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L_k-2-TYPE HYPERSURFACES IN HYPERBOLIC SPACES

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Abstract. In this article, we study L_k-finite-type hypersurfaces M^n of a hyperbolic space $\mathbb{H}^{n+1} \subset \mathbb{R}_1^{n+2}$, for $k \geq 1$. In the 3-dimensional case, we obtain the following classification result. Let $\psi : M^3 \to \mathbb{H}^4 \subset \mathbb{R}_1^5$ be an orientable hypersurface with constant k-th mean curvature H_k, which is not totally umbilical. Then M^3 is of L_k-2-type if and only if M^3 is an open portion of a standard Riemannian product $\mathbb{H}^1(r_1) \times \mathbb{S}^2(r_2)$ or $\mathbb{H}^2(r_1) \times \mathbb{S}^1(r_2)$, with $-r_1^2 + r_2^2 = -1$. In the n-dimensional case, we show that a hypersurface $M^n \subset \mathbb{H}^{n+1}$, with constant k-th mean curvature H_k and having at most two distinct principal curvatures, is of L_k-2-type if and only if M^n is an open portion of a Riemannian product $\mathbb{H}^m(r_1) \times \mathbb{S}^{n-m}(r_2)$, with $-r_1^2 + r_2^2 = -1$, for some integer $m \in \{1, \ldots, n-1\}$. In the case $k = n-1$ we drop the condition on the principal curvatures of the hypersurface M^n, and prove that if $M^n \subset \mathbb{H}^{n+1}$ is an orientable H_{n-1}-hypersurface of L_{n-1}-2-type then its Gauss-Kronecker curvature H_n is a nonzero constant.

1. INTRODUCTION

Submanifolds of finite type were introduced by B.Y. Chen, whose first results were gathered in his book [7] (see also [8]). Although the first definition was given for a compact submanifold in the Euclidean space, Chen extended the concept to non-compact submanifolds in Euclidean or pseudo-Euclidean spaces, [9, 10]. A detailed survey of the results on this subject, up to 1996, was given by Chen in [14], and in a recent article [15], the author provides a detailed account of recent development on the problems and conjectures about finite type submanifolds.

The Laplacian operator Δ can be seen as the first one of a sequence of n operators $L_0 = \Delta, L_1, \ldots, L_{n-1}$, where L_k stands for the linearized operator of the first variation

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of the $(k+1)$-th mean curvature arising from normal variations of the hypersurface (see, for instance, [24]). These operators L_k are given by $L_k(f) = \text{tr}(P_k \circ \nabla^2 f)$ for a smooth function f on M, where P_k denotes the k-th Newton transformation associated to the second fundamental form of the hypersurface and $\nabla^2 f$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f.

As an extension of finite type theory, S.M.B. Kashani [17] introduced the notion of L_k-finite-type hypersurface in the Euclidean space. In general, a submanifold M^n in $\mathbb{R}^m$ is said to be of L_k-finite-type if the position vector $\psi : M^n \to \mathbb{R}^m$ of M^n into $\mathbb{R}^m$ admits the following finite spectral decomposition

$$\psi = a + \psi_1 + \cdots + \psi_q, \quad L_k\psi_t = \lambda_t\psi_t,$$

where a is a constant vector, λ_t are constants and ψ_t are non-constant $\mathbb{R}^m$-valued maps on M^n. If all λ_t's are mutually different, M^n is said to be of L_k-q-type, and if one of λ_t is zero M^n is said to be of L_k-null-q-type. Naturally, that definition is also valid for a pseudo-Riemannian submanifold M_t^n into the pseudo-Euclidean space $\mathbb{R}_s^m$.

In [21], the authors, by using results from [1], show that k-minimal Euclidean hypersurfaces and open portions of hyperspheres are the only L_k-1-type hypersurfaces in $\mathbb{R}^{n+1}$. As for hypersurfaces of L_k-2-type in $\mathbb{R}^{n+1}$, the authors show that if M^n is a hypersurface with at most two distinct principal curvatures, then (i) M^n is not of L_{n-1}-null-2-type (Theorem 3.5); and (ii) M^n is of L_k-null-2-type ($k \neq n-1$) if and only if M is locally isometric to a generalized cylinder (Theorems 3.11 and 3.12).

In [20], the authors study L_k-2-type hypersurfaces in a hypersphere $\mathbb{S}^4 \subset \mathbb{R}^5$. Since the case $k = 0$ corresponds to the classical one, which has been well studied (see, e.g., [11], [12] and [16], among others), the authors concentrate in cases $k = 1$ and $k = 2$, and show the following result:

Theorem A. *Let $\psi : M^3 \to \mathbb{S}^4 \subset \mathbb{R}^5$ be an orientable H_k-hypersurface, which is not an open portion of a hypersphere. Then M^3 is of L_k-2-type if and only if M^3 is a Clifford tori $\mathbb{S}^1(r_1) \times \mathbb{S}^2(r_2)$, $r_1^2 + r_2^2 = 1$, for appropriate radii, or a tube $T^r(V^2)$ of appropriate constant radius r around the Veronese embedding V^2 of the real projective plane $\mathbb{R}P^2(\sqrt{3})$.*

In this paper we extend this result to hypersurfaces in a hyperbolic space. The case $k = 0$ was studied by Chen, [13], in the n-dimensional case. He proved (i) that every 2-type hypersurface of the hyperbolic space has nonzero constant mean curvature and constant scalar curvature, and (ii) that there exists no compact 2-type hypersurfaces in the hyperbolic space.

After a section devoted to preliminaries and basic results we proceed, in the third section, to compute some formulae required to present the examples. In section 4 we present the main results in dimension three, which we can gather in the following classification theorem:

Theorem B. *Let $\psi : M^3 \to \mathbb{H}^4 \subset \mathbb{R}^5_1$ be an orientable H_k-hypersurface, which is not totally umbilical. Then M^3 is of L_k-2-type if and only if M^3 is a standard Riemannian product $\mathbb{H}^1(r_1) \times \mathbb{S}^2(r_2)$ or $\mathbb{H}^2(r_1) \times \mathbb{S}^1(r_2)$, with $-r_1^2 + r_2^2 = -1$.*

In the final section, we extend the previous result to n-dimensional hypersurfaces in the hyperbolic space $\mathbb{H}^{n+1}$ as follows.

Theorem C. *Let $\psi : M^n \to \mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}_1$ be an orientable H_k-hypersurface and assume that M^n has at most two distinct principal curvatures. Then M^n is of L_k-2-type if and only if M^n is an open portion of $\mathbb{H}^m(-\sqrt{1+r^2}) \times \mathbb{S}^{n-m}(r)$, for some positive integer m, $1 \leq m \leq n-1$, and for some positive number r.*

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2. Preliminaries and Lemma

Let $\mathbb{R}^5_1$ be the 5-dimensional Lorentzian space with the standard flat metric g given by

$$g = -\mathrm{d}x_1^2 + \sum_{i=2}^{5} \mathrm{d}x_i^2,$$

where $(x_1, \ldots, x_5)$ is a rectangular coordinate system of $\mathbb{R}^5_1$. For a positive number r and a point $c \in \mathbb{R}^5_1$ we denote by $\mathbb{H}^4(c, -r)$ the (connected) hyperbolic space centered at c with radius r, which is embedded standardly in $\mathbb{R}^5_1$ by

$$\mathbb{H}^4(c, -r) = \left\{x \in \mathbb{R}^5_1 \,|\, \langle x - c, x - c\rangle = -r^2, \text{ and } x_1 > 0\right\},$$

where $\langle , \rangle$ denotes the Lorentzian inner product on $\mathbb{R}^5_1$. To simplify the notation, we write $\mathbb{H}^4(-r) \equiv \mathbb{H}^4(0, -r)$ and $\mathbb{H}^4 \equiv \mathbb{H}^4(0, -1)$. We will also use $\langle , \rangle$ to denote the flat metric g. Without loss of generality, we assume that $c = 0$ and $r = 1$.

Let $\psi : M^3 \to \mathbb{H}^4 \subset \mathbb{R}^5_1$ be an isometric immersion of a connected orientable hypersurface M^3 with Gauss map N. We denote by ∇^0, $\overline{\nabla}$ and ∇ the Levi-Civita connections on $\mathbb{R}^5_1$, $\mathbb{H}^4$ and M^3, respectively. Then the Gauss and Weingarten formulae are given by [22]

$$\nabla^0_X Y = \nabla_X Y + \langle SX, Y\rangle N + \langle X, Y\rangle \psi,$$

$$SX = -\overline{\nabla}_X N = -\nabla^0_X N,$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M^3)$, where $S : \mathfrak{X}(M^3) \longrightarrow \mathfrak{X}(M^3)$ stands for the shape operator (or Weingarten endomorphism) of M^3, with respect to the chosen orientation N.

