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On Generalized Folkman Numbers

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Abstract. For graphs G, G_1 and G_2, let $G \to (G_1, G_2)$ signify that any red/blue edge-coloring of G contains a red G_1 or a blue G_2, and let $f(G_1, G_2)$ be the minimum N such that there is a graph G of order N with $\omega(G) = \max\{\omega(G_1), \omega(G_2)\}$ and $G \to (G_1, G_2)$. It is shown that $c_1(n/\log n)^{(m+1)/2} \le f(K_m, K_{n,n}) \le c_2 n^{m-1}$, where $c_i = c_i(m) > 0$ are constants. In particular, $cn^2/\log n \le f(K_3, K_{n,n}) \le 2n^2 + 2n - 1$. Moreover, $f(K_m, T_n) \le m^2(n-1)$ for all $n \ge m \ge 2$, where T_n is a tree on n vertices.

1. Introduction

For graphs G, G_1 and G_2, let $G \to (G_1, G_2)$ signify that any red/blue edge-coloring of G contains a red G_1 or a blue G_2. The Ramsey number $r(G_1, G_2)$ is the smallest N such that $K_N \to (G_1, G_2)$, for which $r(K_m, K_n)$ is written as $r(m, n)$ for short. Define

$$\mathcal{F}(G_1, G_2; p) = \{G : \omega(G) \le p, G \to (G_1, G_2)\},$$

where $\omega(G)$ is the clique number of G. We call

$$f(G_1, G_2; p) = \min\{|V(G)| : G \in \mathcal{F}(G_1, G_2; p)\}$$

Folkman number. We admit $f(G_1, G_2; p) = \infty$ if $\mathcal{F}(G_1, G_2; p) = \emptyset$. Let us write $\mathcal{F}(m, n; p)$ and $f(m, n; p)$ for $\mathcal{F}(K_m, K_n; p)$ and $f(K_m, K_n; p)$, respectively; and call $f(m, n; p)$ *classical Folkman number*, and $f(G_1, G_2; p)$ *generalized Folkman number* if one of G_1 and G_2 is non-complete. Let us point out that the above *classical Folkman number* $f(3, 3; 3)$ is always instead denoted by $f(2, 3, 4)$, see [3] for example. However, in this note, it maybe convenient to use $f(G_1, G_2; p)$ to denote the *generalized Folkman number*.

The investigation of Folkman number was motivated by a question of Erdős and Hajnal [6] who asked what was the minimum p such that $\mathcal{F}(3, 3; p) \ne \emptyset$. Folkman [9] proved that $\mathcal{F}(m, n; p) \ne \emptyset$ for $p \ge \max\{m, n\}$. Subsequently, Nešetřil and Rödl [18] generalized it by showing that $\mathcal{F}(G_1, G_2; p) \ne \emptyset$ when $p \ge \max\{\omega(G_1), \omega(G_2)\}$.

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Lemma 1.1. [9, 18] *If* $p \geq \max\{\omega(G_1), \omega(G_2)\}$, *then*

$$\mathcal{F}(G_1, G_2; p) \neq \emptyset.$$

It is easy to see that $f(G_1, G_2; p)$ is decreasing on p, and $f(G_1, G_2; p) = r(G_1, G_2)$ if $p \geq r(G_1, G_2)$. Consequently, for any p,

$$f(G_1, G_2; p) \geq r(G_1, G_2). \tag{1.1}$$

For $p = r(m, n) - 1$, Lin [13] proved that $f(m, n; p) = r(m, n) + 2$ in some cases. It is known that $f(3, 3; 5) = 8$ and $f(3, 3; 4) = 15$ due to Graham [11], Lin [13], and Piwakowski, Radziszowski and Urbanski [19], respectively.

Clearly, among all $f(G_1, G_2; p)$ with different parameters p, the crucial case is $p = \max\{\omega(G_1), \omega(G_2)\}$. So we write

$$f(G_1, G_2) = f(G_1, G_2; p), \quad \text{where } p = \max\{\omega(G_1), \omega(G_2)\}.$$

It is known that $f(3, 3) \leq 3 \times 10^9$ due to Spencer [21], which improved an upper bound 7×10^{11} of Frankl and Rödl [10]. Chung and Graham [3] conjectured that $f(3, 3) < 1000$, which was confirmed by Dudek and Rödl [5] with a computer assisted proof, and independently Lu [16] obtained that $f(3, 3) \leq 9697$. Recently, Lange, Radziszowski and Xu [12] obtained $f(3, 3) \leq 786$.

