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THE LIOUVILLE PROPERTY FOR PSEUDOHARMONIC MAPS WITH FINITE DIRICHLET ENERGY

Ting-Hui Chang and Yen-Chang Huang

Abstract. In this paper, we first derive the CR Bochner formula and the CR Kato's inequality for pseudoharmonic maps. Secondly, by applying the CR Bochner formula and the CR Kato's inequality we are able to prove the Liouville property for pseudoharmonic maps with finite Dirichlet energy in a complete $(2n+1)$-pseudohermitian manifold. This is served as CR analogue to the Liouville theorem for harmonic maps in Riemannian Geometry.

1. INTRODUCTION

In the papers of [19] and [9], S.-Y. Cheng and S.-T. Yau derived a well known gradient estimate for positive harmonic functions in a complete noncompact Riemannian manifold. As a consequence, Liouville-type theorem holds for complete noncompact Riemannian manifolds of nonnegative Ricci curvature. The Liouville- type theorem is also studied by a series papers of P. Li and J. Wang ([15, 16]). In particular, in the paper of Li and Wang ([16]), they extended their results to complete manifolds with the condition (P_ρ) (see Definition 1.2). Recently, S.-C. Chang, J.-T. Chen and S.-W. Wei ([7]) considered the p-harmonic functions in a complete manifold with (P_ρ) and the Ricci curvature bounded below depending on ρ, then the Liouville-type properties are still valid on these manifolds. In the paper of Chang, Chen and Kuo ([4]), they applied the method as in the paper [7] and obtained the Liouville-type theorem for p-pseudoharmonic functions with finite Dirichlet p-energy in a complete $(2n+1)$-pseudohermitian manifold. In 1980, Cheng ([5]) extended the result in [9] and obtained the Liouville-type theorem for harmonic maps. In this paper, we study

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the Liouville-type theorem for pseudoharmonic maps with finite Dirichlet energy in a complete pseudohermitian $(2n+1)$-manifold (M^{2n+1}, J, θ).

Let (M^{2n+1}, J, θ) be a complete pseudohermitian manifold and (N^m, g) be a Riemannian manifold. We now recall the definition of the Dirichlet energy $E(\varphi)$ of a C^2-map $\varphi : M \to N$. At each point $p \in M$, we may take a local coordinate chart $U_p \subset M$ of p and a local coordinate chart $V_{\varphi(p)} \subset N$ of $\varphi(p)$ such that $\varphi(U_p) \subset V_{\varphi(p)}$. We define the energy density $e(\varphi)$ of φ at the point $x \in U_p$ by

$$e(\varphi)(x) = \frac{1}{2} h^{\alpha\overline{\beta}}(x) g_{ij}(\varphi(x)) \varphi^i_\alpha \varphi^j_{\overline{\beta}}.$$

Here $h_{\alpha\overline{\beta}}$ is the Levi metric on (M^{2n+1}, J, θ) and we may assume $h_{\alpha\overline{\beta}} = \delta_{\alpha\beta}$ (see Section 2). It can be checked that the energy density is intrinsically defined, i.e., independent of the choice of local coordinates. The Dirichlet energy $E(\varphi)$ of φ is defined by

$$E(\varphi) = \int_M e(\varphi) dv,$$

where $dv = \theta \wedge (d\theta)^n$ is the volume element of M. We also define an extra energy $E^0(\varphi)$ by

$$E^0(\varphi) = \int_M e^0(\varphi) dv.$$

Here the extra energy density $e^0(\varphi)$ is given by $e^0(\varphi) := g_{ij}\varphi^i_0 \varphi^j_0$ which will help us to deal with the term $\langle J\nabla_b \varphi^k, \nabla_b \varphi^k_0 \rangle$ in the CR Bochner formula.

In the paper of E. Barletta, S. Dragomir and H. Urakawa [1], they introduced a notion of the pseudoharmonic map from a pseudohermitian $(2n+1)$-manifold (M^{2n+1}, J, θ) into a Riemannian m-manifold (N^m, g) as following:

Definition 1.1. A C^2-map $\varphi : (M^{2n+1}, J, \theta) \to (N^m, g)$ is said to be a pseudoharmonic map if it is a critical point of the energy functional E.

Definition 1.2. We say that M satisfies the condition (P_ρ) if there exists a positive function $\rho(x)$ a.e. such that, for every smooth function Ψ with compact support on M, the inequality

$$\int_M \rho \Psi^2 dv \le \int_M |\nabla_b \Psi|^2 \, dv$$

holds on M.

Note that if the first eigenvalue λ_1 with respect to Δ_b is positive on $\left(M^{2n+1}, J, \theta\right)$, then there holds the condition (P_{λ_1}). We refer to [16] in case of complete Riemannian manifolds.

We now state our main theorem as follows.

Theorem 1.1. *Let* (M^{2n+1}, J, θ) *be a complete noncompact pseudohermitian* $(2n+1)$*-manifold,* (N^m, g) *be a Riemannian manifold with nonpositive sectional curvature. Suppose that* M *satisfies* (P_ρ) *and*

$$(1.1) \qquad \big(2Ric - (n-2)Tor\big)(Z, Z) \geq -2\tau\rho\,|Z|^2$$

for some fixed constant $\tau \in (0,1)$ *and for all* $Z \in T_{1,0}$*. If* $\varphi : M \to N$ *is a pseudoharmonic map with finite Dirichlet energy* $E(\varphi)$ *and*

$$(1.2) \qquad [\Delta_b, T]\,\varphi^k = 0, \ \ k = 1, \cdots m,$$

then φ *must be a constant map.*

Remark 1.1.

(1) In [11], Graham and Lee defined the purely holomorphic second-order operator Q by

$$Qu = 2i \sum_{\alpha,\beta=1}^{n} (A_{\bar{\alpha}\bar{\beta}} u_\beta)_{,\alpha}\,.$$

They showed that for any smooth function u, $[\Delta_b, T]\,u = 2\mathrm{Im}(Qu)$. Therefore, if (M, J, θ) is a complete pseudohermitian $(2n+1)$-manifold with vanishing pseudohermitian torsion, that is, $A_{\alpha\beta} = 0$, then condition (1.2) holds. However, it is not true vice versa.