As is well known, for every point $p \in M^3$, S defines a linear self-adjoint endomorphism on the tangent space T_pM^3, and its eigenvalues $\kappa_1(p), \kappa_2(p)$ and $\kappa_3(p)$ are the principal curvatures of the hypersurface. The characteristic polynomial $Q_S(t)$ of S is defined by

$$Q_S(t) = \det(tI - S) = (t-\kappa_1)(t-\kappa_2)(t-\kappa_3) = t^3 + a_1t^2 + a_2t + a_3,$$

where the coefficients of $Q_S(t)$ are given by

$$a_1 = -(\kappa_1+\kappa_2+\kappa_3), \qquad a_2 = \kappa_1\kappa_2+\kappa_1\kappa_3+\kappa_2\kappa_3, \qquad a_3 = -\kappa_1\kappa_2\kappa_3.$$

These coefficients can be expressed in terms of the traces of S^j as follows:

$$\begin{aligned} a_1 &= -\mathrm{tr}(S), \\ a_2 &= -\frac{1}{2}\mathrm{tr}(S^2) + \frac{1}{2}\mathrm{tr}(S)^2, \\ a_3 &= -\frac{1}{3}\mathrm{tr}(S^3) + \frac{1}{2}\mathrm{tr}(S^2)\mathrm{tr}(S) - \frac{1}{6}\mathrm{tr}(S)^3. \end{aligned} \tag{1}$$

The *k-th mean curvature* H_k or *mean curvature of order* k of M^3 in $\mathbb{H}^4$ is defined by

$$\binom{3}{k}H_k = (-1)^k a_k, \quad \text{with } H_0 = 1.$$

We say that M^3 is an H_k-hypersurface if its k-th mean curvature H_k is constant. If $H_{k+1} = 0$, we then say that M^3 is a k-minimal hypersurface; a 0-minimal hypersurface is nothing but a minimal hypersurface in $\mathbb{H}^4$.

The k-th Newton transformation of M^3 is the operator $P_k : \mathfrak{X}(M^3) \to \mathfrak{X}(M^3)$ defined by

$$P_k = \sum_{j=0}^{k}(-1)^j\binom{3}{k-j}H_{k-j}S^j = (-1)^k\sum_{j=0}^{k}a_{k-j}S^j.$$

In particular,

$$P_0 = I, \quad P_1 = 3HI - S, \quad P_2 = 3H_2I - S\circ P_1, \quad P_3 = H_3I - S\circ P_2. \tag{2}$$

Note that by Cayley-Hamilton theorem we have $P_3 = 0$. Let us recall that, for every point $p \in M^3$, each $P_k(p)$ is also a self-adjoint linear operator on the tangent hyperplane T_pM which commutes with $S(p)$. Indeed, $S(p)$ and $P_k(p)$ can be simultaneously diagonalized: if $\{e_1, e_2, e_3\}$ are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\kappa_1(p), \kappa_2(p), \kappa_3(p)$, respectively, then they are also the eigenvectors of $P_k(p)$ with corresponding eigenvalues given by

$$\mu_k^i(p) = \sum_{\substack{i_1<\cdots<i_k \\ i_j\notin i}}^{3} \kappa_{i_1}\cdots\kappa_{i_k}, \qquad \text{for every } i = 1,2,3 \text{ and } k = 1,2.$$

In particular,

$$\mu_1^1 = \kappa_2 + \kappa_3, \qquad \mu_1^2 = \kappa_1 + \kappa_3, \qquad \mu_1^3 = \kappa_1 + \kappa_2,$$
$$\mu_2^1 = \kappa_2\kappa_3, \qquad \mu_2^2 = \kappa_1\kappa_3, \qquad \mu_2^3 = \kappa_1\kappa_2.$$

According to [22, p. 86], the divergence of a vector field X is the differentiable function defined as the contraction of the operator ∇X, where $\nabla X(Y) := \nabla_Y X$, that is,

$$\mathrm{div}(X) = C(\nabla X) = \mathrm{tr}(\nabla X) = \sum_{i,j} g^{ij} \langle \nabla_{E_i} X, E_j \rangle,$$

$\{E_i\}$ being any local frame of tangent vectors fields, where (g^{ij}) represents the inverse of the metric $(g_{ij}) = (\langle E_i, E_j \rangle)$. For an operator $T : \mathfrak{X}(M^3) \longrightarrow \mathfrak{X}(M^3)$ we have two divergences: one associated to the (1,1)-contraction C_1^1, and another associated to the metric contraction C_{12}; the first contraction produces a 1-form and the second contraction produces a vector field. We consider here the second one, so that the divergence of an operator T will be the vector field $\mathrm{div}(T) \in \mathfrak{X}(M^3)$ defined as

$$\mathrm{div}(T) = C_{12}(\nabla T) = \sum_{i,j} g^{ij} (\nabla_{E_i} T) E_j,$$

where $\nabla T(X, Y) = (\nabla_X T)Y = \nabla_X(TY) - T(\nabla_X Y)$.

In the following lemma (see [19] for details) we present some interesting properties of the Newton transformations. The proof of the first four is merely algebraic and straightforward.

Lemma 1. *The Newton transformations P_k, $k = 1, 2$, satisfy the following properties:*

(a) $\mathrm{tr}(P_k) = c_k H_k$,

(b) $\mathrm{tr}(S \circ P_k) = c_k H_{k+1}$,

(c) $\mathrm{tr}(S^2 \circ P_1) = 9HH_2 - 3H_3$,

(d) $\mathrm{tr}(S^2 \circ P_2) = 3HH_3$,

(e) $\mathrm{tr}(\nabla_X S \circ P_k) = \binom{3}{k+1} \langle \nabla H_{k+1}, X \rangle$,

(f) $\mathrm{div}(P_k) = 0$,

where $c_1 = 6$ and $c_2 = 3$.

Keeping in mind this lemma we obtain

$$\mathrm{div}(P_k(\nabla f)) = \mathrm{tr}\big(P_k \circ \nabla^2 f\big),$$

where $\nabla^2 f : \mathfrak{X}(M^3) \longrightarrow \mathfrak{X}(M^3)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f, given by $\big\langle \nabla^2 f(X), Y \big\rangle = \langle \nabla_X(\nabla f), Y \rangle$, for vector fields $X, Y \in \mathfrak{X}(M^3)$. Associated to each Newton transformation P_k, we can define the second-order linear differential operator $L_k : \mathcal{C}^\infty(M^3) \longrightarrow \mathcal{C}^\infty(M^3)$ by

$$L_k(f) = \mathrm{tr}\big(P_k \circ \nabla^2 f\big). \tag{3}$$

An interesting property of L_k is the following:

$$L_k(fg) = gL_k(f) + fL_k(g) + 2\,\langle P_k(\nabla f), \nabla g\rangle\,, \tag{4}$$

for every couple of differentiable functions $f, g \in C^\infty(M^3)$.

3. First Formulas and Examples

First we will calculate L_k acting on the coordinate components of the immersion ψ, that is, a function given by $\langle \psi, e\rangle$, where $e \in \mathbb{R}^5_1$ is an arbitrary fixed vector. An easy computation shows that

$$\nabla\,\langle \psi, e\rangle = e^\top = e - \langle N, e\rangle\, N + \langle \psi, e\rangle\, \psi, \tag{5}$$

where $e^\top \in \mathfrak{X}(M^3)$ denotes the tangential component of e. Taking covariant derivative in (5), and using the Gauss and Weingarten formulae, we obtain

$$\nabla_X \nabla\,\langle \psi, e\rangle = \nabla_X e^\top = \langle N, e\rangle\, SX + \langle \psi, e\rangle\, X, \tag{6}$$

for every vector field $X \in \mathfrak{X}(M^3)$. Finally, by using (3) and Lemma 1, we obtain

$$L_k\,\langle \psi, e\rangle = c_k H_{k+1}\,\langle N, e\rangle + c_k H_k\,\langle \psi, e\rangle\,. \tag{7}$$

This expression allows us to extend operator L_k to vector functions $F = (f_1, \ldots, f_5)$, $f_i \in \mathcal{C}^\infty(M^3)$, as follows: $L_k F := \big(L_k f_1, \ldots, L_k f_5\big)$. Then $L_k\psi$ can be computed as

$$L_k\psi = c_k H_{k+1} N + c_k H_k \psi, \tag{8}$$

where $\{e_1, \ldots, e_5\}$ stands for an orthonormal basis in $\mathbb{R}^5_1$.

