TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 2, pp. 509-557, April 2014
DOI: 10.11650/tjm.18.2014.3651
This paper is available online at http://journal.taiwanmathsoc.org.tw

GENERALIZED FRACTIONAL INTEGRALS AND THEIR COMMUTATORS OVER NON-HOMOGENEOUS METRIC MEASURE SPACES

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Abstract. Let $(\mathcal{X}, d, \mu)$ be a metric measure space satisfying both the upper doubling and the geometrically doubling conditions. In this paper, the authors establish some equivalent characterizations for the boundedness of fractional integrals over $(\mathcal{X}, d, \mu)$. The authors also prove that multilinear commutators of fractional integrals with $\mathrm{RBMO}(\mu)$ functions are bounded on Orlicz spaces over $(\mathcal{X}, d, \mu)$, which include Lebesgue spaces as special cases. The weak type endpoint estimates for multilinear commutators of fractional integrals with functions in the Orlicz-type space $\mathrm{Osc}_{\exp L^r}(\mu)$, where $r \in [1, \infty)$, are also presented. Finally, all these results are applied to a specific example of fractional integrals over non-homogeneous metric measure spaces.

1. INTRODUCTION

During the past ten to fifteen years, considerable attention has been paid to the study of the classical theory of harmonic analysis on Euclidean spaces with non-doubling measures only satisfying the polynomial growth condition (see, for example, [11, 10, 37, 38, 39, 40, 41, 42, 5, 29, 14, 15, 16, 17, 4, 44]). Recall that a Radon measure μ on $\mathbb{R}^d$ is said to only satisfy the polynomial growth condition, if there exists a positive constant C_0 such that, for all $x \in \mathbb{R}^d$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leq C_0 r^{\kappa}, \tag{1.1}$$

Received August 11, 2013, accepted September 17, 2013.
Communicated by Der-Chen Chang.
2010 *Mathematics Subject Classification*: Primary 47B06; Secondary 47B47, 42B25, 42B35, 30L99.
Key words and phrases: Non-homogeneous metric measure space, Fractional integral, Commutator, Orlicz space, Hardy space, $\mathrm{RBMO}(\mu)$, $\mathrm{Osc}_{\exp L^r}(\mu)$.
This project is supported by the National Natural Science Foundation of China (Grant Nos. 11171027, 11361020 & 11101038), the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20120003110003) and the Fundamental Research Funds for Central Universities of China (Grant No. 2012LYB26).
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where κ is some fixed number in $(0, d]$ and $B(x, r) := \{y \in \mathbb{R}^d : |y - x| < r\}$. The analysis associated with such non-doubling measures μ as in (1.1) has proved to play a striking role in solving the long-standing open Painlevé's problem and Vitushkin's conjecture by Tolsa [40, 41, 42].

Obviously, the non-doubling measure μ as in (1.1) may not satisfy the well-known doubling condition, which is a key assumption in harmonic analysis on spaces of homogeneous type in the sense of Coifman and Weiss [6, 7]. To unify both spaces of homogeneous type and the metric spaces endowed with measures only satisfying the polynomial growth condition, Hytönen [18] introduced a new class of metric measure spaces satisfying both the so-called geometrically doubling and the upper doubling conditions (see also, respectively, Definitions 1.1 and 1.3 below), which are called *non-homogeneous metric measure spaces*. Recently, many classical results have been proved still valid if the underlying spaces are replaced by the non-homogeneous metric measure spaces (see, for example, [18, 22, 2, 19, 20, 21, 25, 8, 24]). It is now also known that the theory of the singular integral operators on non-homogeneous metric measure spaces arises naturally in the study of complex and harmonic analysis questions in several complex variables (see [43, 20]). More progresses on the Hardy space H^1 and the boundedness of operators on non-homogeneous metric measure spaces can be found in the survey [45] and the monograph [46].

Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space in the sense of Hytönen [18]. In this paper, we establish some equivalent characterizations for the boundedness of fractional integrals over $(\mathcal{X}, d, \mu)$. We also prove that multilinear commutators of fractional integrals with $\mathrm{RBMO}(\mu)$ functions are bounded on Orlicz spaces over $(\mathcal{X}, d, \mu)$, which include Lebesgue spaces as special cases. The weak type endpoint estimates for multilinear commutators of fractional integrals with functions in the Orlicz-type space $\mathrm{Osc}_{\exp L^r}(\mu)$, where $r \in [1, \infty)$, are also presented. Finally, all these results are applied to a specific example of fractional integrals over non-homogeneous metric measure spaces. The results of this paper round out the picture on fractional integrals and their commutators over non-homogeneous metric measure spaces.

Recall that the well-known Hardy-Littlewood-Sobolev theorem (see, for example, [34, pp. 119-120, Theorem 1]) states that the classical fractional integral I_α, with $\alpha \in (0, d)$, is bounded from $L^p(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$, for all $p \in (1, d/\alpha)$ and $1/q = 1/p - \alpha/d$, and bounded from $L^1(\mathbb{R}^d)$ to weak $L^{d/(d-\alpha)}(\mathbb{R}^d)$. Chanillo [3] further showed that the commutator $[b, I_\alpha]$, generated by $b \in \mathrm{BMO}(\mathbb{R}^d)$ and I_α, which is defined by

$$[b, I_\alpha](f)(x) := b(x) I_\alpha(f)(x) - I_\alpha(bf)(x), \quad x \in \mathbb{R}^d,$$

is bounded from $L^p(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$ for all $\alpha \in (0, d)$, $p \in (1, d/\alpha)$ and $1/q = 1/p - \alpha/d$. These results, when the d-dimensional Lebesgue measure is replaced by the non-doubling measure μ as in (1.1), were obtained by García-Cuerva and Martell [11] and by Chen and Sawyer [5], respectively. Moreover, also in this setting with the non-doubling measure μ as in (1.1), some equivalent characterizations for the boundedness of fractional integrals were established in [17] and the boundedness for the multilinear commutators of fractional integrals with $\mathrm{RBMO}(\mu)$ or $\mathrm{Osc}_{\exp L^r}(\mu)$ functions was presented in [14]. Notice also that Nakai [27, 28] introduced a class of generalized fractional integrals and obtained their boundedness on Orlicz spaces over $\mathbb{R}^d$ with the d-dimensional Lebesgue measure and also over spaces of homogeneous type.

On the other hand, due to the request of applications, as a natural extension of Lebesgue spaces, the Orlicz spaces were introduced by Birnbaum-Orlicz in [1] and Orlicz in [30]. Since then, the theory of Orlicz spaces and its applications have been well developed (see, for example, [32, 33, 26]).

To state the main results of this paper, we first recall some necessary notions.

The following notion of the geometrically doubling is well known in analysis on metric spaces, which was originally introduced by Coifman and Weiss in [6, pp. 66-67] and is also known as *metrically doubling* (see, for example, [13, p. 81]).

Definition 1.1. A metric space $(\mathcal{X}, d)$ is said to be *geometrically doubling* if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.

Remark 1.2. Let $(\mathcal{X}, d)$ be a metric space. In [18], Hytönen showed that the following statements are mutually equivalent:

(i) $(\mathcal{X}, d)$ is geometrically doubling.

(ii) For any $\epsilon \in (0, 1)$ and any ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, \epsilon r)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0\epsilon^{-n}$, here and in what follows, N_0 is as in Definition 1.1 and $n := \log_2 N_0$.

(iii) For every $\epsilon \in (0, 1)$, any ball $B(x, r) \subset \mathcal{X}$ contains at most $N_0\epsilon^{-n}$ centers of disjoint balls $\{B(x_i, \epsilon r)\}_i$.

(iv) There exists $M \in \mathbb{N}$ such that any ball $B(x, r) \subset \mathcal{X}$ contains at most M centers $\{x_i\}_i$ of disjoint balls $\{B(x_i, r/4)\}_{i=1}^M$.

Recall that spaces of homogeneous type are geometrically doubling, which was proved by Coifman and Weiss in [6, pp. 66-68].

The following notion of upper doubling metric measure spaces was originally introduced by Hytönen [18] (see also [19, 25]).

Definition 1.3. A metric measure space $(\mathcal{X}, d, \mu)$ is said to be *upper doubling* if μ is a Borel measure on $\mathcal{X}$ and there exist a *dominating function* $\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$ and a positive constant C_λ, depending on λ, such that, for each $x \in \mathcal{X}$, $r \to \lambda(x, r)$ is non-decreasing and, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leq \lambda(x, r) \leq C_\lambda \lambda(x, r/2). \tag{1.2}$$

A metric measure space $(\mathcal{X}, d, \mu)$ is called a *non-homogeneous metric measure space* if $(\mathcal{X}, d)$ is geometrically doubling and $(\mathcal{X}, d, \mu)$ upper doubling.

Remark 1.4.

(i) Obviously, a space of homogeneous type is a special case of upper doubling spaces, where we take the dominating function $\lambda(x,r) := \mu(B(x,r))$. On the other hand, the Euclidean space $\mathbb{R}^d$ with any Radon measure μ as in (1.1) is also an upper doubling space by taking the dominating function $\lambda(x,r) := C_0 r^\kappa$.

