\lim_{n\to\infty} \alpha_n = 0$, we have

$$\lim_{n\to\infty} \|J_{r_n}x_n - x_n\| = 0. \tag{5}$$

So, by

$$\begin{aligned}\|J_{r_n}x_n - J_1J_{r_n}x_n\| &= \|(I-J_1)J_{r_n}x_n\| = \|A_1J_{r_n}x_n\| \\ &\le \inf\{\|z\| : z \in AJ_{r_n}x_n\} \\ &\le \|A_{r_n}x_n\| \\ &= \left\|\frac{x_n - J_{r_n}x_n}{r_n}\right\|\end{aligned}$$

and $\underline{\lim}_{n\to\infty} r_n > 0$, we have

$$\lim_{n\to\infty}\|J_{r_n}x_n - J_1J_{r_n}x_n\| = 0. \tag{6}$$

We remark that C is weakly compact. Letting v be a weak subsequential limit of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v$. Then, we also obtain that $J_{r_{n_k}}x_{n_k} \rightharpoonup v$ by (5). Then, it follows from (6) and Proposition 2.1 that $v \in F(J_1) = A^{-1}0$. Since $\{x_{n_k}\}$ is arbitrary, we have the desired result. ■

Now, we can prove a weak convergence theorem in a Banach space satisfying Opial's condition without strict convexity (see also [12]).

Theorem 3.3. *Let C be a weakly compact convex subset of a Banach space E which satisfies Opial's condition. Let $A \subset E \times E$ be an m-accretive operator such that $D(A) \subset C$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty}\alpha_n = 0$ and let $\{r_n\}$ be a sequence of positive real numbers such that $\underline{\lim}_{n\to\infty} r_n > 0$. Let $x \in C$ and let $\{x_n\}$ be the sequence defined by $x_0 = x$ and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)J_{r_n}x_n$$

for every $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

Proof. Since $\{x_n\} \subset C$ and C is weakly compact, $\{x_n\}$ must contain a subsequence of $\{x_n\}$ which converges weakly to a point in C. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ which converge weakly to y and z, respectively. By Lemma 3.2, we have $y, z \in A^{-1}0$. We will show $y = z$. Suppose not. Then from Lemma 3.1 and Opial's condition, we have

$$\lim_{n\to\infty}\|x_n - y\| = \lim_{i\to\infty}\|x_{n_i} - y\|$$

$$< \lim_{i\to\infty} \|x_{n_i} - z\| = \lim_{n\to\infty} \|x_n - z\|$$

$$= \lim_{j\to\infty} \|x_{n_j} - z\|$$

$$< \lim_{j\to\infty} \|x_{n_j} - y\| = \lim_{j\to\infty} \|x_n - y\|.$$

This is a contradiction. Hence, we obtain $y = z$. This implies that $\{x_n\}$ converges weakly to an element of $A^{-1}0$. ■

4. Strong Convergence Theorems

In this section, we study the strong convergence of the iterates defined by (4) in general Banach spaces (see also [6, 7]). Futher, we prove a strong convergence theorem which is connected with the metric projections.

Theorem 4.1. *Let C be a compact convex subset of a Banach space E. Let $A \subset E \times E$ be an m-accretive operator such that $D(A) \subset C$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \alpha_n = 0$ and let $\{r_n\}$ be a sequence of positive real numbers such that $\varliminf_{n\to\infty} r_n > 0$. Let $x \in C$ and let $\{x_n\}$ be the sequence defined by $x_0 = x$ and*

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) J_{r_n} x_n$$

for every $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to an element of $A^{-1}0$.

Proof. By the definition of $\{x_n\}$, we have

$$(x_n - J_{r_n} x_n) = \alpha_n (x_{n-1} - J_{r_n} x_n).$$

So, it follows that

$$\|x_n - J_{r_n} x_n\| = \alpha_n \|x_{n-1} - J_{r_n} x_n\| \le 2\alpha_n M$$

where $M = \sup_{z\in C} \|z\|$. By $\lim_{n\to\infty} \alpha_n = 0$, we have

$$\lim_{n\to\infty} \|J_{r_n} x_n - x_n\| = 0. \tag{7}$$

By

$$\|J_{r_n} x_n - J_1 J_{r_n} x_n\| = \|(I - J_1) J_{r_n} x_n\| = \|A_1 J_{r_n} x_n\|$$

