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EXISTENCE OF UNCONDITIONAL WAVELET PACKET BASES FOR THE SPACES $L^p(\mathbb{R})$ AND $\mathcal{H}^1(\mathbb{R})$

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Abstract. It is proved that the system $\{\omega_{\ell,n,k} : \ell = j - m; n = 2^m, 2^m + 1, \ldots, 2^{m+1} - 1; \quad j, k \in \mathbb{Z}\}$ of wavelet packets is an unconditional basis for $L^p(\mathbb{R})$, $1 < p < \infty$ and $\mathcal{H}^1(\mathbb{R})$, where $m = 0$ if $j \leq 0$ and $m = 0, 1, 2, \ldots, j$ if $j > 0$, provided the orthonormal wavelet packets ω_n and its derivative ω_n' have a common radial decreasing L^1-*majorant* satisfying the condition $\int_0^\infty sH(s)\,ds < \infty$.

1. INTRODUCTION

In his paper, Mallat [10] first formulated the remarkable idea of multiresolution analysis (MRA) that deals with a general formalism for construction of an orthogonal basis of wavelet bases. A multiresolution analysis consists of a sequence of embedded closed subspaces $\{V_j : j \in \mathbb{Z}\}$ for approximating $L^2(\mathbb{R})$ functions (see Debnath [6]). On the other hand, wavelet packets represent a simple but powerful extension of wavelets and MRA. Orthogonal wavelet packets (also simply called *wave packets*) introduced by Coifman et al. [3] are used to further decompose wavelet components. Wickerhauser [14] thoroughly investigated discrete wavelet packets and developed computer programs and implemented them. Several authors including Mallat [10], Daubechies [5], and Meyer [11] have laid the foundation of wavelet theory and its diverse applications through multiresolution analysis.

Gripenberg [7] introduced the subject of unconditionality of wavelet bases for Lebesgue spaces $L^p(\mathbb{R})$, $1 < p < \infty$. The constructive proofs of the unconditional basis for $\mathcal{H}^1(\mathbb{R})$ have been given by Carleson [2] and Wojtaszczyk [15], where the latter author has given an example of an unconditional basis for the Hardy space $\mathcal{H}^1(\mathbb{R})$ as the Franklin system. In fact, a large class of wavelets which have

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unconditional basis for the Hardy space $\mathcal{H}^1(\mathbb{R})$ was discovered by Meyer [11]. It was Strömberg [13] who first discovered unconditional bases for the spaces $\mathcal{H}^1(\mathbb{R})$ and $L^p(\mathbb{R})$, $1 < p < \infty$, and they are spline systems of higher order.

Motivated by the study of unconditionality of wavelet bases for the spaces $L^p(\mathbb{R})$, $1 < p < \infty$, by Gripenberg [7], we are interested in extending the results on unconditional wavelet packet basis for spaces in the context of wavelet packets. In this paper, we prove the results on the existence of unconditional wavelet packet bases for spaces $\mathcal{H}^1(\mathbb{R})$ and $L^p(\mathbb{R})$, $1 < p < \infty$ based on an approach similar to that of Hernández and Weiss [9] and Meyer [11]. We have also used the atomic decomposition of $\mathcal{H}^1(\mathbb{R})$ described by Coifman [4] and the Calderon-Zygmund theory for boundedness of certain operators by Han and Swayer [8] to prove our results.

2. Basic Ideas and Results

For basic ideas, results on wavelets and multiresoltuion analysis, we refer to Chapters 6 and 7 of Debnath [6].

We construct wavelet packets from multiresolution analysis. In general, consider two sequences $\{\alpha_n\}_{n\epsilon\mathbb{Z}}$ and $\{\beta_n\}_{n\epsilon\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$. Let $\mathbb{H}$ be a Hilbert space with orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$. Then, the sequences

$$f_{2n} = \sqrt{2}\sum_{k\epsilon\mathbb{Z}}\alpha_{2n-k}\ e_k, \qquad f_{2n+1} = \sqrt{2}\sum_{k\epsilon\mathbb{Z}}\beta_{2n-k}\ e_k$$

are orthonormal bases of two orthogonal closed subspaces $\mathbb{H}_1$ and $\mathbb{H}_0$, respectively, such that

$$\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_0.$$

Using this "splitting trick" we now define the basic wavelet packets associated with a scaling function ϕ as defined in MRA.