Now, we will compute $L_k N$, and in order to do that we are going to compute the operator L_k acting on the coordinate functions of the Gauss map N, that is, the functions $\langle N, e\rangle$ where $e \in \mathbb{R}^5_1$ is an arbitrary fixed vector. A straightforward computation yields

$$\nabla\,\langle N, e\rangle = -Se^\top,$$

that jointly with the Weingarten formula and (6), leads to

$$\nabla_X \nabla\,\langle N, e\rangle = -(\nabla_{e^\top} S)X - \langle N, e\rangle\, S^2 X - \langle \psi, e\rangle\, SX,$$

for every tangent vector field X. This equation, combined with (3) and Lemma 1, yields

$$\tag{9} \begin{aligned} L_k \langle N, e\rangle &= -\mathrm{tr}(\nabla_{e^\top} S \circ P_k) - \langle N, e\rangle \,\mathrm{tr}(S^2 \circ P_k) - \langle \psi, e\rangle \,\mathrm{tr}(S \circ P_k) \\ &= -\tbinom{3}{k+1} \langle \nabla H_{k+1}, e\rangle - \mathrm{tr}(S^2 \circ P_k) \langle N, e\rangle - c_k H_{k+1} \langle \psi, e\rangle, \end{aligned}$$

which is equivalent to

$$L_k N = -\tbinom{3}{k+1} \nabla H_{k+1} - \mathrm{tr}(S^2 \circ P_k) N - c_k H_{k+1} \psi.$$

On the other hand, equations (4) and (7) lead to

$$\begin{aligned} L_k^2 \langle \psi, e\rangle = c_k H_{k+1} L_k \langle N, e\rangle + L_k(c_k H_{k+1}) \langle N, e\rangle + 2c_k \big\langle P_k(\nabla H_{k+1}), \nabla \langle N, e\rangle \big\rangle \\ + c_k H_k L_k \langle \psi, e\rangle + L_k(c_k H_k) \langle \psi, e\rangle + 2c_k \big\langle P_k(\nabla H_k), \nabla \langle \psi, e\rangle \big\rangle, \end{aligned}$$

and by using again (7) and (9) we get

$$\begin{aligned} & L_k^2 \langle \psi, e\rangle \\ &= -c_k \tbinom{3}{k+1} H_{k+1} \langle \nabla H_{k+1}, e\rangle - 2c_k \langle (S \circ P_k)(\nabla H_{k+1}), e\rangle + 2c_k \langle P_k(\nabla H_k), e\rangle \\ &\quad + \Big[c_k L_k(H_{k+1}) - \big(\mathrm{tr}(S^2 \circ P_k) - c_k H_k\big) c_k H_{k+1}\Big] \langle N, e\rangle \\ &\quad + \Big[- c_k^2 H_{k+1}^2 + c_k^2 H_k^2 + c_k L_k(H_k)\Big] \langle \psi, e\rangle . \end{aligned}$$

Finally, we obtain

$$\tag{10} \begin{aligned} L_k^2 \psi &= -\frac{c_k}{2} \tbinom{3}{k+1} \nabla H_{k+1}^2 - 2c_k (S \circ P_k)(\nabla H_{k+1}) + 2c_k P_k(\nabla H_k) \\ &\quad + \Big[c_k L_k(H_{k+1}) - \big(\mathrm{tr}(S^2 \circ P_k) - c_k H_k\big) c_k H_{k+1}\Big] N \\ &\quad + \Big[- c_k^2 H_{k+1}^2 + c_k^2 H_k^2 + c_k L_k(H_k)\Big] \psi. \end{aligned}$$

Example 1. k-minimal H_k-hypersurfaces in $\mathbb{H}^4$ are of L_k-1-type or L_k-null-1-type. In fact, from (8) we obtain that $L_k\psi = \lambda\psi$, with $\lambda = c_k H_k$, and then M^3 is of L_k-1-type if $H_k \neq 0$; otherwise, M^3 is of L_k-null-1-type.

Example 2. Non-flat totally umbilical hypersurfaces in $\mathbb{H}^4$ are of L_k-1-type. As is well known, totally umbilical hypersurfaces in $\mathbb{H}^4$ are obtained as the intersection of $\mathbb{H}^4$ with a hyperplane of $\mathbb{R}^5_1$, and the causal character of the hyperplane determines the type of the hypersurface. More precisely, let $a \in \mathbb{R}^5_1$ be a non-zero constant vector with $\langle a, a\rangle \in \{1, 0, -1\}$, and take the differentiable function $f_a : \mathbb{H}^4 \subset \mathbb{R}^5_1 \to \mathbb{R}$ defined by

$f_a(x) = \langle x, a\rangle$. It is not difficult to see that for every $\tau \in \mathbb{R}$ with $\langle a, a\rangle + \tau^2 = \delta^2 > 0$, the set

$$M_\tau = f_a^{-1}(\tau) = \{x \in \mathbb{H}^4 \mid \langle x, a\rangle = \tau\}$$

is a totally umbilical hypersurface in $\mathbb{H}^4$, with Gauss map

$$N(x) = \frac{1}{\delta}\,(a + \tau x),$$

and shape operator

$$SX = -\frac{\tau}{\delta}X. \tag{11}$$

It is easy to see, from (11), that M_τ has constant mean curvature $H = -\tau/\delta$ and constant Gauss-Kronecker curvature $K = -1 + H^2 = -\langle a, a\rangle\,\delta^{-2}$. Therefore, H_k and H_{k+1} are also nonzero constants.

Now we will consider all different possibilities:

(i) If $\langle a, a\rangle = -1$, then $|\tau| > 1$, $K = 1/(\tau^2 - 1)$ is positive, and $M_\tau \equiv \mathbb{S}^3(\sqrt{\tau^2 - 1})$.

(ii) If $\langle a, a\rangle = 0$, then $\tau \neq 0$, $K = 0$, and $M_\tau \equiv \mathbb{R}^3$.

(iii) If $\langle a, a\rangle = 1$, then $K = -1/(\tau^2 + 1)$ is negative, and $M_\tau \equiv \mathbb{H}^3(-\sqrt{\tau^2 + 1})$.

Bearing (8) in mind we find that $L_k\psi = \lambda\psi + b$, where $\lambda = c_k H^k(1 - H^2)$ and $b = c_k H^{k+1}\delta^{-1}a$. We distinguish three cases:

(i) If $H = 0$, then M^3 is of L_k-null-1-type.

(ii) If $|H| = 1$, then $\langle a, a\rangle = 0$ and M^3 is flat.

(iii) Otherwise, $\lambda \neq 0$ and we can write

$$\psi = \psi_0 + \psi_1, \quad \psi_0 = -\frac{b}{\lambda} \quad \text{and} \quad \psi_1 = \psi + \frac{b}{\lambda},$$

where ψ_0 is constant and $L_k\psi_1 = \lambda\psi_1$. Therefore, M^3 is L_k-1-type in $\mathbb{R}^5_1$.

The following proposition shows that the hypersurfaces exhibited in Examples 1 and 2 are the only hypersurfaces in $\mathbb{H}^4$ of L_k-1-type in $\mathbb{R}^5_1$.

Proposition 2. *k-minimal H_k-hypersurfaces in $\mathbb{H}^4$ and open portions of a non-flat totally umbilical hypersurface in $\mathbb{H}^4$ are the only L_k-1-type hypersurfaces in $\mathbb{H}^4$.*

Proof. Let M^3 be a L_k-1-type hypersurface in $\mathbb{H}^4$, then its position vector ψ can be put as $\psi = \psi_0 + \psi_1$, where ψ_0 is a constant vector and $L_k\psi_1 = \lambda\psi_1$. Hence we deduce $L_k\psi = \lambda\psi + b$, with $b = -\lambda\psi_0$. From (8) we get

$$b = c_k H_{k+1} N + (c_k H_k - \lambda)\psi,$$

and taking covariant derivative here we obtain

$$0 = -c_k H_{k+1} SX + (c_k H_k - \lambda)X + c_k X(H_{k+1})N + c_k X(H_k)\psi,$$

for every vector field $X \in \mathfrak{X}(M^3)$. The previous equation implies that H_k and H_{k+1} are both constant. If $H_{k+1} \neq 0$ then we get $SX = \mu X$, for a certain constant μ, i.e. M^3 is totally umbilical, and then the result follows from Example 2. ■

Example 3. Standard Riemannian products $\mathbb{H}^1(r_1) \times \mathbb{S}^2(r_2)$ and $\mathbb{H}^2(r_1) \times \mathbb{S}^1(r_2)$, with $-r_1^2 + r_2^2 = -1$, are hypersurfaces in $\mathbb{H}^4$ of L_k-2-type in $\mathbb{R}^5_1$.