$$\le \inf\{\|z\| : z \in A J_{r_n} x_n\}$$

$$\le \|A_{r_n} x_n\|$$

$$= \left\| \frac{x_n - J_{r_n} x_n}{r_n} \right\|$$

and $\underline{\lim}_{n\to\infty} r_n > 0$, we have

$$\lim_{n\to\infty} \|J_{r_n}x_n - J_1J_{r_n}x_n\| = 0. \tag{8}$$

Since C is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in C$ such that $x_{n_k} \to z$. Then, we also obtain that $J_{r_{n_k}}x_{n_k} \to z$ by (7) and hence

$$0 = \lim_{n\to\infty} \|J_{r_n}x_n - J_1J_{r_n}x_{n_k}\| = \lim_{k\to\infty} \|J_{r_{n_k}}x_{n_k} - J_1J_{r_{n_k}}x_{n_k}\| = \|z - J_1z\|.$$

So, we have $z \in F(J_1) = A^{-1}0$. By Lemma 3.1, we obtain

$$\lim_{n\to\infty} \|x_n - z\| = \lim_{i\to\infty} \|x_{n_k} - z\| = 0.$$

This completes the proof. ■

The following is a strong convergence theorem which is connected with the metric projections.

Theorem 4.2. *Let E be a uniformly convex Banach space E. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$ for every $n \in \mathbb{N}$ and let $\{r_n\}$ be a sequence of positive real numbers. Suppose $A^{-1}0 \neq \emptyset$. Let P be the metric projection of E onto $A^{-1}0$. Let $x \in E$ and let $\{x_n\}$ be the sequence defined by $x_0 = x$ and*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)J_{r_n}x_n$$

for every $n \in \mathbb{N}$. Then, Px_n converges strongly to a unique element z_0 of $A^{-1}0$ such that

$$\lim_{n\to\infty} \|x_n - z_0\| = \inf\{\lim_{n\to\infty} \|x_n - w\| : w \in A^{-1}0\}.$$

Proof. First, we remark again that for each $w \in A^{-1}0$, $\{\|x_n - w\|\}$ is a nonincreasing sequence by Lemma 3.1. Set $R = \inf\{\lim_{n\to\infty} \|x_n - w\| : w \in A^{-1}0\}$ and $K = \{w \in A^{-1}0 : \lim_{n\to\infty} \|x_n - w\| = R\}$. Since E is a uniformly convex Banach space, K consists of the exact one point, say z, i.e., z is the unique point satisfying $\lim_{n\to\infty} \|x_n - z\| = R$. Suppose that $\{Px_n\}$ does not converge strongly to the point z, i.e., $\overline{\lim}_{n\to\infty} \|Px_n - z\| > 0$. In this case, $R > 0$. By $\|x_n - Px_n\| \le \|x_n - z\|$ for each $n \in \mathbb{N}$, we have $\overline{\lim}_{n\to\infty} \|x_n - Px_n\| \le R$. By the uniform convexity of E, we have $\underline{\lim}_{n\to\infty} \|x_n - (Px_n + z)/2\| < R$ (see [26]). Since $(Px_n + z)/2 \in A^{-1}0$, we have $\|x_{m+n} - (Px_n + z)/2\| \le \|x_n - (Px_n + z)/2\|$ and hence

$$\begin{aligned} R \le \lim_{m\to\infty} \left\| x_m - \frac{(Px_n + z)}{2} \right\| &= \lim_{m\to\infty} \left\| x_{m+n} - \frac{(Px_n + z)}{2} \right\| \\ &\le \left\| x_n - \frac{(Px_n + z)}{2} \right\| \end{aligned}$$

for each $n \in \mathbb{N}$. Then, we obtain

$$R \leq \varliminf_{n\to\infty} \lim_{m\to\infty} \left\| x_m - \frac{Px_n + z}{2} \right\| \leq \varliminf_{n\to\infty} \left\| x_n - \frac{Px_n + z}{2} \right\| < R,$$

which is a contradiction. Therefore $\{Px_n\}$ converges strongly to z. ∎

Remark 4.3. From the proof above, we know that the following holds: *Let W be a nonempty, closed, convex subset of a uniformly convex Banach space E and let P be the metric projection from E onto W. Let $\{y_n\}$ be a sequence of points in E such that $\{\|y_n - w\|\}$ is nonincreasing for each $w \in W$. Then $\{Py_n\}$ converges strongly to the unique point z satisfying* $\lim_{n\to\infty} \|y_n - z\| = \inf\{\lim_{n\to\infty} \|y_n - w\| : w \in W\}$ (see 3, 26).