Let $\omega_0 = \phi$. The basic wavelet packets ω_n, $n = 0, 1, 2, \ldots$ associated with the scaling function ϕ are defined recursively by

$$\omega_{2n}(x) = \sqrt{2} \sum_{k\epsilon\mathbb{Z}} h_k\ \omega_n(2x-k)\,, \tag{2.1}$$

and

$$\omega_{2n+1}(x) = \sqrt{2} \sum_{k\in\mathbb{Z}} g_k\ \omega_n(2x-k)\,. \tag{2.2}$$

It follows from the above definition that $\omega_1 = \psi$ is a mother wavelet and the set $\{\omega_n(x-k) : n = 0, 1, ..., k \in \mathbb{Z}\}$ is an orthonormal basis for the Hilbert space $L^2(\mathbb{R})$.

Corresponding to some orthonormal scaling function $\phi = \omega_0$, the family of wavelet packets $\{\omega_n\}$ defines a family of subspaces of $L^2(\mathbb{R})$as follows :

$$U_j^n = span\left\{2^j\ \omega_n\left(2^j x - k\right)\ :\ k \in \mathbb{Z}\right\};\ j \in \mathbb{Z},\ n = 0, 1, 2, ... \tag{2.3}$$

Observe that

$$U_j^0 = V_j, \qquad U_j^1 = W_j$$

so that the orthogonal decomposition can be written as

$$U_{j+1}^0 = U_j^0 \oplus U_j^1. \tag{2.4}$$

A generalization of this result for other values of n can be written as

$$U_{j+1}^n = U_j^{2n} \oplus U_j^{2n+1},\ j \in \mathbb{Z}. \tag{2.5}$$

Now, we state a lemma which will be used in the proof of the preceding results.

Lemma 2.1. *For each $j = 1, 2, 3, \ldots$, the decomposition trick (2.5) gives*

$$\begin{aligned} W_j &= U_j^1 = U_{j-1}^2 \oplus U_{j-1}^3 \\ &= U_{j-2}^4 \oplus U_{j-2}^5 \oplus U_{j-2}^6 \oplus U_{j-2}^7 \\ &\ \ \vdots \\ &\ \ \vdots \\ &= U_{j-k}^{2^k} \oplus U_{j-k}^{2^k+1} \oplus U_{j-k}^{2^k+2} \oplus \cdots \oplus U_{j-k}^{2^{k+1}-1} \\ &\ \ \vdots \\ &\ \ \vdots \\ &= U_0^{2^j} \oplus U_0^{2^j+1} \oplus U_0^{2^j+2} \oplus \cdots \oplus U_0^{2^{j+1}-1} \end{aligned} \tag{2.6}$$

where U_j^n is defined by (2.3). Moreover, for each $j = 1, 2, \ldots$; $k = 1, 2, \ldots, j$ and $m = 0, 1, 2, \ldots, 2^k - 1$, and the set $\left\{2^{\frac{j-k}{2}}\omega_p\left(2^{j-k}x - \ell\right) : \ell \in \mathbb{Z}\right\}$ *is as orthonormal basis of U_{j-k}^p where $p = 2^k + m$.*

Let Q_j^n be the orthogonal projection onto U_j^n with kernel $Q_j^n(x, y)$ defined by

$$Q_j^n(x, y) = \sum_{k \in \mathbb{Z}} \omega_{j,n,k}\ (x)\ \overline{\omega_{j,n,k}\ (y)}\,;\ j \in \mathbb{Z}, \quad n = 0, 1, 2, 3, \ldots, \tag{2.7}$$

where $\omega_{j,n,k}$ are the wavelet packets.