(ii) Let $(\mathcal{X}, d, \mu)$ be upper doubling with λ being the dominating function on $\mathcal{X} \times (0,\infty)$ as in Definition 1.3. It was proved in [21] that there exists another dominating function $\widetilde{\lambda}$ such that $\widetilde{\lambda} \le \lambda$, $C_{\widetilde{\lambda}} \le C_\lambda$ and, for all $x,\ y \in \mathcal{X}$ with $d(x,y) \le r$,

$$\widetilde{\lambda}(x,r) \le C_{\widetilde{\lambda}} \widetilde{\lambda}(y,r). \tag{1.3}$$

(iii) It was shown in [35] that the upper doubling condition is equivalent to the *weak growth condition*: there exist a dominating function $\lambda : \mathcal{X} \times (0,\infty) \to (0,\infty)$, with $r \to \lambda(x,r)$ non-decreasing, positive constants C_λ, depending on λ, and ϵ such that

(a) for all $r \in (0,\infty)$, $t \in [0,r]$, $x,\ y \in \mathcal{X}$ and $d(x,y) \in [0,r]$,

$$|\lambda(y, r+t) - \lambda(x,r)| \le C_\lambda \left[\frac{d(x,y)+t}{r}\right]^\epsilon \lambda(x,r);$$

(b) for all $x \in \mathcal{X}$ and $r \in (0,\infty)$,

$$\mu(B(x,r)) \le \lambda(x,r).$$

Based on Remark 1.4(ii), from now on, we *always assume that* $(\mathcal{X}, d, \mu)$ *is a non-homogeneous metric measure space with the dominating function* λ *satisfying* (1.3).

We now recall the notion of the coefficient $K_{B,S}$ introduced by Hytönen [18], which is analogous to the quantity $K_{Q,R}$ introduced by Tolsa [38, 39]. It is well known that $K_{B,S}$ well characterizes the geometrical properties of balls B and S.

Definition 1.5. For any two balls $B \subset S$, define

$$K_{B,S} := 1 + \int_{2S\setminus B} \frac{1}{\lambda(c_B, d(x,c_B))}\, d\mu(x),$$

where c_B is the center of the ball B.

Remark 1.6. The following discrete version, $\widetilde{K}_{B,S}$, of $K_{B,S}$ defined in Definition 1.5, was first introduced by Bui and Duong [2] in non-homogeneous metric measure spaces, which is more close to the quantity $K_{Q,R}$ introduced by Tolsa [37] in the setting of non-doubling measures. For any two balls $B \subset S$, let $\widetilde{K}_{B,S}$ be defined by

$$\widetilde{K}_{B,S} := 1 + \sum_{k=1}^{N_{B,S}} \frac{\mu(6^k B)}{\lambda(c_B, 6^k r_B)},$$

where r_B and r_S respectively denote the *radii* of the balls B and S, and $N_{B,S}$ the *smallest integer* satisfying $6^{N_{B,S}} r_B \geq r_S$. Obviously, $K_{B,S} \lesssim \widetilde{K}_{B,S}$. As was pointed out by Bui and Duong [2], in general, it is not true that $K_{B,S} \sim \widetilde{K}_{B,S}$.

Though the measure doubling condition is not assumed uniformly for all balls in the non-homogeneous metric measure space $(\mathcal{X}, d, \mu)$, it was shown in [18] that there exist still many balls which have the following (η, β)-doubling property.

Definition 1.7. Let $\eta, \beta \in (1, \infty)$. A ball $B \subset \mathcal{X}$ is said to be (η, β)-*doubling* if $\mu(\eta B) \leq \beta \mu(B)$.

To be precise, it was proved in [18, Lemma 3.2] that, if a metric measure space $(\mathcal{X}, d, \mu)$ is upper doubling and $\eta, \beta \in (1, \infty)$ satisfying $\beta > C_\lambda^{\log_2 \eta} =: \eta^\nu$, then, for any ball $B \subset \mathcal{X}$, there exists some $j \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ such that $\eta^j B$ is (η, β)-doubling. Moreover, let $(\mathcal{X}, d)$ be geometrically doubling, $\beta > \eta^n$ with $n := \log_2 N_0$ and μ a Borel measure on $\mathcal{X}$ which is finite on bounded sets. Hytönen [18, Lemma 3.3] also showed that, for μ-almost every $x \in \mathcal{X}$, there exist arbitrary small (η, β)-doubling balls centered at x. Furthermore, the radii of these balls may be chosen to be the form $\eta^{-j} B$ for $j \in \mathbb{N}$ and any preassigned number $r \in (0, \infty)$. Throughout this paper, for any $\eta \in (1, \infty)$ and ball B, the *smallest* (η, β_η)-*doubling ball of the form* $\eta^j B$ *with* $j \in \mathbb{N}$ is denoted by $\widetilde{B}^\eta$, where

$$\beta_\eta := \max\{\eta^{3n}, \eta^{3\nu}\} + 30^n + 30^\nu = \eta^{3(\max\{n,\nu\})} + 30^n + 30^\nu. \tag{1.4}$$

In what follows, by a *doubling ball* we mean a $(6, \beta_6)$-doubling ball and $\widetilde{B}^6$ is *simply denoted* by $\widetilde{B}$.

Now we recall the following notion of $\mathrm{RBMO}(\mu)$ from [18].

Definition 1.8. Let $\rho \in (1, \infty)$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the *space* $\mathrm{RBMO}(\mu)$ if there exist a positive constant C and, for any ball $B \subset \mathcal{X}$, a number f_B such that

$$\frac{1}{\mu(\rho B)} \int_B |f(x) - f_B| \, d\mu(x) \leq C \tag{1.5}$$

and, for any two balls B and B_1 such that $B \subset B_1$,

$$|f_B - f_{B_1}| \leq C K_{B,B_1}. \tag{1.6}$$

The infimum of the positive constants C satisfying both (1.5) and (1.6) is defined to be the $\mathrm{RBMO}(\mu)$ *norm* of f and denoted by $\|f\|_{\mathrm{RBMO}(\mu)}$.

From [18, Lemma 4.6], it follows that the space $\mathrm{RBMO}(\mu)$ is independent of the choice of $\rho \in (1, \infty)$.

In this paper, we consider a variant of the generalized fractional integrals from [10, Definition 4.1] (see also [17, (1.4)]).

Definition 1.9. Let $\alpha \in (0, 1)$. A function $K_\alpha \in L^1_{\mathrm{loc}}(\mathcal{X} \times \mathcal{X} \setminus \{(x, x) : x \in \mathcal{X}\})$ is called a *generalized fractional integral kernel* if there exists a positive constant C_{K_α}, depending on K_α, such that

(i) for all $x, y \in \mathcal{X}$ with $x \neq y$,

$$|K_\alpha(x, y)| \leq C_{K_\alpha} \frac{1}{[\lambda(x, d(x, y))]^{1-\alpha}}; \tag{1.7}$$

(ii) there exist positive constants $\delta \in (0, 1]$ and $c_{K_\alpha} \in (0, \infty)$ such that, for all $x, \widetilde{x}, y \in \mathcal{X}$ with $d(x, y) \geq c_{K_\alpha} d(x, \widetilde{x})$,

$$\begin{aligned} &|K_\alpha(x, y) - K_\alpha(\widetilde{x}, y)| + |K_\alpha(y, x) - K_\alpha(y, \widetilde{x})| \\ &\leq C_{K_\alpha} \frac{[d(x, \widetilde{x})]^\delta}{[d(x, y)]^\delta [\lambda(x, d(x, y))]^{1-\alpha}}. \end{aligned} \tag{1.8}$$

Let $L^\infty_b(\mu)$ be the *space of all $L^\infty(\mu)$ functions with bounded support*. A linear operator T_α is called a *generalized fractional integral* with kernel K_α satisfying (1.7) and (1.8) if, for all $f \in L^\infty_b(\mu)$ and $x \notin \operatorname{supp} f$,

$$T_\alpha f(x) := \int_{\mathcal{X}} K_\alpha(x, y) f(y)\, d\mu(y). \tag{1.9}$$

Remark 1.10.

(i) Without loss of generality, for the simplicity, we may assume in (1.8) that $c_{K_\alpha} \equiv 2$.

(ii) If a kernel K_α satisfies (1.7) and (1.8) with $\alpha = 0$, then K_α is called a *standard kernel* and the associated operator T_α as in (1.9) is called a *Calderón-Zygmund operator* on non-homogeneous metric measure spaces (see [20, Subsetion 2.3]).

(iii) We give a specific example of the generalized fractional integrals, which is a natural variant of the so-called "Bergman-type" operators from [43, Section 2.1] (see also [20, Section 12] and [36, Section 2.2]). Let $\mathcal{X} := \mathbb{B}_{2d}$ be the open unit ball in $\mathbb{C}^d$. Suppose that the measure μ satisfies the upper power bound $\mu(B(x, r)) \leq r^m$ with $m \in (0, 2d]$ except the case when $B(x, r) \subset \mathbb{B}_{2d}$. However, in the exceptional case it holds true that $r \leq \widetilde{d}(x) := d(x, \mathbb{C}^d \setminus \mathbb{B}_{2d})$, where $d(x, y) := ||x| - |y|| + |1 - \bar{x} \cdot y/|x||y||$ for all $x, y \in \overline{\mathbb{B}}_{2d} \subset \mathbb{C}^d$, and hence $\mu(B(x, r)) \leq \max\{[\widetilde{d}(x)]^m, r^m\} =: \lambda(x, r)$. By similar arguments to those used in the proofs of [36, Proposition 2.13] and [20, Section 2], we conclude that, if

$\alpha \in (0,1)$, then the kernel $K_{m,\alpha}(x,y) := (1-\bar{x}\cdot y)^{-m(1-\alpha)}$, $x, y \in \overline{\mathbb{B}}_{2d} \subset \mathbb{C}^d$, satisfies the conditions (1.7) and (1.8). So, when $\alpha \in (0,1)$, the fractional integral $T_{m,\alpha}$, associated with $K_{m,\alpha}$, is an example of the generalized fractional integrals as in Definition 1.9. Recall that, when $\alpha = 0$, the operator $T_{m,0}$, associated with $K_{m,0}$, is just the so-called "Bergman-type" operator (see [36, 43, 20]).