Combining Theorems 3.3 and 4.2, we obtain the following results.

Theorem 4.4. *Let H be a Hilbert space and let $A \subset H \times H$ be a maximal monotone operator. Let J_r be the resolvent of A for $r > 0$. Let $\{\alpha_n\}$ and $\{r_n\}$ be as in Theorem 3.3. Let $x \in H$ and let $\{x_n\}$ be the sequence defined by $x_0 = x$ and*

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) J_{r_n} x_n$$

for every $n \in \mathbb{N}$. If $A^{-1}0 \neq \emptyset$ and P is the metric projection of H onto $A^{-1}0$, then $\{x_n\}$ converges weakly to $v \in A^{-1}0$, where $v = \lim_{n\to\infty} Px_n$.

Proof. By Theorem 3.3, $\{x_n\}$ converges weakly to $v \in A^{-1}0$ and by Theorem 4.2, $\{Px_n\}$ converges strongly to a unique element z_0 of $A^{-1}0$ such that

$$\lim_{n\to\infty} \|x_n - z_0\| = \inf\{\lim_{n\to\infty} \|x_n - w\| : w \in A^{-1}0\}.$$

Since P is the metric projection of H onto $A^{-1}0$, from [26], we have

$$\langle x_n - Px_n, w - Px_n \rangle \leq 0$$

for all $w \in A^{-1}0$. Then,

$$0 \geq \lim_{n\to\infty} \langle x_n - Px_n, w - Px_n \rangle = \langle v - z_0, w - z_0 \rangle$$

for all $w \in A^{-1}0$. Putting $w = v$, we have $\|v - z_0\|^2 \leq 0$ and hence $v = z_0$. This completes the proof. ∎

5. Deduced Theorems from Main Results

In this section, we give applications of Theorems 3.3 and 4.1. Throughout this section, we assume that H is a Hilbert space. We consider the variational inequality problem. Let X be a nonempty closed convex subset of H and let T be a single valued operator of X into H. We denote by

$$VI(X,T) = \{w \in X : \langle u - w, Tw \rangle \geq 0, \forall u \in X\}.$$

A single valued operator T is called hemicontinuous if T is continuous from each line segment of X to H with the weak topology. Let F be a single valued, monotone and hemicontinuous operator of X into H and let $N_X z$ be the normal cone to X at $z \in X$, i.e.,

$$N_X z = \{w \in H : \langle z - u, w \rangle \geq 0, \quad \forall u \in X\}.$$

Letting

$$Az = \begin{cases} Fz + N_X z, & z \in X \\ \emptyset, & z \in H \backslash X, \end{cases}$$

we have that A is a maximal monotone operator (see [21, Theorem 3]). We can also check that $0 \in Av$ if and only if $v \in VI(X, F)$ and that $J_r x = VI(X, F_{r,x})$ for all $r > 0$ and $x \in H$, where $F_{r,x} z = Fz + (z - x)/r$ for all $z \in X$. Then, we have the following results.

Corollary 5.1. *Let X be a nonempty closed convex subset of H and let F be a single valued, monotone and hemicontinuous operator of X into H. Let $\{\alpha_n\}$ and $\{r_n\}$ be as in Theorem 3.3. Let $x \in X$ and let $\{x_n\}$ be the sequence defined by $x_0 = x$ and*

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) VI(X, F_{r_n, x_n})$$

for each $n \in \mathbb{N}$. If $VI(X, F) \neq \emptyset$, and P is the metric projection of H onto $VI(X, F)$, then $\{x_n\}$ converges weakly to $v \in VI(X, F)$, where $v = \lim_{n\to\infty} Px_n$.

Corollary 5.2. *Let X be a nonempty compact convex subset of H and let F be a single valued, monotone and hemicontinuous operator of X into H. Let $\{\alpha_n\}$ and $\{r_n\}$ be as in Theorem 4.1. Let $x \in X$ and let $\{x_n\}$ be the sequence defined by $x_0 = x$ and*

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) VI(X, F_{r_n, x_n})$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $v \in VI(X, F)$.

REFERENCES

1. S. Atsushiba, Strong convergence theorems for finite nonexpansive mappings, *Comm. Appl. Nonlinear Anal.*, **9** (2002), 57-68.

2. S. Atsushiba, *Strong convergence theorems for finite nonexpansive mappings in Banach spaces*, Proceedings of the Third International Conference on Nonlinear Analysis and Convex Analysis (W. Takahashi and T. Tanaka Eds.), pp. 9-16, Yokohama Publishers, Yokohama, 2004.