(2) In the paper of [8], they observe that condition (1.2) is related to existence of pseudo-Einstein contact forms in a closed pseudohermitian $(2n+1)$-manifold with $n \geq 2$.

Corollary 1.2. *Let* (M^{2n+1}, J, θ) *be a complete noncompact pseudohermitian* $(2n+1)$*-manifold with vanishing pseudohermitian torsion and* (N^m, g) *be a Riemannian manifold with nonpositive sectional curvature. Suppose that* M *satisfies* (P_ρ) *and*

$$Ric\,(Z, Z) \geq -\tau\rho\,|Z|^2$$

for some fixed constant $\tau \in (0,1)$ *and for all* $Z \in T_{1,0}$*. If* $\varphi : M \to N$ *is a pseudoharmonic map with finite Dirichlet energy* $E(\varphi)$*, then* φ *must be a constant map.*

In particular, if M has positive spectrum $\lambda > 0$, then condition (P_λ) holds and we have

Corollary 1.3. *Let* (M, J, θ) *be a complete noncompact pseudohermitian* $(2n+1)$*-manifold with vanishing pseudohermitian torsion and* (N^m, g) *be a Riemannian manifold with nonpositive sectional curvature. Suppose that* M *satisfies* (P_λ) *and*

$$Ric\,(Z, Z) \geq -\tau\,|Z|^2$$

for some fixed constant $\tau \in (0, \lambda)$ *and for all* $Z \in T_{1,0}$. *If* $\varphi : M \to N$ *is a pseudoharmonic map with finite Dirichlet energy* $E(\varphi)$*, then* φ *must be a constant map.*

The organization of this paper is as follows. In section 2, we first introduce some basic materials in a pseudohermitian $(2n+1)$-manifold. In section 3, we derive the CR Bochner type formula and the CR Kato's inequality. The CR Bochner formula derived in section 3 consists the term $\left\langle J \bigtriangledown_b \varphi^k, \bigtriangledown_b \varphi_0^k \right\rangle$ which is hard to estimate. However, by deriving the CR Bochner type formula (see (3.2)) for $e^0(\varphi)$ for a pseudoharmonic map φ, we can overcome the difficulty (see Remark 4.2) and the Liouville property for pseudoharmonic maps with finite Dirichlet energy can be obtained in section 4.

2. Preliminaries

In this section, we give a brief introduction to pseudohermitian geometry (see [13], [14] for more details). Let (M, ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure ξ, $\dim_R \xi = 2n$. A CR structure compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the integrability condition $J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]$, where $X, Y \in \xi$. We can extend J in a natural way to $\mathbb{C}\otimes\xi$ and decomposes $\mathbb{C}\otimes\xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$ which are eigenspaces of J with respect to i and $-i$, respectively.

A manifold M with a CR structure is called a CR manifold. A pseudohermitian structure compatible with ξ is a CR structure J compatible with ξ together with a choice of contact form θ. Such a choice determines a unique real vector field T transverse to ξ , which is called the characteristic vector field of θ, such that $\theta(T) = 1$ and $\mathcal{L}_T\theta = 0$ or $d\theta(T, \cdot) = 0$.

Let $\{T, Z_\alpha, Z_{\bar\alpha}\}$ be a frame of $TM \otimes \mathbb{C}$, where Z_α is any local frame of $T_{1,0}$, $Z_{\bar\alpha} = \overline{Z_\alpha} \in T_{0,1}$ and $\alpha = 1, \cdots, n$. Then the dual coframe $\{\theta, \theta^\alpha, \theta^{\bar\alpha}\}$ satisfies

$$d\theta = i h_{\alpha\bar\beta} \theta^\alpha \wedge \theta^{\bar\beta}, \tag{2.1}$$

for some positive definite hermitian matrix of functions $(h_{\alpha\bar\beta})$. Actually we can always choose Z_α such that $h_{\alpha\bar\beta} = \delta_{\alpha\beta}$; hence, throughout this paper, we assume $h_{\alpha\bar\beta} = \delta_{\alpha\beta}$.

The Levi form $h = \langle\ ,\ \rangle$ is the Hermitian form on $T_{1,0}$ defined by

$$\langle Z, W \rangle = -i \left\langle d\theta, Z \wedge \overline{W} \right\rangle .$$

This can be extend $\langle\ ,\ \rangle$ to $T_{0,1}$ by defining $\left\langle \overline{Z}, \overline{W} \right\rangle = \overline{\langle Z, W \rangle}$ for all $Z, W \in T_{1,0}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, also denoted by $\langle\ ,\ \rangle$, and hence on all the induced tensor bundles.

The pseudohermitian connection of (J, θ) is the connection ∇ on $TM \otimes \mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_\alpha \in T_{1,0}$ by

$$\nabla Z_\alpha = {\omega_\alpha}^\beta \otimes Z_\beta, \qquad \nabla Z_{\bar\alpha} = {\omega_{\bar\alpha}}^{\bar\beta} \otimes Z_{\bar\beta}, \qquad \nabla T = 0,$$

where ${\omega_\alpha}^\beta$ are the 1-forms uniquely determined by the following equations:

$$
\begin{aligned}
d\theta^\beta &= \theta^\alpha \wedge {\omega_\alpha}^\beta + \theta \wedge \tau^\beta, \\
0 &= \tau_\alpha \wedge \theta^\alpha, \\
0 &= {\omega_\alpha}^\beta + {\omega_{\overline{\beta}}}^{\overline{\alpha}}
\end{aligned}
\tag{2.2}
$$

By Cartan lemma, we write $\tau_\alpha = A_{\alpha\gamma}\theta^\gamma$, where $A_{\alpha\gamma} = A_{\gamma\alpha}$ are so called the pseudo-hermitian torsion of (M, J, θ).