Now we recall the notion of the atomic Hardy space from [21].

Definition 1.11. Let $\rho \in (1,\infty)$ and $p \in (1,\infty]$. A function $b \in L^1_{\mathrm{loc}}(\mu)$ is called a $(p,1)_\lambda$-*atomic block* if

(i) there exists a ball B such that $\operatorname{supp} b \subset B$;

(ii) $\int_{\mathcal{X}} b(x)\,d\mu(x) = 0$;

(iii) for any $j \in \{1,\,2\}$, there exist a function a_j supported on ball $B_j \subset B$ and a number $\lambda_j \in \mathbb{C}$ such that $b = \lambda_1 a_1 + \lambda_2 a_2$ and $\|a_j\|_{L^p(\mu)} \le [\mu(\rho B_j)]^{1/p-1} K^{-1}_{B_j,B}$. Moreover, let $|b|_{H^{1,p}_{\mathrm{atb}}(\mu)} := |\lambda_1| + |\lambda_2|$.

A function $f \in L^1(\mu)$ is said to belong to the *atomic Hardy space* $H^{1,p}_{\mathrm{atb}}(\mu)$ if there exist $(p,1)_\lambda$-atomic blocks $\{b_i\}_{i=1}^\infty$ such that $f = \sum_{i=1}^\infty b_i$ in $L^1(\mu)$ and $\sum_{i=1}^\infty |b_i|_{H^{1,p}_{\mathrm{atb}}(\mu)} < \infty$. The $H^{1,p}_{\mathrm{atb}}(\mu)$ *norm* of f is defined by

$$\|f\|_{H^{1,p}_{\mathrm{atb}}(\mu)} := \inf\left\{\sum_{i=1}^\infty |b_i|_{H^{1,p}_{\mathrm{atb}}(\mu)}\right\},$$

where the infimum is taken over all the possible decompositions of f as above.

Remark 1.12.

(i) It was proved in [21] that, for each $p \in (1,\infty]$, the atomic Hardy space $H^{1,p}_{\mathrm{atb}}(\mu)$ is independent of the choice of ρ and that, for all $p \in (1,\infty]$, the spaces $H^{1,p}_{\mathrm{atb}}(\mu)$ and $H^{1,\infty}_{\mathrm{atb}}(\mu)$ coincide with equivalent norms. Thus, in what follows, we *denote* $H^{1,p}_{\mathrm{atb}}(\mu)$ *simply by* $H^1(\mu)$ and, unless explicitly pointed out, we *always assume* that $\rho = 2$ in Definition 1.11.

(ii) It was proved in [25, Remark 1.3(ii)] that the atomic Hardy space introduced by Bui and Duong [2] and the atomic Hardy space in Definition 1.11 coincide with equivalent norms.

Then we state the first main theorem of this paper.

Theorem 1.13. *Let* $\alpha \in (0,1)$ *and* T_α *be as in* (1.9) *with kernel* K_α *satisfying* (1.7) *and* (1.8). *Then the following statements are equivalent:*

(I) T_α *is bounded from* $L^p(\mu)$ *into* $L^q(\mu)$ *for all* $p \in (1,\,1/\alpha)$ *and* $1/q = 1/p - \alpha$*;*

(II) T_α *is bounded from* $L^1(\mu)$ *into* $L^{1/(1-\alpha),\infty}(\mu)$*;*

(III) *There exists a positive constant* C *such that, for all* $f \in L^{1/\alpha}(\mu)$ *with* $T_\alpha f$ *being finite almost everywhere,* $\|T_\alpha f\|_{\mathrm{RBMO}(\mu)} \le C\|f\|_{L^{1/\alpha}(\mu)}$*;*

(IV) T_α *is bounded from* $H^1(\mu)$ *into* $L^{1/(1-\alpha)}(\mu)$*;*
(V) T_α *is bounded from* $H^1(\mu)$ *into* $L^{1/(1-\alpha),\infty}(\mu)$*.*

Remark 1.14. Theorem 1.13 covers [17, Theorem 1.1] by taking $\mathcal{X} := \mathbb{R}^d$, d being the usual Euclidean metric and μ as in (1.1). The difference between Theorem 1.13 and [17, Theorem 1.1] exists in that no conclusion of Theorem 1.13 is known to be true, while all conclusions of [17, Theorem 1.1] are true.

Let Φ be a *convex Orlicz function* on $[0,\infty)$, namely, a convex increasing function satisfying $\Phi(0) = 0$, $\Phi(t) > 0$ for all $t \in (0,\infty)$ and $\Phi(t) \to \infty$ as $t \to \infty$. Let

$$a_\Phi := \inf_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)} \quad \text{and} \quad b_\Phi := \sup_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}. \tag{1.10}$$

We refer to [26] for more properties of a_Φ and b_Φ.

The *Orlicz space* $L^\Phi(\mu)$ is defined to be the space of all measurable functions f on $(\mathcal{X}, d, \mu)$ such that $\int_{\mathcal{X}} \Phi(|f(x)|)\,d\mu(x) < \infty$; moreover, for any $f \in L^\Phi(\mu)$, its *Luxemburg norm* in $L^\Phi(\mu)$ is defined by

$$\|f\|_{L^\Phi(\mu)} := \inf\left\{t \in (0,\infty) : \int_{\mathcal{X}} \Phi(|f(x)|/t)\,d\mu(x) \le 1\right\}.$$

For any sequence $\vec{b} := (b_1, \ldots, b_k)$ of functions, the *multilinear commutator* $T_{\alpha,\vec{b}}$ of the generalized fractional integral T_α with $\vec{b}$ is defined by setting, for all suitable functions f,

$$T_{\alpha,\vec{b}} f := [b_k, \cdots, [b_1, T_\alpha] \cdots] f, \tag{1.11}$$

where

$$[b_1, T_\alpha] f := b_1 T_\alpha f - T_\alpha(b_1 f). \tag{1.12}$$

The second main result of this paper is the following boundedness of the multilinear commutator $T_{\alpha,\vec{b}}$ on Orlicz spaces.

Theorem 1.15. *Let* $\alpha \in (0,1)$*,* $k \in \mathbb{N}$ *and* $b_j \in \mathrm{RBMO}(\mu)$ *for all* $j \in \{1, \ldots, k\}$*. Let* Φ *be a convex Orlicz function and* Ψ *defined, via its inverse, by setting, for all* $t \in (0,\infty)$*,* $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha}$*, where* $\Phi^{-1}(t) := \inf\{s \in (0,\infty) : \ \Phi(s) > t\}$*. Suppose that* T_α *is as in* (1.9)*, with kernel* K_α *satisfying* (1.7) *and* (1.8)*, which is bounded from* $L^p(\mu)$ *into* $L^q(\mu)$ *for all* $p \in (1, 1/\alpha)$ *and* $1/q = 1/p - \alpha$*. If* $1 < a_\Phi \le b_\Phi < \infty$ *and* $1 < a_\Psi \le b_\Psi < \infty$*, then the multilinear commutator* $T_{\alpha,\vec{b}}$ *as in* (1.11) *is bounded from* $L^\Phi(\mu)$ *to* $L^\Psi(\mu)$*, namely, there exists a positive constant* C *such that, for all* $f \in L^\Phi(\mu)$*,*

$$\|T_{\alpha,\vec{b}} f\|_{L^\Psi(\mu)} \le C \prod_{j=1}^k \|b_j\|_{\mathrm{RBMO}(\mu)} \|f\|_{L^\Phi(\mu)}.$$

Remark 1.16.

(i) Let all the notation be the same as in Theorem 1.15. By Theorem 1.13, we can, in Theorem 1.15, replace the assumption that T_α is bounded from $L^p(\mu)$ into $L^q(\mu)$ for all $p \in (1, 1/\alpha)$ and $1/q = 1/p - \alpha$ by any one of the statements (II)-(V) in Theorem 1.13.

(ii) In Theorem 1.15, if $p \in (1, 1/\alpha)$ and $\Phi(t) := t^p$ for all $t \in (0, \infty)$, then $\Psi(t) = t^q$ and $1/q = 1/p - \alpha$. In this case, $a_\Phi = b_\Phi = p \in (1, \infty)$, $a_\Psi = b_\Psi = q \in (1, \infty)$, $L^\Phi(\mu) = L^p(\mu)$ and $L^\Psi(\mu) = L^q(\mu)$. Thus, Theorem 1.15, even when $\mathcal{X} := \mathbb{R}^d$, d being the usual Euclidean metric and μ as in (1.1), also contains [14, Theorem 1.1] as a special case. In the non-homogenous setting, Theorem 1.15, even when $k = 1$, is also new.

(iii) If a convex function $\Phi : [0, \infty) \to [0, \infty)$ satisfies that $\Phi(0) = 0$, $\Phi(t) > 0$ for all $t \in (0, \infty)$ and $\Phi(t) \to \infty$ as $t \to \infty$, then Φ is absolutely continuous on any closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself. Therefore, Φ is differentiable almost everywhere and $\Phi^{-1}(s) := \inf\{t \in [0, \infty) : \ \Phi(t) > s\}$ is the usual inverse function.

The end point counterpart of Theorem 1.15 is also considered in this paper. To this end, we first recall the following Orlicz type space $\mathrm{Osc}_{\exp L^r}(\mu)$ of functions (see, for example, Pérez and Trujillo-González [31] for Euclidean spaces and [14] for non-doubling measures).

