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HOMOCLINIC ORBITS FOR THE FIRST-ORDER HAMILTONIAN SYSTEM WITH SUPERQUADRATIC NONLINEARITY

Wen Zhang, Xianhua Tang* and Jian Zhang

Abstract. In this paper, we consider the following first-order Hamiltonian system

$$\dot{z} = \mathcal{J} H_z(t, z),$$

where $H \in C^1(\mathbb{R}\times\mathbb{R}^{2N}, \mathbb{R})$ is the form $H(t, z) = \frac{1}{2}L(t)z\cdot z+R(t, z)$. By applying the variant generalized weak linking theorem for strongly indefinite problem developed by Schechter and Zou, we establish nontrivial and ground state solutions for the above system under conditions weaker than those in [39].

1. Introduction and Main Results

We consider the following first-order Hamiltonian system

$$\dot{z} = \mathcal{J} H_z(t, z), \tag{1.1}$$

where $z = (p, q) \in \mathbb{R}^N \times\mathbb{R}^N = \mathbb{R}^{2N}$, $\mathcal{J} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$, and $H \in C^1(\mathbb{R}\times\mathbb{R}^{2N}, \mathbb{R})$ is the form

$$H(t, z) = \frac{1}{2}L(t)z \cdot z + R(t, z)$$

with $L(t) \in C(\mathbb{R}, \mathbb{R}^{4N^2})$ being a $2N \times 2N$ symmetric matrix valued function, and $R \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ is superquadratic at infinity. In this paper, we are concerned with the existence of homoclinic orbits. Here by a homoclinic orbit of system (1.1) we mean a solution of the system satisfying $z(t) \not\equiv 0$ and $z(t) \to 0$ as $|t| \to \infty$.

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*Corresponding author.

For convenience, we first introduce the following Hamiltonian operator

$$A = -(\mathcal{J}\frac{d}{dt} + L).$$

As a special case of dynamical systems, Hamiltonian systems are very important in the study of gas dynamics, finance, fluid mechanics, relativistic mechanics and nuclear physics (see [1]). During the last decades, many authors were devoted to the existence of periodic and homoclinic solutions for Hamiltonian systems via modern variational methods. For example, see [3, 5, 6, 8, 16, 18, 30, 31, 32, 33, 34, 38] for the second order systems, and [2, 4, 7, 9, 10, 11, 12, 13, 15, 17, 19, 20, 21, 24, 26, 27, 28, 35, 36, 37, 39, 40, 41, 42] for the first order systems and infinite dimensional systems. Coti-Zelati, Ekeland and Séré first considered the system (1.1) in [2]. Under the classical Ambrosetti-Rabinowitz growth condition, they proved the existence and multiplicity of homoclinic orbits for strictly convex Hamiltonian system. The existence of infinitely many homoclinic orbits was established in Séré [20], which generalized the result in [2]. Subsequently, Hofer and Wysocki [15] removed the convexity assumption and obtained the existence of homoclinic orbits. Using the subharmonic method, Tanaka [26] also removed the convexity assumption, and proved that the system (1.1) has at least one homoclinic orbit.

Recently, suppose that $R(t, z)$ and $L(t)$ depend periodically on t, the existence and multiplicity of homoclinic orbit for system (1.1) was considered in [4, 7, 9, 13, 23, 24, 35, 36]. However, without the assumption of periodicity, the problem is quite different in nature, and the main difficulty of such type problem is the lack of compactness of the Sobolev embeddings. By applying a variety of techniques, some authors considered the non-periodic case, we refer the readers to [10, 11, 12, 17, 29, 37, 39] and references therein.

To continue the discussion, we define some notations. For any real function $q(x)$ will be regarded as a symmetric matrix $q(x)I_{2N\times 2N}$ and $\mathcal{J}_0 := \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}$, for two given matrix valued functions $M_1(t)$ and $M_2(t)$, we say that $M_1(t) \le M_2(t)$ if and only if $\max_{\xi\in\mathbb{R}^{2N},|\xi|=1}(M_1(t) - M_2(t))\xi \cdot \xi \le 0$, and $M_1(t) > M_2(t)$ if and only if $M_1(t) \le M_2(t)$ does not hold. Here we will mention the recent work of Zhang et al.[39]. Based on the generalized Nehari manifold method developed by Szulkin and Weth [21](see also [22]), the authors obtained the existence of ground state solution under the following assumptions:

(L_0) $L(t) \in C(\mathbb{R} \times \mathbb{R}^{2N\times 2N})$, there exists $r_0 > 0$ such that, for any $h > 0$,

$$meas\,(\{t \in \mathbb{R} : |t - t_1| \le r_0,\quad \mathcal{J}_0 L(t) < h\}) \to 0,\ as\ |t_1| \to \infty,$$

where $meas$ denotes the Lebesgue measure;

(R_1) $R \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, [0, \infty))$, $R(t, z) > 0$ for all $z \ne 0$ and $|R_z(t, z)| \le c(1 + |z|^{p-1})$ for some $c > 0$, $p > 2$;

(R_2) $R_z(t,z) = o(|z|)$ as $|z| \to 0$ uniformly in t;

(R_3) $\frac{R(t,z)}{|z|^2} \to \infty$ as $|z| \to \infty$ uniformly in t, and $\hat{R}(t,z) := \frac{1}{2}R_z(t,z)\cdot z - R(t,z) > 0$ for all $z \neq 0$;

(R_4) $(R_z(t,z)\cdot w)(z\cdot w) \geq 0$ uniformly in t for all $z, w \in \mathbb{R}^{2N}$, where $z \cdot w$ denotes the usual Euclidean scalar product;

(R_5) $R(t,z) = R(t,w)$ and $R_z(t,z)\cdot w \leq R_z(t,z)\cdot z$ uniformly in t if $|z| = |w|$, if in addition $z \neq w$, then $R_z(t,z)\cdot w < R_z(t,z)\cdot z$;

(R_6) $R_z(t,z)\cdot w \neq R_z(t,z)\cdot z$ uniformly in t if $|z| = |w|$ and $z \cdot w \neq 0$.