Let $\theta = \theta^0$. The curvature of the Webster-Stanton connection, expressed in terms of the coframe $\{\theta^0, \theta^\alpha, \theta^{\bar{\alpha}}\}$, is

$$
\begin{aligned}
{\Pi_\beta}^\alpha &= \overline{{\Pi_{\bar{\beta}}}^{\bar{\alpha}}} = d{\omega_\beta}^\alpha - {\omega_\beta}^\gamma \wedge {\omega_\gamma}^\alpha, \\
{\Pi_0}^\alpha &= {\Pi_\alpha}^0 = {\Pi_0}^{\bar{\beta}} = {\Pi_{\bar{\beta}}}^0 = {\Pi_0}^0 = 0.
\end{aligned}
$$

Webster showed that ${\Pi_\beta}^\alpha$ can be written

$$
{\Pi_\beta}^\alpha = {R_\beta}^\alpha{}_{\rho\bar{\sigma}}\theta^\rho \wedge \theta^{\bar{\sigma}} + {W_\beta}^\alpha{}_\rho \theta^\rho \wedge \theta - W^\alpha{}_{\beta\bar{\rho}}\theta^{\bar{\rho}} \wedge \theta + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha,
$$

where the coefficients satisfy

$$
R_{\beta\bar{\alpha}\rho\bar{\sigma}} = \overline{R_{\alpha\bar{\beta}\sigma\bar{\rho}}} = R_{\bar{\alpha}\beta\bar{\sigma}\rho} = R_{\rho\bar{\alpha}\beta\bar{\sigma}}, \quad W_{\beta\bar{\alpha}\gamma} = W_{\gamma\bar{\alpha}\beta}.
$$

We will denote components of covariant derivatives with indices preceded by comma; thus write $A_{\alpha\beta,\gamma}$. The indices $\{0, \alpha, \bar{\alpha}\}$ indicate derivatives with respect to $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $f_\alpha = Z_\alpha f,\ f_{\alpha\bar{\beta}} = Z_{\bar{\beta}}Z_\alpha f - {\omega_\alpha}^\gamma(Z_{\bar{\beta}})Z_\gamma f,\ f_0 = Tf$ for a (smooth) function.

For a smooth function u, the Cauchy-Riemann operator ∂_b be defined locally by $\partial_b u = u_\alpha \theta^\alpha$, and $\overline{\partial}_b$ be the conjugate of ∂_b. For a real function f, the subgradient ∇_b is defined by $\nabla_b f \in \xi$ and $\langle Z, \nabla_b f\rangle = df(Z)$ for all vector fields Z tangent to contact plane. Locally $\nabla_b f = f_{\overline{\alpha}} Z_\alpha + f_\alpha Z_{\bar{\alpha}}$. We can use the connection to define the subhessian as the complex linear map $\nabla_b^2 f : T_{1,0} \oplus T_{0,1} \to T_{1,0} \oplus T_{0,1}$ by

$$
(\nabla_b^2 f)(Z) = \nabla_Z \nabla_b f.
$$

In particular, we have in local coordinates,

$$
|\nabla_b f|^2 = 2 f_\alpha f_{\overline{\alpha}} \quad \text{and} \quad \left|\nabla_b^2 f\right|^2 = 2\,|f_{\alpha\beta}|^2 + 2\left|f_{\alpha\overline{\beta}}\right|^2.
$$

The sub-Laplacian $\Delta_b f$ is the differential operator defined to be the trace of the subhessian

$$
\Delta_b f = Tr\left(\nabla_b^2 f\right) = f_{\alpha\overline{\alpha}} + f_{\overline{\alpha}\alpha}.
$$

The pseudohermitian Ricci tensor and the torsion tensor on $T_{1,0}$ are defined by

$$Ric(X,Y) = R_{\alpha\bar{\beta}} X^{\alpha} Y^{\bar{\beta}},$$
$$Tor(X,Y) = i \sum_{\alpha,\beta} \big(A_{\bar{\alpha}\bar{\beta}} X^{\bar{\alpha}} Y^{\bar{\beta}} - A_{\alpha\beta} X^{\alpha} Y^{\beta} \big),$$

where $X = X^{\alpha} Z_{\alpha},\ Y = Y^{\beta} Z_{\beta},\ R_{\alpha\bar{\beta}} = R_{\gamma}{}^{\gamma}{}_{\alpha\bar{\beta}}$.

Definition 2.1. Let (M, J, θ) be a pseudohermitian $(2n+1)$-manifold. A piecewise smooth curve $\gamma : [0,1] \to M$ is said to be horizontal if $\gamma'(t) \in \xi$ whenever $\gamma'(t)$ exists. The length of γ is then defined by

$$l(\gamma) = \textstyle\int_0^1 \langle \gamma'(t), \gamma'(t) \rangle_{L_\theta}^{\frac{1}{2}} dt.$$

The Carnot-Carathéodory distance between two points $p,\ q \in M$ is

$$d_c(p,q) = \inf\{ l(\gamma) | \gamma \in C_{p,q} \}$$

where $C_{p,q}$ is the set of all horizontal curves joining p and q. We say M is complete if it is complete as a metric space. By Chow connectivity theorem [6], there always exists a horizontal curve joining p and q, so the distance is finite. Furthermore, there is a minimizing geodesic joining p and q so that its length is equal to the distance $d_c(p,q)$.

3. CR Bochner Type Formula and Kato's Inequality

In this section, we derive some key lemmas. First, we derive the CR version of Bochner formula for pseudoharmonic maps.