In what follows, let $L^1_{\rm loc}(\mu)$ be the *space of all locally μ-integrable functions on $\mathcal{X}$*. For all balls B and $f \in L^1_{\rm loc}(\mu)$, $m_B(f)$ denotes the *mean value of f on ball B*, namely,

$$m_B(f) := \frac{1}{\mu(B)} \int_B f(y)\, d\mu(y). \tag{1.13}$$

Definition 1.17. Let $r \in [1, \infty)$. A function $f \in L^1_{\rm loc}(\mu)$ is said to belong to the *space* $\mathrm{Osc}_{\exp L^r}(\mu)$ if there exists a positive constant C_1 such that

(i) for all balls B,

$$\begin{aligned}&\|f - m_{\widetilde{B}}(f)\|_{\exp L^r,\, B,\, \mu/\mu(2B)}\\ &:= \inf\left\{\lambda \in (0, \infty) : \ \frac{1}{\mu(2B)} \int_B \exp\left(\frac{|f(y) - m_{\widetilde{B}}(f)|}{\lambda}\right)^r d\mu(y) \le 2\right\} \le C_1;\end{aligned}$$

(ii) for all doubling balls $Q \subset R$, $|m_Q(f) - m_R(f)| \le C_1 K_{Q,R}$.

The $\mathrm{Osc}_{\exp L^r}(\mu)$ *norm* of f, $\|f\|_{\mathrm{Osc}_{\exp L^r}(\mu)}$, is then defined to be the infimum of all positive constants C_1 satisfying (i) and (ii).

Remark 1.18. Obviously, for any $r \in [1, \infty)$, $\mathrm{Osc}_{\exp L^r}(\mu) \subset \mathrm{RBMO}(\mu)$. Moreover, from [18, Corollary 6.3], it follows that $\mathrm{Osc}_{\exp L^1}(\mu) = \mathrm{RBMO}(\mu)$.

We recall some notation from [15]. For $i \in \{1, \ldots, k\}$, the *family of all finite subsets* $\sigma := \{\sigma(1), \ldots, \sigma(i)\}$ *of* $\{1, \ldots, k\}$ *with* i *different elements* is denoted by C_i^k. For any $\sigma \in C_i^k$, the *complementary sequence* σ' is defined by $\sigma' := \{1, \ldots, k\} \setminus \sigma$. For any $\sigma := \{\sigma(1), \ldots, \sigma(i)\} \in C_i^k$ and k-tuple $r := (r_1, \ldots, r_k)$, we write that $1/r_\sigma := 1/r_{\sigma(1)} + \cdots + 1/r_{\sigma(i)}$ and $1/r_{\sigma'} := 1/r - 1/r_\sigma$, where $1/r := 1/r_1 + \cdots + 1/r_k$.

Now we state the third main result of this paper.

Theorem 1.19. *Let* $\alpha \in (0,1)$, $k \in \mathbb{N}$, $r_i \in [1,\infty)$ *and* $b_i \in \mathrm{Osc}_{\exp L^{r_i}}(\mu)$ *for* $i \in \{1, \ldots, k\}$. *Let* T_α *and* $T_{\alpha,\vec{b}}$ *be, respectively, as in* (1.9) *and* (1.11) *with kernel* K_α *satisfying* (1.7) *and* (1.8). *Suppose that* T_α *is bounded from* $L^p(\mu)$ *into* $L^q(\mu)$ *for all* $p \in (1, 1/\alpha)$ *and* $1/q = 1/p - \alpha$. *Then, there exists a positive constant* C *such that, for all* $\lambda \in (0,\infty)$ *and* $f \in L_b^\infty(\mu)$,

$$\begin{aligned}
&\mu(\{x \in \mathcal{X} : |T_{\alpha,\vec{b}}f(x)| > \lambda\}) \\
&\leq C \left[\Phi_{1/r}\left(\prod_{j=1}^k \|b_j\|_{\mathrm{Osc}_{\exp L^{r_j}}(\mu)}\right)\right] \left[\sum_{j=0}^k \sum_{\sigma \in C_j^k} \Phi_{1/r_\sigma}\left(\|\Phi_{1/r_{\sigma'}}(\lambda^{-1}|f|)\|_{L^1(\mu)}\right)\right],
\end{aligned}$$

where $\Phi_s(t) := t \log^s(2+t)$ *for all* $t \in (0,\infty)$ *and* $s \in (0,\infty)$.

Remark 1.20. Theorem 1.19 covers [17, Theorem 1.1] by taking $\mathcal{X} := \mathbb{R}^d$, d being the usual Euclidean metric and μ as in (1.1).

The organization of this paper is as follows.

In Section 2, we show Theorem 1.13 by first establishing a new interpolation theorem (see Theorem 2.7 below), which, when $p_0 = \infty$, is just [23, Theorem 1.1] and whose version on the linear operators over the non-doubling setting is just [17, Lemma 2.3]. Moreover, we prove Theorem 2.7 by borrowing some ideas from the proof of [23, Theorem 1.1], which seals some gaps existing in the proof of [17, Lemma 2.3]. The key tool for the proof of Theorem 2.7 is the Calderón-Zygmund decomposition in the non-homogeneous setting obtained by Bui and Duong [2] (see also Lemma 2.6 below). Again, using the Calderón-Zygmund decomposition (Lemma 2.6) and the interpolation theorem (Theorem 2.7), together with the full applications of the geometrical properties of $K_{B,S}$ and the underlying space $(\mathcal{X}, d, \mu)$, we then complete the proof of Theorem 1.13.

Section 3 is devoted to proving Theorems 1.15 and 1.19. We first prove, in Theorem 3.9 below, that, if the generalized fractional integral T_α ($\alpha \in (0,1)$) is bounded from $L^p(\mu)$ into $L^q(\mu)$ for some $p \in (1, 1/\alpha)$ and $1/q = 1/p - \alpha$, then so is its commutator with any $\mathrm{RBMO}(\mu)$ function, by borrowing some ideas of [5, Theorem 1]. The main new ingredient appearing in our approach used for the proof of Theorem 3.9 is that we introduce a quantity $\widetilde{K}_{B,S}^{(\alpha)}$, which is a fractional variant of $\widetilde{K}_{B,S}$ and, in the setting of non-doubling measures, was introduced by Chen and Sawyer in [5, Section 1]. As the case $\widetilde{K}_{B,S}$, $\widetilde{K}^{(\alpha)}_{B,S}$ also well characterizes the geometrical properties of balls B and S and, moreover, it preserves all the properties of $K^{(\beta)}_{Q,R}$ in [5, Lemma 3]. To prove Theorem 3.9, we also need to introduce the maximal operator $\widetilde{M}^{\#,\alpha}$, associated with $\widetilde{K}^{(\alpha)}_{B,S}$, adapted from the maximal operator $M^{\#,(\beta)}$ in [5, Section 2]. Then we complete the proof of Theorem 1.15 by the interpolation theorem in [8] on Orlicz spaces and borrowing some ideas from the proof of [15, Theorem 2]. To obtain the weak type endpoint estimates of multilinear commutators in Theorem 1.19, we need to use the generalized Hölder inequality over the non-homogeneous setting from [8, Lemma 4.1] and the Calderón-Zygmund decomposition mentioned above.

In Section 4, under some weak reverse doubling condition of the dominating function λ (see Section 4 below), which is weaker than the assumption introduced by Bui and Duong in [2, Subsection 7.3]: there exists $m \in (0,\infty)$ such that, for all $x \in \mathcal{X}$ and $a,\ r \in (0,\infty)$, $\lambda(x, ar) = a^m\lambda(x, r)$, we construct a non-trivial example of generalized fractional integrals satisfying all the assumptions of this article. The key tool is the weak growth condition (see Remark 1.4(iii)) introduced by Tan and Li [35], which is equivalent to the upper doubling condition.

Finally, we make some conventions on notation. Throughout the whole paper, C stands for a *positive constant* which is independent of the main parameters, but it may vary from line to line. Moreover, we use $C_{\rho,\gamma,\dots}$ or $C_{(\rho,\gamma,\dots)}$ to denote a positive constant depending on the parameters $\rho,\ \gamma,\ \dots$. For any ball B and $f \in L^1_{\rm loc}(\mu)$, $m_B(f)$ denotes the *mean value of f over B* as in (1.13); the center and the radius of B are denoted, respectively, by c_B and r_B. If $f \le Cg$, we then write $f \lesssim g$; if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any subset E of $\mathcal{X}$, we use χ_E to denote its *characteristic function*.

2. Proof of Theorem 1.13

In this section, we prove Theorem 1.13. We begin with recalling some useful properties of $K_{B,S}$ in Definition 1.5 (see, for example, [18, Lemmas 5.1 and 5.2] and [21, Lemma 2.2]).

Lemma 2.1.

(i) *For all balls $B \subset R \subset S$, $K_{B,R} \le K_{B,S}$.*

(ii) *For any $\rho \in [1,\infty)$, there exists a positive constant $C_{(\rho)}$, depending on ρ, such that, for all balls $B \subset S$ with $r_S \le \rho r_B$, $K_{B,S} \le C_{(\rho)}$.*

(iii) *For any $\alpha \in (1,\infty)$, there exists a positive constant $C_{(\alpha)}$, depending on α, such that, for all balls B, $K_{B,\widetilde{B}^\alpha} \le C_{(\alpha)}$.*

(iv) *There exists a positive constant c such that, for all balls $B \subset R \subset S$,*

$$K_{B,S} \le K_{B,R} + cK_{R,S}.$$

In particular, if B and R are concentric, then $c = 1$.

(v) *There exists a positive constant* $\widetilde{c}$ *such that, for all balls* $B \subset R \subset S$, $K_{R,S} \leq \widetilde{c} K_{B,S}$; *moreover, if* B *and* R *are concentric, then* $K_{R,S} \leq K_{B,S}$.

Now we recall the following equivalent characterization of $\mathrm{RBMO}(\mu)$ established in [21, Proposition 2.10].