Motivated by the above facts, in the present paper, we continue to consider the non-periodic system (1.1) without Ambrosetti-Rabinowitz condition. Our main purpose is to weaken the above conditions to generalize and improve the result in [39]. More precisely, we make the following assumptions for nonlinearity:

(H_1) $R \in C^1(\mathbb{R}\times\mathbb{R}^{2N}, [0,\infty))$ and $|R_z(t,z)| \leq c(1+|z|^{p-1})$ for some $c > 0$, $p > 2$;

(H_2) $|R(t,z)| \leq \frac{1}{2}\gamma|z|^2$ if $|z| < \delta$ for some $0 \leq \gamma < \lambda_1$, where $\delta > 0$, $z \in \mathbb{R}^{2N}$, and λ_1 will be defined later in (2.1);

(H_3) $\frac{R(t,z)}{|z|^2} \to \infty$ as $|z| \to \infty$ uniformly in t;

(H_4) $R(t, z+u) - R(t,z) - rR_z(t,z)u + \frac{(r-1)^2}{2}R_z(t,z)z \geq -W_1(t)$, $r \in [0,1]$, $W_1(t) \in L^1(\mathbb{R})$ and $u, z \in \mathbb{R}^{2N}$.

The main results of this paper are the following theorems.

Theorem 1.1. *Let (L_0) and $(H_1) - (H_4)$ be satisfied, then system (1.1) has at least one solution.*

Theorem 1.2. *Let $\mathcal{M}$ be the collection of solutions of system (1.1). Then there is a solution that minimizes the energy functional*

$$\Phi(z) := \int_{\mathbb{R}} \left(\frac{1}{2}Az\cdot z - R(t,z)\right) dt,\ z \in E$$

over $\mathcal{M}$, where E will be defined later. In addition, if

$$|R_z(t,z)| = o(|z|),\ \textit{as}\ |z| \to 0$$

uniformly in t, then there is a nontrivial solution that minimize the energy functional over $\mathcal{M} \setminus \{0\}$.

Remark 1.3. It is obvious that conditions $(H_1)-(H_3)$ are weaker than conditions $(R_1)-(R_3)$. Now we show that conditions $(R_4)-(R_6)$ imply (H_4). In fact, under conditions $(R_4)-(R_6)$, Lemma 3.3 in [39] shows the following relation holds

$$R(t,(s+1)z+w)-R(t,z)-R_z(t,z)\left(s(\frac{s}{2}+1)z+(s+1)w\right)\geq 0,\ s\geq -1. \tag{1.2}$$

If we take $r=s+1$ and $w=(1-r)z+u$, then

$$R(t,z+u)-R(t,z)-rR_z(t,z)u+\frac{(r-1)^2}{2}R_z(t,z)z\geq 0,\ r\geq 0,$$

which implies (H_4) holds if we take $W(x)=0$ and $r\in[0,1]$.

Observe that the energy functional of system (1.1) is strongly indefinite, in order to obtain the existence of ground states, Zhang et al.[39] used the generalized Nehari manifold method in [21, 22]. It is well known that (1.2) plays a very important role in generalized Nehari manifold method, see [21, Lemma 2.2]. However, (1.2) is no longer valid under the conditions we considered in this paper, hence their arguments collapses in this case. In order to successfully carry out our work, the tool we used is the variant generalized weak linking theorem for strongly indefinite problem developed by Schechter and Zou [25]. Therefore, from the above argument and Remark 1.3, we see that our results improve and generalize the result in [39] by weakening the corresponding conditions.

The rest of the present paper is organized as follows. In Section 2, we establish the variational framework associated with (1.1), and we also give some preliminary lemmas, which are useful in the proofs of our main results. In Section 3, we give the detailed proofs of our main results.

2. Variational Setting and Preliminary Lemmas

Below by $\|\cdot\|_q$ we denote the usual L^q- norm, $(\cdot,\cdot)_2$ denote the usual L^2 inner product, c_i, C, C_i stand for different positive constants. Let $\sigma(A)$, $\sigma_d(A)$ be the spectrum of A, the discrete spectrum of A, respectively. Observe that, since we have assumed (L_0) on $L(t)$, A is a selfdajoint operater on $L^2:=L^2(\mathbb{R},\mathbb{R}^{2N})$ with $\mathcal{D}(A)\subset H^1(\mathbb{R},\mathbb{R}^{2N})$. In order to establish a variational setting for system (1.1), we have the following Lemma due to [17].