Lemma 3.1. (CR Bochner formula). *Let (M^{2n+1}, J, θ) be a complete pseudohermitian manifold and (N^m, g) be a Riemannian manifold. If $\varphi : M \to N$ is a pseudoharmonic map, then we have*

$$\begin{aligned} \Delta_b\big(2e(\varphi)\big) &= |\nabla_b^2 \varphi^k|^2 + \big(2Ric - (n-2)Tor\big)\big((\nabla_b \varphi^k)_C, (\nabla_b \varphi^k)_C\big) \\ &\quad + 2\langle J\nabla_b \varphi^k, \nabla_b \varphi_0^k \rangle - 2\big[\widetilde{R}_{ijk\ell} \varphi_\alpha^i \varphi_\beta^j \varphi_{\overline{\alpha}}^k \varphi_{\overline{\beta}}^\ell + \widetilde{R}_{ijk\ell} \varphi_\alpha^i \varphi_{\overline{\beta}}^j \varphi_{\overline{\alpha}}^k \varphi_\beta^\ell \big]. \end{aligned} \tag{3.1}$$

Moreover, assume that condition (1.2) holds, we then have

$$\frac{1}{2}\Delta_b\big(e^0(\varphi)\big) = |\nabla_b \varphi_0^k|^2 - 2\widetilde{R}_{ijk\ell} \varphi_\alpha^i \varphi_0^j \varphi_{\overline{\alpha}}^k \varphi_0^\ell. \tag{3.2}$$

Here for any smooth function u, $(\nabla_b u)_C = u_{\overline{\alpha}} Z_\alpha$ is the corresponding complex $(1,0)$-vector of $\nabla_b u$ and $\widetilde{R}_{ijk\ell} = \big\langle \widetilde{R}(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_\ell})\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_i} \big\rangle$ with $\widetilde{R}$ the Riemannian curvature tensor of (N, g).

Proof. For each point $p \in M$ we may choose a normal coordinate chart U of p and a normal coordinate chart V of $\varphi(p)$ such that $\varphi(U) \subset V$ and fulfill the following computations at the point p.

1° To prove equation (3.1), we first recall the following version of Bochner formula for smooth functions (see [12, 3]). For any smooth function u, there holds

$$\begin{aligned}\frac{1}{2}\Delta_b|\nabla_b u|^2 &= |\nabla_b^2 u|^2 + \langle \nabla_b u, \nabla_b \Delta_b u\rangle + 2\langle J\nabla_b u, \nabla_b u_0\rangle \\ &\quad + \big(2Ric - (n-2)Tor\big)\big((\nabla_b u)_C, (\nabla_b u)_C\big).\end{aligned} \tag{3.3}$$

Thus at the point p,

$$\begin{aligned}&\Delta_b(g_{ij}\varphi^i_\alpha\varphi^j_{\overline{\alpha}}) \\ &= g_{ij}\Delta_b(\varphi^i_\alpha\varphi^j_{\overline{\alpha}}) + \varphi^i_\alpha\varphi^j_{\overline{\alpha}}\Delta_b(g_{ij}) \\ &= \frac{1}{2}\Delta_b|\nabla_b\varphi^k|^2 + \varphi^i_\alpha\varphi^j_{\overline{\alpha}}\Delta_b(g_{ij}) \\ &= |\nabla_b^2\varphi^k|^2 + \langle\nabla_b\varphi^k, \nabla_b\Delta_b\varphi^k\rangle + 2\langle J\nabla_b\varphi^k, \nabla_b\varphi^k_0\rangle \\ &\quad + \big(2Ric - (n-2)Tor\big)\big((\nabla_b\varphi^k)_C, (\nabla_b\varphi^k)_C\big) + \varphi^i_\alpha\varphi^j_{\overline{\alpha}}\Delta_b(g_{ij}).\end{aligned} \tag{3.4}$$

Here we used equation (3.3) in the last equality. Furthermore, at the point p, we have by direct computations that

$$2\big[\widetilde{R}_{ijk\ell}\varphi^i_\alpha\varphi^j_\beta\varphi^k_{\overline{\alpha}}\varphi^\ell_{\overline{\beta}} + \widetilde{R}_{ijk\ell}\varphi^i_\alpha\varphi^j_{\overline{\beta}}\varphi^k_{\overline{\alpha}}\varphi^\ell_\beta\big] = -\langle\nabla_b\varphi^k, \nabla_b\Delta_b\varphi^k\rangle - \varphi^i_\alpha\varphi^j_{\overline{\alpha}}\Delta_b(g_{ij}).$$

The reader may refer to the proof of Lemma 3.2 in [2] for details of the above equation. Therefore, equation (3.1) follows immediately from (3.4).

2° To prove equation (3.2), one has, by direct computations, that at the point p,

$$\begin{aligned}\frac{1}{2}\Delta_b\big(e^0(\varphi)\big) &= \frac{1}{2}\Delta_b(g_{ij}\varphi^i_0\varphi^j_0) \\ &= |\nabla_b\varphi^k_0|^2 + \varphi^k_0\Delta_b(\varphi^k_0) + \frac{1}{2}\varphi^i_0\varphi^j_0\Delta_b(g_{ij}) \\ &= |\nabla_b\varphi^k_0|^2 + \varphi^k_0(\Delta_b\varphi^k)_0 + \frac{1}{2}\varphi^i_0\varphi^j_0\Delta_b(g_{ij}) \\ &= |\nabla_b\varphi^k_0|^2 - 2\widetilde{\Gamma}^k_{ij,\ell}\varphi^i_\alpha\varphi^j_{\overline{\alpha}}\varphi^k_0\varphi^\ell_0 + \frac{1}{2}\varphi^i_0\varphi^j_0\Delta_b(g_{ij}).\end{aligned}$$

Here we used the assumption that $[\Delta_b, T]\varphi^k = 0$, for $k = 1, \cdots, m$, in the third equality. Again, at the point p, direct computations as in the proof of Lemma 3.2 in [2] show that

$$2\widetilde{R}_{ijk\ell}\varphi_\alpha^i\varphi_0^j\varphi_{\bar{\alpha}}^k\varphi_0^\ell = 2\widetilde{\Gamma}_{ij,\ell}^k\varphi_\alpha^i\varphi_{\bar{\alpha}}^j\varphi_0^k\varphi_0^\ell - \frac{1}{2}\varphi_0^i\varphi_0^j\Delta_b(g_{ij}).$$

Therefore, equation (3.2) follows immediately. ■

Next, we recall the following remark, which is an important fact that should be observed.

