TAIWANESE JOURNAL OF MATHEMATICS
Vol. 11, No. 3, pp. 661-682, August 2007
This paper is available online at http://www.math.nthu.edu.tw/tjm/

STRONG CONVERGENCE OF A HYBRID VISCOSITY APPROXIMATION METHOD WITH PERTURBED MAPPINGS FOR NONEXPANSIVE AND ACCRETIVE OPERATORS

Lu-Chuan Ceng and Hong-Kun Xu

Abstract. Recently, H. K. Xu [J. Math. Anal. Appl. 314 (2006) 631-643] considered the iterative method for approximation to zeros of an m-accretive operator A in a Banach space X. In this paper, we propose a hybrid viscosity approximation method with perturbed mapping that generates the sequence $\{x_n\}$ by the algorithm $x_{n+1} = \alpha_n(u + f(x_n)) + (1 - \alpha_n)[J_{r_n}x_n - \lambda_n F(J_{r_n}x_n)]$, where $\{\alpha_n\}$, $\{r_n\}$ and $\{\lambda_n\}$ are three sequences satisfying certain conditions, f is a contraction on X, J_r denotes the resolvent $(I + rA)^{-1}$ for $r > 0$, and F is a perturbed mapping which is both δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda \geq 1$. Under the assumption that X either has a weakly continuous duality map or is uniformly smooth, we establish some strong convergence theorems for this hybrid viscosity approximation method with perturbed mapping.

1. INTRODUCTION

Let X be a real Banach space whose dual space is denoted by X^*. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for each $x \in X$. Moreover, it

Received December 1, 2006.
Communicated by J. C. Yao.
2000 *Mathematics Subject Classification*: 47H09, 47H10, 47H17.
Key words and phrases: Hybrid viscosity approximation method with perturbed mapping, m-Accretive operator, Nonexpansive mapping, Weakly continuous duality map, Uniformly smooth Banach space.
L. C. Ceng was partially supported by the Teaching and Research Award Fund for Outstanding Young Teachers in Higher Education Institutions of MOE, China and the Dawn Program Foundation in Shanghai.
H. K. Xu was supported in part by the National Research Foundation of South Africa.

is known that J is single-valued if and only if X is smooth, while if X is uniformly smooth, then the mapping J is uniformly continuous on bounded subsets of X. Recall that an operator $A : D(A) \to 2^X$ is said to be accretive, where $D(A)$ is the domain of A, if for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there exists $j \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2, j\rangle \geq 0. \tag{1.1}$$

An accretive operator A is m-accretive if the range of $I + rA$ is precisely X for all $r > 0$, where I denotes the identity operator of X. Denote by $\mathrm{N}(A)$ the zero set of A; i.e.,

$$\mathrm{N}(A) := A^{-1}(0) = \{x \in D(A) : 0 \in Ax\}.$$

Denote by J_r the resolvent of A for $r > 0$:

$$J_r = (I + rA)^{-1}.$$

It is well known that the resolvent $J_r = (I + rA)^{-1}$ is a single-valued nonexpansive mapping whose domain is all X (e.g., Jung and Morales [17, p. 232]); see [1] for more details.

Let C be a nonempty closed convex subset of X. Recall that a self-mapping $f : C \to C$ is said to be α-contractive if for all $x, y \in C$

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|$$

for some $\alpha \in (0, 1)$. Note that each contraction $f : C \to C$ has a unique fixed point in C. Let now $T : C \to C$ be a nonexpansive mapping, i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Denote by $\mathrm{Fix}(T)$ the set of fixed points of T, i.e., $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$. Take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1 - t)Tx, \quad x \in C,$$

where $u \in C$ is a fixed point. Whenever $\mathrm{Fix}(T) \neq \emptyset$, Browder [2] proved that if X is a Hilbert space, then $\{x_t\}$ does converges strongly to the fixed point of T that is nearest to u. Reich [8] extended Browder's result to the setting of Banach spaces and proved that if X is a uniformly smooth Banach space, then $\{x_t\}$ converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $\mathrm{Fix}(T)$. Further, in the first result of [11] Xu pointed out that Reich's result holds in a Banach space which has a weakly continuous duality map. Subsequently, Zeng and Yao [15] proposed a new implicit iteration scheme with perturbed mapping for approximation of common fixed points of a finite family of nonexpansive mappings.

Recently, Xu [18] studied the viscosity approximation methods for nonexpansive mappings. Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a nonexpansive self-mapping with $\mathrm{Fix}(T) \neq \emptyset$. For a contraction f on C and $t \in (0,1)$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tf(x) + (1-t)Tx$. Consider also the iteration process $\{x_n\}$, where $x_0 \in C$ is arbitrary and $x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n$ for $n \geq 1$, where $\{\alpha_n\} \subset (0,1)$. If X is either a Hilbert space or a uniformly smooth Banach space, then it is shown in [18] that $\{x_t\}$ or, under certain appropriate conditions on $\{\alpha_n\}$, $\{x_n\}$ converges strongly to a fixed point of T which solves some variational inequality.

Motivated by Xu [11, 18] and Zeng and Yao [15], we define a mapping $T_t : X \to X$ by

$$T_t x = t(u + f(x)) + (1-t)[Tx - \theta_t F(Tx)], \quad x \in X,$$

where $\theta_t \in [0,1)$ for all $t \in (0,1)$, $u \in X$ is a fixed point, $T : X \to X$ is a nonexpansive mapping, $f : X \to X$ is a contraction, and $F : X \to X$ is a perturbed mapping which is both δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda \geq 1$. Then $T_t : X \to X$ is a contraction; see the proof in the third section. Banach's Contraction Mapping Principle guarantees that T_t has a unique fixed point x_t in X. In this paper, under Xu's assumption of a weakly continuous duality map or uniform smoothness of X we prove that $\{x_t\}$ converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from X onto $\mathrm{Fix}(T)$.

On the other hand, in [4] the authors studied iterative solutions of m-accretive operator A in a Banach space that is uniformly smooth and has a weakly continuous duality map. The iterative method studied in [4] generates the sequence $\{x_n\}$ by the algorithm

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J_{r_n} x_n, \quad n \geq 0, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{r_n\}$ is a sequence of positive numbers, and the initial guess $x_0 \in C$ is arbitrarily chosen. Theorem 2.5 of [4] asserts that if X is uniformly smooth and has a weakly continuous duality map, then the sequence $\{x_n\}$ given in (1.2) converges strongly to a point in $\mathrm{N}(A)$ provided the sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy certain conditions.

In [11], Xu proved that the above mentioned result remains valid under the lack of either the uniform smoothness assumption or the assumption of a weakly continuous duality map.

Motivated by Xu [11, 18], and Zeng and Yao [15], we propose a hybrid viscosity approximation method with perturbed mapping that generates the sequence $\{x_n\}$ by the algorithm

$$x_{n+1} = \alpha_n(u + f(x_n)) + (1-\alpha_n)[J_{r_n} x_n - \lambda_n F(J_{r_n} x_n)],$$

where $\{\alpha_n\},\ \{r_n\}$ and $\{\lambda_n\}$ are three sequences satisfying certain conditions. Such an iterative method with perturbed mapping includes the method (1.2) as a special case. Under Xu's assumption that X has a weakly continuous duality map, we establish some strong convergence theorems for this iterative method with perturbed mapping.

2. Preliminaries

We need the following lemmas which will be used in the sequel.

