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WEIGHTED LIPSCHITZ ESTIMATES FOR COMMUTATORS ON WEIGHTED MORREY-HERZ SPACES

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Abstract. In this paper, the authors establish the boundedness of commutators generated by Calderón-Zygmund singular integral operators and weighted Lipschitz functions on weighted Morrey-Herz spaces.

1. Introduction

The standard singular integral operator is defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)f(y)dy.$$

A well-known result of Stein in [13] states that if T is bounded on $L^q(\mathbb{R}^n)$, $1 < q < \infty$, and

$$|K(x)| \leq \frac{C}{|x|^n}, \quad \forall x \neq 0, \tag{1.1}$$

then T is also bounded on the weighted spaces $L^q_{|x|^\beta}(\mathbb{R}^n)$, $-n < \beta < n(q-1)$, where the range of β is the best.

In 1994, the above Stein's result was developed by Soria and Weiss [12] in the following way. The singular integral operator satisfying (1.1) will be replaced by any sublinear operator T satisfying the following size condition: For any $f \in L^1(\mathbb{R}^n)$ with compact support and for $x \notin \text{supp} f$,

$$|Tf(x)| \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy.$$

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Originated from the definition of Coifman, Rochberg and Weiss [1], throughout this paper we focus on the Calderón-Zygmund singular integral operator,

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)f(y)dy,$$

which satisfies

(1) $|K(x)| \leq C|x|^{-n}$, $x \neq 0$;

(2) $|K(x-y) - K(x)| \leq C\dfrac{|y|}{|x|^{n+1}}$, $2|y| \leq |x|$;

(3) T can be extended into a continuous operator on $L^2(\mathbb{R}^n)$.

Let b be a locally integrable function on $\mathbb{R}^n$ and let T be a Calderón-Zygmund singular integral operator. The commutator $[b,T]$ generated by b and T is defined by

$$[b,T]f(x) = bT(f)(x) - T(bf)(x).$$

Janson [4] proved that $[b,T]$ is bounded on L^p for $1 < p < \infty$ if and only if $b \in BMO$; see also [1]. Paluszyński [11] showed that $b \in Lip_\beta$ if and only if the commutator $[b,T]$ is bounded from L^p to L^q, where $1 < p < q < \infty$, $0 < \beta < 1$ and $1/q = 1/p - \beta/n$. Lu and Yang [10] obtained the boundedness of commutators generated by singular integrals and Lipschitz functions on Herz spaces. And Lin [6] proved the boundedness of commutators generated by strongly singular Calderón-Zygmund operators and Lipschitz functions on Morrey type spaces.

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $E_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{E_k}$ for $k \in \mathbb{Z}$, where by χ_E we denote the characteristic function of a set E. Let $f_B = \frac{1}{|B|}\int_B f(x)dx$.

Definition 1.1. Let $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^q(\mathbb{R}^n)} \right)^{1/p}$$

with usual modifications made when $p = \infty$.

Definition 1.2. Let $1 \leq p \leq q < \infty$. The Morrey space $M_p^q(\mathbb{R}^n)$ is defined by

$$M_p^q(\mathbb{R}^n) = \{f \in L^q_{\text{loc}}(\mathbb{R}^n) : \|f\|_{M_p^q(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{M_p^q(\mathbb{R}^n)} = \sup_{B \subset \mathbb{R}^n} |B|^{\frac{1}{q}-\frac{1}{p}} \left(\int_B |f(x)|^p dx \right)^{1/p}$$

with usual modifications made when $p = \infty$.

Definition 1.3. Let $0 \le \lambda < \infty$, $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$. A function f is said to belong to the Morrey-Herz space $MK_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$ provided that

$$\|f\|_{MK_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p \right)^{1/p} < \infty$$

with usual modifications made when $p = \infty$.

A non-negative function μ defined on $\mathbb{R}^n$ is called a weight if it is locally integrable. A weight μ is said to belong to the Muckenhoupt class $A_p(\mathbb{R}^n)$ for $1 < p < \infty$, if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B|} \int_B \mu(x)dx \right) \left(\frac{1}{|B|} \int_B \mu(x)^{-\frac{1}{p-1}} dx \right)^{p-1} dx \le C,$$

for every ball $B \subset \mathbb{R}^n$. The class $A_1(\mathbb{R}^n)$ is defined replacing the above inequality by

$$\frac{1}{|B|} \int_B \mu(x)dx \le C\mu(x), \qquad a.e.\, x \in \mathbb{R}^n.$$

A function $\mu \in A_\infty$ if it satisfies the condition of A_p for some $p > 1$. It is well-known that if $1 < p < q < \infty$, then we have $A_1 \subset A_p \subset A_q$.

Definition 1.4. Let $0 \le \lambda < \infty$, $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$, and let μ_1 and μ_2 be non-negative weighted functions. A function f is said to belong to the weighted Morrey-Herz space $MK_{p,q}^{\alpha,\lambda}(\mu_1, \mu_2)$ provided that

$$\|f\|_{MK_{p,q}^{\alpha,\lambda}(\mu_1,\mu_2)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} \mu_1(B_k)^{\alpha p/n} \|f\chi_k\|_{L^q(\mu_2)}^p \right)^{1/p} < \infty.$$

It is easy to see that $\dot{K}_q^{0,q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ for $0 < q \le \infty$, $M_p^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with $1 \le p < \infty$, so Herz spaces and Morrey spaces are the generalization of Lebesgue spaces. Meanwhile $MK_{p,q}^{\alpha,0}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, and $MK_{q,q}^{0,\lambda}(\mathbb{R}^n) \supseteq M_q^{\frac{nq}{n-\lambda q}}(\mathbb{R}^n)$, where $1 \le q < \infty$, $0 \le \lambda < n/q$, so the special cases of Morrey-Herz spaces are Morrey spaces and Herz spaces. Obviously, when $\mu_1 = \mu_2 \equiv 1$, $MK_{p,q}^{\alpha,\lambda}(\mu_1, \mu_2) = MK_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$.