It is trivial that $f(2, n) = n$. For $n \geq m \geq 3$, the upper bounds for $f(m, n)$ and $f(G_1, G_2)$ deduced from [9] and [18] are huge. In particular, such an upper bound $g(n)$ for $f(n, n)$ or even $f(3, n)$ is a tower, whose height is larger than the value of $g(n-1)$. It is widely believed that such huge upper bounds for Folkman numbers are far away from the truth. However, for bipartite graphs B_1 and B_2, the upper bound for $f(B_1, B_2)$ is more reasonable. Let *bipartite Ramsey number* $\mathrm{br}(G_1, G_2)$ be the smallest N such that $K_{N,N} \to (B_1, B_2)$. The following relationship says that $f(B_1, B_2)$ is close to $\mathrm{br}(B_1, B_2)$:

$$\mathrm{br}(B_1, B_2) \leq f(B_1, B_2) \leq 2\,\mathrm{br}(B_1, B_2).$$

Indeed, let $N = \mathrm{br}(B_1, B_2)$. We have $f(B_1, B_2) \leq 2N$ since $K_{N,N} \in \mathcal{F}(B_1, B_2; 2)$. On the other hand, if B is a graph of order $N = f(B_1, B_2)$, then the fact that $B \to (B_1, B_2)$ implies that $K_{N,N} \to (B_1, B_2)$, and so $\mathrm{br}(B_1, B_2) \leq N = f(B_1, B_2)$.

In this paper, we have the following results.

Theorem 1.2. *For fixed* $m \geq 3$,

$$c\left(\frac{n}{\log n}\right)^{(m+1)/2} \leq f(K_m, K_{n,n}) \leq (m-1)(n^{m-1} + n - 1) + 1,$$

where $c = c(m) > 0$.

Note that $f(K_3, K_{n,n}) \geq r(K_3, K_{n,n})$, and $r(K_3, K_{n,n}) \geq cn^2/\log n$ by Lin and Li [14] which extended a method of Bohman [2], and hence we have the following result.

Corollary 1.3. *There exists a constant $c > 0$ such that*

$$\frac{cn^2}{\log n} \leq f(K_3, K_{n,n}) \leq 2n(n+1) - 1$$

for sufficiently large n.

However, there still exists a gap between the lower bound and the upper bound.

Theorem 1.4. *Let T_n be a tree of order n. If $m, n \geq 2$, then*

$$f(K_m, T_n) \leq m^2(n-1).$$

Remark 1.5. From the well-known result by Chvátal [4] that $r(K_m, T_n) = (m-1)(n-1)+1$, we have $f(K_m, T_n) \geq (m-1)(n-1)+1$ immediately. We do not know which direction is right.

2. Proofs for the main results

Let us denote by $K_m(n_1, \ldots, n_m)$ the complete m-partite graph, in which the ith part has n_i vertices. For convenience, write $K_{m,n}$ for $K_2(m, n)$ and $K_m(n)$ for $K_m(n, \ldots, n)$.

Proof of the upper bound of Theorem 1.2. The upper bound comes from the fact that $\mathcal{F}(K_m, K_{n,n}; m)$ contains a graph of order at most $(m-1)(n^{m-1}+n-1)+1$, which we shall prove. □

Lemma 2.1. *Let $m \geq 2$ and $n \geq 1$ be integers, and let $N = n^{m-1}$. Then*

$$K_m(N, \ldots, N, (m-1)(n-1)+1) \to (K_m, K_{n,n}).$$

Proof. We shall prove the lemma by induction on m. As it is trivial for $m = 2$, we assume that $m \geq 3$ and the assertion holds for $m-1$. Let $N = n^{m-1}$, and let $V_1, V_2, \ldots, V_m$ be the parts of vertex set of $K_m(N, \ldots, N, (m-1)(n-1)+1)$, where

$$|V_1| = |V_2| = \cdots = |V_{m-1}| = N, \quad |V_m| = (m-1)(n-1)+1.$$

Let (R, B) be an edge-coloring of $K_m(N, \ldots, N, (m-1)(n-1)+1)$ by red and blue. Assume that there is neither red K_m nor blue $K_{n,n}$. We will show that this assumption leads to a contradiction.