Lemma 2.1. *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n\beta_n + \gamma_n, \quad \forall n \ge 0,$$

where $\{\alpha_n\},\ \{\beta_n\}$, and $\{\gamma_n\}$ satisfy the conditions:

(*i*) $\{\alpha_n\} \subset [0,1],\ \sum_{n=0}^{\infty} \alpha_n = \infty$, *or equivalently,* $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$;

(*ii*) $\limsup_{n\to\infty} \beta_n \le 0$;

(*iii*) $\gamma_n \ge 0\ (n \ge 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.2. *In a smooth Banach space X there holds the inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle, \quad x, y \in X.$$

Lemma 2.3. (The Resolvent Identity). *For $\lambda, \mu > 0$, there holds the identity:*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + (1-\frac{\mu}{\lambda})J_\lambda x\right), \quad x \in X.$$

Lemma 2.4. *Assume that $c_2 \ge c_1 > 0$. Then $\|J_{c_1}x - x\| \le 2\|J_{c_2}x - x\|$ for all $x \in X$.*

The proof of Lemma 2.1 can be found in [9, 10]. Lemma 2.2 is an immediate consequence of the subdifferential inequality of the function $\frac{1}{2}\|\cdot\|^2$. Lemma 2.3 is the resolvent identity which can be found in [1]. Lemma 2.4 can be found in [7].

Recall that a gauge is a continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Associated to a gauge φ is the duality map $J_\varphi : X \to X^*$ defined by

$$J_\varphi(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|\varphi(\|x\|),\ \|x^*\| = \varphi(\|x\|)\}, \quad x \in X.$$

(Note that the duality map J_φ corresponding to the gauge $\varphi(t) = t$ for all ≥ 0 is exactly the normalized duality map J introduced in the Introduction.)

Following Browder [3], we say that a Banach space X has a weakly continuous duality map if there exists a gauge φ for which the duality map J_φ is single-valued and weak-to-weak* sequentially continuous (i.e., if $\{x_n\}$ is a sequence in X weakly convergent to a point x, then the sequence $\{J_\varphi(x_n)\}$ converges weak*ly to $J_\varphi(x)$). It is known that l^p has a weakly continuous duality map for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad t \geq 0.$$

Then

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad x \in X,$$

where ∂ denotes the subdifferential in the sense of convex analysis. In [19, p. 194], Xu and Roach gave the following relation between J_φ and J:

$$J_\varphi(\lambda x) = \operatorname{sign}(\lambda)(\varphi(|\lambda|\|x\|)/\|x\|)J(x), \quad \forall \lambda \in (-\infty, \infty),\ x \in X \text{ with } x \neq 0.$$

The first part of the next lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [6].

Lemma 2.5. *Assume that X has a weakly continuous duality map J_φ with gauge φ.*

(i) *For all $x, y \in X$, there holds the inequality*

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y)\rangle.$$

(ii) *Assume a sequence $\{x_n\}$ in X is weakly convergent to a point x. Then there holds the identity*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad x, y \in X.$$

Notation: " $\rightharpoonup$ " *stands for weak convergence and* " $\to$ " *for strong convergence.*

Recall that a mapping $F : X \to X$ is said to be δ-strongly accretive if for each $x, y \in X$ there exists $j \in J(x - y)$ such that

$$\langle Fx - Fy, j\rangle \geq \delta\|x - y\|^2 \tag{2.1}$$

for some $\delta \in (0, 1)$. $F : X \to X$ is said to be λ-strictly pseudocontractive [16] if for each $x, y \in X$ there exists $j \in J(x - y)$ such that

$$\langle Fx - Fy, j\rangle \leq \|x - y\|^2 - \lambda\|x - y - (Fx - Fy)\|^2 \tag{2.2}$$

for some $\lambda \in (0,1)$.

Proposition 2.1. *Let X be a Banach space and $F : X \to X$ be a mapping.*

(i) *If F is λ-strictly pseudocontractive then F is Lipschitz continuous with constant $L \le 1 + 1/\lambda$.*

(ii) *If X is smooth and if F is both λ-strictly pseudocontractive and δ-strongly accretive with $\lambda + \delta \ge 1$, then $I - F$ is nonexpansive.*

Proof. (i) From (2.2) we derive

$$\begin{aligned}\lambda\|(I-F)x-(I-F)y\|^2 &\le \langle (I-F)x-(I-F)y, j\rangle \\ &\le \|(I-F)x-(I-F)y\|\|x-y\|,\end{aligned}$$

which implies that

$$\|(I-F)x-(I-F)y\| \le \frac{1}{\lambda}\|x-y\|.$$

Hence

$$\|Fx-Fy\| \le \|(I-F)x-(I-F)y\| + \|x-y\| \le (1+\frac{1}{\lambda})\|x-y\|$$

and F is Lipschitz continuous.

(ii) By (2.1) and (2.2), we get

$$\begin{aligned}\lambda\|(I-F)x-(I-F)y\|^2 &\le \|x-y\|^2 - \langle Fx-Fy, J(x-y)\rangle \\ &\le (1-\delta)\|x-y\|^2.\end{aligned}$$

Since $\lambda + \delta \ge 1$,

$$\|(I-F)x-(I-F)y\| \le \sqrt{\frac{1-\delta}{\lambda}}\|x-y\| \le \|x-y\|$$

and $I - F$ is nonexpansive. ■

3. Fixed Points of Nonexpansive Mappings

Let X be a reflexive and smooth Banach space, and C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a nonexpansive mapping with a nonempty fixed point set. Recall also that for $t \in (0,1)$ and $u \in C$, x_t is the unique solution to the fixed point equation

$$x_t = tu + (1-t)Tx_t. \tag{3.1}$$

It is known that (Reich [8]) if X is a uniformly smooth Banach space, then $\{x_t\}$ converges strongly to a fixed point of T and the limit defines the sunny nonexpansive retraction from C onto $\mathrm{Fix}(T)$. Recently, Xu [11, Theorem 3.1] proved that Reich's result holds in a Banach space which has a weakly continuous duality map.

In this section, let $T : X \to X$ be nonexpansive, $f : X \to X$ be α-contractive with $\alpha \in (0,1)$, and $F : X \to X$ be both δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda \geq 1$. Now, take $t \in (0,1)$. For given $\theta_t \in [0,1)$ we define a mapping $T_t : X \to X$ by

$$T_t x = t(u + f(x)) + (1-t)[Tx - \theta_t F(Tx)], \quad x \in X,$$

where $u \in X$ is a fixed point. Then $T_t : X \to X$ is a contraction. Indeed, observe that for all $x, y \in X$

$$\begin{aligned}
\lambda\|(I-F)Tx-(I-F)Ty\|^2 &\leq \langle (I-F)Tx-(I-F)Ty, J(Tx-Ty)\rangle \\
&= \|Tx-Ty\|^2 - \langle F(Tx)-F(Ty), J(Tx-Ty)\rangle \\
&\leq (1-\delta)\|Tx-Ty\|^2.
\end{aligned}$$

Hence we have

$$\|(I-F)Tx-(I-F)Ty\| \leq \sqrt{\frac{1-\delta}{\lambda}}\|Tx-Ty\|, \quad x, y \in X. \tag{3.2}$$