For a positive number r, let us denote $M^3_m(r) = \mathbb{H}^m(-\sqrt{1+r^2}) \times \mathbb{S}^{3-m}(r) \subset \mathbb{H}^4$, $m = 1, 2$. In the case $m = 1$, observe that the hypersurface $M^3_1(r)$ is defined by the equation

$$M^3_1(r) = \{x \in \mathbb{H}^4 \mid x_3^2 + x_4^2 + x_5^2 = r^2\},$$

and its Gauss map is given by

$$N(x) = \left(\frac{r}{\sqrt{1+r^2}} x_1, \frac{r}{\sqrt{1+r^2}} x_2, \frac{\sqrt{1+r^2}}{r} x_3, \frac{\sqrt{1+r^2}}{r} x_4, \frac{\sqrt{1+r^2}}{r} x_5 \right).$$

Then its principal curvatures in $\mathbb{H}^4$ are

$$\kappa_1 = \frac{-r}{\sqrt{1+r^2}} \quad \text{and} \quad \kappa_2 = \kappa_3 = \frac{-\sqrt{1+r^2}}{r}.$$

Hence we get

$$H_1 = -\frac{2+3r^2}{3r\sqrt{1+r^2}}, \qquad H_2 = \frac{1+3r^2}{3r^2}, \qquad H_3 = -\frac{\sqrt{1+r^2}}{r}.$$

If we put $\psi_1 = (x_1, x_2, 0, 0, 0)$ and $\psi_2 = (0, 0, x_3, x_4, x_5)$, then $\psi = \psi_1 + \psi_2$ and by using (8) we obtain:

(a) $L_0\psi_1 = \lambda_1\psi_1$ and $L_0\psi_2 = \lambda_2\psi_2$, where $\lambda_1 = \frac{1}{1+r^2}$ and $\lambda_2 = -\frac{2}{r^2}$. Therefore, $M^3_1(r)$ is of L_0-2-type in $\mathbb{R}^5_1$ for any r (see [11, Example 1]).

(b) $L_1\psi_1 = \lambda_1\psi_1$ and $L_1\psi_2 = \lambda_2\psi_2$, where $\lambda_1 = -\frac{2}{r\sqrt{1+r^2}}$ and $\lambda_2 = \frac{2(1+2r^2)}{r^3\sqrt{1+r^2}}$. Therefore, $M^3_1(r)$ is of L_1-2-type in $\mathbb{R}^5_1$ for any r.

(c) $L_2\psi_1 = \lambda_1\psi_1$ and $L_2\psi_2 = \lambda_2\psi_2$, where $\lambda_1 = \frac{1}{r^2}$ and $\lambda_2 = -\frac{2}{r^2}$. Therefore, $M^3_1(r)$ is of L_2-2-type in $\mathbb{R}^5_1$ for any r.

In the case $m = 2$, note that the hypersurface $M^3_2(r)$ is defined by the equation

$$M^3_2(r) = \{x \in \mathbb{H}^4 \mid x_4^2 + x_5^2 = r^2\}.$$

In this case, the Gauss map on $M_2^3(r)$ in $\mathbb{H}^4$ is given by

$$N(x) = \left(\frac{r}{\sqrt{1+r^2}}\, x_1, \frac{r}{\sqrt{1+r^2}}\, x_2, \frac{r}{\sqrt{1+r^2}}\, x_3, \frac{\sqrt{1+r^2}}{r}\, x_4, \frac{\sqrt{1+r^2}}{r}\, x_5 \right),$$

and its principal curvatures in $\mathbb{H}^4$ are

$$\kappa_1 = \kappa_2 = \frac{-r}{\sqrt{1+r^2}} \quad \text{and} \quad \kappa_3 = \frac{-\sqrt{1+r^2}}{r}.$$

Consequently, we get

$$H_1 = -\frac{1+3r^2}{3r\sqrt{1+r^2}}, \qquad H_2 = \frac{2+3r^2}{3(1+r^2)}, \qquad H_3 = -\frac{r}{\sqrt{1+r^2}}.$$

If we put as before $\psi_1 = (x_1, x_2, x_3, 0, 0)$ and $\psi_2 = (0, 0, 0, x_4, x_5)$, then $\psi = \psi_1 + \psi_2$ and by using (8) we obtain:

(a) $L_0\psi_1 = \lambda_1\psi_1$ and $L_0\psi_2 = \lambda_2\psi_2$, where $\lambda_1 = \frac{2}{1+r^2}$ and $\lambda_2 = -\frac{1}{r^2}$. Therefore, $M_2^3(r)$ is of L_0-2-type in $\mathbb{R}_1^5$ for any r (see [11, Example 1]).

(b) $L_1\psi_1 = \lambda_1\psi_1$ and $L_1\psi_2 = \lambda_2\psi_2$, where $\lambda_1 = -\frac{2(1+2r^2)}{r(1+r^2)^{3/2}}$ and $\lambda_2 = \frac{2}{r\sqrt{1+r^2}}$. Therefore, $M_2^3(r)$ is of L_1-2-type in $\mathbb{R}_1^5$ for any r.

(c) $L_2\psi_1 = \lambda_1\psi_1$ and $L_2\psi_2 = \lambda_2\psi_2$, where $\lambda_1 = \frac{2}{1+r^2}$ and $\lambda_2 = -\frac{1}{1+r^2}$. Therefore, $M_2^3(r)$ is of L_2-2-type in $\mathbb{R}_1^5$ for any r.

4. The Three-dimensional Case

Let us suppose that a hypersurface M^3 in $\mathbb{H}^4$ is of L_k-2-type in $\mathbb{R}_1^5$, that is, its position vector ψ can be written as follows

$$\psi = a + \psi_1 + \psi_2, \qquad L_k\psi_1 = \lambda_1\psi_1, \qquad L_k\psi_2 = \lambda_2\psi_2,$$

where a is a constant vector in $\mathbb{R}_1^5$ and ψ_1, ψ_2 are $\mathbb{R}_1^5$-valued non-constant differentiable functions defined on M^3.

It is easy to see that $L_k\psi = \lambda_1\psi_1 + \lambda_2\psi_2$ and $L_k^2\psi = \lambda_1^2\psi_1 + \lambda_2^2\psi_2$, and thus

$$L_k^2\psi = (\lambda_1 + \lambda_2)L_k\psi - \lambda_1\lambda_2(\psi - a).$$

By using (8) we get

$$\begin{aligned} L_k^2\psi = \lambda_1\lambda_2 a^\top &+ \big[(\lambda_1+\lambda_2)c_k H_{k+1} + \lambda_1\lambda_2 \langle N, a\rangle\big] N \\ &+ \big[(\lambda_1+\lambda_2)c_k H_k - \lambda_1\lambda_2 - \lambda_1\lambda_2 \langle \psi, a\rangle\big]\psi, \end{aligned}$$

that, jointly with (10), yields the following equations of L_k-2-type,

$$\lambda_1\lambda_2 a^\top = -\frac{c_k}{2}\binom{3}{k+1}\nabla H_{k+1}^2 - 2c_k(S\circ P_k)(\nabla H_{k+1}) + 2c_k P_k(\nabla H_k), \tag{12}$$

$$\lambda_1\lambda_2\langle N, a\rangle = c_k L_k(H_{k+1}) - \big(\mathrm{tr}(S^2\circ P_k) - c_k H_k + \lambda_1 + \lambda_2\big)c_k H_{k+1}, \tag{13}$$

$$\lambda_1\lambda_2\langle \psi, a\rangle = c_k^2 H_{k+1}^2 - (c_k H_k - \lambda_1)(c_k H_k - \lambda_2) - c_k L_k(H_k). \tag{14}$$

In [13], the author shows that if M^n is a hypersurface of the hyperbolic space $\mathbb{H}^{n+1}$ with constant mean curvature and constant scalar curvature, then M^n is either of 1-type or of 2-type. He also proves that every 2-type hypersurface of the hyperbolic space has nonzero constant mean curvature and constant scalar curvature.