Remark 3.1. Let (M^{2n+1}, J, θ) be a complete pseudohermitian manifold, (N^m, g) be a Riemannian manifold and $\varphi : M \to N$ be a smooth map. Suppose that the sectional curvature of N is nonpositive, then

$$\sum_{i,j,k,\ell=1}^{m}\sum_{\alpha,\beta=1}^{n}\widetilde{R}_{ijk\ell}\varphi_\alpha^i\varphi_\beta^j\varphi_{\bar{\alpha}}^k\varphi_{\bar{\beta}}^\ell + \widetilde{R}_{ijk\ell}\varphi_\alpha^i\varphi_{\bar{\beta}}^j\varphi_{\bar{\alpha}}^k\varphi_\beta^\ell \le 0.$$

Moreover, if condition (1.2) holds, we also have

$$\sum_{i,j,k,\ell=1}^{m}\sum_{\alpha=1}^{n}\widetilde{R}_{ijk\ell}\varphi_\alpha^i\varphi_0^j\varphi_{\bar{\alpha}}^k\varphi_0^\ell \le 0.$$

The result of this remark was proved in Theorem 1.1 of [2].

We now show the CR version of Kato's inequality. This is the key point that we can extend the Liouville properties to pseudohermitian manifolds with negative curvatures (see Remark 4.1).

Lemma 3.2. (CR Kato's inequality). *If u is any smooth function on M, then*

$$\left|\nabla_b^2 u\right|^2 \ge \tfrac{n}{2}u_0^2 \tag{3.5}$$

and the following CR version of Kato's inequality

$$f^2\left|\nabla_b^2 u\right|^2 \ge \tfrac{\left|\nabla_b f^2\right|^2}{4} \tag{3.6}$$

hold for all $x \in M$, where $f = |\nabla_b u|$.

Proof.

Fix $x \in M$, then

$$\begin{aligned} nu_0^2 &= \sum_{\alpha=1}^{n}\left(u_{\alpha\bar{\alpha}} - u_{\bar{\alpha}\alpha}\right)\left(u_{\bar{\alpha}\alpha} - u_{\alpha\bar{\alpha}}\right) \\ &= \sum_{\alpha=1}^{n}\left(2\left|u_{\alpha\bar{\alpha}}\right|^2 - u_{\alpha\bar{\alpha}}^2 - u_{\bar{\alpha}\alpha}^2\right) \\ &\le 4\sum_{\alpha=1}^{n}\left|u_{\alpha\bar{\alpha}}\right|^2 \le 2\left|\nabla_b^2 u\right|^2 \end{aligned}$$

and

$$
\begin{aligned}
\left|\nabla_b f^2\right|^2 &= \left|\nabla_b\left(2\sum_{\alpha=1}^{n} u_\alpha u_{\bar\alpha}\right)\right|^2 \\
&= 8\sum_{\alpha,\beta=1}^{n} \left(u_\alpha u_{\bar\alpha}\right)_\beta \left(u_\alpha u_{\bar\alpha}\right)_{\bar\beta} \\
&\le 8\sum_{\alpha,\beta=1}^{n} \left(\left|u_{\alpha\beta}\right|^2 + \left|u_{\bar\alpha\beta}\right|^2 + 2\left|u_{\alpha\beta}\right|\left|u_{\alpha\bar\beta}\right|\right)\left|u_\alpha\right|^2 \\
&\le 8\left|\nabla_b u\right|^2 \sum_{\alpha,\beta=1}^{n}\left(\left|u_{\alpha\beta}\right|^2 + \left|u_{\bar\alpha\beta}\right|^2\right) = 4\left|\nabla_b u\right|^2\left|\nabla_b^2 u\right|^2. \qquad \blacksquare
\end{aligned}
$$

4. Liouville Properties for Pseudoharmonic Maps

In this section, we will prove the Liouville-type theorem for pseudoharmonic maps. Let φ be a pseudoharmonic map on a complete noncompact pseudohermitian $(2n+1)$-manifold (M, J, θ). Let $f^k = |\nabla_b \varphi^k|$ and for each $\varepsilon > 0$, we define

$$f_\varepsilon^k = \sqrt{|\nabla_b \varphi^k|^2 + \varepsilon}, \quad k = 1, \cdots, m.$$

We first derive the following lemma:

Lemma 4.1. *Let (M^{2n+1}, J, θ) be a complete noncompact pseudohermitian manifold and (N^m, g) be a Riemannian manifold with nonpositive sectional curvature. Suppose that $\varphi : M \to N$ is a pseudoharmonic map. Then for any $0 < \varepsilon_1$, ε_2, $\sigma < 1$, we have*

$$
\begin{aligned}
&\Big(\frac{1}{\varepsilon_1} + (1-\sigma-\varepsilon_1)\big(\frac{1}{\varepsilon_2} - 1\big)\Big)\int_M |\nabla_b \eta|^2 (f_\varepsilon^k)^2 dv \\
\ge&(1-\varepsilon_2)(1-\sigma-\varepsilon_1)\int_M \rho\eta^2 (f_\varepsilon^k)^2 dv + \int_M \eta^2\Big[\sigma|\nabla_b^2\varphi^k|^2 + 2\big\langle J\nabla_b\varphi^k, \nabla_b\varphi_0^k\big\rangle\Big] dv \\
&+ \int_M \eta^2\big(2Ric - (n-2)Tor\big)\big((\nabla_b\varphi^k)_C, (\nabla_b\varphi^k)_C\big) dv,
\end{aligned}
$$

where η is a cut-off function on M satisfying

$$
\begin{cases}
\eta(x) = 1 & \textit{if } x \in B(R), \\
0 < \eta(x) < 1 & \textit{if } x \in B(2R)\backslash\overline{B(R)}, \\
\eta(x) = 0 & \textit{if } x \in M\backslash\overline{B(2R)},
\end{cases}
$$

and

$$
\begin{cases}
|\nabla_b\eta(x)| = 0 & \textit{if } x \in B(R) \textit{ or } x \in M\backslash\overline{B(2R)}, \\
|\nabla_b\eta(x)|^2 \le c\eta R^{-2} & \textit{if } x \in B(2R)\backslash\overline{B(R)}.
\end{cases}
$$