Also, observe that for all $x, y \in X$

$$\begin{aligned}
&\|T_t x - T_t y\| \\
&= \|t(f(x)-f(y)) + (1-t)[Tx-\theta_t F(Tx)] - [Ty-\theta_t F(Ty)]\| \\
&= \|t(f(x)-f(y)) + (1-t)[Tx-Ty-\theta_t(F(Tx)-F(Ty))]\| \\
&\leq t\|f(x)-f(y)\| + (1-t)\|(1-\theta_t)(Tx-Ty) \\
&\quad +\theta_t[(I-F)(Tx)-(I-F)(Ty)]\| \\
&\leq t\alpha\|x-y\| + (1-t)[(1-\theta_t)\|Tx-Ty\| \\
&\quad +\theta_t\|(I-F)(Tx)-(I-F)(Ty)\|] \\
&\leq t\alpha\|x-y\| + (1-t)[(1-\theta_t)\|Tx-Ty\| \\
&\quad +\theta_t\sqrt{\tfrac{1-\delta}{\lambda}}\|Tx-Ty\|] \\
&\leq t\alpha\|x-y\| + (1-t)[1-\theta_t(1-\sqrt{\tfrac{1-\delta}{\lambda}})]\|Tx-Ty\| \\
&\leq t\alpha\|x-y\| + (1-t)[1-\theta_t(1-\sqrt{\tfrac{1-\delta}{\lambda}})]\|x-y\| \\
&\leq (1-(1-\alpha)t)\|x-y\|,
\end{aligned} \tag{3.3}$$

which hence implies that $T_t : X \to X$ is a contraction. Utilizing Banach's Contraction Mapping Principle we conclude that T_t has a unique fixed point x_t in X; that is, x_t is the unique solution to the fixed point equation

$$x_t = t(u + f(x_t)) + (1-t)[Tx_t - \theta_t F(Tx_t)]. \tag{3.4}$$

Define another mapping $S_t : X \to X$ by

$$S_t x = T_t x - \theta_t F(Tx), \quad \forall x \in X.$$

According to (3.3) it is easy to see that

$$\|S_t x - S_t y\| \le \|x - y\|, \quad \forall x, y \in X,$$

that is, S_t is nonexpansive.

We now state and prove our first result.

Theorem 3.1. *Let X be a reflexive Banach space and have a weakly continuous duality map J_φ with gauge φ. Let $T : X \to X$ be nonexpansive, $f : X \to X$ be α-contractive with $\alpha \in (0,1)$, and $F : X \to X$ be both δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda \ge 1$. Fix $u \in X$ and $t \in (0,1)$. Let $x_t \in X$ be the unique solution in X to Eq. (3.4), where $\theta_t \in [0,1), \forall t \in (0,1)$ and $\lim_{t\to 0^+} \theta_t/t = 0$. Then $\mathrm{Fix}(T) \ne \emptyset$ if and only if*

$$\limsup_{t\to 0^+} \|x_t\| < \infty,$$

and in this case, $\{x_t\}$ converges as $t \to 0^+$ strongly to an element of $\mathrm{Fix}(T)$.

Proof. Assume first that $\mathrm{Fix}(T) \ne \emptyset$. Take $p \in \mathrm{Fix}(T)$. According to (3.4) we deduce that for $t \in (0,1)$

$$\begin{aligned}
\|x_t - p\| &= \|t(u + f(x_t) - p) + (1-t)[Tx_t - \theta_t F(Tx_t) \\
&\quad -(p - \theta_t F(p))] - (1-t)\theta_t F(p)\| \\
&\le t\|u + f(p) - p\| + t\|f(x_t) - f(p)\| \\
&\quad +(1-t)\|S_t x_t - S_t p\| + (1-t)\theta_t\|F(p)\| \\
&\le t\|u + f(p) - p\| + t\alpha\|x_t - p\| + (1-t)\|x_t - p\| + \theta_t\|F(p)\| \\
&\le t\|u + f(p) - p\| + (1-(1-\alpha)t)\|x_t - p\| + \theta_t\|F(p)\|,
\end{aligned}$$

which hence implies that

$$\|x_t - p\| \le \frac{\|u + f(p) - p\|}{1-\alpha} + \frac{\theta_t}{(1-\alpha)t}\|F(p)\|.$$

So we have

$$\|x_t\| \le \|p\| + \frac{\|u + f(p) - p\|}{1-\alpha} + \frac{\theta_t}{(1-\alpha)t}\|F(p)\|.$$

Since $\lim_{t\to 0^+} \theta_t/t = 0$, we get

$$\limsup_{t\to 0^+} \|x_t\| \le \|p\| + \frac{\|u + f(p) - p\|}{1-\alpha} < \infty.$$

Next assume that $\limsup_{t\to 0^+} \|x_t\| < \infty$. Assume that $t_n \to 0^+$ and $\{x_{t_n}\}$ is bounded. Since X is reflexive, we may assume that $x_{t_n} \rightharpoonup z$ for some $z \in X$. Since J_φ is weakly continuous, we have by Lemma 2.5,

$$\limsup_{n\to\infty} \Phi(\|x_{t_n} - x\|) = \limsup_{n\to\infty} \Phi(\|x_{t_n} - z\|) + \Phi(\|x - z\|), \quad \forall x \in X.$$

Put

$$\rho(x) = \limsup_{n\to\infty} \Phi(\|x_{t_n} - x\|), \quad \forall x \in X.$$

It follows that

$$\rho(x) = \rho(z) + \Phi(\|x - z\|), \quad \forall x \in X.$$

Note that the boundedness of $\{x_{t_n}\}$ implies the boundedness of $\{Tx_{t_n}\}, \{f(x_{t_n})\}$ and $\{F(Tx_{t_n})\}$. Since, from (3.4),

$$\|x_{t_n} - Tx_{t_n}\| \le \|t_n\|u + f(x_{t_n}) - Tx_{t_n}\| + \theta_{t_n}\|F(Tx_{t_n})\| \to 0, \tag{3.5}$$

we obtain

$$\begin{aligned}\rho(Tz) &= \limsup_{n\to\infty} \Phi(\|x_{t_n} - Tz\|) = \limsup_{n\to\infty} \Phi(\|Tx_{t_n} - Tz\|)\\ &\le \limsup_{n\to\infty} \Phi(\|x_{t_n} - z\|) = \rho(z).\end{aligned}$$

On the other hand, however,

$$\rho(Tz) = \rho(z) + \Phi(\|Tz - z\|). \tag{3.6}$$

Combining Eqs. (3.5) with (3.6) yields

$$\Phi(\|Tz - z\|) \le 0.$$

Hence, $Tz = z$ and $z \in \mathrm{Fix}(T)$.

We next show that $x_{t_n} \to z$. Indeed, since $\{x_{t_n}\}$ is bounded, $\{\varphi(\|x_{t_n} - z\|)\}$ and $\{F(Tx_{t_n})\}$ are bounded. Note that $\|J_\varphi(x)\| = \varphi(\|x\|)$ for all $x \in X$. Utilizing Lemma 2.5, we have

$$\begin{aligned}
\Phi(\|x_{t_n} - z\|) &= \Phi(\|t_n(u + f(x_{t_n}) - z) + (1 - t_n)(Tx_{t_n} - z - \theta_{t_n} F(Tx_{t_n}))\|) \\
&= \Phi(\|(1 - t_n)(Tx_{t_n} - z) + t_n(f(x_{t_n}) - f(z)) \\
&\quad -(1 - t_n)\theta_{t_n} F(Tx_{t_n}) + t_n(u + f(z) - z)\|) \\
&\le \Phi(\|(1 - t_n)(Tx_{t_n} - z) + t_n(f(x_{t_n}) - f(z))\|) \\
&\quad +\langle -(1 - t_n)\theta_{t_n} F(Tx_{t_n}) + t_n(u + f(z) - z), J_\varphi(x_{t_n} - z)\rangle \\
&\le \Phi((1 - t_n)\|Tx_{t_n} - z\| + t_n\|f(x_{t_n}) - f(z)\|) \\
&\quad +t_n\langle u + f(z) - z, J_\varphi(x_{t_n} - z)\rangle + \theta_{t_n}|\langle F(Tx_{t_n}), J_\varphi(x_{t_n} - z)\rangle| \\
&\le \Phi((1 - t_n)\|x_{t_n} - z\| + \alpha t_n\|x_{t_n} - z\|) \\
&\quad +t_n\langle u + f(z) - z, J_\varphi(x_{t_n} - z)\rangle + \theta_{t_n}\|F(Tx_{t_n})\|\varphi(\|x_{t_n} - z\|) \\
&\le (1 - (1 - \alpha)t_n)\Phi(\|x_{t_n} - z\|) + t_n\langle u + f(z) - z, J_\varphi(x_{t_n} - z)\rangle \\
&\quad +\theta_{t_n}\|F(Tx_{t_n})\|\varphi(\|x_{t_n} - z\|),
\end{aligned}$$

which hence implies that

$$\Phi(\|x_{t_n} - z\|) \le \frac{1}{1-\alpha}\langle u + f(z) - z, J_\varphi(x_{t_n} - z)\rangle + \frac{\theta_{t_n}}{t_n(1-\alpha)}\|F(Tx_{t_n})\|\varphi(\|x_{t_n} - z\|).$$