Our goal in this section is to prove similar results for operators L_1 and L_2.

Theorem 3. *Let $\psi : M^3 \to \mathbb{H}^4 \subset \mathbb{R}^5_1$ be an orientable H_2-hypersurface. If M^3 is of L_2-2-type then the Gauss-Kronecker curvature H_3 is a nonzero constant.*

Proof. Let $\{E_1, E_2, E_3\}$ be a local orthonormal frame of principal directions of S such that $SE_i = \kappa_i E_i$ for every $i = 1, 2, 3$, and consider the open set

$$\mathcal{U}_3 = \Big\{p \in M^3 \mid \nabla H_3^2(p) \neq 0\Big\}.$$

Let us suppose that $\mathcal{U}_3$ is not empty. Since we are assuming that M^3 is of L_2-2-type and H_2 is constant, then by taking covariant derivative in (14) we have $\lambda_1\lambda_2 a^\top = 9\nabla H_3^2$, and putting this into (12) yields

$$(S\circ P_2)(\nabla H_3^2) = -\frac{7}{2}H_3\nabla H_3^2 \quad \text{on } \mathcal{U}_3. \tag{15}$$

Since $P_3 = 0$ then $S\circ P_2 = H_3 I$ and so $(S\circ P_2)(\nabla H_3^2) = H_3\nabla H_3^2$, that jointly with (15) implies $H_3\nabla H_3^2 = 0$ on $\mathcal{U}_3$, which is not possible. ∎

We want to extend the previous theorem for the operator L_1; next theorem is an intermediate step.

Recall that a hypersurface M^n immersed in either the Euclidean space $\mathbb{R}^{n+1}$, the sphere $\mathbb{S}^{n+1}$ or the hyperbolic space $\mathbb{H}^{n+1}$, is called *isoparametric* if all the principal curvatures κ_i are constant functions; this is equivalent to saying that all the mean curvatures H_i are constant functions. An isoparametric hypersurface of the Euclidean space can have at most two distinct principal curvatures, and it must be an open portion of a hyperplane, hypersphere or spherical cylinder $\mathbb{S}^k(r)\times\mathbb{R}^{n-k}$ (see e.g. [26, 25]). A similar result holds for $\mathbb{H}^{n+1}$: an isoparametric hypersurface must be an open part of a totally umbilical hypersurface or hyperbolic cylinder $\mathbb{H}^m(r_1)\times\mathbb{S}^{n-m}(r_2)$ (see [3]). However, the classification of isoparametric hipersurfaces in the sphere $\mathbb{S}^{n+1}$ turns out to be much more complicated, as Elie Cartan showed (see [4, 5, 6]).

Theorem 4. *Let M^3 be an orientable H_k-hypersurface of the hyperbolic space $\mathbb{H}^4$, which is not totally umbilical, and consider the following three conditions:*

(a) H_{k+1} *is a nonzero constant.*

(b) $\text{tr}(S^2 \circ P_k)$ *is constant.*

(c) M^3 *is of L_k-2-type.*

Then any two conditions imply the third one.

Proof. First, we show that conditions $a)$ and $b)$ imply condition $c)$. From Lemma 1 we obtain that M^3 is an isoparametric hypersurface; since M^3 is not totally umbilical then M^3 is a hyperbolic cylinder, and then the claim follows from Example 3.

Secondly, we show that conditions $a)$ and $c)$ imply condition $b)$. By taking covariant differentiation in equation (13), and bearing (14) in mind, we find

$$c_k H_{k+1} X(\text{tr}(S^2 \circ P_k)) = -\lambda_1\lambda_2 X(\langle N, a\rangle) = \lambda_1\lambda_2 \left\langle SX, a^\top \right\rangle = 0,$$

that is, $\text{tr}(S^2 \circ P_k)$ is constant on M^3.

Finally, we show that conditions $b)$ and $c)$ imply condition $a)$. In the case $k = 2$, the proof follows directly from Theorem 3. To prove the claim in the case $k = 1$, let us consider the open set

$$\mathcal{U}_2 = \{p \in M^3 \mid \nabla H_2^2(p) \neq 0\},$$

and assume that it is not empty. Since H is constant, by taking covariant derivative in (14) we obtain that $\lambda_1\lambda_2 a^\top = 36\nabla H_2^2$. Using this in (12) we get

$$(S \circ P_1)(\nabla H_2^2) = -\frac{15}{2} H_2 \nabla H_2^2 \quad \text{on } \mathcal{U}_2, \tag{16}$$

that jointly with equation (2) leads to $P_2(\nabla H_2^2) = \frac{21}{2} H_2 \nabla H_2^2$. Now, by applying the operator S on both sides, we have

$$(S \circ P_2)(\nabla H_2^2) = \frac{21}{2} H_2 S(\nabla H_2^2). \tag{17}$$

Since $P_3 = 0$ we get $S \circ P_2 = H_3 I$, and then $(S \circ P_2)(\nabla H_2^2) = H_3 \nabla H_2^2$, that jointly with (17) implies

$$S(\nabla H_2^2) = \frac{2H_3}{21H_2} \nabla H_2^2.$$

Without loss of generality, let us assume that E_1 is parallel to ∇H_2^2, i.e. the principal curvature $\kappa_1 = \frac{2H_3}{21H_2}$. Then we have

$$(S \circ P_1)(\nabla H_2^2) = \kappa_1 \mu_1^1 \nabla H_2^2 = \frac{2H_3}{21H_2}\big(3H - \frac{2H_3}{21H_2}\big)\nabla H_2^2,$$

that jointly with (16) yields the following equation,

$$6615\, H_2^3 + 252\, H H_2 H_3 - 8 H_3^2 = 0.$$

From Lemma 1 we have that $3H_3 = 9HH_2 - \mathrm{tr}(S^2 \circ P_1)$, and then the previous equation can be rewritten as follows

$$6615\, H_2^3 + 684 H^2\, H_2^2 - 68 H \mathrm{tr}(S^2 \circ P_1)\, H_2 - \tfrac{8}{9} \mathrm{tr}(S^2 \circ P_1) = 0.$$

In other words, H_2 is a root of a polynomial with constant coefficients, and so H_2 has to be constant, which is a contradiction. ∎

An interesting consequence of the last theorem is the following result.

Theorem 5. *Let $\psi : M^3 \to \mathbb{H}^4 \subset \mathbb{R}_1^5$ be an orientable H_2-hypersurface. If M is of L_2-2-type then M^3 is an isoparametric hypersurface.*

Proof. From Theorem 3 we get that H_3 is a nonzero constant, and then Theorem 4 yields that $\mathrm{tr}(S^2 \circ P_2)$ is constant. Now we use Lemma 1(d) to deduce that the mean curvature H is constant, and this concludes the proof. ∎

Since the isoparametric hypersurfaces of the hyperbolic space $\mathbb{H}^4 \subset \mathbb{R}_1^5$ are well known, the following result is clear.

Theorem 6. *Let $\psi : M^3 \to \mathbb{H}^4 \subset \mathbb{R}_1^5$ be an orientable H_2-hypersurface, which is not totally umbilical. Then M^3 is of L_2-2-type if and only if M^3 is a standard Riemannian product $\mathbb{H}^1(r_1) \times \mathbb{S}^2(r_2)$ or $\mathbb{H}^2(r_1) \times \mathbb{S}^1(r_2)$, with $-r_1^2 + r_2^2 = -1$.*

Now, we state the main result of this section.