Lemma 2.2. *Let* $\rho \in (1, \infty)$ *and* $f \in L^1_{\mathrm{loc}}(\mu)$. *The following statements are equivalent:*

(a) $f \in \mathrm{RBMO}(\mu)$;

(b) *there exists a positive constant* C *such that, for all balls* B,

$$\frac{1}{\mu(\rho B)} \int_B \left| f(x) - m_{\widetilde{B}} f \right| \, d\mu(x) \leq C$$

and, for all doubling balls $B \subset S$,

$$|m_B(f) - m_S(f)| \leq C K_{B,S}. \tag{2.1}$$

Moreover, let $\|f\|_*$ *be the infimum of all admissible constants* C *in* (b). *Then there exists a constant* $\widetilde{C} \in [1, \infty)$ *such that, for all* $f \in \mathrm{RBMO}(\mu)$, $\|f\|_* / \widetilde{C} \leq \|f\|_{\mathrm{RBMO}(\mu)} \leq \widetilde{C} \|f\|_*$.

We also need the following conclusion, which is just [8, Corollary 3.3].

Corollary 2.3. *If* $f \in \mathrm{RBMO}(\mu)$, *then there exists a positive constant* C *such that, for any ball* B, $\rho \in (1, \infty)$ *and* $r \in [1, \infty)$,

$$\left\{ \frac{1}{\mu(\rho B)} \int_B \left| f(x) - m_{\widetilde{B}} f \right|^r \, d\mu(x) \right\}^{1/r} \leq C \|f\|_{\mathrm{RBMO}(\mu)}. \tag{2.2}$$

Moreover, the infimum of the positive constants C *satisfying both* (2.2) *and* (2.1) *is an equivalent* $\mathrm{RBMO}(\mu)$ *norm of* f.

The following interpolation result is from [8, Theorem 2.2].

Lemma 2.4. *Let* $\alpha \in [0, 1)$, p_i, $q_i \in (0, \infty)$ *satisfy* $1/q_i = 1/p_i - \alpha$ *for* $i \in \{1, 2\}$, $p_1 < p_2$ *and* T *be a sublinear operator of weak type* (p_i, q_i) *for* $i \in \{1, 2\}$. *Then* T *is bounded from* $L^\Phi(\mu)$ *to* $L^\Psi(\mu)$, *where* Φ *and* Ψ *are convex Orlicz functions satisfying the following conditions:* $1 < p_1 < a_\Phi \leq b_\Phi < p_2 < \infty$, $1 < q_1 < a_\Psi \leq b_\Psi < q_2 < \infty$ *and, for all* $t \in (0, \infty)$, $\Psi^{-1}(t) = \Phi^{-1}(t) t^{-\alpha}$.

We also recall some results in [2, Subsection 4.1] and [18, Corollary 3.6].

Lemma 2.5. (*i*) *Let* $p \in (1, \infty)$, $r \in (1, p)$ *and* $\rho \in [5, \infty)$. *The following maximal operators defined, respectively, by setting, for all* $f \in L^1_{\mathrm{loc}}(\mu)$ *and* $x \in \mathcal{X}$,

$$M_{r,\rho}f(x) := \sup_{Q\ni x}\left[\frac{1}{\mu(\rho Q)}\int_Q |f(y)|^r\,d\mu(y)\right]^{\frac{1}{r}},$$
$$Nf(x) := \sup_{Q\ni x,\,Q\,\mathrm{doubling}}\frac{1}{\mu(Q)}\int_Q |f(y)|\,d\mu(y)$$

and

$$M_{(\rho)}f(x) := \sup_{Q\ni x}\frac{1}{\mu(\rho Q)}\int_Q |f(y)|\,d\mu(y),$$

are bounded on $L^p(\mu)$ and also bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$.

(ii) *For all $f\in L^1_{\rm loc}(\mu)$, it holds true that $|f(x)|\le Nf(x)$ for μ-almost every $x\in\mathcal{X}$.*

Before we prove Theorem 1.13, we establish a new interpolation theorem, which is adapted from [23, Theorem 1.1]. To this end, we first recall the following Calderón-Zygmund decomposition theorem obtained by Bui and Duong [2, Theorem 6.3]. Let γ_0 be a fixed positive constant satisfying that $\gamma_0>\max\{C_\lambda^{3\log_2 6}, 6^{3n}\}$, where C_λ is as in (1.2) and n as in Remark 1.2(ii).

Lemma 2.6. *Let $p\in[1,\infty)$, $f\in L^p(\mu)$ and $t\in(0,\infty)$ $\left(t>\frac{\gamma_0^{1/p}\|f\|_{L^p(\mu)}}{[\mu(\mathcal{X})]^{1/p}}\right.$ when $\left.\mu(\mathcal{X})<\infty\right)$. Then*

(i) *there exists a family of finite overlapping balls $\{6B_j\}_j$ such that $\{B_j\}_j$ is pairwise disjoint,*

$$\frac{1}{\mu(6^2B_j)}\int_{B_j}|f(x)|^p\,d\mu(x)>\frac{t^p}{\gamma_0}\ \text{for all } j, \tag{2.3}$$
$$\frac{1}{\mu(6^2\eta B_j)}\int_{\eta B_j}|f(x)|^p\,d\mu(x)\le\frac{t^p}{\gamma_0}\ \text{for all } j \text{ and all } \eta\in(2,\infty),$$

and

$$|f(x)|\le t \text{ for } \mu\text{-almost every } x\in\mathcal{X}\setminus(\cup_j 6B_j); \tag{2.4}$$

(ii) *for each j, let R_j be a $(3\times6^2, C_\lambda^{\log_2(3\times6^2)+1})$-doubling ball of the family $\{(3\times6^2)^kB_j\}_{k\in\mathbb{N}}$, and $\omega_j:=\chi_{6B_j}/(\sum_k\chi_{6B_k})$. Then there exists a family $\{\varphi_j\}_j$ of functions such that, for each j, $\mathrm{supp}(\varphi_j)\subset R_j$, φ_j has a constant sign on R_j,*

$$\int_{\mathcal{X}}\varphi_j(x)\,d\mu(x)=\int_{6B_j}f(x)\omega_j(x)\,d\mu(x) \tag{2.5}$$

and

$$\sum_j|\varphi_j(x)|\le\gamma t \text{ for } \mu\text{-almost every } x\in\mathcal{X}, \tag{2.6}$$

where γ is a positive constant depending only on $(\mathcal{X},\mu)$ and there exists a positive constant C, independent of f, t and j, such that, if $p=1$, then

$$\|\varphi_j\|_{L^\infty(\mu)}\mu(R_j)\le C\int_{\mathcal{X}}|f(x)\omega_j(x)|\,d\mu(x) \tag{2.7}$$

and, if $p\in(1,\infty)$, then

$$\left\{\int_{R_j}|\varphi_j(x)|^p\,d\mu(x)\right\}^{1/p}[\mu(R_j)]^{1/p'}\le\frac{C}{t^{p-1}}\int_{\mathcal{X}}|f(x)\omega_j(x)|^p\,d\mu(x); \tag{2.8}$$

(*iii*) *when $p\in(1,\infty)$, if, for any j, choosing R_j to be the smallest $(3\times6^2,C_\lambda^{\log_2(3\times6^2)+1})$-doubling ball of the family $\{(3\times6^2)^kB_j\}_{k\in\mathbb{N}}$, then $h:=\sum_j(f\omega_j-\varphi_j)\in H^1(\mu)$ and there exists a positive constant C, independent of f and t, such that*

$$\|h\|_{H^1(\mu)}\le\frac{C}{t^{p-1}}\|f\|^p_{L^p(\mu)}. \tag{2.9}$$

Recall that the *sharp maximal operator* $M^\#$ in [2] is defined by setting, for all $f\in L^1_{\rm loc}(\mu)$ and $x\in\mathcal{X}$,

$$M^\#f(x):=\sup_{B\ni x}\frac{1}{\mu(6B)}\int_B|f(y)-m_{\widetilde{B}}f|\,d\mu(y)+\sup_{(Q,R)\in\Delta_x}\frac{|m_Qf-m_Rf|}{K_{Q,R}},$$

where $\Delta_x:=\{(Q,R):\ x\in Q\subset R \text{ and } Q,\ R \text{ are doubling balls}\}$.

Theorem 2.7. *Let T be a bounded sublinear operator from $L^{p_0}(\mu)$ into $\mathrm{RBMO}(\mu)$ and from $H^1(\mu)$ into $L^{p_0',\infty}(\mu)$, where $p_0\in(1,\infty]$ and $1/p_0+1/p_0'=1$. Then T extends to a bounded linear operator from $L^p(\mu)$ into $L^q(\mu)$, where $p\in(1,p_0)$ and $1/q=1/p-1/p_0$.*

Proof. By the Marcinkiewicz interpolation theorem, it suffices to prove that

$$\mu(\{x\in\mathcal{X}:\ |Tf(x)|>t\})\lesssim[t^{-1}\|f\|_{L^p(\mu)}]^q \tag{2.10}$$

for all $p\in(1,p_0)$ and $1/q=1/p-1/p_0$. We consider the following two cases.