Since J_φ is weak-to-weak* sequentially continuous and $\lim_{t\to 0^+} \theta_t/t = 0$, we conclude from the last inequality that

$$\Phi(\|x_{t_n} - z\|) \to 0.$$

Hence $x_{t_n} \to z$.

We finally prove that the entire net $\{x_t\}$ converges strongly. Towards this end, we assume that there exists another sequence $\{s_n\}$ in (0,1) such that $s_n \to 0$ and $x_{s_n} \to z'$. Then $z' \in \mathrm{Fix}(T)$. It remains to prove that $z' = z$. Towards this end, we observe that

$$x_t - p = (1 - t)(S_t x_t - S_t p) + t(u + f(x_t) - p) - (1 - t)\theta_t F(p)$$

for $p \in \mathrm{Fix}(T)$. It follows that

$$\begin{aligned}
\|x_t - p\|\varphi(\|x_t - p\|) &= \langle x_t - p, J_\varphi(x_t - p)\rangle \\
&= (1 - t)\langle S_t x_t - S_t p, J_\varphi(x_t - p)\rangle \\
&\quad +t\langle u + f(x_t) - p, J_\varphi(x_t - p)\rangle - (1 - t)\theta_t\langle F(p), J_\varphi(x_t - p)\rangle \\
&\le (1 - t)\|x_t - p\|\varphi(\|x_t - p\|) \\
&\quad +t\langle u + f(x_t) - p, J_\varphi(x_t - p)\rangle + \theta_t\|F(p)\|\varphi(\|x_t - p\|).
\end{aligned}$$

Therefore,

$$(3.7)\quad \|x_t-p\|\varphi(\|x_t-p\|) \le \langle u+f(x_t)-p, J_\varphi(x_t-p)\rangle + \frac{\theta_t}{t}\|F(p)\|\varphi(\|x_t-p\|).$$

In particular,

$$\|x_{t_n}-p\|\varphi(\|x_{t_n}-p\|) \le \langle u+f(x_{t_n})-p, J_\varphi(x_{t_n}-p)\rangle + \frac{\theta_{t_n}}{t_n}\|F(p)\|\varphi(\|x_{t_n}-p\|),$$

and

$$\|x_{s_n}-p\|\varphi(\|x_{s_n}-p\|) \le \langle u+f(x_{s_n})-p, J_\varphi(x_{s_n}-p)\rangle + \frac{\theta_{s_n}}{s_n}\|F(p)\|\varphi(\|x_{s_n}-p\|).$$

By passing on to the limits as $n \to \infty$ we obtain

$$\|z-p\|\varphi(\|z-p\|) \le \langle u+f(z)-p, J_\varphi(z-p)\rangle$$

and

$$\|z'-p\|\varphi(\|z'-p\|) \le \langle u+f(z')-p, J_\varphi(z'-p)\rangle.$$

Putting $p=z'$ and $p=z$ in the last two inequalities, respectively, and then adding up them, we obtain

$$\begin{aligned} 2\|z-z'\|\varphi(\|z-z'\|) &\le \langle z-z', J_\varphi(z-z')\rangle + \langle f(z)-f(z'), J_\varphi(z-z')\rangle \\ &\le (1+\alpha)\|z-z'\|\varphi(\|z-z'\|), \end{aligned}$$

which implies that

$$(1-\alpha)\|z-z'\|\varphi(\|z-z'\|) \le 0.$$

Hence $\|z-z'\|\varphi(\|z-z'\|)=0$ and we must have $z=z'$. This shows that $\{x_t\}$ converges strongly to an element of $\mathrm{Fix}(T)$. ■

We next establish the version of Theorem 3.1 in a uniformly smooth Banach space.

Theorem 3.2. *Let X be a uniformly smooth Banach space. Let $T: X \to X$ be nonexpansive, $f: X \to X$ be α-contractive with $\alpha \in (0,1)$, and $F: X \to X$ be both δ-strongly accretive and λ-strictly pseudocontractive with $\delta+\lambda \ge 1$. Fix $u \in X$ and $t \in (0,1)$. Let $x_t \in X$ be the unique solution in X to Eq. (3.4), where $\theta_t \in [0,1), \forall t \in (0,1)$ and $\lim_{t\to 0^+}\theta_t/t = 0$. Then $\mathrm{Fix}(T) \ne \emptyset$ if and only if*

$$\limsup_{t\to 0^+}\|x_t\| < \infty,$$

and in this case, $\{x_t\}$ converges as $t \to 0^+$ strongly to an element of $\mathrm{Fix}(T)$.

Proof. The proof of the necessity of $\mathrm{Fix}(T) \neq \emptyset$ is the same as that in Theorem 3.1.

To prove the sufficiency part we assume that $\limsup_{t\to 0^+} \|x_t\| < \infty$. Let now $\{t_n\}$ be a sequence in $(0,1)$ such that $t_n \to 0$ as $n \to \infty$. Define a function $\rho : X \to [0,\infty)$ on X by

$$\rho(x) = \mathrm{LIM}_n \frac{1}{2}\|x_{t_n} - x\|^2, \quad \forall x \in X. \tag{3.8}$$

(Here LIM denotes a Banach limit on l^∞.)

Let D be the set of minimizers of ρ over X; that is,

$$D = \{x \in X : \rho(x) = \min_{y\in X} \rho(y)\}.$$

Since ρ is continuous and convex, $\rho(z) \to \infty$ as $\|z\| \to \infty$, and X is reflexive, ρ attains its infimum over X. Hence the set D is a closed bounded convex nonempty subset of X. Because of (3.5), D is also T-invariant (i.e., $TD \subset D$). Since a uniformly smooth Banach space has the fixed point property for nonexpansive mappings, T admits a fixed point in D. Denote by v such a fixed point of T. Since v is a minimizer of ϱ over X, it follows that, for $x \in X$,

$$\begin{aligned} 0 &\le [\rho(v + \lambda(x-v)) - \rho(v)]/\lambda \\ &= \mathrm{LIM}_n \tfrac{1}{2}(\|(x_{t_n} - v) + \lambda(v-x)\|^2 - \|x_{t_n} - v\|^2)/\lambda. \end{aligned}$$