Recently the necessary and sufficient conditions for boundedness of some commutators of singular integrals with weighted Lipschitz functions on weighted Lebesgue spaces are established by Hu and Gu in [3]. After this, the boundedness of commutators generated by weighted Lipschitz functions and singular integrals or fractional integrals with rough kernels on weighted Lebesgue spaces was established in [7, 8]. And the boundedness of commutators generated by singular integrals and weighted Lipschitz functions on weighted Herz spaces has been obtained in [9]. Since Morrey-Herz spaces are generalizations of Herz spaces, a natural question is whether this kind of commutators also have boundedness on weighted Morrey-Herz spaces. The answer is affirmative. The main purpose of this paper is to generalize the above results and establish the corresponding boundedness on weighted Morrey-Herz spaces.

2. Main Results

In order to obtain our main results, first we need introduce some necessary notations and requisite lemmas.

Definition 2.1. [2]. We say that a locally integrable function f belongs to the weighted Lipschitz space $Lip^p_{\beta,\mu}$ for $1 \le p \le \infty$, $0 < \beta < 1$ and $\mu \in A_\infty$, if

$$\sup_B \frac{1}{\mu(B)^{\beta/n}} \left[\frac{1}{\mu(B)} \int_B |f(x) - f_B|^p \mu(x)^{1-p} dx \right]^{1/p} < \infty,$$

where B is any ball in $\mathbb{R}^n$.

Modulo constants, the Banach space of such functions is denoted by $Lip^p_{\beta,\mu}$. The smallest bound C satisfying conditions above is taken to be the norm of f in this space, and is denoted by $\|f\|_{Lip^p_{\beta,\mu}}$. Put $Lip_{\beta,\mu} = Lip^1_{\beta,\mu}$. Obviously, for the case $\mu \equiv 1$, the space $Lip_{\beta,\mu}$ is the classical Lipschitz space Lip_β. Thus, weighted Lipschitz spaces are generalizations of classical Lipschitz spaces.

If $\mu \in A_1(\mathbb{R}^n)$, García-Cuerva in [2] proved that the space $Lip^p_{\beta,\mu}$ coincide, and the norm of $\|\cdot\|_{Lip^p_{\beta,\mu}}$ are equivalent with respect to different values of p provided that $1 \le p \le \infty$. That is $\|f\|_{Lip^p_{\beta,\mu}} \sim \|f\|_{Lip_{\beta,\mu}}$, where $1 \le p \le \infty$.

Lemma 2.1. [5]. *Let $\mu \in A_1$, then there are constants C_1, C_2 and $0 < \delta < 1$ depending only on A_1-constant of μ, such that for any measurable subset E of a ball B,*

$$C_1 \frac{|E|}{|B|} \le \frac{\mu(E)}{\mu(B)} \le C_2 \left(\frac{|E|}{|B|} \right)^\delta.$$

Lemma 2.2. [9]. *Let $\mu \in A_1$ and $b \in Lip_{\beta,\mu}$, then there is a constant C such that for all j, $k \in \mathbb{Z}$ with $j > k$,*

$$|b_{B_j} - b_{B_k}| \le C(j-k)\|b\|_{Lip_{\beta,\mu}} \mu(B_j)^{\frac{\beta}{n}} \frac{\mu(B_k)}{|B_k|}.$$

Lemma 2.3. [9]. *Let $\mu \in A_1$, then there is a constant C such that for $1 < p < \infty$ and any ball B,*

$$\int_B \mu(x)^{1-p'} dx \le C|B|^{p'} \mu(B)^{1-p'},$$

where $1/p + 1/p' = 1$.

Recently, the authors of [3] discussed the boundedness of commutators generated by singular integrals and weighted Lipschitz functions on weighted Lebesgue spaces.

Theorem A. [3]. *Let T be a Calderón-Zygmund singular integral operator. Let $\mu \in A_1$, $1/q = 1/p - \beta/n$ for $0 < \beta < 1$ and $1 < p < q < \infty$. Let $b \in Lip_{\beta,\mu}$. Then the commutator $[b, T]$ is bounded from $L^p(\mu)$ to $L^q(\mu^{1-q})$.*

The purpose of this paper is to state the boundedness of commutators generated by singular integrals and weighted Lipschitz functions on weighted Morrey-Herz spaces. The main result is as follows:

Theorem 2.1. *Let T be a Calderón-Zygmund singular integral operator and δ be defined as in Lemma 2.1. Let $b \in Lip_{\beta,\mu}$, $\mu \in A_1$, $0 < \beta < 1$, then the commutator $[b, T]$ is bounded from $MK^{\alpha,\lambda}_{p,q_1}(\mu, \mu)$ to $MK^{\alpha,\lambda}_{p,q_2}(\mu, \mu^{1-q_2})$, where $0 \le \lambda < \infty$, $0 < p \le \infty$, $1 < q_1, q_2 < \infty$, $1/q_2 = 1/q_1 - \beta/n$, $1/\widetilde{q}_1 = 1 - \min\{1 - \frac{1}{\delta q_1}, \delta - \frac{1}{q_1}\}$, $1/\widetilde{q}_2 = \min\{\frac{1}{q_1} - \frac{\beta}{n\delta}, \frac{\delta}{q_1} - \frac{\beta}{n}\}$, and $\lambda/\delta - n/\widetilde{q}_2 < \alpha < n(1 - 1/\widetilde{q}_1)$.*