Note that $K_{m-1}(N', \ldots, N', (m-2)(n-1)+1)$ is a subgraph of $K_{m-1}(N', \ldots, N')$, where $N' = n^{m-2} \geq (m-2)(n-1)+1$ (This is a routing proof by induction on $m \geq 2$), and the inductive assumption implies that

$$K_{m-1}(N', \ldots, N') \to (K_{m-1}, K_{n,n}). \tag{2.1}$$

For each vertex v, denote by $d_r^{(i)}(v)$ and $d_b^{(i)}(v)$ the number of red-neighbors and blue-neighbors of v in V_i, respectively. If there is some vertex $v \in V_m$, such that $d_r^{(i)}(v) \geq N'$ for each i with $1 \leq i \leq m-1$, then, by (2.1), we have either a red K_{m-1} or a blue $K_{n,n}$ in the red-neighborhood of v in $V_1 \cup V_2 \cup \cdots \cup V_{m-1}$. Since there is no blue $K_{n,n}$, we have a red K_{m-1}, which together with the vertex v form a red K_m. This is impossible. Thus for each vertex v of V_m, there is some i with $1 \leq i \leq m-1$ such that $d_r^{(i)}(v) \leq N'-1$, which implies that

$$d_b^{(i)}(v) \geq N - N' + 1 \quad \text{for some } i \text{ with } 1 \leq i \leq m-1.$$

For $1 \leq i \leq m-1$, let $U_i = \left\{v \in V_m : d_b^{(i)}(v) \geq N - N' + 1\right\}$. Then $V_m = \bigcup_{i=1}^{m-1} U_i$. Since $|V_m| = (m-1)(n-1)+1$, there is some U_i, say U_1, such that $|U_1| \geq n$. Labeling these n vertices of $U_1 \subseteq V_m$ as $v_1, v_2, \ldots, v_n$. Then

$$d_b^{(1)}(v_j) \geq N - N' + 1 \quad \text{for } 1 \leq j \leq n.$$

If the number of common blue-neighbors of $v_1, v_2, \ldots, v_n$ in V_1 is at least n, then we can find a blue $K_{n,n}$. This can be seen as follows. As each v_i is blue-adjacent to all but at most $N'-1$ vertices of V_1, and v_1, v_2 are commonly blue-adjacent to at least $N-2(N'-1)$ vertices in V_1. Similarly, $v_1, v_2, \ldots, v_n$ are commonly blue-adjacent to at least $N-n(N'-1)$ vertices in V_1. Note that

$$N - n(N'-1) = n^{m-1} - n(n^{m-2}-1) = n,$$

hence we indeed obtain a blue $K_{n,n}$ and reach the desired contradiction. □

Now, let us turn to the lower bound for Theorem 1.2. In fact, this can be deduced from (1.1) and the lower bound for $r(K_m, K_{n,n})$, whose proof is similar to that for $r(m,n)$ by using Lovász local lemma, see [7, 20]. Here we shall have a slightly easier proof with a slightly better multiplicative constant. We will adopt the form of the lemma obtained by Erdős and Spencer [8], see also Alon and Spencer [1, p. 70], or Lu and Székely [17].

A graph F on $[n]$ (the set of indices for the events) is called *negative dependency graph* (see [17], which is called *lopsidependency graph* in [8]) of events $A_1, A_2, \ldots, A_n$ if for each $i \in [n]$ and any set $S \subseteq [n] \setminus N[i]$,

$$\Pr\left(A_i \,\middle|\, \bigcap_{j \in S} \overline{A}_j\right) \leq \Pr(A_i),$$

where $N[i] = N(i) \cup \{i\}$ is the closed neighborhood of i in F.