Since the duality map J is uniformly continuous over bounded sets of X, we can take

$$\mathrm{LIM}_n\langle x - v, J(x_{t_n} - v)\rangle \le 0, \quad x \in X.$$

In particular, when $x = u + f(v)$,

$$\mathrm{LIM}_n\langle u + f(v) - v, J(x_{t_n} - v)\rangle \le 0. \tag{3.9}$$

Since $J = J_\varphi$ with $\varphi(t) = t$ for all $t \in (-\infty,\infty)$, it follows from (3.7) that for each $p \in \mathrm{Fix}(T)$

$$\begin{aligned} \|x_t - p\|^2 &\le \langle u + f(x_t) - p, J(x_t - p)\rangle + \tfrac{\theta_t}{t}\|F(p)\|\|x_t - p\| \\ &= \langle u + f(p) - p, J(x_t - p)\rangle + \langle f(x_t) - f(p), J(x_t - p)\rangle + \tfrac{\theta_t}{t}\|F(p)\|\|x_t - p\| \\ &\le \langle u + f(p) - p, J(x_t - p)\rangle + \alpha\|x_t - p\|^2 + \tfrac{\theta_t}{t}\|F(p)\|\|x_t - p\|, \end{aligned}$$

which hence implies that

$$(1-\alpha)\|x_{t_n} - v\|^2 \le \langle u + f(v) - v, J(x_{t_n} - v)\rangle + \frac{\theta_{t_n}}{t_n}\|F(v)\|\|x_{t_n} - v\|. \tag{3.10}$$

Noting $\lim_{n\to\infty} \theta_{t_n}/t_n = 0$, we obtain

$$\mathrm{LIM}_n \|x_{t_n} - v\|^2 \le 0.$$

Hence there exists a subsequence of $\{x_{t_n}\}$, still denoted $\{x_{t_n}\}$, converging strongly to v.

To see that the entire net $\{x_t\}$ actually converges strongly as $t \to 0^+$, we assume that there exists another sequence $\{s_n\}$ in $(0,1)$, $s_n \to 0$ as $n \to \infty$, such that $x_{s_n} \to z$. Then we have $z \in \mathrm{Fix}(T)$. From (3.7) with $\varphi(t) = t, \forall t \in (-\infty, \infty)$ and $J_\varphi = J$, we derive

$$\|x_{t_n} - z\|^2 \le \langle u + f(x_{t_n}) - z, J(x_{t_n} - z)\rangle + \frac{\theta_{t_n}}{t_n}\|F(z)\|\|x_{t_n} - z\|, \tag{3.11}$$

and

$$\|x_{s_n} - v\|^2 \le \langle u + f(x_{s_n}) - v, J(x_{s_n} - v)\rangle + \frac{\theta_{s_n}}{s_n}\|F(v)\|\|x_{s_n} - v\|. \tag{3.12}$$

Letting $n \to \infty$ we deduce from (3.11) and (3.12) that

$$\|v - z\|^2 \le \langle u + f(v) - z, J(v - z)\rangle, \tag{3.13}$$

and

$$\|z - v\|^2 \le \langle u + f(z) - v, J(z - v)\rangle. \tag{3.14}$$

Adding up (3.11) and (3.12) yields

$$\begin{aligned} 2\|z - v\|^2 &\le \langle z - v, J(z - v)\rangle + \langle f(z) - f(v), J(z - v)\rangle \\ &\le (1 + \alpha)\|z - v\|^2. \end{aligned}$$

Hence $z = v$ and $\{x_t\}$ converges as $t \to 0^+$ strongly to an element of $\mathrm{Fix}(T)$. ∎

4. Zeros of m-Accretive Operators

Let C be a nonempty subset of X, let K be a nonempty subset of C and let Q be a mapping of C onto K. Then Q is said to be sunny if

$$Q(Qx + \tau(x - Qx)) = Qx,$$

whenever $Qx + \tau(x - Qx) \in C$ for $x \in C$ and $\tau \ge 0$. A mapping Q of C into itself is said to be a retraction if $Q = Q^2$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for each $z \in R(Q)$. A subset K of C is said to be a (sunny) nonexpansive retract if there exists a (sunny) nonexpansive retraction of C onto K. For a sunny nonexpansive retraction, there exists the following useful characterization:

Lemma 4.1. [13, Proposition 4, p. 59]. *Let C be a convex subset of a smooth Banach space X, let K be a nonempty subset of C and let Q be a retraction from C onto K. Then Q is sunny and nonexpansive if and only if for all $x \in C$ and $y \in K$,*

$$\langle x - Qx, J(y - Qx)\rangle \leq 0.$$

Hence there is at most one sunny nonexpansive retraction from C onto K.

More details involving sunny nonexpansive retractions can be found in [5, 14].

Recall that an operator $A : D(A) \subset X \to 2^X$ is said to be accretive, where $D(A)$ is the domain of A, if for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there exists $j \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2, j\rangle \geq 0.$$

Furthermore, A is said to be m-accretive if A is accretive and the range of $I+rA$ is precisely X for all $r > 0$. For this class of operators, the resolvent $J_r = (I+rA)^{-1}$ is a single-valued nonexpansive mapping whose domain is all X; see, e.g., Jung and Morales [17, p. 232]. Recall also that the Yosida approximation of A is defined by

$$A_r = \frac{1}{r}(I - J_r).$$

In this section, consider the problem of finding a zero of m-accretive operator A in a Banach space X,

$$0 \in Ax. \tag{4.1}$$

Moreover, assume always that

$$\mathrm{N}(A) := \{x \in X : 0 \in Ax\} = A^{-1}(0) \neq \emptyset.$$

Let $f : X \to X$ be α-contractive with $\alpha \in (0, 1)$ and $F : X \to X$ be both δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda \geq 1$. Consider the following algorithm

$$x_{n+1} = \alpha_n(u + f(x_n)) + (1 - \alpha_n)[J_{r_n}x_n - \lambda_n F(J_{r_n}x_n)], \quad \forall n \geq 0, \tag{4.2}$$

where $u \in X$ is arbitrarily fixed, $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{r_n\}$ is a sequence of positive numbers, and $\{\lambda_n\}$ is a sequence in $[0, 1)$.

Whenever $f \equiv 0$ and $\lambda_n = 0$ for all $n \geq 0$, algorithm (4.2) reduces to the following algorithm

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \quad \forall n \geq 0. \tag{4.3}$$

Algorithm (4.3) has been investigated in [4] in which strong convergence is proved provided the space X is uniformly smooth and has a weakly continuous duality map J_φ for some gauge φ. Recently, Xu [11] also studied it under the weaker assumption that X is reflexive and has a weakly continuous duality map J_φ with gauge φ. Next we state and prove the main results in this section.

Theorem 4.1. *Let X be reflexive and have a weakly continuous duality map J_φ with gauge φ. Suppose that A is an m-accretive operator in X and that $f : X \to X$ is α-contractive with $\alpha \in (0,1)$. Assume that*

- (i) $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
- (ii) $r_n \geq \varepsilon$ *for all* n *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$;
- (iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ *and* $\lim_{n\to\infty} \lambda_n/\alpha_n = 0$.

Then there hold the following:

- (I) *the sequence $\{x_n\}$ generated by algorithm (4.2) is bounded, and each weak limit point of $\{x_n\}$ lies in $N(A)$;*
- (II) *$\{x_n\}$ converges strongly to an element of $\mathrm{N}(A)$ if $\{f(x_n)\}$ is strongly convergent;*
- (III) *$\{x_n\}$ converges strongly to an element of $\mathrm{N}(A)$ if*

$$\lim_{n\to\infty} \langle f(x_n), J_\varphi(x_n - Q(u))\rangle \leq 0,$$

where Q is the unique sunny nonexpansive retraction from X onto $N(A)$.