Note that if $\delta = 1$, then $\widetilde{q}_1 = q_1$ and $\widetilde{q}_2 = q_2$, and hence when $\mu \equiv 1$, we have $Lip_{\beta,\mu}(\mathbb{R}^n) = Lip_\beta(\mathbb{R}^n)$, $MK^{\alpha,\lambda}_{p,q_1}(\mu, \mu) = MK^{\alpha,\lambda}_{p,q_1}(\mathbb{R}^n)$, and $MK^{\alpha,\lambda}_{p,q_2}(\mu, \mu^{1-q_2}) = MK^{\alpha,\lambda}_{p,q_2}(\mathbb{R}^n)$. Thus, we can obtain the boundedness of commutators generated by singular integrals and classical Lipschitz functions on Morrey-Herz spaces as a corollary of Theorem 2.1.

Corollary 2.1. *Let T be a Calderón-Zygmund singular integral operator. Let $b \in Lip_\beta(\mathbb{R}^n)$, $0 < \beta < 1$, then the commutator $[b, T]$ is bounded from $MK^{\alpha,\lambda}_{p,q_1}(\mathbb{R}^n)$ to $MK^{\alpha,\lambda}_{p,q_2}(\mathbb{R}^n)$, where $0 \le \lambda < \infty$, $0 < p \le \infty$, $1 < q_1, q_2 < \infty$, $1/q_2 = 1/q_1 - \beta/n$, and $\lambda - n/q_2 < \alpha < n(1 - 1/q_1)$.*

When $\lambda = 0$, the weighted Morrey-Herz space $MK^{\alpha,0}_{p,q}(\mu_1, \mu_2)$ is the homogeneous weighted Herz space $\dot{K}^{\alpha,p}_q(\mu_1, \mu_2)$. Thus, we can obtain the boundedness of commutators generated by singular integrals and weighted Lipschitz functions on weighted Herz spaces as another corollary of Theorem 2.1.

Corollary 2.2. *Let T be a Calderón-Zygmund singular integral operator and δ be defined as in Lemma 2.1. Let $b \in Lip_{\beta,\mu}$, $\mu \in A_1$, $0 < \beta < 1$, then the commutator $[b, T]$ is bounded from $\dot{K}^{\alpha,p}_{q_1}(\mu, \mu)$ to $\dot{K}^{\alpha,p}_{q_2}(\mu, \mu^{1-q_2})$, where $0 < p \le \infty$, $1 < q_1, q_2 <$*

∞, $1/q_2 = 1/q_1 - \beta/n$, $1/\widetilde{q}_1 = 1-\min\{1-\frac{1}{\delta q_1}, \delta - \frac{1}{q_1}\}$, $1/\widetilde{q}_2 = \min\{\frac{1}{q_1} - \frac{\beta}{n\delta}, \frac{\delta}{q_1} - \frac{\beta}{n}\}$, *and* $-n/\widetilde{q}_2 < \alpha < n(1-1/\widetilde{q}_1)$.

Remark 2.1. Actually, the result of Corollary 2.2 has been obtained in [9], however there are some calculation errors in the proof of Theorem 2.5 in [9]. The exact range of α there should be $-n/\widetilde{q}_2 < \alpha < n(1-1/\widetilde{q}_1)$.

3. Proof of Theorem 2.1

In this section, we will give the proof of Theorem 2.1.

Proof. We only consider the case $0 < p < \infty$ and omit the details of the case $p = \infty$ since their similarity. Set

$$f = \sum_{j=-\infty}^{\infty} f\chi_j := \sum_{j=-\infty}^{\infty} f_j.$$

Then, by the Minkowski inequality, we write

$$\begin{aligned}
&\|[b,T]f\|_{MK^{\alpha,\lambda}_{p,q_2}(\mu,\mu^{1-q_2})}\\
&= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0} \mu(B_k)^{\alpha p/n}\|([b,T]f)\chi_k\|^p_{L^{q_2}(\mu^{1-q_2})}\Bigg)^{\frac{1}{p}}\\
&= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0} \mu(B_k)^{\alpha p/n}\Bigg\|\sum_{j=-\infty}^{\infty}([b,T]f\chi_j)\chi_k\Bigg\|^p_{L^{q_2}(\mu^{1-q_2})}\Bigg)^{\frac{1}{p}}\\
&\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0} \mu(B_k)^{\alpha p/n}\Bigg(\sum_{j=-\infty}^{k-2}\|([b,T]f\chi_j)\chi_k\|_{L^{q_2}(\mu^{1-q_2})}\Bigg)^p\Bigg)^{\frac{1}{p}}\\
&\quad +C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0} \mu(B_k)^{\alpha p/n}\Bigg(\sum_{j=k-1}^{k+1}\|([b,T]f\chi_j)\chi_k\|_{L^{q_2}(\mu^{1-q_2})}\Bigg)^p\Bigg)^{\frac{1}{p}}\\
&\quad +C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0} \mu(B_k)^{\alpha p/n}\Bigg(\sum_{j=k+2}^{\infty}\|([b,T]f\chi_j)\chi_k\|_{L^{q_2}(\mu^{1-q_2})}\Bigg)^p\Bigg)^{\frac{1}{p}}\\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