Proof. Using (3.6) we have

$$\begin{aligned}
|\nabla_b^2\varphi^k|^2 &= (1-\sigma)|\nabla_b^2\varphi^k|^2+\sigma|\nabla_b^2\varphi^k|^2\\
&\geq (1-\sigma)(f_\varepsilon^k)^{-2}(f^k)^2|\nabla_b^2\varphi^k|^2+\sigma|\nabla_b^2\varphi^k|^2\\
&\geq \frac{1-\sigma}{4}(f_\varepsilon^k)^{-2}|\nabla_b(f^k)^2|^2+\sigma|\nabla_b^2\varphi^k|^2\\
&= \frac{1-\sigma}{4}(f_\varepsilon^k)^{-2}|\nabla_b(f_\varepsilon^k)^2|^2+\sigma|\nabla_b^2\varphi^k|^2\\
&= (1-\sigma)|\nabla_b f_\varepsilon^k|^2+\sigma|\nabla_b^2\varphi^k|^2,
\end{aligned}$$

where $0<\sigma<1$. Thus, by multiplying both sides of (3.1) by the cut-off function $\eta^2\in C_0^\infty(M)$ mentioned above and integrating over M, one has

$$\begin{aligned}
&\int_M \eta^2\Delta_b\big(2e(\varphi)\big)dv\\
&\geq (1-\sigma)\int_M \eta^2|\nabla_b f_\varepsilon^k|^2dv+\int_M\eta^2\big(2Ric^{(k)}-(n-2)Tor^{(k)}\big)dv \qquad (4.1)\\
&\quad+\int_M\eta^2\big[\sigma|\nabla_b^2\varphi^k|^2+2\langle J\nabla_b\varphi^k,\nabla_b\varphi_0^k\rangle\big]dv.
\end{aligned}$$

Here we used the expression

$$2Ric^{(k)}-(n-2)Tor^{(k)}=\big(2Ric-(n-2)Tor\big)\big((\nabla_b\varphi^k)_C,(\nabla_b\varphi^k)_C\big)$$

for convenient and used the fact that

$$\sum_{i,j,k,\ell=1}^{m}\sum_{\alpha,\beta=1}^{n}\widetilde{R}_{ijk\ell}\varphi_\alpha^i\varphi_\beta^j\varphi_{\overline{\alpha}}^k\varphi_{\overline{\beta}}^\ell+\widetilde{R}_{ijk\ell}\varphi_\alpha^i\varphi_{\overline{\beta}}^j\varphi_{\overline{\alpha}}^k\varphi_\beta^\ell\leq 0$$

under the assumption that the sectional curvature of N is nonpositive (see Remark 3.1).

On the other hand, for each point $p\in M$ we may choose a normal coordinate chart U of p and a normal coordinate chart V of $\varphi(p)$ such that $\varphi(U)\subset V$ and fulfill the following computations at the point p.

$$\begin{aligned}
\mathrm{div}_b\big(\eta^2\nabla_b(g_{ij}\varphi_\alpha^i\varphi_{\overline{\alpha}}^j)\big)&=\eta^2\Delta_b(g_{ij}\varphi_\alpha^i\varphi_{\overline{\alpha}}^j)+\big\langle\nabla_b\eta^2,\nabla_b(g_{ij}\varphi_\alpha^i\varphi_{\overline{\alpha}}^j)\big\rangle\\
&=\eta^2\Delta_b\big(2e(\varphi)\big)+\big\langle\nabla_b\eta^2,\frac{1}{2}\nabla_b|\nabla_b\varphi^k|^2\big\rangle.
\end{aligned}$$

By integrating over M and using the divergence theorem (the reader may refer to Proposition 5.2 in [10]), we have

$$\begin{aligned}
\int_M\eta^2\Delta_b\big(2e(\varphi)\big)dv&=-\int_M\big\langle\nabla_b\eta^2,\frac{1}{2}\nabla_b|\nabla_b\varphi^k|^2\big\rangle dv\\
&\leq 2\int_M\eta f_\varepsilon^k|\nabla_b\eta||\nabla_b f_\varepsilon^k|dv \qquad (4.2)\\
&\leq \varepsilon_1\int_M\eta^2|\nabla_b f_\varepsilon^k|^2dv+\frac{1}{\varepsilon_1}\int_M|\nabla_b\eta|^2(f_\varepsilon^k)^2dv,
\end{aligned}$$

where $\varepsilon_1 \in (0,1)$ is a constant. Combining (4.1) and (4.2) one obtains

$$\begin{aligned}\frac{1}{\varepsilon_1}\int_M |\nabla_b\eta|^2(f_\varepsilon^k)^2dv \geq (1-\sigma-\varepsilon_1)&\int_M \eta^2|\nabla_b f_\varepsilon^k|^2dv\\ &+\int_M \eta^2\big[\sigma|\nabla_b^2\varphi^k|^2+2\langle J\nabla_b\varphi^k,\nabla_b\varphi_0^k\rangle\big]dv\\ &+\int_M \eta^2\big(2Ric^{(k)}-(n-2)Tor^{(k)}\big)dv.\end{aligned}\tag{4.3}$$

By using the weighted Poincaré inequality $\int_M \rho\Psi^2dv \leq \int_M |\nabla_b\Psi|^2\,dv$ with $\Psi = \eta f_\varepsilon^k$, we have

$$\begin{aligned}\int_M \eta^2|\nabla_b f_\varepsilon^k|^2dv &= \int_M |\nabla_b(\eta f_\varepsilon)-(\nabla_b\eta)f_\varepsilon^k|^2dv\\ &\geq (1-\varepsilon_2)\int_M |\nabla_b(\eta f_\varepsilon^k)|^2dv+\big(1-\frac{1}{\varepsilon_2}\big)\int_M|\nabla_b\eta|^2(f_\varepsilon^k)^2dv\\ &\geq (1-\varepsilon_2)\int_M \rho\eta^2(f_\varepsilon^k)^2dv+\big(1-\frac{1}{\varepsilon_2}\big)\int_M|\nabla_b\eta|^2(f_\varepsilon^k)^2dv,\end{aligned}\tag{4.4}$$

where $\varepsilon_2 \in (0,1)$ is a constant. The lemma now follows from (4.3) and (4.4). ∎