Lemma 2.2. (Ahamd et al. [1]). *If $\omega_n \in L^2(\mathbb{R})$ are wavelet packets related to the scaling function $\phi\,(\omega_0 = \phi)$, then $\hat{\omega}_n\ (0) = 0, \quad n = 1, 2, 3, \ldots$.*

We refer to Singer [12] for the concepts of basis and unconditional basis in Banach spaces.

Lemma 2.3. (see Hernández and Weiss [9]). *For a basis $\{x_j : j \in \mathbb{N}\}$ of a Banach space $(\mathbb{B}, \|.\|)$ the following statements are equivalent:*

(i) *$\{x_j : j \in \mathbb{N}\}$ is an unconditional basis for $\mathbb{B}$.*

(ii) *There exists a constant $C > 0$ such that $\|S_\beta(x)\| \le C\|x\|$ for all sequences $\beta = \{\beta_j\}_{j\in\mathbb{N}}$ with $|\beta_j| \le 1$, where $S_\beta(x) = \sum_{j\in\mathbb{N}} \beta_j f_j\ (x)\, x_j,\ for\ all\ x \in \mathbb{B}$.*

(iii) *There exists a constant $C > 0$ such that $\|S_\varepsilon(x)\| \le C\|x\|$ for all sequences $\varepsilon = \{\varepsilon_j\}_{j\in\mathbb{N}}$ with $\varepsilon_j = \pm 1$.*

(iv) *There exists a constant $C > 0$ such that $\|S_\beta(x)\| \le C\|x\|$ for all finitely non-zero sequences $\beta = \{\beta_j\}_{j\in\mathbb{N}}$ with $\beta_j = 1$ or 0.*

Definition 2.4. For a given function f defined on $\mathbb{R}$, we say that a bounded function $H : [0, \infty) \to \mathbb{R}^+$ is a radial decreasing L^1-majorant of f if $|f(x)| \le H(|x|)$ and H satisfies the following conditions:

(2.8) (i) $H \in L^1[0, \infty)$, (ii) H is decreasing, (iii) $H(0) < \infty$.

Let RB denote a set of all radially bounded decreasing functions.

Lemma 2.5. (see Han and Sawyer [8]). *Let H be a function on $[0, \infty)$ satisfying the conditions (2.8). Then*

$$\sum_{k\in\mathbb{Z}} H(|x-k|)\ H(|y-k|) \le CH\left(\frac{|x-y|}{2}\right), \quad \text{for all } x, y \in \mathbb{R},$$

where C is a constant depending on H.

3. Unconditional Wavelet Packet Bases for the Spaces $\mathcal{H}^1(\mathbb{R})$ and $L^p(\mathbb{R})$, $(1 < p < \infty)$

Throughout this section we consider the set

$$\left\{\omega_{\ell,n,k} : \ell = j - m\,;\ n = 2^m,\ 2^m + 1,\ 2^m + 2, \ldots, 2^{m+1} - 1\,;\ j,\, k \in \mathbb{Z}\right\}$$

an orthonormal basis for the space which is to be considered, where $m = 0$ if $j \leq 0$ and $m = 0, 1, 2, \ldots, j$ if $j > 0$. Now, we define a radial decreasing L^1-majorant H related to wavelet packets. Also, consider a bounded function $H : [0, \infty) \to \mathbb{R}^+$ a common radial decreasing L^1-majorant of ω_n, for all n, satisfying (2.8).