For I_2, by Theorem A and Lemma 2.1, we find

$$\begin{aligned}
I_2 &\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0} \mu(B_k)^{\alpha p/n}\Bigg(\sum_{j=k-1}^{k+1}\|[b,T](f\chi_j)\|_{L^{q_2}(\mu^{1-q_2})}\Bigg)^p\Bigg)^{\frac{1}{p}}\\
&\le C\|b\|_{Lip_{\beta},\mu}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0} \mu(B_k)^{\alpha p/n}\Bigg(\sum_{j=k-1}^{k+1}\|f\chi_j\|_{L^{q_1}(\mu)}\Bigg)^p\Bigg)^{\frac{1}{p}}
\end{aligned}$$

$$\le C\|b\|_{Lip_\beta,\mu}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p/n}\|f\chi_k\|^p_{L^{q_1}(\mu)}\bigg)^{\frac{1}{p}}$$
$$= C\|b\|_{Lip_\beta,\mu}\|f\|_{MK^{\alpha,\lambda}_{p,q_1}(\mu,\mu)}.$$

To obtain the estimates for I_1 and I_3, we first observe that for $j,k\in\mathbb{Z}$ with $|j-k|\ge 2$,

$$([b,T]f\chi_j)(x) = (b(x)-b_{B_j})T(f\chi_j)(x) - T((b-b_{B_j})f\chi_j)(x).$$

Thus,

$$\begin{aligned}
&\|([b,T]f\chi_j)\chi_k\|_{L^{q_2}(\mu^{1-q_2})}\\
=&\bigg(\int_{E_k}|[b,T](f\chi_j)(x)|^{q_2}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}\\
=&\bigg(\int_{E_k}|(b(x)-b_{B_j})T(f\chi_j)(x)-T((b-b_{B_j})f\chi_j)(x)|^{q_2}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}\\
\le&\bigg(\int_{E_k}|b(x)-b_{B_j}|^{q_2}|T(f\chi_j)(x)|^{q_2}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}\\
&\quad+\bigg(\int_{E_k}|T((b-b_{B_j})f\chi_j)(x)|^{q_2}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}\\
=&\bigg(\int_{E_k}|b(x)-b_{B_j}|^{q_2}\bigg|\int_{E_j}K(x-y)(f\chi_j)(y)dy\bigg|^{q_2}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}\\
&\quad+\bigg(\int_{E_k}\bigg|\int_{E_j}K(x-y)(b(y)-b_{B_j})(f\chi_j)(y)dy\bigg|^{q_2}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}\\
\le&\bigg(\int_{E_k}|b(x)-b_{B_j}|^{q_2}\bigg(\int_{E_j}\frac{1}{|x-y|^n}|(f\chi_j)(y)|dy\bigg)^{q_2}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}\\
&\quad+\bigg(\int_{E_k}\bigg(\int_{E_j}\frac{1}{|x-y|^n}|b(y)-b_{B_j}||(f\chi_j)(y)|dy\bigg)^{q_2}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}\\
:=&D_1(j,k)+D_2(j,k).
\end{aligned}$$

Now, let us estimate I_1. If $j\le k-2$, then

$$D_1(j,k)\le C2^{-kn}\bigg(\int_{E_k}|b(x)-b_{B_j}|^{q_2}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}\bigg(\int_{E_j}|(f\chi_j)(y)|dy\bigg),$$

and

$$D_2(j,k)\le C2^{-kn}\bigg(\int_{E_k}\mu(x)^{1-q_2}dx\bigg)^{\frac{1}{q_2}}\bigg(\int_{E_j}|b(y)-b_{B_j}||(f\chi_j)(y)|dy\bigg).$$

By Lemma 2.2, Lemma 2.3 and Lemma 2.1, for $j \le k-2$ we have

$$\begin{aligned}
&\left(\int_{E_k} |b(x)-b_{B_j}|^{q_2}\mu(x)^{1-q_2}dx\right)^{\frac{1}{q_2}}\\
\le&\left(\int_{B_k} |b(x)-b_{B_k}|^{q_2}\mu(x)^{1-q_2}dx\right)^{\frac{1}{q_2}}+\left(\int_{B_k} |b_{B_k}-b_{B_j}|^{q_2}\mu(x)^{1-q_2}dx\right)^{\frac{1}{q_2}}\\
\le&\ C\|b\|_{Lip_{\beta,\,\mu}}\mu(B_k)^{\frac{\beta}{n}}\mu(B_k)^{\frac{1}{q_2}}+|b_{B_k}-b_{B_j}|\left(\int_{B_k}\mu(x)^{1-q_2}dx\right)^{\frac{1}{q_2}}\\
\le&\ C\|b\|_{Lip_{\beta,\,\mu}}\mu(B_k)^{\frac{\beta}{n}}\mu(B_k)^{\frac{1}{q_2}}+C(k-j)\|b\|_{Lip_{\beta,\,\mu}}\mu(B_k)^{\frac{\beta}{n}}\frac{\mu(B_j)}{|B_j|}|B_k|\mu(B_k)^{\frac{1-q_2}{q_2}}\\
=&\ C\|b\|_{Lip_{\beta,\,\mu}}\mu(B_k)^{\frac{1}{q_1}}+C\|b\|_{Lip_{\beta,\,\mu}}(k-j)\mu(B_k)^{\frac{1}{q_1}}\frac{|B_k|}{|B_j|}\frac{\mu(B_j)}{\mu(B_k)}\\
\le&\ C\|b\|_{Lip_{\beta,\,\mu}}\mu(B_k)^{\frac{1}{q_1}}+C\|b\|_{Lip_{\beta,\,\mu}}(k-j)\mu(B_k)^{\frac{1}{q_1}}\frac{|B_k|}{|B_j|}\left(\frac{|B_j|}{|B_k|}\right)^{\delta}\\
\le&\ C\|b\|_{Lip_{\beta,\,\mu}}(k-j)\mu(B_k)^{\frac{1}{q_1}}2^{(k-j)n(1-\delta)}.
\end{aligned}$$