Lemma 2.1. ([17], Lemma 2.2). *Suppose (L_0) holds. Then $\sigma(A)=\sigma_d(A)$.*

From Lemma 2.1, we know that the Hamiltonian operator A has a sequence of eigenvalues

$$\cdots\lambda_{-k}\leq\cdots\leq\lambda_{-1}\leq 0<\lambda_1\leq\cdots\leq\lambda_k\cdots, \tag{2.1}$$

with $\lambda_{\pm k} \to \pm\infty$ as $k \to \infty$, and corresponding eigenfunctions $\{e_{\pm k}\}_{k\in\mathbb{N}}$ form an orthogonal basis in L^2. Observe that we have an orthogonal decomposition

$$L^2 = L^- \oplus L^0 \oplus L^+ \text{ and } z = z^- + z^0 + z^+,$$

such that A is negative definite on L^- and positive definite on L^+ and $L^0 = \ker A$. Let $P^0 : L^2 \to L^0$ be the projection. Set $E := \mathcal{D}(|A|^{\frac{1}{2}})$ be the domain of the selfadjoint operator $|A|^{\frac{1}{2}}$ which is a Hilbert space equipped with the inner product

$$\langle z, w\rangle = (|A|^{\frac{1}{2}}z, |A|^{\frac{1}{2}}w)_2 + (P^0 z, P^0 w)_2$$

and the norm $\|z\| = \langle z, z\rangle^{\frac{1}{2}}$. Let $E^{\pm} := \overline{\text{span}\{e_{\pm k}\}}_{k\in\mathbb{N}}$, $E^0 = \ker A$. Clearly, E^-, E^0 and E^+ are orthogonal with respect to the products $(\cdot,\cdot)_2$ and $\langle\cdot,\cdot\rangle$. Hence

$$E = E^- \oplus E^0 \oplus E^+$$

is an orthogonal decomposition of E. Moreover, it is easy to prove the following embedding theorem by Lemma 2.1.

Lemma 2.2. ([17], Lemma 2.3). *E embeds continuously into* $H^{\frac{1}{2}} := H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}^{2N})$. *Moreover, E embeds compactly into* $L^p := L^p(\mathbb{R}, \mathbb{R}^{2N})$ *for all* $p \in [2, \infty)$.

Next, on E we define the following functional

$$\Phi(z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \Psi(z), \tag{2.2}$$

where $\Psi(z) = \int_{\mathbb{R}} R(t, z)dt$. Lemma 2.1 implies that Φ is strongly indefinite, and our hypotheses imply that $\Phi \in C^1(E, \mathbb{R})$, and a standard argument shows that critical points of Φ are solutions of system (1.1) (see [14, 43]).

The following abstract critical point theorem plays an important role in proving our main result. Let E be a Hilbert space with norm $\|\cdot\|$ and have an orthogonal decomposition $E = N \oplus N_\perp, N \subset E$ being a closed and separable subspace. There exists a norm $|v|_\omega \le \|v\|$ for all $v \in N$ and induces a topology equivalent to the weak topology of N on a bounded subset of N. For $z = v + w \in E = N \oplus N^\perp$ with $v \in N, w \in N^\perp$, we define $|z|_\omega^2 = |v|_\omega^2 + \|w\|^2$. Particularly, if $z_n = v_n + w_n$ is $|\cdot|_\omega$-bounded and $z_n \xrightarrow{|\cdot|_\omega} z$, then $v_n \rightharpoonup v$ weakly in N, $w_n \to w$ strongly in $N^\perp$, $z_n \rightharpoonup v + w$ weakly in E ([25]).

Let $E = E^- \oplus E^0 \oplus E^+$, $e \in E^+$ with $\|e\| = 1$. Let $N := E^- \oplus E^0 \oplus \mathbb{R}e$ and $E_1^+ := N^\perp = (E^- \oplus E^0 \oplus \mathbb{R}e)^\perp$. For $R > 0$, let

$$Q := \{z := z^- + z^0 + se : s \in \mathbb{R}^+, z^- + z^0 \in E^- \oplus E^0, \|z\| < R\}.$$

For $0 < s_0 < R$, we define

$$D := \{z := se + w^+ : s \geq 0, w^+ \in E_1^+, \|se + w^+\| = s_0\}.$$

For $\Phi \in C^1(E, \mathbb{R})$, define

$$\Gamma := \left\{ h : \begin{array}{c} h : \mathbb{R} \times \bar{Q} \to E \text{ is } |\cdot|_\omega \text{ -continuous;} \\ h(0, z) = z \text{ and } \Phi(h(s, z)) \leq \Phi(z) \text{ for all } z \in \bar{Q}; \\ \text{For any}(s_0, z_0) \in \mathbb{R} \times \bar{Q}, \text{ there is a } |\cdot|_\omega \text{ -neighborhood;} \\ U(s_0, z_0) \ s.t. \ \{z - h(t, z) : (t, z) \in U(s_0, z_0) \cap (\mathbb{R} \times \bar{Q})\} \subset E_{fin}. \end{array} \right\}$$

where E_{fin} denotes various finite-dimensional subspaces of E; $\Gamma \neq 0$ since $id \in \Gamma$.

The variant weak linking theorem is:

Lemma 2.3. *The family of C^1-functional Φ_λ has the form*

$$\Phi_\lambda(z) := \lambda K(z) - J(z), \ \forall \lambda \in [1, \lambda_0],$$

where $\lambda_0 > 1$. Assume that

(a) $K(z) \geq 0, \forall\ z \in E, \Phi_1 = \Phi$*;*

(b) $|J(z)| + K(z) \to \infty$ *as* $\|z\| \to \infty$*;*

(c) Φ_λ *is* $|\cdot|_\omega$*-upper semicontinuous,* Φ'_λ *is weakly sequentially continuous on* E*,* Φ_λ *maps bounded sets to bounded sets;*

(d) $\sup_{\partial Q} \Phi_\lambda < \inf_D \Phi_\lambda, \forall\ \lambda \in [1, \lambda_0]$.