Theorem 7. *Let $\psi : M^3 \to \mathbb{H}^4 \subset \mathbb{R}_1^5$ be an orientable H_k-hypersurface. If M is of L_k-2-type then H_{k+1} is a nonzero constant.*

Proof. Case $k = 0$ is shown in [13] and case $k = 2$ has been proved in Theorem 3, so we can assume $k = 1$. Let us consider $\{E_1, E_2, E_3\}$ a local orthonormal frame of principal directions of S such that $SE_i = \kappa_i E_i$ for every $i = 1, 2, 3$. Let us define the open set

$$\mathcal{U}_2 = \{p \in M^3 \mid \nabla H_2^2(p) \neq 0\},$$

and suppose that $\mathcal{U}_2$ is not empty. Since we are assuming that M^3 is L_1-2-type and H is constant, then equation (14) leads to

$$\lambda_1 \lambda_2 a^\top = 36 \nabla H_2^2. \tag{18}$$

Using this equation in (12) we have that $(S \circ P_1)(\nabla H_2^2) = -\frac{15}{2} H_2 \nabla H_2^2$ on $\mathcal{U}_2$, and substituting this into (2) we obtain

$$P_2(\nabla H_2^2) = \frac{21}{2} H_2 \nabla H_2^2 \quad \text{on } \mathcal{U}_2. \tag{19}$$

The vector field ∇H_2^2 can be written as $\nabla H_2^2 = E_1(H_2^2)E_1+E_2(H_2^2)E_2+E_3(H_2^2)E_3$, and then

$$P_2(\nabla H_2^2) = E_1(H_2^2)\mu_2^1 E_1 + E_2(H_2^2)\mu_2^2 E_2 + E_3(H_2^2)\mu_2^3 E_3.$$

Therefore equation (19) is equivalent to

$$E_i(H_2^2)\left(\mu_2^i - \frac{21}{2}H_2\right) = 0 \quad \text{on } \mathcal{U}_2, \tag{20}$$

for every $i = 1, 2, 3$. An immediate and important consequence of this equation is that $E_i(H_2^2) = 0$ for some i. Otherwise, we deduce that

$$\text{tr}(P_2) = \mu_2^1 + \mu_2^2 + \mu_2^3 = \frac{63}{2}H_2,$$

that jointly with Lemma 1 leads to $H_2 = 0$ on $\mathcal{U}_2$, which is a contradiction.

Bearing in mind the previous consequence, and without loss of generality, we have to analyze the following two possible cases.

Case 1. $E_1(H_2^2) \neq 0$, $E_2(H_2^2) \neq 0$ and $E_3(H_2^2) = 0$.

From (20) we have $\mu_2^1 = \mu_2^2 = \frac{21}{2}H_2$, then $(\kappa_1-\kappa_2)\kappa_3 = 0$, and therefore $\kappa_1 = \kappa_2$. Observe that $\kappa_i \neq 0$ for all i, otherwise $H_2 = 0$. It is easy to see that

$$\kappa_2\kappa_3 = \mu_2^1 = \frac{21}{2}H_2 = \frac{7}{2}(\kappa_2^2 + 2\kappa_2\kappa_3),$$

and so $7\kappa_2 + 12\kappa_3 = 0$. On the other hand, we know that $3H = 2\kappa_2 + \kappa_3$ and then we get that the principal curvatures κ_2 and κ_3 are constant. So H_2 is also constant, which can not be possible.

Case 2. $E_1(H_2^2) \neq 0$, $E_2(H_2^2) = 0$ and $E_3(H_2^2) = 0$.

We know that $3H_2 = \kappa_1\mu_1^1 + \mu_2^1$ and $\mu_2^1 = \frac{21}{2}H_2$ (see (20)), then we have

$$H_2 = \frac{2}{15}\left(\kappa_1^2 - 3H\kappa_1\right) \qquad \text{and} \qquad H_2^2 = p(\kappa_1), \tag{21}$$

where $p(x) = \left(\frac{2}{15}\right)^2\left(x^4 - 6Hx^3 + 9H^2x^2\right)$. Observe that $H \neq 0$; otherwise, $\kappa_2+\kappa_3 = -\kappa_1$ and from (21) we get $\kappa_2\kappa_3 = \frac{7}{5}\kappa_1^2$. Then κ_2 and κ_3 are the roots of the equation $t^2 + \kappa_1 t + \frac{7}{5}\kappa_1^2 = 0$, but this is not possible since the discriminant of this equation is negative.

We claim that

$$E_1(H_2^2) = p'(\kappa_1)E_1(\kappa_1), \tag{22}$$

$$\lambda_1\lambda_2\langle\psi, a\rangle = 36\,p(\kappa_1) + A_0, \tag{23}$$

$$\lambda_1\lambda_2\langle N, a\rangle = q(\kappa_1) + B_0, \tag{24}$$

where $q(x) = -\left(\frac{4}{5}\right)^2\left(\frac{4}{5}x^5 - \frac{9H}{2}x^4 + 6H^2x^3\right)$, and A_0, B_0 are two constants. First, (22) and (23) follow directly from (21) and (14), respectively. On the other hand, bearing (18) in mind we find that

$$\begin{aligned} X(\lambda_1\lambda_2\langle N, a\rangle) &= -\lambda_1\lambda_2\left\langle SX, a^\top\right\rangle = -36\kappa_1\left\langle X, \nabla H_2^2\right\rangle \\ &= -36\kappa_1 X(H_2^2) = X\big(q(\kappa_1)\big), \end{aligned}$$

for any tangent vector field X, and this implies equation (24).

Now, by taking covariant differentiation in (18) in the direction of an arbitrary tangent vector field X, we have

$$\lambda_1\lambda_2\nabla_X a^\top = 36X\big(E_1(H_2^2)\big)E_1 + 36E_1(H_2^2)\nabla_X E_1,$$

that jointly with (6) yields

$$36E_1(H_2^2)\nabla_X E_1 = -36X\big(E_1(H_2^2)\big)E_1 + \lambda_1\lambda_2\big(\langle N, a\rangle SX + \langle\psi, a\rangle X\big), \tag{25}$$

or equivalently

$$\begin{aligned} &36E_1(H_2^2)\langle\nabla_X E_1, E_i\rangle \\ &= -36X\big(E_1(H_2^2)\big)\delta_{1i} + \lambda_1\lambda_2\big(\langle N, a\rangle\kappa_i + \langle\psi, a\rangle\big)\langle X, E_i\rangle, \end{aligned} \tag{26}$$

for $i = 1, 2, 3$. If we take $X = E_1$, then (26) reduces to the following equations

$$\begin{aligned} 36E_1(E_1(H_2^2)) &= \lambda_1\lambda_2\big(\langle N, a\rangle\kappa_1 + \langle\psi, a\rangle\big), \\ E_1(H_2^2)\langle\nabla_{E_1}E_1, E_i\rangle &= 0, \qquad i = 2, 3. \end{aligned}$$

From the last equation we conclude that $\nabla_{E_1}E_1 = 0$, that is, the integral curves of E_1 on $\mathcal{U}_2$ are geodesics of M^3.

Let X be a tangent vector field orthogonal to E_1. Then equation (26) for $i = 1$ leads to $X(E_1(H_2^2)) = 0$ and thus (25) yields

$$36E_1(H_2^2)\nabla_X E_1 = \lambda_1\lambda_2\big(\langle N, a\rangle SX + \langle\psi, a\rangle X\big), \qquad \forall\ X \perp E_1. \tag{27}$$

From the Codazzi equation $(\nabla_{E_j}S)E_1 = (\nabla_{E_1}S)E_j$, we get

$$E_1\big(\kappa_j\big) = (\kappa_1 - \kappa_j)\left\langle\nabla_{E_j}E_1, E_j\right\rangle, \quad j = 2, 3,$$

that jointly with (27) for $X = E_j$ yields

$$\begin{aligned} &36E_1(H_2^2)E_1(\kappa_j) \\ &= (\kappa_1 - \kappa_j)\big[\lambda_1\lambda_2\langle N, a\rangle\kappa_j + \lambda_1\lambda_2\langle\psi, a\rangle\big] \\ &= -\lambda_1\lambda_2\langle N, a\rangle\kappa_j^2 + \lambda_1\lambda_2\langle N, a\rangle\kappa_1\kappa_j - \lambda_1\lambda_2\langle\psi, a\rangle\kappa_j + \lambda_1\lambda_2\langle\psi, a\rangle\kappa_1. \end{aligned}$$

Last equation implies

$$\begin{aligned}36E_1(H_2^2)(E_1(\kappa_2)+E_1(\kappa_3)) &= -\lambda_1\lambda_2\langle N,a\rangle(\kappa_2^2+\kappa_3^2)+\lambda_1\lambda_2\langle N,a\rangle\kappa_1(\kappa_2+\kappa_3)\\ &\quad-\lambda_1\lambda_2\langle\psi,a\rangle(\kappa_2+\kappa_3)+2\lambda_1\lambda_2\langle\psi,a\rangle\kappa_1,\end{aligned}$$

that is,

$$\begin{aligned}36E_1(H_2^2)E_1(3H-\kappa_1) &= -\lambda_1\lambda_2\langle N,a\rangle(\mathrm{tr}(S^2)-\kappa_1^2)+\lambda_1\lambda_2\langle N,a\rangle\kappa_1(3H-\kappa_1)\\ &\quad-\lambda_1\lambda_2\langle\psi,a\rangle(3H-\kappa_1)+2\lambda_1\lambda_2\langle\psi,a\rangle\kappa_1.\end{aligned}$$