Proof. Let, for each n, S_n be defined by

$$S_n x = J_{r_n} x - \lambda_n F(J_{r_n} x), \quad \forall x \in X.$$

Then there hold the following

(a) The algorithm (4.2) is rewritten as

$$x_{n+1} = \alpha_n (u + f(x_n)) + (1 - \alpha_n) S_n x_n, \quad \forall n \geq 0. \tag{4.4}$$

(b) By Proposition 2.1, S_n is nonexpansive.

(c) $S_n p = p - \lambda_n F(p)$ for all $p \in \mathrm{N}(A)$.

We now show that $\{x_n\}$ is bounded. As a matter of fact, since $\mathrm{N}(A) = \mathrm{Fix}(J_r)$ for all $r > 0$, we derive that, for $p \in \mathrm{N}(A)$,

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(u + f(x_n) - p) + (1 - \alpha_n)(S_n x_n - p)\| \\
&\leq \alpha_n\|u + f(p) - p\| + \alpha_n\|f(x_n) - f(p)\| \\
&\quad +(1 - \alpha_n)\|x_n - p\| + \lambda_n\|F(p)\| \\
&\leq \alpha_n\|u + f(p) - p\| + \alpha_n\alpha\|x_n - p\| \\
&\quad +(1 - \alpha_n)\|x_n - p\| + \lambda_n\|F(p)\| \\
&= \alpha_n\|u+f(p)-p\|+(1-(1-\alpha)\alpha_n)\|x_n-p\|+\lambda_n\|F(p)\| \\
&\leq (1-(1-\alpha)\alpha_n)\|x_n-p\|+\alpha_n\|u+f(p)-p\,|+\lambda_n\|F(p)\|.
\end{aligned} \tag{4.5}
$$

Since $\lim_{n\to\infty} \lambda_n/\alpha_n = 0$, we may assume without loss of generality that $\lambda_n \leq \alpha_n$ for all n. Hence, from (4.5) we get

$$
\|x_{n+1}-p\| \leq (1-\alpha)\alpha_n\cdot\frac{\|u + f(p) - p\| + \|F(p)\|}{1-\alpha}+(1-(1-\alpha)\alpha_n)\|x_n-p\|, \quad n \geq 0.
$$

By induction, we infer that

$$
\|x_n - p\| \leq \max\{(\|u + f(p) - p\| + \|F(p)\|)/(1 - \alpha), \|x_0 - p\|\}, \quad \forall n \geq 0.
$$

Therefore, $\{x_n\}$ is bounded, so are the sequences $\{f(x_n)\},\ \{J_{r_n}x_n\}$ and $\{F(J_{r_n}x_n)\}$.

By definition of x_n and x_{n+1} we obtain

$$
\begin{aligned}
x_{n+1}-x_n &= (\alpha_n-\alpha_{n-1})f(x_{n-1}) + \alpha_n(f(x_n) - f(x_{n-1})) \\
&\quad +(\alpha_n-\alpha_{n-1})(u-J_{r_{n-1}}x_{n-1})+(1-\alpha_n)(J_{r_n}x_n-J_{r_{n-1}}x_{n-1}) \\
&\quad -[(1 - \alpha_n)\lambda_n F(J_{r_n}x_n) - (1 - \alpha_{n-1})\lambda_{n-1}F(J_{r_{n-1}}x_{n-1})].
\end{aligned} \tag{4.6}
$$

Observe that

$$
\|(\alpha_n-\alpha_{n-1})f(x_{n-1})+\alpha_n(f(x_n)-f(x_{n-1}))\| \leq M|\alpha_n-\alpha_{n-1}|+\alpha_n\alpha\|x_n-x_{n-1}\|,
$$

and

$$
\begin{aligned}
&\|(1 - \alpha_n)\lambda_n F(J_{r_n}x_n) - (1 - \alpha_{n-1})\lambda_{n-1}F(J_{r_{n-1}}x_{n-1})\| \\
&= \|(\lambda_{n-1} - \lambda_n)(1 - \alpha_{n-1})F(J_{r_{n-1}}x_{n-1}) \\
&\quad +\lambda_n[(1 - \alpha_{n-1})F(J_{r_{n-1}}x_{n-1}) - (1 - \alpha_n)F(J_{r_n}x_n)]\| \\
&\leq |\lambda_n - \lambda_{n-1}|\|F(J_{r_{n-1}}x_{n-1})\| + \lambda_n\|(1 - \alpha_{n-1})F(J_{r_{n-1}}x_{n-1}) - (1 - \alpha_n)F(J_{r_n}x_n)\| \\
&\leq |\lambda_n - \lambda_{n-1}|\|F(J_{r_{n-1}}x_{n-1})\| + \lambda_n(\|F(J_{r_{n-1}}x_{n-1})\| + \|F(J_{r_n}x_n)\|) \\
&\leq |\lambda_n - \lambda_{n-1}|M + \lambda_n M
\end{aligned}
$$

for some constant $M > 0$. Moreover, if $r_{n-1} \le r_n$, using the resolvent identity

$$J_{r_n} x_n = J_{r_{n-1}} \left(\frac{r_{n-1}}{r_n} x_n + \left(1 - \frac{r_{n-1}}{r_n} \right) J_{r_n} x_n \right),$$

we obtain

$$\begin{aligned} \|J_{r_n} x_n - J_{r_{n-1}} x_{n-1}\| &\le \frac{r_{n-1}}{r_n} \|x_n - x_{n-1}\| + (1 - \frac{r_{n-1}}{r_n}) \|J_{r_n} x_n - x_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + (\frac{r_n - r_{n-1}}{r_n}) \|J_{r_n} x_n - x_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + (1/\varepsilon) |r_{n-1} - r_n| \|J_{r_n} x_n - x_{n-1}\|. \end{aligned}$$

It follows from (4.6) that

$$\begin{aligned} \|x_{n+1} - x_n\| &\le (1 - (1-\alpha)\alpha_n) \|x_n - x_{n-1}\| + \lambda_n \hat{M} + (|\alpha_n - \alpha_{n-1}| \\ &\quad + |r_n - r_{n-1}| + |\lambda_n - \lambda_{n-1}|) \hat{M} \\ &= (1 - (1-\alpha)\alpha_n) \|x_n - x_{n-1}\| + (1-\alpha)\alpha_n \cdot \frac{\lambda_n}{\alpha_n (1-\alpha)} \hat{M} \\ &\quad + (|\alpha_n - \alpha_{n-1}| + |r_n - r_{n-1}| + |\lambda_n - \lambda_{n-1}|) \hat{M} \end{aligned} \tag{4.7}$$

for some constant $\hat{M} \ge M$. Similarly we can prove (4.7) if $r_{n-1} \ge r_n$. By assumptions (i)-(iii) and Lemma 2.1, we conclude that

$$\|x_{n+1} - x_n\| \to 0.$$

This implies that

$$\|x_n - J_{r_n} x_n\| \le \|x_{n+1} - x_n\| + \|x_{n+1} - J_{r_n} x_n\| \to 0 \tag{4.8}$$

since

$$\|x_{n+1} - J_{r_n} x_n\| \le \alpha_n \|u + f(x_n) - J_{r_n} x_n\| + (1 - \alpha_n) \lambda_n \|F(J_{r_n} x_n)\| \to 0.$$

It follows that

$$\|A_{r_n} x_n\| = \frac{1}{r_n} \|x_n - J_{r_n} x_n\| \le \frac{1}{\varepsilon} \|x_n - J_{r_n} x_n\| \to 0.$$