Then for almost all $\lambda \in [1, \lambda_0]$, there exists a sequences a sequences $\{z_n\}$ such that

$$\sup_n \|z_n\| < \infty, \ \Phi'_\lambda(z_n) \to 0, \ \Phi_\lambda(z_n) \to c_\lambda,$$

where

$$c_\lambda := \inf_{h \in \Gamma} \sup_{z \in \bar{Q}} \Phi_\lambda(h(1, z)) \in [\inf_D \Phi_\lambda, \sup_{\bar{Q}} \Phi_\lambda].$$

In order to apply Lemma 2.3, we shall prove a few Lemmas. We pick λ_0 such that $1 < \lambda_0 < \min[2, \frac{\lambda_1}{\gamma}]$. For $1 \leq \lambda \leq \lambda_0$, we consider

$$\Phi_\lambda(z) := \frac{\lambda}{2}\|z^+\|^2 - \left(\frac{1}{2}\|z^-\|^2 + \int_{\mathbb{R}} R(t, z(t))dt\right) := \lambda K(z) - J(z). \tag{2.3}$$

It is easy to see that Φ_λ satisfies condition (a) in Lemma 2.3. To see (c), if $z_n \xrightarrow{|\cdot|_\omega} z$, and $\Phi_\lambda(z_n) \geq c$, then $z_n^+ \to z^+$ and $z_n^- \rightharpoonup z^-$ in E, $z_n \to z$ $a.e.$ on $\mathbb{R}$, going to a subsequence if necessary. Using Fatou's lemma, we know $\Phi_\lambda(z) \geq c$, which means that Φ_λ is $|\cdot|_\omega$-upper semicontinuous; Φ'_λ is weakly sequentially continuous on E is due to [43].

Lemma 2.4. *Under the assumptions of Theorem* 1.1*, then*

$$J(z) + K(z) \to \infty \text{ as } \|z\| \to \infty.$$

Proof. Suppose to the contrary that there exists $\{z_n\}$ with $\|z_n\| \to \infty$ such that $J(z_n) + K(z_n) \leq M$ for some $M > 0$. Let $w_n = \frac{z_n}{\|z_n\|} = w_n^- + w_n^0 + w_n^+$, then $\|w_n\| = 1$ and

$$\begin{aligned} \frac{M}{\|z_n\|^2} &\geq \frac{K(z_n) + J(z_n)}{\|z_n\|^2} \\ &= \frac{1}{2}(\|w_n^+\|^2 + \|w_n^-\|^2) + \int_{\mathbb{R}} \frac{R(t, z_n)}{\|z_n\|^2} dt \\ &= \frac{1}{2}(\|w_n\|^2 - \|w_n^0\|^2) + \int_{\mathbb{R}} \frac{R(t, z_n)}{\|z_n\|^2} dt. \end{aligned} \tag{2.4}$$

Going to a subsequence if necessary, we may assume $w_n \rightharpoonup w$, $w_n^- \rightharpoonup w^-$, $w_n^+ \rightharpoonup w^+$, $w_n^0 \to w^0$ and $w_n(x) \to w(x)$ on $\mathbb{R}$. If $w^0 = 0$, by (H_1) and (2.4) we have

$$\frac{1}{2}\|w_n\|^2 + \int_{\mathbb{R}} \frac{R(t, z_n)}{\|z_n\|^2} dt \leq \frac{1}{2}\|w_n^0\|^2 + \frac{M}{\|z_n\|^2},$$

which implies $\|w_n\| \to 0$, this contradicts with $\|w_n\| = 1$. If $w^0 \neq 0$, then $w \neq 0$. Therefore, $|z_n| = |w_n|\|z_n\| \to \infty$. By (H_1), (H_3) and Fatou's lemma we have

$$\int_{\mathbb{R}} \frac{R(t, z_n)}{|z_n|^2} |w_n| dt \to \infty.$$

Hence by (2.4) again, we obtain $0 \geq +\infty$, a contradiction. The proof is complete. ∎

Therefore, Lemma 2.4 implies condition (b) holds. To continue the discussion, we still need to verify condition (d), that is, the following two Lemmas:

Lemma 2.5. *Under the assumptions of Theorem* 1.1*, there are two positive constants* $\kappa, \rho > 0$ *such that*

$$\Phi_\lambda(z) \geq \kappa, \ z \in E^+, \ \|z\| = \rho, \ \lambda \in [1, \lambda_0].$$

Proof. It is easy to see that

$$\|z\|^2 = (Az, z)_2 \geq \lambda_1 \|z\|_2^2, \ \forall z \in E^+, \tag{2.5}$$

where λ_1 is defined in (2.1).

For any $z \in E^+$, by (H_1), (H_2), (2.5) and Lemma 2.2, we have

$$\begin{aligned}
\Phi_\lambda(z) &= \frac{\lambda}{2}\|z\|^2 - \int_{\mathbb{R}} R(t,z)dt \\
&\geq \frac{1}{2}\|z\|^2 - \int_{\{|z|<\delta\}} R(t,z)dt - \int_{\{|z|\geq\delta\}} R(t,z)dt \\
&\geq \frac{1}{2}\|z\|^2 - \frac{1}{2}\gamma\int_{\{|z|<\delta\}} |z|^2 dt - c\int_{\{|z|\geq\delta\}} (|z|^2 + |z|^p)dt \\
&\geq \frac{1}{2}\|z\|^2 - \frac{\gamma}{\lambda_1}\frac{1}{2}\|z\|^2 - C'\|z\|^p \\
&= \frac{1}{2}\|z\|^2(1 - \frac{\gamma}{\lambda_1} - 2C'\|z\|^{p-2}),\ 0 \leq \gamma < \lambda_1.
\end{aligned}$$