From (1) and (21) we have that $\mathrm{tr}(S^2)=9H^2-\frac{3}{5}H\kappa_1-\frac{4}{5}\kappa_1^2$. By using this and (22), last equation can be written as

$$\begin{aligned}&36p'(\kappa_1)\big[E_1(\kappa_1)\big]^2\\ &=-\frac{1}{5}\lambda_1\lambda_2\langle N,a\rangle(4\kappa_1^2+3H\kappa_1-45H^2)+3\lambda_1\lambda_2\langle\psi,a\rangle(H-\kappa_1).\end{aligned}\tag{28}$$

On the other hand, a direct computation shows

$$\begin{aligned}36^2\big[p'(\kappa_1)E_1(\kappa_1)\big]^2 &= 36^2\big[E_1(H_2^2)\big]^2=36^2\left\langle\nabla H_2^2,\nabla H_2^2\right\rangle=\lambda_1^2\lambda_2^2|a^\top|^2\\ &=\lambda_1^2\lambda_2^2|a|^2-(\lambda_1\lambda_2\langle N,a\rangle)^2+(\lambda_1\lambda_2\langle\psi,a\rangle)^2.\end{aligned}\tag{29}$$

From equations (28) and (29), and taking into account (23) and (24), we find a polynomial $T(x)$ with constant coefficients given by

$$\begin{aligned}T(x) &= \big[q(x)+B_0\big]^2-\big[36p(x)+A_0\big]^2\\ &\quad-\tfrac{36}{5}\big[q(x)+B_0\big](4x+15H)(x-3H)p'(x)\\ &\quad+108\big[36p(x)+A_0\big](H-x)p'(x)-\lambda_1^2\lambda_2^2|a|^2,\end{aligned}\tag{30}$$

and satisfying $T(\kappa_1)=0$. Therefore, κ_1 is locally constant on $\mathcal{U}_2$, and so H_2 is also constant, which is a contradiction with the definition of $\mathcal{U}_2$. This finishes the proof. ■

An interesting consequence is the following result, similar to Theorem 5.

Theorem 8. *Let $\psi: M^3\to\mathbb{H}^4\subset\mathbb{R}_1^5$ be an orientable H-hypersurface. If M^3 is of L_1-2-type then M^3 is an isoparametric hypersurface.*

Proof. From Theorem 7 we get that H_2 is a non-zero constant, and then Theorem 4 yields that $\mathrm{tr}(S^2\circ P_1)$ is constant. Now we use Lemma 1(c) to deduce that the Gauss-Kronecker curvature H_3 is constant, and this concludes the proof. ■

Bearing in mind Theorems 8 and 4, and the classification of isoparametric hypersurfaces in the hyperbolic space $\mathbb{H}^4$, the following result, that extends Theorem 6, is clear.

Theorem 9. *Let $\psi : M^3 \to \mathbb{H}^4 \subset \mathbb{R}^5_1$ be an orientable H-hypersurface, which is not totally umbilical. Then M^3 is of L_1-2-type if and only if M^3 is a standard Riemannian product $\mathbb{H}^1(r_1) \times \mathbb{S}^2(r_2)$ or $\mathbb{H}^2(r_1) \times \mathbb{S}^1(r_2)$, with $-r_1^2 + r_2^2 = -1$.*

5. The n-Dimensional Case

Let $\psi : M^n \to \mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}_1$ denote an isometric immersion of an orientable hypersurface M^n in the hyperbolic space $\mathbb{H}^{n+1} \equiv \mathbb{H}^{n+1}(0, -1)$. The goal of this section is to classify L_k-2-type hypersurfaces with constant k-th mean curvature H_k and having at most two distinct principal curvatures.

Suppose that ψ is of L_k-2-type, then we can write

$$\psi = a + \psi_1 + \psi_2, \qquad L_k\psi_1 = \lambda_1\psi_1, \qquad L_k\psi_2 = \lambda_2\psi_2,$$

where $a \in \mathbb{R}^{n+2}_1$ is a constant vector and $\psi_1, \psi_2 : M^n \to \mathbb{R}^{n+2}_1$ are non-constant differentiable functions.

Performing calculations similar to those made in Sections 3 and 4, the following equations can be obtained:

$$\lambda_1\lambda_2 a^\top = -\frac{c_k}{2}\binom{n}{k+1}\nabla H_{k+1}^2 - 2c_k(S \circ P_k)(\nabla H_{k+1}) + 2c_k P_k(\nabla H_k), \tag{31}$$

$$\lambda_1\lambda_2 \langle N, a\rangle = c_k L_k(H_{k+1}) - \big(\mathrm{tr}(S^2 \circ P_k) - c_k H_k + \lambda_1 + \lambda_2\big)c_k H_{k+1}, \tag{32}$$

$$\lambda_1\lambda_2 \langle \psi, a\rangle = c_k^2 H_{k+1}^2 - (c_k H_k - \lambda_1)(c_k H_k - \lambda_2) - c_k L_k(H_k), \tag{33}$$

where $c_k = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1}$.

The following example exhibits hypersurfaces of L_k-2-type in the hyperbolic space $\mathbb{H}^{n+1}$.

Example 4. For each positive number r and each integer m, $1 \leq m \leq n-1$, let $M^n_m(r)$ be the n-dimensional submanifold of $\mathbb{R}^{n+2}_1$ defined by

$$M^n_m(r) = \Big\{(x_1, \ldots, x_{n+2}) \,\Big|\, -x_1^2 + \sum_{i=2}^{m+1} x_i^2 = -1 - r^2,\ \sum_{j=m+2}^{n+2} x_j^2 = r^2\Big\}.$$

It is well known that $M^n_m(r)$ is a complete and non-compact hypersurface of the hyperbolic space $\mathbb{H}^{n+1}$; in fact, $M^n_m(r)$ is isometric to the standard Riemannian product $\mathbb{H}^m(-\sqrt{1+r^2}) \times \mathbb{S}^{n-m}(r)$.

The Gauss map of $M^n_m(r)$ in $\mathbb{H}^{n+1}$ is given by

$$N(x) = \left(\frac{r}{\sqrt{1+r^2}}\,x_1, \ldots, \frac{r}{\sqrt{1+r^2}}\,x_{m+1}, \frac{\sqrt{1+r^2}}{r}\,x_{m+2}, \ldots, \frac{\sqrt{1+r^2}}{r}\,x_{n+2}\right),$$

and then $M^n_m(r)$ has two constant distinct principal curvatures given by

$$\kappa_1 = \cdots = \kappa_m = \frac{-r}{\sqrt{1+r^2}} \quad \text{and} \quad \kappa_{m+1} = \cdots = \kappa_n = \frac{-\sqrt{1+r^2}}{r}.$$

Hence, the mean curvature H_k is constant for every k.

If we put $\psi_1 = (x_1, \ldots, x_{m+1}, 0, \ldots, 0)$ and $\psi_2 = (0, \ldots, 0, x_{m+2}, \ldots, x_{n+2})$, then $\psi = \psi_1 + \psi_2$ and, by using (7), we obtain $L_k\psi_1 = \lambda_1\psi_1$ and $L_k\psi_2 = \lambda_2\psi_2$, where

$$\lambda_1 = \frac{c_k}{\sqrt{1+r^2}}\Big(rH_{k+1} + \sqrt{1+r^2}H_k\Big) \qquad \text{and} \qquad \lambda_2 = \frac{c_k}{r}\Big(\sqrt{1+r^2}H_{k+1} + rH_k\Big).$$

Therefore, $M^n_m(r)$ is a hypersurface of L_k-2-type of the hyperbolic space $\mathbb{H}^{n+1}$.

Now, we are ready to prove the following classification result.

Theorem 10. *Let $\psi : M^n \to \mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}_1$ be an orientable H_k-hypersurface and assume that M^n has at most two distinct principal curvatures. Then M^n is of L_k-2-type if and only if M^n is an open portion of $M^n_m(r)$, for some positive integer m, $1 \le m \le n-1$, and for some positive number r.*

Proof. Let us assume that M^n is a hypersurface of L_k-2-type. Let κ_1 and κ_2 denote the principal curvatures of M^n, with multiplicities m and $n-m$, respectively. Consider $\{E_1, E_2, \ldots, E_n\}$ a local orthonormal frame of principal directions of S such that $SE_i = \kappa_1 E_i$, for $i = 1, \ldots, m$, and $SE_j = \kappa_2 E_j$, $j = m+1, \ldots, n$. Without loss of generality, we can distinguish two cases according to the multiplicity m.