Now if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ converging weakly to a point $\tilde{x}$, then taking the limit as $k \to \infty$ in the relation

$$[J_{r_{n_k}} x_{n_k}, A_{r_{n_k}} x_{n_k}] \in A,$$

we get $[\tilde{x}, 0] \in A$; i.e., $\tilde{x} \in \mathrm{N}(A)$. We therefore conclude that all weak limit points of $\{x_n\}$ are zeros of A. Utilizing Lemma 2.5 we get for each $p \in N(A)$

$$
\begin{aligned}
& \Phi(\|x_{n+1} - p\|) \\
&= \Phi(\|(1-\alpha_n)(J_{r_n}x_n - p) + \alpha_n(u + f(x_n) - p) \\
&\quad -(1-\alpha_n)\lambda_n F(J_{r_n}x_n)\|) \\
&\le \Phi((1-\alpha_n)\|J_{r_n}x_n - p\|) + \alpha_n\langle u + f(x_n) - p, J_\varphi(x_{n+1} - p)\rangle \\
&\quad -(1-\alpha_n)\lambda_n\langle F(J_{r_n}x_n), J_\varphi(x_{n+1} - p)\rangle \\
&\le (1-\alpha_n)\Phi(\|x_n - p\|) + \alpha_n\langle u + f(x_{n+1}) - p, J_\varphi(x_{n+1} - p)\rangle \\
&\quad +\alpha_n\langle f(x_n) - f(x_{n+1}), J_\varphi(x_{n+1} - p)\rangle \\
&\quad -(1-\alpha_n)\lambda_n\langle F(J_{r_n}x_n), J_\varphi(x_{n+1} - p)\rangle \\
&\le (1-\alpha_n)\Phi(\|x_n - p\|) + \alpha_n\langle u + f(x_{n+1}) - p, J_\varphi(x_{n+1} - p)\rangle \\
&\quad +\alpha_n\|f(x_n) - f(x_{n+1})\|\varphi(\|x_{n+1} - p\|) \\
&\quad +\lambda_n\|F(J_{r_n}x_n)\|\varphi(\|x_{n+1} - p\|) \\
&\le (1-\alpha_n)\Phi(\|x_n - p\|) + \alpha_n\langle u + f(x_{n+1}) - p, J_\varphi(x_{n+1} - p)\rangle \\
&\quad +\alpha_n[\alpha\|x_n - x_{n+1}\| + \tfrac{\lambda_n}{\alpha_n}\|F(J_{r_n}x_n)\|]\varphi(\|x_{n+1} - p\|).
\end{aligned} \tag{4.9}
$$

Since $N(A)$ is the fixed point set of the nonexpansive mapping J_r, we know from Xu [11, Theorem 3.1] that there exists a unique sunny nonexpansive retraction Q from X onto $N(A)$.

Next discuss two possible cases for the convergence of $\{x_n\}$.

Case 1. $\{f(x_n)\}$ is strongly convergent. In this case, let $f(x_n) \to v_0 \in X$. Then we write $q = Q(u + v_0)$. Putting $p = q$ in (4.9) we have

$$
\begin{aligned}
\Phi(\|x_{n+1} - q\|) \le\ & (1-\alpha_n)\Phi(\|x_n - q\|) \\
& +\alpha_n\langle u + f(x_{n+1}) - q, J_\varphi(x_{n+1} - q)\rangle \\
& +\alpha_n[\alpha\|x_n - x_{n+1}\| + \tfrac{\lambda_n}{\alpha_n}\|F(J_{r_n}x_n)\|]\varphi(\|x_{n+1} - q\|).
\end{aligned} \tag{4.10}
$$

Now take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u + f(x_n) - q, J_\varphi(x_n - q)\rangle = \lim_{k\to\infty}\langle u + f(x_{n_k}) - q, J_\varphi(x_{n_k} - q)\rangle.$$

Since X is reflexive, we may further assume that $x_{n_k} \rightharpoonup \tilde{x}$. Note that J_φ is sequentially continuous from the weak topology of X to the weak* topology of X^*.

Hence we deduce that

$$\begin{aligned}&\limsup_{n\to\infty}\langle u+f(x_n)-q, J_\varphi(x_n-q)\rangle\\&=\lim_{k\to\infty}\langle u+f(x_{n_k})-q, J_\varphi(x_{n_k}-q)\rangle\\&=\langle u+v_0-Q(u+v_0), J_\varphi(\tilde{x}-Q(u+v_0))\rangle\\&\le 0,\end{aligned}$$

since Q is a sunny nonexpansive retraction from X onto $N(A)$. Since

$$\limsup_{n\to\infty}\langle u+f(x_{n+1})-q, J_\varphi(x_{n+1}-q)\rangle\le 0,\ \lim_{n\to\infty}\|x_n-x_{n+1}\|=0,\ \lim_{n\to\infty}\lambda_n/\alpha_n=0,$$

and since both $\{\|F(J_{r_n}x_n)\|\}$ and $\{\varphi(\|x_{n+1}-q\|)\}$ are bounded, from Lemma 2.1 we obtain $\Phi(\|x_n-q\|)\to 0$; that is, $\|x_n-q\|\to 0$.

Case 2. $\limsup_{n\to\infty}\langle f(x_n), J_\varphi(x_n-Q(u))\rangle\le 0$. In this case, putting $p=Q(u)$ in (4.9) we have

$$\begin{aligned}&\Phi(\|x_{n+1}-Q(u)\|)\\&\le(1-\alpha_n)\Phi(\|x_n-Q(u)\|)+\alpha_n\langle u+f(x_{n+1})-Q(u), J_\varphi(x_{n+1}-Q(u))\rangle\\&\quad+\alpha_n[\alpha\|x_n-x_{n+1}\|+\frac{\lambda_n}{\alpha_n}\|F(J_{r_n}x_n)\|]\varphi(\|x_{n+1}-Q(u)\|).\end{aligned}\tag{4.11}$$

Now take a subsequence $\{x_{m_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u+f(x_n)-Q(u), J_\varphi(x_n-Q(u))\rangle=\lim_{k\to\infty}\langle u+f(x_{m_k})-Q(u), J_\varphi(x_{m_k}-Q(u))\rangle.$$

Since X is reflexive, we may further assume that $x_{m_k}\rightharpoonup\hat{x}$. Note that J_φ is sequentially continuous from the weak topology of X to the weak* topology of X^*. Hence we deduce that

$$\begin{aligned}&\limsup_{n\to\infty}\langle u+f(x_n)-Q(u), J_\varphi(x_n-Q(u))\rangle\\&=\lim_{k\to\infty}\langle u+f(x_{m_k})-Q(u), J_\varphi(x_{m_k}-Q(u))\rangle\\&=\lim_{k\to\infty}\langle u-Q(u), J_\varphi(x_{m_k}-Q(u))\rangle+\lim_{k\to\infty}\langle f(x_{m_k}), J_\varphi(x_{m_k}-Q(u))\rangle\\&=\langle u-Q(u), J_\varphi(\hat{x}-Q(u))\rangle+\lim_{k\to\infty}\langle f(x_{m_k}), J_\varphi(x_{m_k}-Q(u))\rangle\\&\le\langle u-Q(u), J_\varphi(\hat{x}-Q(u))\rangle+\limsup_{n\to\infty}\langle f(x_n), J_\varphi(x_n-Q(u))\rangle\\&\le\langle u-Q(u), J_\varphi(\hat{x}-Q(u))\rangle\\&\le 0,\end{aligned}$$

since Q is a sunny nonexpansive retraction from X onto $N(A)$. Again since

$$\limsup_{n\to\infty}\langle u+f(x_{n+1})-Q(u), J_\varphi(x_{n+1}-Q(u))\rangle \leq 0,$$
$$\lim_{n\to\infty}\|x_n-x_{n+1}\|=0,\ \lim_{n\to\infty}\lambda_n/\alpha_n=0,$$

and both $\{\|F(J_{r_n}x_n)\|\}$ and $\{\varphi(\|x_{n+1}-Q(u)\|)\}$ are bounded, so, from (4.11) and Lemma 2.1 it follows that $\Phi(\|x_n-Q(u)\|)\to 0$; that is, $\|x_n-Q(u)\|\to 0$. ■

Next consider the variational inequality problem:

(VI$(N(A),f)$) find $x\in N(A)$ such that $\langle f(x), J(y-x)\rangle \leq 0,\ \forall y\in N(A)$,

where $f: X\to X$ is a given mapping.