Lemma 4.2. *Let (M^{2n+1}, J, θ) be a complete noncompact pseudohermitian manifold, (N^m, g) be a Riemannian manifold with nonpositive sectional curvature and $\varphi : M \to N$ be a pseudoharmonic map. Assume that condition (1.2) holds, then*

$$\int_M \eta^3|\nabla_b\varphi_0^k|^2dv \leq 3\int_M \eta^2|\nabla_b\eta||\varphi_0^k||\nabla_b\varphi_0^k|dv.$$

Here η is the cut-off function mentioned in Lemma 4.1.

Proof. By multiplying both sides of equation (3.2) by η^3 and integrating over M gives

$$\frac{1}{2}\int_M \eta^3\Delta_b\big(e^0(\varphi)\big)dv \geq \int_M \eta^3|\nabla_b\varphi_0^k|^2dv.\tag{4.5}$$

Here we used the fact that

$$\sum_{i,j,k,\ell=1}^{m}\sum_{\alpha=1}^{n}\widetilde{R}_{ijk\ell}\varphi_\alpha^i\varphi_0^j\varphi_{\overline{\alpha}}^k\varphi_0^\ell \leq 0$$

under assumption that the sectional curvature of N is nonpositive.

On the other hand,

$$\begin{aligned}\frac{1}{2}\int_M \eta^3\Delta_b\big(e^0(\varphi)\big)dv &= \frac{1}{2}\int_M \eta^3\Delta_b(g_{ij}\varphi_0^i\varphi_0^j)dv\\ &= -\frac{1}{2}\int_M \langle\nabla_b\eta^3,\nabla_b(g_{ij}\varphi_0^i\varphi_0^j)\rangle dv\\ &\leq 3\int_M \eta^2|\nabla_b\eta||\varphi_0^k||\nabla_b\varphi_0^k|dv.\end{aligned}\tag{4.6}$$

Then the lemma follows immediately from (4.5) and (4.6). ∎

The Proof of Theorem 1.1. Combining Lemma 4.1, Lemma 4.2 and the assumption (1.2), we obtain

$$
\begin{aligned}
&d\Big(\frac{1}{\varepsilon_1}+(1-\sigma-\varepsilon_1)\big(\frac{1}{\varepsilon_2}-1\big)\Big)\int_M |\nabla_b\eta|^2(f_\varepsilon^k)^2dv\\
\geq&(1-\varepsilon_2)(1-\sigma-\varepsilon_1)\int_M \rho\eta^2(f_\varepsilon^k)^2dv\\
&+\int_M \eta^2\Big[\sigma|\nabla_b^2\varphi^k|^2+2\big\langle J\nabla_b\varphi^k,\nabla_b\varphi_0^k\big\rangle\Big]dv\\
&+\int_M \eta^2\big(2Ric-(n-2)Tor\big)\big((\nabla_b\varphi^k)_C,(\nabla_b\varphi^k)_C\big)dv\\
&+\int_M \Big[a\eta^3|\nabla_b\varphi_0^k|^2-3a\eta^2|\nabla_b\eta||\varphi_0^k||\nabla_b\varphi_0^k|\Big]dv,
\end{aligned}
\tag{4.7}
$$

where $a\in\mathbb{R}$ is to be determined. Let

$$
\begin{aligned}
D=&\int_M \eta^2\Big[\sigma|\nabla_b^2\varphi^k|^2+2\big\langle J\nabla_b\varphi^k,\nabla_b\varphi_0^k\big\rangle\Big]dv\\
&+\int_M \Big[a\eta^3|\nabla_b\varphi_0^k|^2-3a\eta^2|\nabla_b\eta||\varphi_0^k||\nabla_b\varphi_0^k|\Big]dv,
\end{aligned}
$$

and we will give a lower bound of D.

From (3.5), one has

$$
\begin{aligned}
D\geq&\frac{\sigma n}{2}\int_M \eta^2(\varphi_0^k)^2dv-2\int_M \eta^2 f^k|\nabla_b\varphi_0^k|dv\\
&+a\int_M \eta^3|\nabla_b\varphi_0^k|^2dv-3a\int_M \eta^2|\nabla_b\eta||\varphi_0^k||\nabla_b\varphi_0^k|dv.
\end{aligned}
\tag{4.8}
$$

Also by Young's inequality, we have

$$
\begin{aligned}
2\int_M \eta^2 f^k|\nabla_b\varphi_0^k|dv&\leq 2\int_M \eta^2 f_\varepsilon^k|\nabla_b\varphi_0^k|dv\\
&\leq \delta_1\int_M \eta(f_\varepsilon^k)^2dv+\frac{1}{\delta_1}\int_M \eta^3|\nabla_b\varphi_0^k|^2dv,
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
&3a\int_M \eta^2|\nabla_b\eta||\varphi_0^k||\nabla_b\varphi_0^k|dv\\
\leq&\frac{9a^2\delta_2}{2}\int_M \eta|\nabla_b\eta|^2(\varphi_0^k)^2dv+\frac{1}{2\delta_2}\int_M \eta^3|\nabla_b\varphi_0^k|^2dv,
\end{aligned}
\tag{4.10}
$$

where $\delta_1, \delta_2 \in (0,1)$ are constants to be determined. Substituting (4.9) and (4.10) into (4.8) implies

$$\begin{aligned} D \geq \frac{\sigma n}{2} \int_M \eta^2 (\varphi_0^k)^2 dv - \delta_1 \int_M \eta (f_\varepsilon^k)^2 dv + \big(a - \frac{1}{\delta_1} - \frac{1}{2\delta_2}\big) \int_M \eta^3 |\nabla_b \varphi_0^k|^2 dv \\ - \frac{9a^2\delta_2}{2} \int_M \eta |\nabla_b \eta|^2 (\varphi_0^k)^2 dv. \end{aligned}$$