To study that wavelet packets form an unconditional basis for $\mathcal{H}^1(\mathbb{R})$ and $L^p(\mathbb{R})$, $1 < p < \infty$, we first define an operator T_β by

$$T_\beta f = \sum_{j\in\mathbb{Z}} \sum_{n=2^m}^{2^{m+1}-1} \sum_{k\in\mathbb{Z}} \beta_{\ell,n,k} < f, \, \omega_{\ell,n,k} > \omega_{\ell,n,k} \tag{3.1}$$

in $\mathcal{H}^1(\mathbb{R})$ and in $L^p(\mathbb{R})$, $1 < p < \infty$; where $\ell = j - m$, $m = 0$ if $j \leq 0$ and $m = 0, 1, 2, ..., j$ if $j > 0$. Also, $\beta = \{\beta_{\ell,n,k}\}$ is a sequence such that $\beta_{\ell,n,k} = 1$ for finite number of indices and $\beta_{\ell,n,k} = 0$ for remaining indices. Now, we examine the boundedness of this operator T_β. The operator can be written as an integral operator of the form

$$(T_\beta f)(x) = \int_{\mathbb{R}} K_\beta(x, y) f(y) \, dy, \tag{3.2}$$

where

$$K_\beta(x, y) = \sum_{j\in\mathbb{Z}} \sum_{n=2^m}^{2^{m+1}-1} \sum_{k\in\mathbb{Z}} \beta_{\ell,n,k} \, \omega_{\ell,n,k} \ (x) \tag{3.3}$$

where ℓ and m are defined in the beginning of this section.

If the wavelet packets ω_n are bounded by a radial decreasing L^1-majorant H, then by using Lemma 2.5, we obtain the following estimate

$$\begin{aligned} |K_\beta(x, y)| &\leq \sum_{j\in\mathbb{Z}} \sum_{n=2^m}^{2^{m+1}-1} \sum_{k\in\mathbb{Z}} 2^\ell \left|\omega_n\left(2^\ell x - k\right) \ \omega_n\left(2^\ell y - k\right)\right| \\ &\leq \sum_{j\in\mathbb{Z}} \sum_{n=2^m}^{2^{m+1}-1} \sum_{k\in\mathbb{Z}} 2^\ell \, H\left(\left|2^\ell x - k\right|\right) H\left(\left|2^\ell y - k\right|\right) \\ &\leq \sum_{j\in\mathbb{Z}} \sum_{n=2^m}^{2^{m+1}-1} 2^\ell \, CH\left(\tfrac{2^\ell|x-y|}{2}\right) \\ &= C \sum_{j\in\mathbb{Z}} 2^m 2^\ell \, H\left(2^{\ell-1}|x-y|\right) \\ &= C \sum_{j\in\mathbb{Z}} 2^j H\left(2^{\ell-1}|x-y|\right) \end{aligned} \tag{3.4}$$

where C depends only on H and $\ell = j - m$, $m = 0$ if $j \leq 0$ and $m = 0, 1, 2, 3, \ldots, j$ if $j > 0$.

Lemma 3.1. *A Calderon-Zygmund operator T is a bounded linear operator on $L^2(\mathbb{R})$ such that*

$$(Tf)(x) = \int_{\mathbb{R}} K(x,y) f(y)\, dy,$$

where $x \notin supp(f)$ and the kernel K is a jointly measurable function satisfying

$$|K(x,y)| \leq \frac{C_1}{|x-y|}; \tag{3.5}$$

$$|K(x_0,y) - K(x,y)| \leq \frac{C_2 |x - x_0|}{|x-y|^2}, \quad \text{if} \quad |x - x_0| \leq \frac{1}{2}|x-y|; \tag{3.6}$$

$$|K(x, y_0) - K(x, y)| \leq \frac{C_3 |y - y_0|}{|x-y|^2}, \quad \text{if} \quad |y - y_0| \leq \frac{1}{2}|x-y|. \tag{3.7}$$

Lemma 3.2. *Let T be a Calderon-Zygmund operator such that*

$$\int_{\mathbb{R}} T f(x)\, dx = 0 \quad \textit{and} \quad \int_{\mathbb{R}} T^* f(x)\, dx = 0,$$

whenever $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} f(x)\, dx = 0$, where T^* is the dual of T. Then, T extends to a bounded operator on $\mathcal{H}^1(\mathbb{R})$, $BMO(\mathbb{R})$ and on $L^p(\mathbb{R})$, $1 < p < \infty$, with operator norm depending only on $\|T\|_{L^2(\mathbb{R})}$ and the constants involved in the inequalities (3.5), (3.6) and (3.7).