By Hölder's inequality and Lemma 2.3, denoting that $1/q_1+1/q_1'=1$, for all $j\in\mathbb{Z}$, we obtain

$$\begin{aligned}
\int_{E_j}|(f\chi_j)(y)|dy &\le \left(\int_{B_j}|(f\chi_j)(y)|^{q_1}\mu(y)dy\right)^{\frac{1}{q_1}}\left(\int_{B_j}\mu(y)^{1-q_1'}dy\right)^{\frac{1}{q_1'}}\\
&\le C\|f\chi_j\|_{L^{q_1}(\mu)}|B_j|\mu(B_j)^{-\frac{1}{q_1}}\\
&= C2^{jn}\|f\chi_j\|_{L^{q_1}(\mu)}\mu(B_j)^{-\frac{1}{q_1}}.
\end{aligned}$$

By the above two estimates and Lemma 2.1, for $j\le k-2$ we have

$$\begin{aligned}
D_1(j,k) &\le C\|b\|_{Lip_{\beta,\,\mu}}2^{-kn}(k-j)\mu(B_k)^{\frac{1}{q_1}}2^{(k-j)n(1-\delta)}2^{jn}\|f\chi_j\|_{L^{q_1}(\mu)}\mu(B_j)^{-\frac{1}{q_1}}\\
&= C\|b\|_{Lip_{\beta,\,\mu}}(k-j)2^{-(k-j)n\delta}\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\frac{1}{q_1}}\|f\chi_j\|_{L^{q_1}(\mu)}\\
&\le C\|b\|_{Lip_{\beta,\,\mu}}(k-j)2^{-(k-j)n\delta}\left(\frac{|B_k|}{|B_j|}\right)^{\frac{1}{q_1}}\|f\chi_j\|_{L^{q_1}(\mu)}\\
&= C\|b\|_{Lip_{\beta,\,\mu}}(k-j)2^{(k-j)n(\frac{1}{q_1}-\delta)}\|f\chi_j\|_{L^{q_1}(\mu)}.
\end{aligned}$$

By Hölder's inequality, Lemma 2.3 and Lemma 2.1, for $j\le k-2$ we have

$$
\begin{aligned}
D_2(j,k) &\le C2^{-kn}|B_k|\mu(B_k)^{\frac{1}{q_2}-1}\left(\int_{B_j}|b(y)-b_{B_j}|^{q_1'}\mu(y)^{\frac{1}{-q_1}q_1'}dy\right)^{\frac{1}{q_1'}}\\
&\quad\times\left(\int_{B_j}|(f\chi_j)(y)|^{q_1}\mu(y)dy\right)^{\frac{1}{q_1}}\\
&\le C\|b\|_{Lip_{\beta,\mu}}\mu(B_k)^{\frac{1}{q_2}-1}\mu(B_j)^{\frac{1}{q_1'}}\mu(B_j)^{\frac{\beta}{n}}\|f\chi_j\|_{L^{q_1}(\mu)}\\
&= C\|b\|_{Lip_{\beta,\mu}}\left(\frac{\mu(B_j)}{\mu(B_k)}\right)^{1-\frac{1}{q_2}}\|f\chi_j\|_{L^{q_1}(\mu)}\\
&\le C\|b\|_{Lip_{\beta,\mu}}\left(\frac{|B_j|}{|B_k|}\right)^{\delta(1-\frac{1}{q_2})}\|f\chi_j\|_{L^{q_1}(\mu)}\\
&= C\|b\|_{Lip_{\beta,\mu}}2^{(j-k)n\delta(1-\frac{1}{q_2})}\|f\chi_j\|_{L^{q_1}(\mu)}.
\end{aligned}
$$

Combining the estimates for $D_1(j,k)$ and $D_2(j,k)$, we obtain that if $j \le k-2$, then

$$
\begin{aligned}
&\|([b,T]f\chi_j)\chi_k\|_{L^{q_2}(\mu^{1-q_2})}\\
&\le C\|b\|_{Lip_{\beta,\mu}}(k-j)2^{(k-j)n(\frac{1}{q_1}-\delta)}\|f\chi_j\|_{L^{q_1}(\mu)}\\
&\quad+C\|b\|_{Lip_{\beta,\mu}}2^{(k-j)n\delta(\frac{1}{q_2}-1)}\|f\chi_j\|_{L^{q_1}(\mu)}\\
&\le C\|b\|_{Lip_{\beta,\mu}}(k-j)2^{(k-j)n(\frac{1}{q_1}-\delta)}\|f\chi_j\|_{L^{q_1}(\mu)}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
I_1 &\le C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p/n}\right.\\
&\quad\left.\times\left(\sum_{j=-\infty}^{k-2}\|b\|_{Lip_{\beta,\mu}}(k-j)2^{(k-j)n(\frac{1}{q_1}-\delta)}\|f\chi_j\|_{L^{q_1}(\mu)}\right)^p\right)^{\frac{1}{p}}\\
&= C\|b\|_{Lip_{\beta,\mu}}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}\left(\sum_{j=-\infty}^{k-2}\mu(B_j)^{\frac{\alpha}{n}}\right.\right.\\
&\quad\left.\left.\times\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\frac{\alpha}{n}}(k-j)2^{(k-j)n(\frac{1}{q_1}-\delta)}\|f\chi_j\|_{L^{q_1}(\mu)}\right)^p\right)^{\frac{1}{p}}.
\end{aligned}
$$