This implies the conclusion if we take $\|z\|$ sufficiently small. ∎

Lemma 2.6. *Under the assumptions of Theorem* 1.1*, there exists a constant* $R > 0$ *such that*

$$\Phi_\lambda(z) \leq 0,\ z \in \partial Q_R,\ \lambda \in [1, \lambda_0],$$

where

$$Q_R := \{z := v + se : s \geq 0, v \in E^- \oplus E^0, e \in E^+ \textit{ with } \|e\| = 1, \|z\| \leq R\}.$$

Proof. By contradiction, we suppose that there exit $R_n \to \infty$, $\lambda_n \in [1, \lambda_0]$ and $z_n = v_n + s_n e = v_n^- + v_n^0 + s_n e \in \partial Q_{R_n}$ such that $\Phi_{\lambda_n}(z_n) > 0$. If $s_n = 0$, by (H_1), we get

$$\Phi_{\lambda_n}(z_n) = -\frac{1}{2}\|v_n^-\|^2 - \int_{\mathbb{R}} R(t, z_n)dt \leq -\frac{1}{2}\|v_n^-\|^2 \leq 0.$$

Therefore,

$$s_n \neq 0 \text{ and } \|z_n\|^2 = \|v_n\|^2 + s_n^2.$$

Let

$$\tilde{z}_n = \frac{z_n}{\|z_n\|} = \tilde{s}_n e + \tilde{v}_n,$$

then

$$\|\tilde{z}_n\|^2 = \|\tilde{v}_n\|^2 + \tilde{s}_n^2 = 1.$$

Thus, passing to a subsequence, we may assume

$$\begin{aligned}
&\tilde{s}_n \to \tilde{s},\ \lambda_n \to \lambda, \\
&\tilde{z}_n = \frac{z_n}{\|z_n\|} = \tilde{s}_n e + \tilde{v}_n \rightharpoonup \tilde{z} \text{ in } E, \\
&\tilde{z}_n \to \tilde{z} \text{ a.e. on } \mathbb{R}.
\end{aligned}$$

It follows from $\Phi_{\lambda_n}(z_n) > 0$ and the definition of Φ that

$$
\begin{aligned}
0 < \frac{\Phi_{\lambda_n}(z_n)}{\|z_n\|^2} &= \frac{1}{2}(\lambda_n \tilde{s}_n^2 - \|\tilde{v}_n\|^2) - \int_{\mathbb{R}} \frac{R(t, z_n)}{|z_n|^2} |\tilde{z}_n|^2 dt \\
&= \frac{1}{2}[(\lambda_n + 1)\tilde{s}_n^2 - 1] - \int_{\mathbb{R}} \frac{R(t, z_n)}{|z_n|^2} |\tilde{z}_n|^2 dt.
\end{aligned} \tag{2.6}
$$

From (H_1) and (2.6), we know that

$$(\lambda + 1)\tilde{s}^2 - 1 \geq 0,$$

that is

$$\tilde{s}^2 \geq \frac{1}{1 + \lambda} \geq \frac{1}{1 + \lambda_0} > 0.$$

Thus $\tilde{z} \neq 0$. It follows from (H_3) and Fatou's lemma that

$$\int_{\mathbb{R}} \frac{R(t, z_n)}{|z_n|^2} |\tilde{z}_n|^2 dt \to \infty \text{ as } n \to \infty,$$

which contradicts to (2.6). The proof is complete. ■

Hence, Lemmas 2.5 and 2.6 imply condition (d) of Lemma 2.3 holds. Applying Lemma 2.3, we soon obtain the following fact:

Lemma 2.7. *Under the assumptions of Theorem* 1.1, *for almost all* $\lambda \in [1, \lambda_0]$, *there exists a sequence* $\{z_n\}$ *such that*

$$\sup_n \|z_n\| < \infty, \ \Phi'_\lambda(z_n) \to 0, \ \Phi_\lambda(z_n) \to c_\lambda,$$

where the definition of c_λ *is given in Lemma* 2.3.

Lemma 2.8. *Under the assumptions of Theorem* 1.1, *for almost all* $\lambda \in [1, \lambda_0]$, *there exists a* $z_\lambda \in E$ *such that*

$$\Phi'_\lambda(z_\lambda) = 0, \ \Phi_\lambda(z_\lambda) = c_\lambda.$$

Proof. Let $\{z_n\}$ be the sequence obtained in Lemma 2.7. Since $\{z_n\}$ is bounded, we can assume $z_n \rightharpoonup z_\lambda$ in E and $z_n \to z_\lambda$ a.e. on $\mathbb{R}$. By Lemma 2.7 and the fact Φ'_λ is weakly sequentially continuous, we have

$$\langle \Phi'_\lambda(z_\lambda), \varphi \rangle = \lim_{n\to\infty} \langle \Phi'_\lambda(z_n), \varphi \rangle = 0, \ \forall \varphi \in E.$$

That is $\Phi'_\lambda(z_\lambda) = 0$. By Lemma 2.7, we have

$$\Phi_\lambda(z_n) - \frac{1}{2}\langle \Phi'_\lambda(z_n), z_n \rangle = \int_{\mathbb{R}} \left[\frac{1}{2} R_z(t, z_n) z_n - R(t, z_n) \right] dt \to c_\lambda.$$

On the other hand, by Lemma 2.2, it is easy to prove that

$$\int_{\mathbb{R}} \frac{1}{2} R_z(t, z_n) z_n dt \to \int_{\mathbb{R}} \frac{1}{2} R_z(t, z_\lambda) z_\lambda dt \tag{2.7}$$

and

$$\int_{\mathbb{R}} R(t, z_n) dt \to \int_{\mathbb{R}} R(t, z_\lambda) dt, \tag{2.8}$$