3. S. Atsushiba, N. Shioji and W. Takahashi, Approximating common fixed points by the Mann iteration procedure in Banach spaces, *J. Nonlinear Convex Anal.*, **1** (2000), 351-361.

4. S. Atsushiba and W. Takahashi, Approximating common fixed points of two nonexpansive mappings in Banach spaces, *Bull. Austral. Math. Soc.*, **57** (1998), 117-127.

5. S. Atsushiba and W. Takahashi, Weak and strong convergence theorems for nonexpansive semigroups in Banach spaces, *Fixed Point Theory and Applications,* **2005** (2005), 343-354.

6. S. Atsushiba and W. Takahashi, Strong convergence of Mann's type iterations for nonexpansive semigroups in general Banach spaces, *Nonlinear Anal.*, **61** (2005), 881-899.

7. S. Atsushiba and W. Takahashi, Weak and strong convergence theorems for nonexpansive semigroups in a Banach space satisfying Opial's condition, *Sci. Math. Jpn.*, **65** (2006), 147-160.

8. F. E. Browder, Convergence of approximants to fixed points of non-expansive nonlinear mappings in Banach spaces, *Arch. Rational. Mech. Anal.*, **24** (1967), 82-90.

9. D. Van Dulst, Equivalent norms and the fixed point property for nonexpansive mappings, *J. London. Math. Soc.*, **25** (1982), 139-144.

10. K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge Univ. Press, Cambridge, 1990.

11. B. Halpern, Fixed points of nonexpansive mappings, *Bull. Amer. Math. Soc.*, **73** (1967), 957-961.

12. S. Kamimura and W. Takahashi, Approximating solution of maximal monotone operatos in Hilbert spaces, *J. Approx. Theory*, **106** (2000), 226-240.

13. S. Kamimura and W. Takahashi, Weak and strong convergence of solutions to accretive operator inclusions and applications, *Set-Valued Anal.*, **8** (2000), 361-374.

14. J. P. Gossez and E. Lami Dozo, Some geometric properties related to the fixed point theory for nonexpansive mappings, *Pacific. J. Math.*, **40** (1972), 565-573.

15. W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506-510.

16. B. Martinet, Regularisation d'inequations variationnelles par approximations successive, *Rev. Franc. Inform. Rech. Opéer.*, **4** (1970), 154-159.

17. B. Martinet, Determination appochee D'un point fixe d'une application pseudo-contractante. Cas de l'application prox, *C.R. Acad. Sci. Paris. Ser. A-B*, **274** (1972), A163-165.

18. Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.*, **73** (1967), 591-597.

19. S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, *J. Math. Anal. Appl.*, **75** (1980), 287-292.

20. R. T. Rockafellar, Characterization of the subdifferentials of convex functions, *Pacific J. Math.*, **17** (1966), 497-510.

21. R. T. Rockafellar, On the maximality of sums of nonlinear monotome operators, *Trans. Amer. Math. Sor.*, **149** (1970), 75-88.

22. R. T. Rockafellar, *Monotome operators and the proximal point algorithm,*, SIAM J.Control Optim.,**14** (1976), 877-898.

23. N. Shioji and W. Takahashi, Strong convergence theorems for asymptotically nonexpansive semigroups in Banach spaces, *J. Nonlinear Convex Anal.*, **1** (2000), 73-87.

24. Z. H. Sun, C.He and Y. Q. Ni, Strong convergence of an implicit iteration process for nonexpansive mappings, *Nonlinear Funct. Anal. Appl.*, **8** (2003), 595-602.

25. W. Takahashi, A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space, *Proc. Amer. Math. Soc.*, **81** (1981), 253-256.

26. W. Takahashi, *Nonlinear functional analysis*, Yokohama Publishers, Yokohama, 2000.

27. W. Takahashi and Y. Ueda, On Reich's strong convergence theorems for resolvents of accretive operators, *J. Math. Anal. Appl.*, **104** (1984), 546-553.

28. H. K. Xu and R. G. Ori, An implicit iteration process for nonexpansive mappings, *Numer. Funct. Anal. Optim.*, **22** (2001), 767-773.

Sachiko Atsushiba
Department of Mathematics,
Shibaura Institute of Technology,
Fukasaku, Minuma-ku,
Saitama-City, Saitama 337–8570
Japan
E-mai: atusiba@sic.shibaura-it.ac.jp