Case (i) $\mu(\mathcal{X})=\infty$. Let $L^\infty_{b,0}(\mu):=\{f\in L^\infty_b(\mu):\ \int_{\mathcal{X}}f(x)\,d\mu(x)=0\}$. Then, by a standard argument, we know that $L^\infty_{b,0}(\mu)$ is dense in $L^p(\mu)$ for all $p\in(1,p_0)$. Let $r\in(0,1)$. Define $N_r(g):=[N(|g|^r)]^{1/r}$ for all $g\in L^r_{\rm loc}(\mu)$. By Lemma 2.5(ii) and a standard density argument, to prove (2.10), it suffices to prove that, for any $f\in L^\infty_{b,0}(\mu)$, $p\in(1,p_0)$ and $1/q=1/p-1/p_0$,

$$\sup_{t\in(0,\infty)}t^q\mu(\{x\in\mathcal{X}:\ |N_r(Tf)(x)|>t\})\lesssim\|f\|^q_{L^p(\mu)}. \tag{2.11}$$

To this end, for any given $f \in L^\infty_{b,0}(\mu)$, applying Lemma 2.6 to f with t replaced by $t^{q/p}$, and letting R_j be as in Lemma 2.6(iii), we see that $f = g + h$, where $g := f\chi_{\mathcal{X}\setminus\cup_j 6B_j} + \sum_j \varphi_j$ and $h := \sum_j(\omega_j f - \varphi_j)$. By Minkowski's inequality, Hölder's inequality and $1/q = 1/p - 1/p_0$, together with (2.4), (2.6) and (2.8) with t replaced by $t^{q/p}$, we conclude that

$$
\begin{aligned}
\|g\|_{L^{p_0}(\mu)} &\le \left\|f\chi_{\mathcal{X}\setminus\cup_j 6B_j}\right\|_{L^{p_0}(\mu)} + \left\|\sum_j \varphi_j\right\|_{L^{p_0}(\mu)}\\
&\lesssim t^{q(\frac{1}{p}-\frac{1}{p_0})}\|f\|^{p/p_0}_{L^p(\mu)} + t^{(q/p)/p_0'}\left[\sum_j \|\varphi_j\|_{L^1(\mu)}\right]^{1/p_0}\\
&\lesssim t\|f\|^{p/p_0}_{L^p(\mu)} + t^{(q/p)/p_0'}\left[\sum_j \|\varphi_j\|_{L^p(\mu)}[\mu(R_j)]^{1/p'}\right]^{1/p_0}\\
&\lesssim t\|f\|^{p/p_0}_{L^p(\mu)} + t^{(q/p)/p_0'}t^{-q/(p'p_0)}\left[\sum_j\int_{\mathcal{X}} |\omega_j(x)f(x)|^p\,d\mu(x)\right]^{1/p_0}\\
&\lesssim t\|f\|^{p/p_0}_{L^p(\mu)}.
\end{aligned}
\tag{2.12}
$$

For each $r \in (0,1)$, define $M^\#_r g := \left\{M^\#(|g|^r)\right\}^{1/r}$. Then, from [23, Lemma 3.1], together with the boundedness of T from $L^{p_0}(\mu)$ into $\mathrm{RBMO}(\mu)$ and (2.12), we deduce that

$$\|M^\#_r Tg\|_{L^\infty(\mu)} \lesssim \|Tg\|_{\,\mathrm{RBMO}(\mu)} \lesssim \|g\|_{L^{p_0}(\mu)} \lesssim t\|f\|^{p/p_0}_{L^p(\mu)}.$$

Hence, if $\widetilde{C}_0$ is chosen to be a sufficiently large positive constant, we then see that

$$\mu\left(\left\{x \in \mathcal{X} :\ M^\#_r(Tg)(x) > \widetilde{C}_0 t\|f\|^{p/p_0}_{L^p(\mu)}\right\}\right) = 0. \tag{2.13}$$

On the other hand, since both f and h belong to $H^1(\mu)$, by (2.9) with t replaced by $t^{q/p}$, we conclude that $g \in H^1(\mu)$ and

$$\|g\|_{H^1(\mu)} \le \|f\|_{H^1(\mu)} + \|h\|_{H^1(\mu)} \lesssim \|f\|_{H^1(\mu)} + \frac{1}{t^{(p-1)q/p}}\|f\|^p_{L^p(\mu)}.$$

From this, together with the boundedness of T from $H^1(\mu)$ into $L^{p_0',\infty}(\mu)$ and [23, Lemma 3.3], we deduce that, for any q satisfying $1/q = 1/p - 1/p_0$ and $R \in (0,\infty)$,

$$
\begin{aligned}
&\sup_{t\in(0,R)} t^q\mu\left(\{x \in \mathcal{X} :\ N_r(Tg)(x) > t\}\right)\\
&\quad\lesssim \sup_{t\in(0,R)} t^{q-p_0'} \sup_{\tau\in[t,\infty)} \tau^{p_0'}\mu\left(\{x \in \mathcal{X} :\ |Tg(x)| > \tau\}\right)\\
&\quad\lesssim R^{q-p_0}\|Tg\|_{L^{p_0',\infty}(\mu)} \lesssim R^{q-p_0}\|g\|_{H^1(\mu)} < \infty.
\end{aligned}
\tag{2.14}
$$

From the fact that $N_r \circ T$ is quasi-linear, (2.14), [23, Lemma 3.2] and (2.13), we deduce that there exists a positive constant $\widetilde{C}$ such that, for all $f \in L_{b,0}^{\infty}(\mu)$,

$$
\begin{aligned}
&\sup_{t\in(0,\infty)} t^q \mu\left(\left\{x\in\mathcal{X}:\ N_r(Tf)(x) > \widetilde{C}\widetilde{C}_0 t\|f\|_{L^p(\mu)}^{p/p_0}\right\}\right)\\
&\lesssim \sup_{t\in(0,\infty)} t^q \mu\left(\left\{x\in\mathcal{X}:\ N_r(Tg)(x) > \widetilde{C}_0 t\|f\|_{L^p(\mu)}^{p/p_0}\right\}\right)\\
&\quad+\sup_{t\in(0,\infty)} t^q \mu\left(\left\{x\in\mathcal{X}:\ N_r(Th)(x) > \widetilde{C}_0 t\|f\|_{L^p(\mu)}^{p/p_0}\right\}\right)\\
&\lesssim \sup_{t\in(0,\infty)} t^q \mu\left(\left\{x\in\mathcal{X}:\ M_r^{\#}(Tg)(x) > \widetilde{C}_0 t\|f\|_{L^p(\mu)}^{p/p_0}\right\}\right)\\
&\quad+\sup_{t\in(0,\infty)} t^q \mu\left(\left\{x\in\mathcal{X}:\ N_r(Th)(x) > \widetilde{C}_0 t\|f\|_{L^p(\mu)}^{p/p_0}\right\}\right)\\
&\sim \sup_{t\in(0,\infty)} t^q \mu\left(\left\{x\in\mathcal{X}:\ N_r(Th)(x) > t\|f\|_{L^p(\mu)}^{p/p_0}\right\}\right).
\end{aligned}
\tag{2.15}
$$

By the boundedness of N from $L^1(\mu)$ into $L^{1,\infty}(\mu)$ (see Lemma 2.5(i)), the layer cake representation, the boundedness of T from $H^1(\mu)$ into $L^{p_0',\infty}(\mu)$ and (2.9) with t replaced by $t^{q/p}$, we conclude that

$$
\begin{aligned}
&\mu\left(\left\{x\in\mathcal{X}: N_r(Th)(x) > t\|f\|_{L^p(\mu)}^{p/p_0}\right\}\right)\\
&=\mu\left(\left\{x\in\mathcal{X}: N(|Th|^r)(x) > t^r\|f\|_{L^p(\mu)}^{rp/p_0}\right\}\right)\\
&\le\mu\left(\left\{x\in\mathcal{X}: N(|Th|^r\chi_{\{y\in\mathcal{X}:\ |Th(y)|>2^{-1/r}t\|f\|_{L^p(\mu)}^{p/p_0}\}})(x) > \frac{t^r}{2}\|f\|_{L^p(\mu)}^{rp/p_0}\right\}\right)\\
&\lesssim t^{-r}\|f\|_{L^p(\mu)}^{-rp/p_0}\int_{\mathcal{X}}|Th(x)|^r\chi_{\left\{x\in\mathcal{X}:\ |Th(x)|>2^{-1/r}t\|f\|_{L^p(\mu)}^{p/p_0}\right\}}(x)\,d\mu(x)\\
&\sim t^{-r}\|f\|_{L^p(\mu)}^{-rp/p_0}\left[\int_0^{2^{-1/r}t\|f\|_{L^p(\mu)}^{p/p_0}} s^{r-1}\right.\\
&\quad\times\mu\left(\left\{x\in\mathcal{X}:\ |Th(x)|>2^{-1/r}t\|f\|_{L^p(\mu)}^{p/p_0}\right\}\right)ds\\
&\quad\left.+\int_{2^{-1/r}t\|f\|_{L^p(\mu)}^{p/p_0}}^{\infty} s^{r-1}\mu\left(\{x\in\mathcal{X}:\ |Th(x)|>s\}\right)ds\right]\\
&\lesssim\mu\left(\left\{x\in\mathcal{X}:\ |Th(x)|>2^{-1/r}t\|f\|_{L^p(\mu)}^{p/p_0}\right\}\right)\\
&\quad+\left[t\|f\|_{L^p(\mu)}^{p/p_0}\right]^{-p_0'}\sup_{s\in(0,\infty)} s^{p_0'}\mu\left(\{x\in\mathcal{X}:\ |Th(x)|>s\}\right)\\
&\lesssim\|h\|_{H^1(\mu)}^{p_0'}\left[t\|f\|_{L^p(\mu)}^{p/p_0}\right]^{-p_0'}\lesssim t^{-q}\|f\|_{L^p(\mu)}^{p},
\end{aligned}
\tag{2.16}
$$

which, together with (2.15), completes the proof of (2.11).