Lemma 2.2. [1, 8, 17] *Let $A_1, A_2, \ldots, A_n$ be events in a probability space $(\Omega, \Pr)$ with negative dependency graph F. If there exist $x_1, x_2, \ldots, x_n$ such that $0 < x_i < 1$ and*

$$\Pr(A_i) \le x_i \prod_{j:ij\in E(F)} (1 - x_j) \tag{2.2}$$

for each i, then $\Pr\left(\bigcap_{i=1}^n \overline{A}_i\right) > 0$.

By taking $y_i = x_i/\Pr(A_i)$, then, (2.2) is equivalent to find positive numbers $y_1, y_2, \ldots,$ y_n such that $0 < y_i \Pr(A_i) < 1$, and

$$\log y_i \ge -\sum_{j:\, ij\in E(F)} \log(1 - y_j \Pr(A_j)). \tag{2.3}$$

By using Lemma 2.2, we can see the following proof of the lower bound for $r(K_m, K_{n,n})$ is slightly simpler.

Proof of the lower bound of Theorem 1.2. Let $m \ge 3$ be fixed integer, and n a sufficiently large integer. We shall prove $r(K_m, K_{n,n}) \ge N$, where $N = N(n)$ is to be chosen. Color the edges of K_N by red and blue randomly and independently, so that each edge is colored red with probability p and blue with probability $q = 1 - p$. For subsets S with $|S| = m$, and $T = T_1 \cup T_2$ with $T_1 \cap T_2 = \emptyset$ and $|T_1| = |T_2| = n$, let A_S be the event that S spans a red K_m and B_T the event that T spans a blue $K_{n,n}$ on color classes T_1 and T_2. Then $\Pr(A_S) = p^{\binom{m}{2}}$ and $\Pr(B_T) = q^{n^2}$.

Suppose S and S' have $r \ge 2$ vertices in common. Then

$$\Pr(A_S \mid \overline{A}_{S'}) = \frac{\Pr(A_S\overline{A}_{S'})}{\Pr(\overline{A}_{S'})} = \frac{\Pr(A_S)\cdot\left(1 - p^{\binom{m}{2}-\binom{r}{2}}\right)}{1 - p^{\binom{m}{2}}} < \Pr(A_S).$$

Similarly, B_T and $\overline{B}_{T'}$ satisfy that $\Pr(B_T \mid \overline{B}_{T'}) < \Pr(B_T)$ if the corresponding subgraphs have an edge in common.

Label such events A_S and B_T as $A_1, \ldots, A_k$ and $B_1, \ldots, B_\ell$, where $k = \binom{N}{m}$ and $\ell = \binom{N}{n}\binom{N-n}{n}$. Define the graph F on those events, in which two events A_i and B_j are adjacent in F if and only if they have an edge in common. From the above observation, it is not difficult to check that F is a negative dependency graph, which is bipartite indeed.

Note that m is fixed and n is sufficiently large. In F, we have that each A-event is adjacent to $d_{AB} \le \binom{m}{2}\binom{N-2}{n-1}\binom{N-n-1}{n-1} \le N^{2(n-1)}$ B-events, and each B-event is adjacent to $d_{BA} \le n^2\binom{N-2}{m-2} \le n^2 N^{m-2}$ A-events. We shall find positive numbers a and b with

$y_i = a$ for each A event and $y_j = b$ for each B event that satisfy (2.3). Namely, $ap^{\binom{m}{2}} < 1$, $bq^{n^2} < 1$ and

$$\log a \geq -d_{AB} \log\left(1 - bq^{n^2}\right), \tag{2.4}$$

$$\log b \geq -d_{BA} \log\left(1 - ap^{\binom{m}{2}}\right). \tag{2.5}$$

If such a and b are available, then there exists a red/blue edge-coloring of K_N such that there is neither red K_m nor blue $K_{n,n}$, implying $f(K_m, K_{n,n}) > N$. To this end, let us set $a = 2$,

$$p = \frac{(m+3)\log n}{n}, \quad b = \exp\{n \log n\}, \quad N = \left(\frac{cn}{\log n}\right)^{(m+1)/2},$$

where $c = c(m) > 0$ is a constant to be determined. Using $q = 1 - p < e^{-p}$ for $p > 0$, we have

$$\begin{aligned} N^{2n} bq^{n^2} &\leq N^{2n} be^{-pn^2} = \exp\left\{2n \log N + \log b - pn^2\right\} \\ &\leq \exp\{-n \log n\} \to 0. \end{aligned}$$