Theorem 3.3. *Let ω_n be othronormal wavelet packets such that ω_n and ω_n' (derivative of ω_n) have a common radial decreasing L^1-majorant H, for all n, satisfying*

$$\int_{\mathbb{R}} s\, H(s)\, ds < \infty.$$

Then, the operators T_β defined by (3.2) and (3.3) are bounded in $\mathcal{H}^1(\mathbb{R})$, $BMO(\mathbb{R})$ (dual of $\mathcal{H}^1(\mathbb{R})$) and $L^p(\mathbb{R})$, $1 < p < \infty$, with norm bounded by a constant independent of the finitely non-zero sequence β consisting of zeroes and ones.

Proof. Since the system $\{\omega_{\ell,n,k}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, it is easy to see that the operator T_β is bounded, i.e.,

$$\begin{aligned}\|T_\beta f\|^2_{L^2(\mathbb{R})} &= \sum_{j\in\mathbb{Z}}\sum_{n=2^m}^{2^{m+1}-1}\sum_{k\in\mathbb{Z}}|\beta_{\ell,n,k} < f,\, \omega_{\ell,n,k} >|^2 \\ &\le \sum_{j\in\mathbb{Z}}\sum_{n=2^m}^{2^{m+1}-1}\sum_{k\in\mathbb{Z}}|< f,\, \omega_{\ell,n,k} >|^2 = \|f\|^2_{L^2(\mathbb{R})},\end{aligned}$$

where $\ell = j - m$, $m = 0$ for $j \le 0$ and $m = 0, 1, 2, ..., j$ for $j > 0$.

For any $f \in L^2(\mathbb{R})$, we have

$$\int_{\mathbb{R}} T_\beta f(x)\,dx = 0 = \int_{\mathbb{R}} T^*_\beta f(x)\,dx,$$

since $T_\beta f$ and $T^*_\beta f$ are finite linear combinations of the $\omega_{\ell,n,k}{}'s$.
Moreover, in view of Lemma 2.2, we have

$$0 = \hat{\omega}_n(0) = \int_{\mathbb{R}} \omega_n(x)\,dx, \qquad \text{for } n > 1.$$

Now, to prove the theorem, it is required to show that T_β is a Calderon-Zygmund operator and then the theorem will follow by using Lemma 3.2. In order to show that T_β is a Calderon-Zygmund operator, it is sufficient to show that K_β satisfies conditions (3.5), (3.6) and (3.7). To prove (3.5), we use (3.4) and obvious estimates, to obtain

$$\begin{aligned}|K_\beta(x,\, y)| &\le C\sum_{j\in\mathbb{Z}} 2^j H\left(2^{j-m-1}|x-y|\right) \\ &\le C\sum_{j=-\infty}^{0} 2^j H\left(2^{j-1}|x-y|\right) \\ &\quad + C\sum_{j=1}^{\infty} 2^j H\left(2^{j-m-1}|x-y|\right).\end{aligned} \tag{3.8}$$

Now, we consider to decompose W_j spaces for some $j = M$, for sufficiently large M. Then, all W_j spaces, for which $j \le M$, will decompose up to last formula in (2.6) and other W_j spaces, for which $j > M$, will decompose according to intermediate formula in (2.6). So inequality (3.8) takes the form