Case (ii) $\mu(\mathcal{X})<\infty$. In this case, assume that $f\in L_b^{\infty}(\mu)$. Notice that, if $t\in(0,t_0]$, where $t_0^q:=\beta_6\|f\|_{L^p(\mu)}^q/\mu(\mathcal{X})$, then (2.10) holds true trivially. Thus, we only need to consider the case when $t \in (t_0, \infty)$. Let N_r and M_r be as in Case (i). For each $t \in (t_0, \infty)$, applying Lemma 2.6 to f with t replaced by $t^{q/p}$, we then see that $f = g + h$ with g and h as in Case (i), which, together with the boundedness of T from $L^{p_0}(\mu)$ into $\mathrm{RBMO}(\mu)$ and [23, Lemma 3.1], shows that (2.13) still holds true for $M_r^{\#}(Tg)$.

We now claim that, for any $r \in (0, 1)$,

$$F := \frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} |Tg(x)|^r \, d\mu(x) \lesssim t^r \|f\|_{L^p(\mu)}^{rp/p_0}, \tag{2.17}$$

where the implicit positive constant only depends on $\mu(\mathcal{X})$ and r. To see this, since $\mu(\mathcal{X}) < \infty$, we may regard $\mathcal{X}$ as a ball, then $g_0 := g - \frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} g(x) \, d\mu(x) \in H^1(\mu)$. Precisely, by (2.12), we see that

$$\|g_0\|_{H^1(\mu)} \lesssim t \|f\|_{L^p(\mu)}^{p/p_0}. \tag{2.18}$$

On the other hand, by Hölder's inequality, the fact that $T1 \in \mathrm{RBMO}(\mu)$ and the locally integrability of $\mathrm{RBMO}(\mu)$ functions, we conclude that

$$\int_{\mathcal{X}} |T1(x)|^r \, d\mu(x) \le \left[\int_{\mathcal{X}} |T1(x)| \, d\mu(x) \right]^r [\mu(\mathcal{X})]^{1-r} < \infty.$$

From this and the layer cake representation, together with $r \in (0, 1)$, Hölder's inequality, (2.12), the boundedness of T from $H^1(\mu)$ into $L^{p_0', \infty}(\mu)$ and (2.18), we deduce that

$$\begin{aligned}
&\int_{\mathcal{X}} |Tg(x)|^r \, d\mu(x) \\
&\le \int_{\mathcal{X}} \left\{ |Tg_0(x)|^r + \left| \frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} g(y) \, d\mu(y) \right|^r |T1(x)|^r \right\} d\mu(x) \\
&\lesssim \int_0^{\|g_0\|_{H^1(\mu)}/\mu(\mathcal{X})} t^{r-1} \mu(\{x \in \mathcal{X} : |Tg_0(x)| > t\}) dt + \int_{\|g_0\|_{H^1(\mu)}/\mu(\mathcal{X})}^{\infty} \cdots + \|g\|_{L^{p_0}(\mu)}^r \\
&\lesssim \int_0^{\|g_0\|_{H^1(\mu)}/\mu(\mathcal{X})} t^{r-1} \, dt + \|g_0\|_{H^1(\mu)}^{p_0'} \int_{\|g_0\|_{H^1(\mu)}/\mu(\mathcal{X})}^{\infty} t^{r-1-p_0'} \, dt + t^r \|f\|_{L^p(\mu)}^{rp/p_0} \\
&\lesssim \|g_0\|_{H^1(\mu)}^r + t^r \|f\|_{L^p(\mu)}^{rp/p_0} \lesssim t^r \|f\|_{L^p(\mu)}^{rp/p_0},
\end{aligned}$$

which implies (2.17).

Observe that $\int_{\mathcal{X}} [|Tg(x)|^r - F] \, d\mu(x) = 0$ and, for any $R \in (0, \infty)$,

$$\sup_{t \in (0, R)} t^q \mu(\{x \in \mathcal{X} : \ N(|Tg|^r - F)(x) > t\}) \le R^q \mu(\mathcal{X}) < \infty.$$

From this and (2.17), together with [23, Lemma 3.2], $M_r^{\#}(F)=0$, (2.13) and some arguments similar to those used in the estimates for (2.15) and (2.16), we deduce that there exists a positive constant $\widetilde{c}$ such that

$$\begin{aligned}
&\sup_{t\in(t_0,\infty)} t^q\mu\left(\left\{x\in\mathcal{X}:\ N_r(Tf)(x)>\widetilde{c}\widetilde{C}_0 t\|f\|_{L^p(\mu)}^{p/p_0}\right\}\right)\\
&\lesssim \sup_{t\in(t_0,\infty)} t^q\mu\left(\left\{x\in\mathcal{X}:\ N(|Tg|^r-F)(x)>(\widetilde{C}_0 t)^r\|f\|_{L^p(\mu)}^{rp/p_0}\right\}\right)\\
&\quad+\sup_{t\in(t_0,\infty)} t^q\mu\left(\left\{x\in\mathcal{X}:\ N_r(Th)(x)>\widetilde{C}_0 t\|f\|_{L^p(\mu)}^{p/p_0}\right\}\right)\\
&\lesssim \sup_{t\in(0,\infty)} t^q\mu\left(\left\{x\in\mathcal{X}:\ M_r^{\#}(Tg)(x)>\widetilde{C}_0 t\|f\|_{L^p(\mu)}^{p/p_0}\right\}\right)\\
&\quad+\sup_{t\in(0,\infty)} t^q\mu\left(\left\{x\in\mathcal{X}:\ N_r(Th)(x)>\widetilde{C}_0 t\|f\|_{L^p(\mu)}^{p/p_0}\right\}\right)\\
&\sim \sup_{t\in(0,\infty)} t^q\mu\left(\left\{x\in\mathcal{X}:\ N_r(Th)(x)>t\|f\|_{L^p(\mu)}^{p/p_0}\right\}\right)\lesssim t^{-q}\|f\|_{L^p(\mu)}^p,
\end{aligned}$$

where $\widetilde{C}_0$ is chosen to be a sufficiently large positive constant, which completes the proof of Theorem 2.7. ■

Proof of Theorem 1.13. (I)⇒(II) Let $f\in L^1(\mu)$. Without loss of generality, we may assume that $\|f\|_{L^1(\mu)}=1$. We denote $1/(1-\alpha)$ by q_0. Applying Lemma 2.6 to f with $p=1$ and t replaced by t^{q_0}, and letting R_j be as in Lemma 2.6(iii), we see that $f=g+h$, where $g:=f\chi_{\mathcal{X}\setminus(\cup_j 6B_j)}+\sum_j\varphi_j$ and $h:=\sum_j(\omega_j f-\varphi_j)$. By (2.7) and the assumption $\|f\|_{L^1(\mu)}=1$, we easily see that

$$\|g\|_{L^1(\mu)}\lesssim\|f\|_{L^1(\mu)}\sim 1. \tag{2.19}$$

From (2.4) and (2.6) with t replaced by t^{q_0}, it follows that, for μ-almost every $x\in\mathcal{X}$,

$$|g(x)|\lesssim t^{q_0}. \tag{2.20}$$

Since T_α is bounded from $L^{p_1}(\mu)$ into $L^{q_1}(\mu)$ for any $p_1\in(1,1/\alpha)$ and $1/q_1=1/p_1-\alpha$, by (2.20) and (2.19), we conclude that

$$\begin{aligned}
\mu(\{x\in\mathcal{X}:\ |T_\alpha g(x)|>t\})&\lesssim t^{-q_1}\|T_\alpha g\|_{L^{q_1}(\mu)}^{q_1}\lesssim t^{-q_1}\|g\|_{L^{p_1}(\mu)}^{q_1}\\
&\lesssim t^{-q_1}(t^{q_0})^{(p_1-1)q_1/p_1}\lesssim t^{-q_0}.
\end{aligned}\tag{2.21}$$

On the other hand, from (2.3) with $p=1$ and t replaced by t^{q_0}, and the fact that $\{B_j\}_j$ is a sequence of pairwise disjoint balls, we deduce that

$$\mu(\cup_j 6^2 B_j)\lesssim t^{-q_0}\int_{\mathcal{X}}|f(y)|\,d\mu(y)\lesssim t^{-q_0}. \tag{2.22}$$

Therefore, to show (II), by $f = g + h$, (2.21) and (2.22), it suffices to prove that

$$\mu\left(\left\{x \in \mathcal{X} \setminus (\cup_j 6^2 B_j) : \ |T_\alpha h(x)| > t\right\}\right) \lesssim t^{-q_0}. \tag{2.23}$$

To this end, denote the center of B_j by x_j, and let N_1 be the positive integer satisfying $R_j = (3 \times 6^2)^{N_1} B_j$. Let θ be a bounded function with $\|\theta\|_{L^{q_0'}(\mu)} \le 1$ whose support is contained in $\mathcal{X} \setminus (\cup_j 6^2 B_j)$. Then

$$\begin{aligned}
&\int_{\mathcal{X}\setminus(\cup_j 6^2 B_j)} |T_\alpha h(x)\theta(x)|\, d\mu(x) \\
&\le \sum_j \int_{\mathcal{X}\setminus 6R_j} |T_\alpha h_j(x)\theta(x)|\, d\mu(x) + \sum_j \int_{6R_j\setminus 6^2 B_j} \cdots \\
&=: \mathrm{F}_1 + \mathrm{F}_2,
\end{aligned}$$

where $h_j := \omega_j f - \varphi_j$. By (2.5), we see that $\int_{\mathcal{X}} h_j(x)\, d\mu(x) = 0$, which, together with (1.8), Hölder's inequality and (2.7), further implies that

$$\begin{aligned}
\mathrm{F}_1 &\le \sum_j \int_{\mathcal{X}\setminus 6R_j} \int_{\mathcal{X}} |\theta(x)||K_\alpha(x,y) - K_\alpha(x,x_j)||h_j(y)|\, d\mu(y)\, d\mu(x) \\
&\lesssim \sum_j \int_{\mathcal{X}} \left[\sum_{i=1}^\infty \int_{6^{i+1}B_j\setminus 6^i B_j} \frac{r_{B_j}^\delta}{(6^i r_{B_j})^\delta [\lambda(x_j, 6^i r_{B_j})]^{1-\alpha}} |\theta(x)|\, d\mu(x)\right] |h_j(y)|\, d\mu(y) \\
&\lesssim \sum_j \int_{\mathcal{X}} |f(y)\omega_j(y)|\, d\mu(y) \sum_{i=1}^\infty 6^{-i\delta} \|\theta\|_{L^{q_0'}(\mu)} \lesssim 1.
\end{aligned}$$