So $\log(1 - x) \sim -x$ for $x = bq^{n^2}$, and the right-hand side of (2.4) is

$$-N^{2(n-1)} \log(1 - bq^{n^2}) \sim N^{2(n-1)} bq^{n^2} \to 0.$$

Thus (2.4) holds for all large n. Finally, note that the right-hand side of (2.5) is asymptotically

$$n^2 N^{m-2} ap^{\binom{m}{2}} = 2c^{(m+1)(m-2)/2}(m+3)^{\binom{m}{2}} n \log n.$$

So (2.5) holds if we choose $c > 0$ such that

$$1 > 2c^{(m+1)(m-2)/2}(m+3)^{\binom{m}{2}}.$$

This completes the proof. $\square$

In the following, we will give a proof of the upper bound for $f(K_m, T_n)$. First, we define a special Turán number. For integers $k \geq 1$ and $r \geq 2$, let $t_r(k)$ be the maximum number of edges of a subgraph of $K_r(k)$ that contains no K_r. Clearly, $t_2(k) = 0$ and $t_r(1) = \binom{r}{2} - 1$. One can find the following result in [15], we include the proof here for completeness.

Lemma 2.3. *Let $t_r(k)$ be defined as above. Then*

$$t_r(k) = \left[\binom{r}{2} - 1\right] k^2.$$

Proof. The lower bound for $t_r(k)$ follows by deleting all edges between a pair of color classes of $K_r(k)$. On the other hand, we shall prove by induction on k that if a subgraph $G = G(V^{(1)}, \ldots, V^{(r)})$ of $K_r(k)$ contains no K_r, then $e(G) \leq \left[\binom{r}{2} - 1\right] k^2$. Suppose $k \geq 2$ and $r \geq 3$ as it is trivial for $k = 1$ or $r = 2$. Now, suppose that G has the maximum possible number of edges subject to this condition. Then G must contain $K_r - e$ as a subgraph, otherwise we could add an edge and the resulting graph would still contain no K_r. Denote the vertex set of this $K_r - e$ by X. We have $\left|X \cap V^{(i)}\right| = 1$ for $i = 1, 2, \ldots, r$. Without loss of generality, suppose $e = \{v_1, v_2\}$, where $v_1 \in V^{(1)}$ and $v_2 \in V^{(2)}$. Let G' be the r-partite subgraph of G that induced by $\left(\bigcup_{i=1}^{r} V^{(i)}\right) \setminus X$. Clearly, G' contains no K_r as a subgraph since G contains no K_r. Hence, from the induction hypothesis, we have $e(G') \leq \left[\binom{r}{2} - 1\right] (k-1)^2$. Moreover, since G contains no K_r, we have that for $i = 1, 2$ there is no vertex in $V^{(i)} \setminus \{v_i\}$ is adjacent to all the vertices of $X \setminus \{v_i\}$. Thus, there are at least

$$(k-1)^2 + 2(k-1) + 1 = k^2$$

edges that should be deleted from $K_r(k)$, which completes the induction step and hence the proof. □

Proof of Theorem 1.4. Consider a red-blue edge-coloring of $K_m(N)$, where $N = m(n-1)$. Let R and B be the subgraphs induced by red edges and blue edges, respectively. Assume that R contains no K_m. Then $e(R) < t_m(N) = \frac{(m+1)(m-2)}{2} N^2$ by Lemma 2.3, and hence

$$e(B) = \binom{m}{2} N^2 - e(R) > N^2 = \frac{N}{m}(mN) = (n-1)(mN).$$

Note that each graph F of order k with at least $(\ell - 1)k$ edges contains T_ℓ as a subgraph. (Indeed, F contains a subgraph F' with minimum degree at least $\ell - 1$. Thus, F' and hence F contains any T_ℓ as a subgraph.) Therefore, B contains T_n as a subgraph as claimed. □

Finally, let us propose the following problem.

Problem 2.4. Prove or disprove that the asymptotic order of $f(K_3, K_{n,n})$ is $n^2/\log n$.

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