Let $W(j,k)=(k-j)2^{(k-j)\delta(\alpha-n(1-1/\widetilde{q}_1))}$ and $M(j,k)=\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\frac{\alpha}{n}}(k-j)2^{(k-j)n(\frac{1}{q_1}-\delta)}$. If $\alpha\ge 0$, then by Lemma 2.1,

$$
\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\alpha/n}\le C\left(\frac{|B_k|}{|B_j|}\right)^{\alpha/n}=C2^{(k-j)\alpha},
$$

so

$$\begin{aligned}M(j,k) &\le C2^{(k-j)\alpha}(k-j)2^{(k-j)n(\frac{1}{q_1}-\delta)}\\ &= C(k-j)2^{(k-j)[\alpha-n(\delta-\frac{1}{q_1})]}\\ &\le C(k-j)2^{(k-j)[\alpha-n(1-\frac{1}{\widetilde{q}_1})]}\\ &\le CW(j,k).\end{aligned}$$

If $\alpha < 0$, then by Lemma 2.1,

$$\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\alpha/n} \le C\left(\frac{|B_k|}{|B_j|}\right)^{\delta\alpha/n} = C2^{(k-j)\delta\alpha},$$

so

$$\begin{aligned}M(j,k) &\le C2^{(k-j)\delta\alpha}(k-j)2^{(k-j)n(\frac{1}{q_1}-\delta)}\\ &= C(k-j)2^{(k-j)\delta[\alpha-n(1-\frac{1}{\delta q_1})]}\\ &\le CW(j,k).\end{aligned}$$

Thus, in both cases $\alpha \ge 0$ and $\alpha < 0$, we have

$$I_1 \le C\|b\|_{Lip_{\beta,\mu}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}\left(\sum_{j=-\infty}^{k-2} W(j,k)\mu(B_j)^{\alpha/n}\|f\chi_j\|_{L^{q_1}(\mu)}\right)^p\right)^{\frac{1}{p}}.$$

If $0 < p \le 1$, then it follows from $\alpha < n(1-1/\widetilde{q}_1)$ that

$$\begin{aligned}I_1 &\le C\|b\|_{Lip_{\beta,\mu}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}\sum_{j=-\infty}^{k-2} W(j,k)^p\mu(B_j)^{\frac{\alpha p}{n}}\|f\chi_j\|^p_{L^{q_1}(\mu)}\right)^{\frac{1}{p}}\\ &\le C\|b\|_{Lip_{\beta,\mu}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{j=-\infty}^{k_0-2}\mu(B_j)^{\frac{\alpha p}{n}}\|f\chi_j\|^p_{L^{q_1}(\mu)}\left(\sum_{k=j+2}^{\infty} W(j,k)^p\right)\right)^{\frac{1}{p}}\\ &\le C\|b\|_{Lip_{\beta,\mu}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{j=-\infty}^{k_0}\mu(B_j)^{\frac{\alpha p}{n}}\|f\chi_j\|^p_{L^{q_1}(\mu)}\right)^{\frac{1}{p}}\\ &= C\|b\|_{Lip_{\beta,\mu}}\|f\|_{MK^{\alpha,\lambda}_{p,q_1}(\mu,\mu)}.\end{aligned}$$

When $1 < p < \infty$, by Hölder's inequality and $\alpha < n(1-1/\widetilde{q}_1)$, we have

$$\begin{aligned}I_1 \le C\|b\|_{Lip_{\beta,\mu}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0}&\left(\sum_{j=-\infty}^{k-2} W(j,k)\right)^{\frac{p}{p'}}\\ &\times\left(\sum_{j=-\infty}^{k-2}\mu(B_j)^{\frac{\alpha p}{n}}\|f\chi_j\|^p_{L^{q_1}(\mu)}W(j,k)\right)\Bigg)^{\frac{1}{p}}\end{aligned}$$

$$= C\|b\|_{Lip_{\beta,\,\mu}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0}\Bigg(\sum_{j=-\infty}^{k-2}\mu(B_j)^{\frac{\alpha p}{n}}\|f\chi_j\|^p_{L^{q_1}(\mu)}W(j,k)\Bigg)\Bigg)^{\frac{1}{p}}$$

$$\le C\|b\|_{Lip_{\beta,\,\mu}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{j=-\infty}^{k_0}\mu(B_j)^{\frac{\alpha p}{n}}\|f\chi_j\|^p_{L^{q_1}(\mu)}\Bigg(\sum_{k=j+2}^{\infty}W(j,k)\Bigg)\Bigg)^{\frac{1}{p}}$$

$$= C\|b\|_{Lip_{\beta,\,\mu}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{j=-\infty}^{k_0}\mu(B_j)^{\frac{\alpha p}{n}}\|f\chi_j\|^p_{L^{q_1}(\mu)}\Bigg)^{\frac{1}{p}}$$