$$\begin{aligned}|K_\beta(x,\,y)| &\leq C\sum_{j=-\infty}^{0}2^jH\left(2^{j-1}|x-y|\right)+C\sum_{j=1}^{M}2^jH\left(2^{-1}|x-y|\right)\\&\quad+C\sum_{j=M+1}^{\infty}2^jH\left(2^{j-M-1}|x-y|\right)\\&\leq C\sum_{j=-\infty}^{0}2^jH\left(2^{j-M-1}|x-y|\right)+C\sum_{j=1}^{M}2^jH\left(2^{j-M-1}|x-y|\right)\\&\quad+C\sum_{j=M+1}^{\infty}2^jH\left(2^{j-M-1}|x-y|\right)\\&= C\sum_{j=-\infty}^{\infty}2^jH\left(2^{j-M-1}|x-y|\right)\\&\leq 2\,C\int_0^\infty H\left(2^{-M-1}\,t\,|x-y|\right)dt\\&= \frac{C\,2^{M+2}}{|x-y|}\,\|H\|_{L^1(0,\infty)}\,.\end{aligned}$$

To prove (3.6) we assume $x < x_0$. Now, we prove

$$\left|\frac{\partial}{\partial x}K_\beta(x,\,y)\right| \leq \frac{C}{|x-y|^2} \quad \text{for } y\in\mathbb{R}. \tag{3.9}$$

It is easy to see that the inequality (3.9) implies (3.6). To see this we apply Mean Value Theorem to obtain a point $x'\in(x_0,\,x)$ such that

$$\begin{aligned}|K_\beta(x_0,\,y)-K_\beta(x,\,y)| &\leq |x_0-x|\left|\frac{\partial K_\beta(x',\,y)}{\partial x}\right|\\&\leq \frac{C\ |x_0-x|}{|x'-y|^2}.\end{aligned}$$

Observe that (3.6) implies that $y\notin(x_0,x)$. If $y\geq x$, it is clear that

$$\left|x'-y\right| \geq |x-y| \geq \frac{1}{2}|x-y|\,.$$

If $y\leq x_0$, we use (3.6) to obtain

$$\left|x'-y\right| \geq |x_0-y| \geq |x-y|-|x-x_0| \geq \frac{1}{2}|x-y|\,.$$

Hence

$$|K_\beta(x_0, y) - K_\beta(x, y)| \leq \frac{4C\,|x_0 - x|}{|x-y|^2},$$

provided $|x - x_0| \leq \frac{1}{2}|x-y|$.

Now, using Lemma 2.5 and the fact that H is decreasing, we obtain

$$\begin{aligned}
\left|\frac{\partial}{\partial x} K_\beta(x, y)\right| &= \left|\sum_{j\in\mathbb{Z}} \sum_{n=2^m}^{2^{m+1}-1} \sum_{k\in\mathbb{Z}} \beta_{\ell,n,k}\; 2^{2\ell}\, \omega_n'\left(2^\ell x - k\right) \overline{\omega_n\left(2^\ell y - k\right)}\right| \\
&\leq \sum_{j\in\mathbb{Z}} \sum_{n=2^m}^{2^{m+1}-1} \sum_{k\in\mathbb{Z}} 2^{2\ell} H\left(\left|2^\ell x - k\right|\right) H\left(\left|2^\ell y - k\right|\right) \\
&\leq C \sum_{j\in\mathbb{Z}} 2^{2\ell+m} \sum_{k\in\mathbb{Z}} H\left(\left|2^\ell x - k\right|\right) H\left(\left|2^\ell y - k\right|\right) \\
&\leq C \sum_{j\in\mathbb{Z}} 2^{2j-m} H\left(2^{j-m-1}|x-y|\right) \\
&\leq C \sum_{j=-\infty}^{0} 2^{2j-M} H\left(2^{j-1}|x-y|\right) + C\sum_{j=1}^{M} 2^{2j} H\left(2^{-1}|x-y|\right) \\
&\quad + C \sum_{j=M+1}^{\infty} 2^{2j-M} H\left(2^{j-M-1}|x-y|\right), \\
&\quad \text{(for } m = M \text{ a large enough value)} \\
&\leq C \sum_{j=-\infty}^{0} 2^{2j} H\left(2^{j-M-1}|x-y|\right) + C\sum_{j=1}^{M} 2^{2j} H\left(2^{j-M-1}|x-y|\right) \\
&\quad + C \sum_{j=M+1}^{\infty} 2^{2j} H\left(2^{j-M-1}|x-y|\right) \\
&= C \sum_{j=-\infty}^{\infty} 2^{2j} H\left(2^{j-M-1}|x-y|\right) \\
&\leq 2\,C \int_0^\infty t\, H\left(2^{-M-1} t\,|x-y|\right) dt \\
&= 2\,C \int_0^\infty \frac{2^{M+1}}{|x-y|}\, s\, H(s)\, \frac{2^{M+1}}{|x-y|}\, ds \\
&= \frac{2^{2M+3}}{|x-y|^2} \int_0^\infty s\, H(s)\, ds
\end{aligned}$$