Case 1. $m = 1$.
Let us consider the open set

$$\mathcal{U}_{k+1} = \Big\{p \in M^n \mid \nabla H^2_{k+1}(p) \neq 0\Big\}.$$

Our goal is to show that $\mathcal{U}_{k+1}$ is empty. Otherwise, since M^n is a L_k-2-type hypersurface and the mean curvature H_k is constant, by taking covariant derivative in (33) we obtain $\lambda_1\lambda_2 a^\top = c_k^2 \nabla H^2_{k+1}$, that jointly with (31) yields

$$(S \circ P_k)(\nabla H^2_{k+1}) = -\tfrac{c_k(2k+3)}{2(k+1)} H_{k+1} \nabla H^2_{k+1} \quad \text{on } \mathcal{U}_{k+1}. \tag{34}$$

From the inductive definition of $P_{k+1} = \binom{n}{k+1}H_{k+1}I - S \circ P_k$ and (34) we obtain

$$P_{k+1}(\nabla H^2_{k+1}) = D_k H_{k+1} \nabla H^2_{k+1} \quad \text{on } \mathcal{U}_{k+1}, \tag{35}$$

where $D_k = \frac{2k+5}{2}\binom{n}{k+1}$. The vector field ∇H^2_{k+1} can be written as $\nabla H^2_{k+1} = \sum_{i=1}^n \left\langle \nabla H^2_{k+1}, E_i \right\rangle E_i$, and then we get

$$P_{k+1}(\nabla H^2_{k+1}) = \sum_{i=1}^n \langle \nabla H_{k+1}, E_i \rangle \mu^i_{_{k+1}} E_i.$$

Hence, Eq. (35) is equivalent to

$$\left\langle \nabla H^2_{k+1}, E_i \right\rangle \left(\mu^i_{_{k+1}} - D_k H_{k+1} \right) = 0 \quad \text{on } \mathcal{U}_{k+1},$$

for every $i = 1, \ldots, n$. Therefore, for every i such that $\left\langle \nabla H^2_{k+1}, E_i \right\rangle \neq 0$ we get

$$\mu^i_{_{k+1}} = D_k H_{k+1}.$$

We will distinguish two cases: (a) $\left\langle \nabla H^2_{k+1}, E_1 \right\rangle \neq 0$, and (b) $\left\langle \nabla H^2_{k+1}, E_i \right\rangle \neq 0$ for some $i > 1$.

(a) First, let us suppose that $\left\langle \nabla H^2_{k+1}, E_1 \right\rangle \neq 0$. Then, we get

$$\mu^1_{_{k+1}} = D_k H_{k+1} = \tfrac{2k+5}{2}\ \mu_{_{k+1}} = \tfrac{2k+5}{2}(\kappa_1 \mu^1_{_k} + \mu^1_{_{k+1}}).$$

This equation, bearing in mind that $\binom{n}{k} H_k = \mu_{_k} = \kappa_1 \mu^1_{_{k-1}} + \mu^1_{_k}$, leads to

$$-(2k+3)\mu^1_{_{k+1}} \mu^1_{_{k-1}} = (2k+5)\left(\tbinom{n}{k} H_k - \mu^1_{_k}\right)\mu^1_{_k}. \tag{36}$$

Now, by using that $\mu^1_{_j} = \binom{n-1}{j}\kappa_2^j$ for $j \in \{1, \ldots, n-1\}$, we can rewrite (36) as follows

$$A\kappa_2^k + B = 0,$$

where A and B are two nonzero constants. Therefore, κ_2 is constant. This implies, since H_k is constant, that the principal curvature κ_1 is constant, and so H_{k+1} is also constant, which is a contradiction.

(b) Now, suppose that $\left\langle \nabla H^2_{k+1}, E_i \right\rangle \neq 0$ for some $i > 1$. Then, we get

$$\kappa_1 \mu^{1,i}_{_k} + \mu^{1,i}_{_{k+1}} = \mu^i_{_{k+1}} = D_k H_{k+1} = \tfrac{2k+5}{2}(\kappa_1 \mu^1_{_k} + \mu^1_{_{k+1}}).$$

It is not difficult to see that this equation is equivalent to

$$\tbinom{n-2}{k}\kappa_1 + \tbinom{n-2}{k+1}\kappa_2 = \tfrac{2k+5}{2}\left(\tbinom{n-1}{k}\kappa_1 + \tbinom{n-1}{k+1}\kappa_2\right).$$

In other words, $C\kappa_1 = D\kappa_2$, where C and D are two nonzero constants given by

$$C = \tfrac{3-n(2k+3)}{2(n-1)}\tbinom{n-1}{k} \qquad \text{and} \qquad D = \tfrac{n(2k+3)-1}{2(n-1)}\tbinom{n-1}{k+1}.$$

By direct computation, we find that

$$C\binom{n}{k}H_k = \left[\binom{n-1}{k}C + \binom{n-1}{k-1}D\right]\kappa_2^k.$$

Therefore, κ_2 is constant. As before, this implies that the $(k+1)$-th mean curvature H_{k+1} is also constant, which is not possible.

Case 2. $1 < m < n-1$ (i.e. the multiplicities of two principal curvatures are greater than one).

Without loss of generality, suppose that $\kappa_1, \kappa_2 \neq 0$. By using a standard reasoning involving the Codazzi equations, we deduce that $E_i(\kappa_1) = 0$, for $i = 1, \dots, m$, and $E_j(\kappa_2) = 0$, for $j = m+1, \dots, n$. Since the number of distinct principal curvatures is two, the distribution corresponding to each principal curvature is smooth and integrable (see, e.g., [2, Paragraph 16.10] and [23]). Hence, we deduce that each principal curvature κ_i is constant on each integral submanifold of the corresponding distribution of the space of principal vectors $V(\kappa_i)$ (see [23]). Therefore, M^n is locally isometric to the Riemannian product $M_1 \times M_2$, where M_i is the maximal integral submanifold corresponding to the distribution of the space $V(\kappa_i)$ (see, e.g., [18, p. 182]).

Since H_k is constant on the hypersurface $M_1 \times M_2$ and κ_1 is constant on M_1, we deduce that κ_2 is also constant on M_1. Similarly, the constancy of H_k and κ_2 on M_2 implies that κ_1 is also constant on M_2. In other words, the principal curvatures κ_1 and κ_2 are constant on the whole hypersurface, and so H_{k+1} is also constant, which is a contradiction.

In conclusion, the mean curvatures H_k and H_{k+1} of the hypersurface M^n are constant. Since M^n has at most two distinct principal curvatures, we get that M^n is an isoparametric hypersurface of the hyperbolic space. Bearing in mind the classification of isoparametric hypersurfaces in $\mathbb{H}^{n+1}$ (see [3]), we deduce that M^n is an open portion of $M_m^n(r)$, for some positive integer m, $1 \leq m \leq n-1$, and for some positive number r. ∎

In the case $k = n-1$ we can drop the condition on the principal curvatures of the hypersurface M^n.

Theorem 11. *Let $\psi : M^n \to \mathbb{H}^{n+1} \subset \mathbb{R}_1^{n+2}$ be an orientable H_{n-1}-hypersurface. If M^n is of L_{n-1}-2-type then its Gauss-Kronecker curvature H_n is a nonzero constant.*

Proof. Let us suppose that H_n is non constant and consider the nonempty open set

$$\mathcal{U}_n = \left\{ p \in M^n \mid \nabla H_n^2(p) \neq 0 \right\}.$$

Since M^n is of L_{n-1}-2-type and H_{n-1} is constant, by taking covariant derivative in (33) we have $\lambda_1\lambda_2 a^\top = c_{n-1}^2 \nabla H_n^2$, and by putting this into Eq. (31) we obtain

$$(S \circ P_{n-1})(\nabla H_n^2) = -\tfrac{2n+1}{2} H_n \nabla H_n^2 \quad \text{on } \mathcal{U}_n. \tag{37}$$

Since $P_n = 0$, we deduce $S \circ P_{n-1} = H_n I$, and so $S \circ P_{n-1}(\nabla H_n^2) = H_n \nabla H_n^2$, that jointly with (37) implies $H_n \nabla H_n^2 = 0$ on $\mathcal{U}_n$, which can not be possible. Therefore, H_n is constant and nonzero. ■

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