Therefore, by (2.7), (2.8) and the fact $\Phi'_\lambda(z_\lambda) = 0$, we obtain

$$\Phi_\lambda(z_\lambda) = \Phi_\lambda(z_\lambda) - \frac{1}{2}\langle \Phi'_\lambda(z_\lambda), z_\lambda \rangle = \int_{\mathbb{R}} \left[\frac{1}{2} R_z(t, z_\lambda) z_\lambda - R(t, z_\lambda) \right] dt = c_\lambda.$$

The proof is complete. ■

Applying Lemma 2.8, we obtain the following fact:

Lemma 2.9. *Under the assumptions of Theorem* 1.1*, for almost all* $\lambda \in [1, \lambda_0]$*, there exists sequences* $z_n \in E$ *and* $\lambda_n \in [1, \lambda_0]$ *with* $\lambda_n \to \lambda$ *such that*

$$\Phi'_{\lambda_n}(z_n) = 0, \ \Phi_{\lambda_n}(z_n) = c_{\lambda_n}.$$

Lemma 2.10. *Under the assumptions of Theorem* 1.1*, then*

$$\int_{\mathbb{R}} \left[R(t, z) - R(t, rw) + r^2 R_z(t, z) w - \frac{1 + r^2}{2} R_z(t, z) z \right] dt \le C,$$

where $z \in E, w \in E^+, 0 \le r \le 1$ *and the constant* C *does not depend on* z, w, r.

Proof. This follows from (H_4) if we take $z = z$ and $u = rw - z$. ■

Lemma 2.11. *Under the assumptions of Theorem* 1.1*, the sequences* $\{z_n\}$ *given in Lemma* 2.9 *are bounded.*

Proof. Suppose to the contrary that $\{z_n\}$ is unbounded. Without loss of generality, we can assume that $\|z_n\| \to \infty$ as $n \to \infty$. Let $v_n = \frac{z_n}{\|z_n\|} = v_n^+ + v_n^0 + v_n^-$, then $\|v_n\| = 1$. Going to a subsequence if necessary, we can assume that $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^p(\mathbb{R})$ for $p \in [2, \infty)$, $v_n \rightharpoonup v$ a.e. on $\mathbb{R}$. For v, we have only the following two cases: $v \neq 0$ or $v = 0$.

Case 1. $v \neq 0$. It follows from (H_3) and Fatou's Lemma that

$$\int_{\mathbb{R}} \frac{R(t, z_n)}{\|z_n\|^2} dt = \int_{\mathbb{R}} \frac{R(t, z_n)}{|z_n|^2} |v_n|^2 dt \to \infty \text{ as } n \to \infty,$$

which, together with Lemma 2.5, Lemma 2.9, thus

$$0 \leq \frac{c_{\lambda_n}}{\|z_n\|^2} = \frac{\Phi_{\lambda_n}(z_n)}{\|z_n\|^2} = \frac{\lambda_n}{2}\|v_n^+\|^2 - \frac{1}{2}\|v_n^-\|^2 - \int_{\mathbb{R}} \frac{R(t,z_n)}{\|z_n\|^2} dt \to -\infty \text{ as } n \to \infty.$$

It is a contradiction.

Case 2. $v = 0$. We claim that there exist a constant c independent of z_n and λ_n such that

$$\Phi_{\lambda_n}(rz_n^+) - \Phi_{\lambda_n}(z_n) \leq c, \ \forall r \in [0,1]. \tag{2.9}$$

Since

$$\frac{1}{2}\langle \Phi'_{\lambda_n}(z_n), \varphi\rangle = \frac{1}{2}\lambda_n(z_n^+, \varphi^+) - \frac{1}{2}(z_n^-, \varphi^-) - \frac{1}{2}\int_{\mathbb{R}} R_z(t,z_n)\varphi dt = 0, \ \forall \varphi \in E,$$

it follows from the definition of Φ that

$$\begin{aligned} &\Phi_{\lambda_n}(rz_n^+) - \Phi_{\lambda_n}(z_n) \\ &= \frac{1}{2}\lambda_n(r^2-1)\|z_n^+\|^2 + \frac{1}{2}\|z_n^-\|^2 + \int_{\mathbb{R}} \left[R(t,z_n) - R(t,rz_n^+)\right] dt \\ &\quad + \frac{1}{2}\lambda_n(z_n^+, \varphi^+) - \frac{1}{2}(z_n^-, \varphi^-) - \frac{1}{2}\int_{\mathbb{R}} R_z(t,z_n)\varphi dt. \end{aligned} \tag{2.10}$$

Take

$$\varphi = (r^2+1)z_n^- - (r^2-1)z_n^+ + (r^2+1)z_n^0 = (r^2+1)z_n - 2r^2 z_n^+,$$

which together with Lemma 2.10 and (2.10) implies that

$$\begin{aligned} &\Phi_{\lambda_n}(rz_n^+) - \Phi_{\lambda_n}(z_n) \\ &= -\frac{1}{2}\|z_n^-\|^2 + \int_{\mathbb{R}} \left[R(t,z_n) - R(t,rz_n^+) + r^2 R_z(t,z_n)z_n^+ - \frac{1+r^2}{2} R_z(t,z_n)z_n\right] dt \\ &\leq C. \end{aligned}$$

Hence, (2.9) holds.