Let $a = \sigma R$. By choosing $\delta_1^{-1} = (2\delta_2)^{-1} = \frac{a}{3}$ and using the fact $|\nabla_b \eta|^2 \leq \frac{c\eta}{R^2}$, then the above inequality becomes

$$\begin{aligned} D &\geq \big(\frac{\sigma n}{2} - \frac{27c}{4}\frac{\sigma}{R}\big) \int_M \eta^2 (\varphi_0^k)^2 dv - \frac{3}{\sigma R} \int_M \eta (f_\varepsilon^k)^2 dv + \frac{\sigma R}{3} \int_M \eta^3 |\nabla_b \varphi_0^k|^2 dv \\ &\geq \frac{\sigma n}{4} \int_M \eta^2 (\varphi_0^k)^2 dv - \frac{3}{\sigma R} \int_M \eta (f_\varepsilon^k)^2 dv + \frac{\sigma R}{3} \int_M \eta^3 |\nabla_b \varphi_0^k|^2 dv, \end{aligned} \tag{4.11}$$

whenever R large enough. This gives a lower bound of D.

Substituting (4.11) into (4.7) and using $|\nabla_b \eta|^2 \leq \frac{c\eta}{R^2}$ again, we have

$$\begin{aligned} &\Big[\Big(\frac{1}{\varepsilon_1} + (1-\sigma-\varepsilon_1)\big(\frac{1}{\varepsilon_2} - 1\big)\Big)\frac{c}{R^2} + \frac{3}{\sigma R}\Big] \int_M \eta (f_\varepsilon^k)^2 dv \\ &\geq (1-\varepsilon_2)(1-\sigma-\varepsilon_1) \int_M \rho \eta^2 (f_\varepsilon^k)^2 dv + \frac{\sigma n}{4} \int_M \eta^2 (\varphi_0^k)^2 dv + \frac{\sigma R}{3} \int_M \eta^3 |\nabla_b \varphi_0^k|^2 dv \\ &\quad + \int_M \eta^2 \big(2Ric - (n-2)Tor\big)\big((\nabla_b \varphi^k)_C, (\nabla_b \varphi^k)_C\big) dv. \end{aligned}$$

Since the second and the third terms on the right-hand side of the above inequality are nonnegative, we may drop it and then we obtain, by letting $\varepsilon \to 0$, that

$$\begin{aligned} &\Big[\Big(\frac{1}{\varepsilon_1} + (1-\sigma-\varepsilon_1)\big(\frac{1}{\varepsilon_2} - 1\big)\Big)\frac{c}{R^2} + \frac{3}{\sigma R}\Big] \int_M \eta (f^k)^2 dv \\ &\geq (1-\varepsilon_2)(1-\sigma-\varepsilon_1) \int_M \rho \eta^2 (f^k)^2 dv \\ &\quad + \int_M \eta^2 \big(2Ric - (n-2)Tor\big)\big((\nabla_b \varphi^k)_C, (\nabla_b \varphi^k)_C\big) dv. \end{aligned}$$

By assumption (1.1), we may write $\big(2Ric - (n-2)Tor\big)(Z,Z) \geq -2\delta\rho|Z|^2$ for $0 < \delta < 1$, and then the above inequality gives

$$\begin{aligned} &\Big[\Big(\frac{1}{\varepsilon_1} + (1-\sigma-\varepsilon_1)\big(\frac{1}{\varepsilon_2} - 1\big)\Big)\frac{c}{R^2} + \frac{3}{\sigma R}\Big] \int_M (f^k)^2 dv \\ &\geq \big[(1-\varepsilon_2)(1-\sigma-\varepsilon_1) - \delta\big] \int_M \rho \eta^2 (f^k)^2 dv. \end{aligned} \tag{4.12}$$

Since $\delta < 1$, we may choose ε_1, ε_2 and σ small enough such that

$$\big[(1-\varepsilon_2)(1-\sigma-\varepsilon_1)-\delta\big] > 0.$$

Finally, let $R \to \infty$ and by the assumption that φ has finite Dirichlet energy

$$\int_M \sum_{k=1}^m |\nabla_b \varphi^k|^2 dv < \infty,$$

we then conclude from (4.12) that $|\nabla_b \varphi^k| = f^k = 0, \;\; k = 1, \cdots, m$. This shows that φ must be a constant map and the proof is now complete. ∎

Remark 4.1. The key point that we can release our assumption in Theorem 1.1 to negative curvatures (see (1.1)) is that we have the positive term

$$(1-\varepsilon_2)(1-\sigma-\varepsilon_1)\int_M \rho\eta^2 (f^k)^2 dv \tag{4.13}$$

on the right-hand side of inequality (4.12). This positive term comes originally from Lemma 4.1, in which we apply the CR Kato's inequality (3.6) to equality (3.1) and then obtain the desired positive term (4.13).

Remark 4.2. In the proof of Theorem 1.1, there is a mixed term $\langle J\nabla_b\varphi^k, \nabla_b\varphi_0^k\rangle$ in (4.7). This mixed term comes from the CR Bochner formula (see (3.1)) and is hard to estimate. However, by deriving the CR Bochner formula for $e^0(\varphi)$ (see (3.2)), we get the positive term

$$\int_M a\eta^3 |\nabla_b \varphi_0^k|^2 dv$$

in (4.7) and then we may deal with the mixed term by estimating the lower bound of the term D instead.

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Ting-Hui Chang
Department of Mathematics
National Taiwan University
Taipei 10617
Taiwan
E-mail: thchang0807@ntu.edu.tw

Yen-Chang Huang
Department of Mathematics
National Tsing Hua University
Hsinchu 30013
Taiwan
E-mail: ychuang2@mx.nthu.edu.tw