This proves (3.9) and, consequently, (3.6) follows. Inequality (3.7) follows from a similar argument as in (3.6). Hence T_β is a Calderon-Zygmund operator. By using Lemma 3.2, the proof of the theorem follows.

Now, we prove the unconditionality of some wavelet packet basis for $L^p(\mathbb{R})$, $1 < p < \infty$.

Theorem 3.4. *Let ω_n be orthonormal wavelet packets such that ω_n and ω_n' have a common radial decreasing L^1-majorant H satisfying*

$$\int_0^\infty s\,H(s)\,ds \quad < \quad \infty.$$

Then, the system $\left\{\omega_{\ell,n,k} : \ell = j - m;\ n = 2^m,\ 2^m + 1, ..., 2^{m+1} - 1;\ j,\ k \in \mathbb{Z}\right\}$ is an unconitional basis for $L^p(\mathbb{R})$, $1 < p < \infty$, where $m = 0$ if $j \leq 0$ and $m = 0, 1, 2, ..., j$ if $j > 0$.

Proof. In order to prove the theorem, first of all we show that the system under consideration is a basis for $L^p(\mathbb{R})$, $1 < p < \infty$. For this let, $S_{r,s}\,f$ be the "rectangular" partial sum of the wavelet packet expansions of f, i.e.,

$$S_{r,s}\,f = \sum_{|j|<r} \sum_{n=2^m}^{2^{m+1}-1} \sum_{|k|<s} < f, \omega_{\ell,n,k} > \omega_{\ell,n,k}, \tag{3.10}$$

where $f \in L_p(\mathbb{R})$, $1 < p < \infty$. This operator is well defined in view of Theorem 3.3. Now, we show that for given $f \in L^p(\mathbb{R})$, $1 < p < \infty$ and $\varepsilon > 0$, we can find r and s large enough so that

$$\|f - S_{r,s}\,f\|_{L^p(\mathbb{R})} \quad < \quad \varepsilon.$$

Let $C = \sup \|T_\beta\| < \infty$, where T_β's are the operators defined in Theorem 3.3 and the superemum is taken over all admissible sequences $\beta = \{\beta_{\ell,n,k}\}$ considered in Theorem 3.3. Since $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, we can find $g \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ such that

$$\|f - g\|_{L^p(\mathbb{R})} \quad < \quad \frac{\varepsilon}{C+3}.$$

We can write

$$\begin{aligned} \|f - S_{r,s}\,f\|_{L^p(\mathbb{R})} &\leq \|f - g\|_{L^p(\mathbb{R})} \\ &\quad + \|g - S_{r,s}\,g\|_{L^p(\mathbb{R})} + \|S_{r,s}\,(g - f)\|_{L^p(\mathbb{R})}. \end{aligned} \tag{3.11}$$

The last summand on the right hand side of (3.11) is smaller than $\frac{\varepsilon C}{C+3}$ in view of Theorem 3.3. Now, we estimate $\|g - S_{r,s}\,g\|_{L^p(\mathbb{R})}$ for $g \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$.