$$= C\|b\|_{Lip_{\beta,\,\mu}}\|f\|_{MK^{\alpha,\lambda}_{p,q_1}(\mu,\mu)}.$$

For I_3, by analogy to the estimates of I_1, we find, for $j \ge k+2$,

$$D_1(j,k) \le C\|b\|_{Lip_{\beta,\,\mu}}(j-k)2^{(k-j)n(\frac{\delta}{q_1}-\frac{\beta}{n})}\|f\chi_j\|_{L^{q_1}(\mu)},$$

and

$$D_2(j,k) \le C\|b\|_{Lip_{\beta,\,\mu}}2^{(k-j)n\frac{1}{q_2}}\|f\chi_j\|_{L^{q_1}(\mu)}.$$

Combining the estimates for $D_1(j,k)$ and $D_2(j,k)$, we obtain that if $j \ge k+2$, then

$$\|([b,T]f\chi_j)\chi_k\|_{L^{q_2}(\mu^{1-q_2})}$$
$$\le C\|b\|_{Lip_{\beta,\,\mu}}(j-k)2^{(k-j)n(\frac{\delta}{q_1}-\frac{\beta}{n})}\|f\chi_j\|_{L^{q_1}(\mu)}+C\|b\|_{Lip_{\beta,\,\mu}}2^{(k-j)n\frac{1}{q_2}}\|f\chi_j\|_{L^{q_1}(\mu)}$$
$$\le C\|b\|_{Lip_{\beta,\,\mu}}(j-k)2^{(k-j)n(\frac{\delta}{q_1}-\frac{\beta}{n})}\|f\chi_j\|_{L^{q_1}(\mu)}.$$

So we have

$$I_3 \le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0}\mu(B_k)^{\alpha p/n}$$
$$\times\Bigg(\sum_{j=k+2}^{\infty}\|b\|_{Lip_{\beta,\,\mu}}(j-k)2^{(k-j)n(\frac{\delta}{q_1}-\frac{\beta}{n})}\|f\chi_j\|_{L^{q_1}(\mu)}\Bigg)^p\Bigg)^{\frac{1}{p}}$$
$$= C\|b\|_{Lip_{\beta,\,\mu}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0}\Bigg(\sum_{j=k+2}^{\infty}\mu(B_j)^{\frac{\alpha}{n}}$$
$$\times\bigg(\frac{\mu(B_k)}{\mu(B_j)}\bigg)^{\frac{\alpha}{n}}(j-k)2^{(k-j)n(\frac{\delta}{q_1}-\frac{\beta}{n})}\|f\chi_j\|_{L^{q_1}(\mu)}\Bigg)^p\Bigg)^{\frac{1}{p}}.$$

Let $V(j,k) = (j-k)2^{(k-j)\delta(\alpha+n/\widetilde{q}_2)}$ and $N(j,k) = \big(\frac{\mu(B_k)}{\mu(B_j)}\big)^{\frac{\alpha}{n}}(j-k)2^{(k-j)n(\frac{\delta}{q_1}-\frac{\beta}{n})}$.

If $\alpha \geq 0$, then by Lemma 2.1,

$$\begin{aligned} N(j,k) &\leq C2^{(k-j)\delta\alpha}(j-k)2^{(k-j)n(\frac{\delta}{q_1}-\frac{\beta}{n})} \\ &= C(j-k)2^{(k-j)\delta[\alpha+n(\frac{1}{q_1}-\frac{\beta}{n\delta})]} \\ &\leq C(j-k)2^{(k-j)\delta(\alpha+\frac{n}{\widetilde{q}_2})} \\ &= CV(j,k). \end{aligned}$$

If $\alpha < 0$, then by Lemma 2.1,

$$\begin{aligned} N(j,k) &\leq C2^{(k-j)\alpha}(j-k)2^{(k-j)n(\frac{\delta}{q_1}-\frac{\beta}{n})} \\ &= C(j-k)2^{(k-j)[\alpha+n(\frac{\delta}{q_1}-\frac{\beta}{n})]} \\ &\leq C(j-k)2^{(k-j)(\alpha+\frac{n}{\widetilde{q}_2})} \\ &\leq CV(j,k). \end{aligned}$$

Thus, in both cases $\alpha \geq 0$ and $\alpha < 0$, we have

$$\begin{aligned} I_3 &\leq C\|b\|_{Lip_{\beta,\mu}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}\bigg(\sum_{j=k+2}^{\infty} V(j,k)\mu(B_j)^{\alpha/n}\|f\chi_j\|_{L^{q_1}(\mu)}\bigg)^p\bigg)^{\frac{1}{p}} \\ &\leq C\|b\|_{Lip_{\beta,\mu}}\bigg(\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}\bigg(\sum_{j=k+2}^{k_0} V(j,k)\mu(B_j)^{\alpha/n}\|f\chi_j\|_{L^{q_1}(\mu)}\bigg)^p\bigg)^{\frac{1}{p}} \\ &\quad + \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}\bigg(\sum_{j=k_0+1}^{\infty} V(j,k)\mu(B_j)^{\alpha/n}\|f\chi_j\|_{L^{q_1}(\mu)}\bigg)^p\bigg)^{\frac{1}{p}}\bigg) \\ &:= C\|b\|_{Lip_{\beta,\mu}}(I_{31}+I_{32}). \end{aligned}$$