By a careful analysis of the proof of Theorem 4.1, we can obtain the following

Theorem 4.2. *Let X be reflexive and have a weakly continuous duality map J_φ with gauge φ. Suppose that A is an m-accretive operator in X and that $f: X\to X$ is α-contractive with $\alpha\in(0,1)$ such that f is sequentially continuous from the weak topology of X to the strong topology of X. Assume that*

(*i*) $\alpha_n\to 0$, $\sum_{n=0}^\infty \alpha_n=\infty$, *and* $\sum_{n=0}^\infty|\alpha_{n+1}-\alpha_n|<\infty$;

(*ii*) $r_n\geq\varepsilon$ *for all* n *and* $\sum_{n=0}^\infty|r_{n+1}-r_n|<\infty$;

(*iii*) $\sum_{n=0}^\infty|\lambda_{n+1}-\lambda_n|<\infty$ *and* $\lim_{n\to\infty}\lambda_n/\alpha_n=0$.

Then there hold the following:

(*I*) *the sequence $\{x_n\}$ generated by algorithm (4.2) is bounded, and each weak limit point of $\{x_n\}$ lies in $N(A)$;*

(*II*) *$\{x_n\}$ converges strongly to an element of $\mathrm{N}(A)$ if $\{f(x_n)\}$ is strongly convergent;*

(*III*) *$\{x_n\}$ converges strongly to an element of $\mathrm{N}(A)$ if each weak limit point of $\{x_n\}$ is a solution of the VI$(N(A),f)$.*

Proof. The proofs of conclusions (I) and (II) are the same as those in Theorem 4.1, so we omit them.

Next we first verify the following inequality for Case 2 in the proof of Theorem 4.1:

$$\limsup_{n\to\infty}\langle u+f(x_n)-Q(u), J_\varphi(x_n-Q(u))\rangle\leq 0.$$

Indeed, take a subsequence $\{x_{m_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u+f(x_n)-Q(u), J_\varphi(x_n-Q(u))\rangle = \lim_{k\to\infty}\langle u+f(x_{m_k})-Q(u), J_\varphi(x_{m_k}-Q(u))\rangle.$$

Since X is reflexive, we may further assume that $x_{m_k} \rightharpoonup \hat{x}$. Then $\hat{x}$ is a solution of the VI$(N(A), f)$. Note that J_φ is sequentially continuous from the weak topology of X to the weak* topology of X^* and that f is sequentially continuous from the weak topology of X to the strong topology of X. Hence we deduce that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle u + f(x_n) - Q(u), J_\varphi(x_n - Q(u))\rangle \\
&= \lim_{k\to\infty}\langle u + f(x_{m_k}) - Q(u), J_\varphi(x_{m_k} - Q(u))\rangle \\
&= \lim_{k\to\infty}\langle u - Q(u), J_\varphi(x_{m_k} - Q(u))\rangle + \lim_{k\to\infty}\langle f(x_{m_k}), J_\varphi(x_{m_k} - Q(u))\rangle \\
&= \langle u - Q(u), J_\varphi(\hat{x} - Q(u))\rangle + \langle f(\hat{x}), J_\varphi(\hat{x} - Q(u))\rangle \\
&\le 0,
\end{aligned}$$

since Q is a sunny nonexpansive retraction from X onto $N(A)$. Since

$$\begin{aligned}
&\limsup_{n\to\infty}\langle u + f(x_{n+1}) - Q(u), J_\varphi(x_{n+1} - Q(u))\rangle \le 0, \\
&\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0, \ \lim_{n\to\infty}\lambda_n/\alpha_n = 0,
\end{aligned}$$

and since both $\{\|F(J_{r_n}x_n)\|\}$ and $\{\varphi(\|x_{n+1} - Q(u)\|)\}$ are bounded, so, from (4.11) and Lemma 2.1 it follows that $\Phi(\|x_n - Q(u)\|) \to 0$; that is, $\|x_n - Q(u)\| \to 0$. ■

REFERENCES

1. V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, 1976.
2. F. E. Browder, Fixed point theorems for noncompact mappings in Hilbert space, *Proc. Natl. Acad. Sci. USA*, **53** (1965), 1272-1276.
3. F. E. Browder, Convergence theorems for sequences of nonlinear operators in Banach spaces, *Math. Z.*, **100** (1967), 201-225.
4. T. Dominguez Benavides, G. Lopez Acedo and H. K. Xu, Iterative solutions for zeros of accretive operators, *Math. Nachr.*, **248-249** (2003), 62-71.
5. K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mapping*, Dekker, 1984.
6. T. C . Lim and H. K. Xu, Fixed point theorems for asymptotically nonexpansive mappings, *Nonlinear Anal.*, **22** (1994), 1345-1355.
7. G. Marino and H. K. Xu, Convergence of generalized proximal point algorithms, *Comm. Pure Appl. Anal.*, **3** (2004), 791-808.
8. S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, *J. Math. Anal. Appl.*, **75** (1980), 287-292.
9. H. K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.*, **66** (2002), 240-256.

10. H. K. Xu, An iterative approach to quadratic optimization, *J. Optim. Theory Appl.*, **116** (2003), 659-678.
11. H. K. Xu, Strong convergence of an iterative method for nonexpansive and accretive operators, *J. Math. Anal. Appl.*, **314** (2006) 631-643.
12. L. C. Zeng, G. M. Lee and N. C. Wong, Ishikawa iteration with errors for approximating fixed points of strictly pseudocontractive mappings of Browder-Petryshyn type, *Taiwanese J. Math.*, **10(1)** (2006), 87-99.
13. N. Shioji and W. Takahashi, Strong convergence theorems for continuous semigroups in Banach spaces, *Math. Japon.*, **50(1)** (1999), 57-66.
14. S. Reich, Product formulas, nonlinear semigroups, and accretive operators, *J. Funct. Anal.*, 3**6(2)** (1980), 147-168.
15. L. C. Zeng and J. C. Yao, Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings, *Nonlinear Anal.*, **64** (2006), 2507-2515.
16. F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, *J. Math. Anal. Appl.*, **20** (1967), 197-228.
17. J. S. Jung and C.H. Morales, The Mann process for perturbed m-accretive operators in Banach spaces, *Nonlinear Anal.*, **46** (2001), 231-243.
18. H. K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.*, **298** (2004), 279-291.
19. Z. B. Xu and G. F. Roach, Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces, *J. Math. Anal. Appl.*, **157** (1991), 189-210.

Lu-Chuan Ceng
Department of Mathematics,
Shanghai Normal University,
Shanghai 200234,
China
E-mail: zenglc@hotmail.com

Hong-Kun Xu
School of Mathematical Scuences,
University of Kwa Zulu-Natal,
Westville Campus,
Private Bag X54001, Durban 4000,
South Africa
E-mail: xuhk@ukzn.ac.za
and
Department of Applied Mathematics,
National Sun Yat-sen University,
Kaohsiung 80424,
Taiwan, R.O.C.
E-mail: xuhk@math.nsysu.edu.tw