By duality, and the density of $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$, (where $\frac{1}{p} + \frac{1}{q} = 1$), we can find $h \in L^2(\mathbb{R}) \cap L^q(\mathbb{R})$ such that

$$\|g - S_{r,s}\, g\|_{L^p(\mathbb{R})} \le \left| \int_{\mathbb{R}} \{g(x) - S_{r,s}\, g(x)\} \overline{h(x)}\, dx \right| + \frac{\varepsilon}{C+3}. \tag{3.12}$$

Using the Schwarz inequality, we deduce

$$\begin{aligned} \left| \int_{\mathbb{R}} \{g(x) - S_{r,s}\, g(x)\} \overline{h(x)}\, dx \right| &= \left| \int_{\mathbb{R}} g(x) \left\{ \overline{h(x)} - \overline{S_{r,s}\, h(x)} \right\} dx \right| \\ &\le \|g\|_{L^2(\mathbb{R})} < \|h - S_{r,s}\, h\|_{L^2(\mathbb{R})}. \end{aligned}$$

Since the system considered is an orthonormal basis for $L^2(\mathbb{R})$, we can find r and s large enough so that

$$\|h - S_{r,s}\, h\|_{L^2(\mathbb{R})} < \frac{\varepsilon}{\|g\|_{L^2(\mathbb{R})}(C+3)}.$$

Hence

$$\|f - S_{r,s}\, f\|_{L^p(\mathbb{R})} \le \frac{\varepsilon}{C+3} + \frac{\varepsilon}{C+3} + \frac{\varepsilon}{C+3} + \frac{\varepsilon\, C}{C+3} = \varepsilon.$$

From the orthonormality of the system under consideration, it follows that the representation

$$f = \sum_{j \in \mathbb{Z}} \sum_{n=2^m}^{2^{m+1}-1} \sum_{k \in \mathbb{Z}} C_{\ell,n,k}\ \omega_{\ell,n,k} \tag{3.13}$$

with convergence in $L^p(\mathbb{R})$, $1 < p < \infty$ is unique. Now, multiplying both the sides by $\overline{\omega_{\ell,n,k}}$ and integrating, we obtain $C_{\ell,n,k} = \langle f, \omega_{\ell,n,k} \rangle$. The unconditionality of the basis follows from Theorem 3.3 and Lemma 2.3. ■

Theorem 3.5. *Let ω_n be orthonormal wavelet packets for $L^2(\mathbb{R})$ such that ω_n and ω_n' have a common radial decreasing L^1-majorant H satisfying*

$$\int_0^\infty s\, H(s)\, ds < \infty.$$

Then, the system $\left\{\omega_{\ell,n,k} : \ \ell = j - m;\ n = 2^m,\ 2^m + 1, \ldots, 2^{m+1} - 1;\ j, k \in \mathbb{Z}\right\}$ is an unconditional basis for $\mathcal{H}^1(\mathbb{R})$, where $m = 0$ if $j \le 0$ and $m = 0, 1, 2, \ldots, j$ if $j > 0$.

Proof. The proof of this theorem is similar to that of Theorem 3.4. For this, we need to show that the system under consideration is a basis for $\mathcal{H}^1(\mathbb{R})$. Inequality

(3.11) is true with L^p-norm replaced by $\mathcal{H}^1$-norm and choosing g to be finite linear combination of atoms. Since $\mathcal{H}^*(\mathbb{R}) = BMO(\mathbb{R})$, we can find a bounded function $h \in BMO(\mathbb{R})$ such that (3.12) is true by replacing $\|.\|_{L^p(\mathbb{R})}$ by $\|.\|_{\mathcal{H}^1(\mathbb{R})}$. By choosing sufficiently large M, we have

$$\|g - S_{r,s}\, g\|_{\mathcal{H}^1(\mathbb{R})} \leq \left| \int_{\mathbb{R}} \{g(x) - S_{r,s}\, g(x)\} \mathcal{X}_{[-M,\, M]}(x) \overline{h(x)}\, dx \right| + \frac{\varepsilon}{C+3}. \tag{3.14}$$

We observe that $\mathcal{X}_{[-M,\, M]}\, h \in L^2(\mathbb{R})$ and, thus, the proof follows from the proof of Theorem 3.4. ■

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