If $0 < p \leq 1$, then it follows from $\alpha > \lambda/\delta - n/\widetilde{q}_2 \geq -n/\widetilde{q}_2$ that

$$\begin{aligned} I_{31} &\leq \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}\sum_{j=k+2}^{k_0} V(j,k)^p\mu(B_j)^{\alpha p/n}\|f\chi_j\|_{L^{q_1}(\mu)}^p\bigg)^{\frac{1}{p}} \\ &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\bigg(\sum_{j=-\infty}^{k_0}\mu(B_j)^{\alpha p/n}\|f\chi_j\|_{L^{q_1}(\mu)}^p\bigg(\sum_{k=-\infty}^{j-2} V(j,k)^p\bigg)\bigg)^{\frac{1}{p}} \\ &= C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\bigg(\sum_{j=-\infty}^{k_0}\mu(B_j)^{\alpha p/n}\|f\chi_j\|_{L^{q_1}(\mu)}^p\bigg)^{\frac{1}{p}} \\ &= C\|f\|_{MK_{p,q_1}^{\alpha,\lambda}(\mu,\mu)}. \end{aligned}$$

When $1 < p < \infty$, by Hölder's inequality and $\alpha > \lambda/\delta - n/\widetilde{q}_2 \geq -n/\widetilde{q}_2$, we have

$$\begin{aligned}
I_{31} &\leq \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0}\Bigg(\sum_{j=k+2}^{k_0} V(j,k)\Bigg)^{\frac{p}{p'}}\Bigg(\sum_{j=k+2}^{k_0}\mu(B_j)^{\alpha p/n}\|f\chi_j\|^p_{L^{q_1}(\mu)}V(j,k)\Bigg)\Bigg)^{\frac{1}{p}}\\
&\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0}\sum_{j=k+2}^{k_0}\mu(B_j)^{\alpha p/n}\|f\chi_j\|^p_{L^{q_1}(\mu)}V(j,k)\Bigg)^{\frac{1}{p}}\\
&= C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{j=-\infty}^{k_0}\mu(B_j)^{\alpha p/n}\|f\chi_j\|^p_{L^{q_1}(\mu)}\Bigg(\sum_{k=-\infty}^{j-2} V(j,k)\Bigg)\Bigg)^{\frac{1}{p}}\\
&= C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{j=-\infty}^{k_0}\mu(B_j)^{\alpha p/n}\|f\chi_j\|^p_{L^{q_1}(\mu)}\Bigg)^{\frac{1}{p}}\\
&= C\|f\|_{MK^{\alpha,\lambda}_{p,q_1}(\mu,\mu)}.
\end{aligned}$$

Thus, in both cases $0 < p \leq 1$ and $1 < p < \infty$, we have

$$I_{31} \leq C\|f\|_{MK^{\alpha,\lambda}_{p,q_1}(\mu,\mu)}.$$

For all $0 < p < \infty$, it follows from $\alpha > \lambda/\delta - n/\widetilde{q}_2$ that

$$\begin{aligned}
I_{32} &\leq \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0}\Bigg(\sum_{j=k_0+1}^{\infty} V(j,k)2^{j\lambda}2^{-j\lambda}\\
&\qquad\times\Bigg(\sum_{l=-\infty}^{j}\mu(B_l)^{\alpha p/n}\|f\chi_l\|^p_{L^{q_1}(\mu)}\Bigg)^{1/p}\Bigg)^p\Bigg)^{\frac{1}{p}}\\
&\leq \|f\|_{MK^{\alpha,\lambda}_{p,q_1}(\mu,\mu)}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0}\Bigg(\sum_{j=k_0+1}^{\infty} V(j,k)2^{j\lambda}\Bigg)^p\Bigg)^{\frac{1}{p}}\\
&= \|f\|_{MK^{\alpha,\lambda}_{p,q_1}(\mu,\mu)}\sup_{k_0\in\mathbb{Z}}\Bigg(\sum_{k=-\infty}^{k_0} 2^{(k-k_0)\lambda p}\Bigg(\sum_{j=k_0+1}^{\infty} V(j,k)2^{(j-k)\lambda}\Bigg)^p\Bigg)^{\frac{1}{p}}\\
&\leq \|f\|_{MK^{\alpha,\lambda}_{p,q_1}(\mu,\mu)}\sup_{k_0\in\mathbb{Z}}\Bigg(\sum_{k=-\infty}^{k_0} 2^{(k-k_0)\lambda p}\Bigg(\sum_{j=k_0}^{\infty}(j-k)2^{(k-j)\delta(\alpha+n/\widetilde{q}_2-\lambda/\delta)}\Bigg)^p\Bigg)^{\frac{1}{p}}\\
&\leq C\|f\|_{MK^{\alpha,\lambda}_{p,q_1}(\mu,\mu)}\sup_{k_0\in\mathbb{Z}}\Bigg(\sum_{k=-\infty}^{k_0}(k_0-k)^p 2^{(k-k_0)\delta(\alpha+n/\widetilde{q}_2)p}\Bigg)^{1/p}\\
&= C\|f\|_{MK^{\alpha,\lambda}_{p,q_1}(\mu,\mu)}.
\end{aligned}$$

Thus,

$$I_3 \leq C\|b\|_{Lip_{\beta,\mu}}\|f\|_{MK^{\alpha,\lambda}_{p,q_1}(\mu,\mu)}.$$

Combining the estimates for I_1, I_2 and I_3, we complete the proof of Theorem 2.1.

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