Let C_0 be a constant and take

$$r_n := \frac{C_0}{\|z_n\|} \to 0 \text{ as } n \to \infty.$$

Therefore, (2.9) implies that

$$\Phi_{\lambda_n}(r_n z_n^+) - \Phi_{\lambda_n}(z_n) \leq C$$

for all sufficiently large n. From $v_n^+ = \frac{z_n^+}{\|z_n\|}$ and Lemma 2.9 that

$$\Phi_{\lambda_n}(C_0 v_n^+) \le C' \tag{2.11}$$

for all sufficiently large n. Note that Lemma 2.5, Lemma 2.9 and (H_1) imply that

$$\begin{aligned} 0 \le \frac{c_{\lambda_n}}{\|z_n\|^2} = \frac{\Phi_{\lambda_n}(z_n)}{\|z_n\|^2} &= \frac{\lambda_n}{2}\|v_n^+\|^2 - \frac{1}{2}\|v_n^-\|^2 - \frac{\int_{\mathbb{R}} R(t, z_n)dt}{\|z_n\|^2} \\ &\le \frac{\lambda_0}{2}\|v_n^+\|^2 - \frac{1}{2}\|v_n^-\|^2, \end{aligned}$$

thus,

$$\lambda_0 \|v_n^+\| \ge \|v_n^-\|.$$

If $v_n^+ \to 0$, then from the above inequality, we have $v_n^- \to 0$, and therefore

$$\|v_n^0\|^2 = 1 - \|v_n^+\|^2 - \|v_n^-\|^2 \to 1.$$

Hence, $v_n^0 \to v^0$ because of $\dim E^0 < \infty$. Thus, $v \ne 0$, a contradiction. Therefore, $v_n^+ \nrightarrow 0$ and $\|v_n^+\|^2 \ge \alpha$ for all n and some $\alpha > 0$. By (H_1) and (H_2), we have

$$\begin{aligned} &\int_{\mathbb{R}} R(t, C_0 v_n^+)dt \\ &\le \frac{1}{2}\gamma C_0^2 \int_{\{|C_0 v_n^+| < \delta\}} |v_n^+|^2 dt + \frac{1}{2}c \int_{\{|C_0 v_n^+| \ge \delta\}} \left(C_0^2 |v_n^+|^2 + C_0^p |v_n^+|^p\right) dt \\ &\le \frac{1}{2}\gamma C_0^2 \int_{\{|C_0 v_n^+| < \delta\}} |v_n^+|^2 dt + C_1' \int_{\{|C_0 v_n^+| \ge \delta\}} |v_n^+|^p dt. \end{aligned} \tag{2.12}$$

For all sufficiently large n, if follows from (2.11), (2.12) and the fact $\lambda_n \to \lambda, v_n^+ \to v^+ = 0$ in $L^p(\mathbb{R})$ for all $[2, \infty)$ that

$$\begin{aligned} \Phi_{\lambda_n}(C_0 v_n^+) &= \frac{1}{2}\lambda_n C_0^2 \|v_n^+\|^2 - \int_{\mathbb{R}} R(t, C_0 v_n^+)dt \\ &\ge \frac{1}{2}\lambda_n C_0^2 \alpha - \frac{1}{2}\gamma C_0^2 \int_{\{|C_0 v_n^+| < \delta\}} |v_n^+|^2 dt - C_1' \int_{\{|C_0 v_n^+| \ge \delta\}} |v_n^+|^p dt \\ &\to \frac{1}{2}\lambda \alpha C_0^2, \text{ as } n \to \infty. \end{aligned}$$

This implies that $\Phi_{\lambda_n}(C_0 v_n^+) \to \infty$ as $C_0 \to \infty$, contrary to (2.11). Therefore, $\{z_n\}$ are bounded. The proof is complete. ∎

3. Proofs of the Main Results

Proof of Theorem 1.1. From Lemma 2.9, there are sequences $1 < \lambda_n \to 1$ and $\{z_n\} \subset E$ such that $\Phi'_{\lambda_n}(z_n) = 0$ and $\Phi_{\lambda_n}(z_n) = c_{\lambda_n}$. By Lemma 2.11, we know $\{z_n\}$ is bounded in E, thus we can assume $z_n \rightharpoonup z$ in E, $z_n \to z$ in $L^p(\mathbb{R})$ for $p \in [2, \infty)$, $z_n \rightharpoonup z$ a.e. on $\mathbb{R}$. Therefore

$$\langle \Phi'_{\lambda_n}(z_n), \varphi\rangle = \lambda_n(z_n^+, \varphi) - (z_n^-, \varphi) - \int_{\mathbb{R}} R_z(t, z_n)\varphi dt = 0, \ \forall \varphi \in E.$$

Hence, in the limit

$$\langle \Phi'(z), \varphi\rangle = (z^+, \varphi) - (z^-, \varphi) - \int_{\mathbb{R}} R_z(t, z)\varphi dt = 0, \ \forall \varphi \in E.$$

Thus $\Phi'(z) = 0$. Note that

$$\Phi_{\lambda_n}(z_n) - \frac{1}{2}\langle \Phi'_{\lambda_n}(z_n), z_n\rangle = \int_{\mathbb{R}} \left[\frac{1}{2}R_z(t, z_n)z_n - R(t, z_n)\right] dt = c_{\lambda_n} \geq c_1. \tag{3.1}$$

Similar to (2.7) and (2.8), we know that

$$\int_{\mathbb{R}} \left[\frac{1}{2}R_z(t, z_n)z_n - R(t, z_n)\right] dt \to \int_{\mathbb{R}} \left[\frac{1}{2}R_z(t, z)z - R(t, z)\right] dt, \text{ as } n \to \infty.$$

It follows from $\Phi'(z) = 0$, (3.1) and Lemma 2.5 that

$$\begin{aligned}\Phi(z) &= \Phi(z) - \frac{1}{2}\langle \Phi'(z), z\rangle \\ &= \int_{\mathbb{R}} \left[\frac{1}{2}R_z(t, z)z - R(t, z)\right] dt \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} \left[\frac{1}{2}R_z(t, z_n)z_n - R(t, z_n)\right] dt \\ &\geq c_1 \geq \kappa > 0.\end{aligned}$$

Therefore, $z \neq 0$. ∎

Proof of Theorem 1.2. By Theorem 1.1, $\mathcal{M} \neq \emptyset$, where $\mathcal{M}$ is the collection of solution of (1.1). Let

$$\theta := \inf_{z\in\mathcal{M}} \Phi(z).$$

If z is a solution of (1.1), by Lemma 2.10, (take $r = 0$)

$$\Phi(z) = \Phi(z) - \frac{1}{2}\langle \Phi'(z), z\rangle = \int_{\mathbb{R}} \left[\frac{1}{2}R_z(t, z)z - R(t, z)\right] dt \geq -C = -\int_{\mathbb{R}} |W_1(t)|dt.$$

Thus, $\theta > -\infty$. Let $\{z_n\}$ be a subsequence in $\mathcal{M}$ such that

$$\Phi(z_n) \to \theta. \tag{3.2}$$

By Lemma 2.11, the sequence $\{z_n\}$ is bounded in E. Thus, $z_n \rightharpoonup z$ in E $z_n \to z$ in $L^p(\mathbb{R})$ for $p \in [2, \infty)$ and $z_n \to z$ a.e. on $\mathbb{R}$, after passing to a subsequence. Therefore

$$\langle \Phi'(z_n), \varphi \rangle = (z_n^+, \varphi) - (z_n^-, \varphi) - \int_{\mathbb{R}} R_z(t, z_n)\varphi dt = 0, \ \forall \varphi \in E.$$

Hence, in the limit

$$\langle \Phi'(z), \varphi \rangle = (z^+, \varphi) - (z^-, \varphi) - \int_{\mathbb{R}} R_z(t, z)\varphi dt = 0, \ \forall \varphi \in E.$$

Thus, $\Phi'(z) = 0$. Similar to (2.7) and (2.8), we have

$$\begin{aligned} \Phi(z_n) - \frac{1}{2}\langle \Phi'(z_n), z_n \rangle &= \int_{\mathbb{R}} \left[\frac{1}{2}R_z(t, z_n)z_n - R(t, z_n)\right] dt \\ &\to \int_{\mathbb{R}} \left[\frac{1}{2}R_z(t, z)z - R(t, z)\right] dt \text{ as } n \to \infty. \end{aligned}$$

If follows from $\Phi'(z) = 0$ and (3.2) that

$$\begin{aligned} \Phi(z) = \Phi(z) - \frac{1}{2}\langle \Phi'(z), z \rangle &= \int_{\mathbb{R}} \left[\frac{1}{2}R_z(t, z)z - R(t, z)\right] dt \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} \left[\frac{1}{2}R_z(t, z_n)z_n - R(t, z_n)\right] dt \\ &= \lim_{n\to\infty} \Phi(z_n) = \theta. \end{aligned}$$

Now suppose that

$$|R_z(t, z)| = o(|z|), \text{ as } |z| \to 0.$$

It follows from (H_1) that for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$|R_z(t, z)| = \varepsilon|z| + C_\varepsilon|z|^{p-1}. \tag{3.3}$$

Let

$$\beta := \inf_{z\in\mathcal{M}'} \Phi(z),$$

where $\mathcal{M}' := \mathcal{M} \setminus \{0\}$. Let $\{z_n\}$ be a sequence in $\mathcal{M} \setminus \{0\}$ such that

$$\Phi(z_n) \to \beta. \tag{3.4}$$

Note that

$$0 = \langle \Phi'(z_n), z_n^+ \rangle = \|z_n^+\|^2 - \int_{\mathbb{R}} R_z(t, z_n) z_n^+ dt,$$

which together with (3.3), Hӧlder inequality and the Sobolev embedding theorem implies

$$\begin{aligned}
\|z_n^+\|^2 &= \int_{\mathbb{R}} R_z(t, z_n) z_n^+ dt \\
&\leq \varepsilon \int_{\mathbb{R}} |z_n||z_n^+| dt + C_\varepsilon \int_{\mathbb{R}} |z_n|^{p-1} |z_n^+| dt \\
&\leq \varepsilon \|z_n\| \|z_n^+\| + C'_\varepsilon \|z_n\|_p^{p-1} \|z_n^+\| \\
&\leq \varepsilon \|z_n\| \|z_n^+\| + C''_\varepsilon \|z_n\|_p^{p-2} \|z_n\| \|z_n^+\| \\
&\leq \varepsilon \|z_n\|^2 + C''_\varepsilon \|z_n\|_p^{p-2} \|z_n\|^2.
\end{aligned} \tag{3.5}$$

Similarly, we get

$$\|z_n^-\|^2 \leq \varepsilon \|z_n\|^2 + C''_\varepsilon \|z_n\|_p^{p-2} \|z_n\|^2. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\|z_n\|^2 \leq 2\varepsilon \|z_n\|^2 + 2C''_\varepsilon \|z_n\|_p^{p-2} \|z_n\|^2,$$

which means $\|z_n\|_p \geq c$ for some constant $c > 0$. Since $z_n \to z$ in $L^p(\mathbb{R})$, we know $z \neq 0$. As before, $\Phi(z_n) \to \Phi(z) = \beta$ as $n \to \infty$. ∎

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Wen Zhang, Xianhua Tang and Jian Zhang
School of Mathematics and Statistics
Central South University
Changsha 410083, Hunan
P. R. China
E-mail: zwmath2011@163.com
tangxh@mail.csu.edu.cn
zhangjian433130@163.com