For F_2, by $h_j := \omega_j f - \varphi_j$, (1.7), Hölder's inequality and an argument similar to that used in the proof of [8, Lemma 3.5(iii)], together with the boundedness of T_α from $L^{p_2}(\mu)$ into $L^{q_2}(\mu)$ with $p_2 \in (1, 1/\alpha)$ and $1/q_2 = 1/p_2 - \alpha$, we have

$$\begin{aligned}
\mathrm{F}_2 &\le \sum_j \int_{6R_j\setminus 6^2 B_j} |\theta(x)||T_\alpha(\omega_j f)(x)|\, d\mu(x) + \sum_j \int_{6R_j} |\theta(x)||T_\alpha \varphi_j(x)|\, d\mu(x) \\
&\lesssim \sum_j \int_{6R_j\setminus 6^2 B_j} \frac{|\theta(x)|}{[\lambda(x_j, d(x,x_j))]^{1-\alpha}}\, d\mu(x) \int_{\mathcal{X}} |f(y)\omega_j(y)|\, d\mu(y) \\
&\quad + \sum_j \left[\int_{6R_j} |T_\alpha \varphi_j(x)|^{q_0}\, d\mu(x)\right]^{1/q_0} \|\theta\|_{L^{q_0'}(\mu)} \\
&\lesssim \sum_j \int_{\mathcal{X}} |f(y)\omega_j(y)|\, d\mu(y) \left[\sum_{k=1}^{N_1+1} \frac{\mu((3\times 6^2)^k B_j)}{\lambda(x_j, (3\times 6^2)^k r_{B_j})}\right]^{1/q_0} \|\theta\|_{L^{q_0'}(\mu)} \\
&\quad + \sum_j \left[\int_{6R_j} |T_\alpha \varphi_j(x)|^{q_2}\, d\mu(x)\right]^{1/q_2} [\mu(6R_j)]^{1/q_0 - 1/q_2} \lesssim 1,
\end{aligned}$$

where we chose p_2 and q_2 such that $p_2 \in (1, 1/\alpha)$ and $1/q_2 = 1/p_2 - \alpha$. The estimates for F_1 and F_2 give (2.23), and hence complete the proof of (I)⇒(II).

(II)⇒(III) Indeed, for any $f \in L^{1/\alpha}(\mu)$, to show $T_\alpha f \in \mathrm{RBMO}(\mu)$, by the assumption that $T_\alpha f$ is finite almost everywhere, it suffices to show that, for any ball Q and $h_Q := m_Q(T_\alpha(f\chi_{\mathcal{X}\setminus(6/5)Q}))$,

$$\frac{1}{\mu(6Q)}\int_Q |T_\alpha f(x) - h_Q|\,d\mu(x) \lesssim \|f\|_{L^{1/\alpha}(\mu)} \tag{2.24}$$

and, for any two balls $Q \subset R$, where R is doubling,

$$|h_Q - h_R| \lesssim K_{Q,R}\|f\|_{L^{1/\alpha}(\mu)}. \tag{2.25}$$

Now we first show (2.24). Write

$$\begin{aligned}
&\frac{1}{\mu(6Q)}\int_Q |T_\alpha f(x) - h_Q|\,d\mu(x)\\
&\quad\le \frac{1}{\mu(6Q)}\int_Q |T_\alpha(f\chi_{(6/5)Q})(x)|\,d\mu(x)\\
&\qquad+\frac{1}{\mu(6Q)}\int_Q |T_\alpha(f\chi_{\mathcal{X}\setminus(6/5)Q})(x) - h_Q|\,d\mu(x) =: \mathrm{H} + \mathrm{I}.
\end{aligned}$$

Notice that Kolmogorov's inequality (see, for example, [12, p. 485, Lemma 2.8]) also holds true in the non-homogeneous setting. By Kolmogorov's inequality, namely, for $0 < p < q < \infty$ and any function f,

$$\|f\|_{L^{q,\infty}(\mu)} \le \sup_E \|f\chi_E\|_{L^p(\mu)}/\|\chi_E\|_{L^s(\mu)} \lesssim \|f\|_{L^{q,\infty}(\mu)},$$

where $1/s = 1/p - 1/q$ and the supremum is taken over all measurable sets E with $0 < \mu(E) < \infty$, together with (II) of Theorem 1.13 and Hölder's inequality, we know that

$$\begin{aligned}
\mathrm{H} &\lesssim \frac{1}{\mu(6Q)}\|\chi_Q\|_{L^{1/\alpha}(\mu)}\|T_\alpha(f\chi_{(6/5)Q})\|_{L^{q_0,\infty}(\mu)}\\
&\lesssim \frac{[\mu(Q)]^\alpha}{\mu(6Q)}\|f\chi_{(6/5)Q}\|_{L^1(\mu)} \lesssim \|f\|_{L^{1/\alpha}(\mu)}.
\end{aligned}$$

To estimate I, we write

$$\begin{aligned}
&\left|T_\alpha(f\chi_{\mathcal{X}\setminus(6/5)Q})(x) - T_\alpha(f\chi_{\mathcal{X}\setminus(6/5)Q})(y)\right|\\
&\quad\le \int_{6Q\setminus(6/5)Q} |K_\alpha(x,z) - K_\alpha(y,z)||f(z)|\,d\mu(z)\\
&\quad= \int_{\mathcal{X}\setminus 6Q} |K_\alpha(x,z) - K_\alpha(y,z)||f(z)|\,d\mu(z) + \int_{\mathcal{X}\setminus(6/5)Q}\cdots =: \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

Let c_Q and r_Q be the center and the radius of Q, respectively. To estimate I_1, from (1.7) and Hölder's inequality, together with (1.2) and (1.3), it follows that

$$\begin{aligned}\mathrm{I}_1 &\lesssim \int_{6Q\setminus(6/5)Q}\left(\frac{1}{[\lambda(x,d(x,z))]^{1-\alpha}}+\frac{1}{[\lambda(y,d(y,z))]^{1-\alpha}}\right)|f(z)|\,d\mu(z)\\ &\lesssim \frac{1}{[\lambda(c_Q,r_Q)]^{1-\alpha}}\int_{6Q}|f(z)|\,d\mu(z)\lesssim \|f\|_{L^{1/\alpha}(\mu)}.\end{aligned}$$

To estimate I_2, by (1.8), (1.2), Hölder's inequality and (1.3), we see that, for any $x,\,y\in Q$,

$$\begin{aligned}\mathrm{I}_2 &\lesssim \sum_{i=1}^{\infty}\int_{2^i(6Q)\setminus 2^{i-1}(6Q)}\frac{[d(x,y)]^{\delta}}{[d(z,y)]^{\delta}[\lambda(y,d(z,y))]^{1-\alpha}}|f(z)|\,d\mu(z)\\ &\lesssim \sum_{i=1}^{\infty}\int_{2^i(6Q)\setminus 2^{i-1}(6Q)}\frac{r_Q^{\delta}}{[2^{i-1}(6r_Q)]^{\delta}[\lambda(y,2^{i-1}6r_Q)]^{1-\alpha}}|f(z)|\,d\mu(z)\\ &\lesssim \sum_{i=1}^{\infty}2^{-(i-1)\delta}\left[\frac{\mu(2^i(6Q))}{\lambda(c_Q,2^i(6r_Q))}\right]^{1-\alpha}\|f\|_{L^{1/\alpha}(\mu)}\lesssim \|f\|_{L^{1/\alpha}(\mu)}.\end{aligned}$$

Therefore, $\mathrm{I}\lesssim \|f\|_{L^{1/\alpha}(\mu)}$.

Combining the estimates for H and I, we obtain (2.24).

Now we show (2.25) for the chosen $\{h_Q\}_Q$. Denote $N_{Q,R}+1$ simply by N_2. Write

$$\begin{aligned}&|h_Q-h_R|\\ &=|m_Q(T_\alpha(f\chi_{\mathcal{X}\setminus(6/5)Q}))-m_R(T_\alpha(f\chi_{\mathcal{X}\setminus(6/5)R}))|\\ &\le |m_Q(T_\alpha(f\chi_{6Q\setminus(6/5)Q}))|+|m_Q(T_\alpha(f\chi_{6^{N_2}Q\setminus 6Q}))|\\ &\quad+|m_Q(T_\alpha(f\chi_{\mathcal{X}\setminus 6^{N_2}Q}))-m_R(T_\alpha(f\chi_{\mathcal{X}\setminus 6^{N_2}Q}))|+|m_R(T_\alpha(f\chi_{6^{N_2}Q\setminus(6/5)R}))|\\ &=:\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3+\mathrm{J}_4.\end{aligned}$$

An argument similar to that used in the estimate for H shows that $\mathrm{J}_4\lesssim\|f\|_{L^{1/\alpha}(\mu)}$. Also, an argument similar to that used in the estimate for I gives us that $\mathrm{J}_3\lesssim \|f\|_{L^{1/\alpha}(\mu)}$.

Next we estimate J_2. For any $x\in Q$, by Hölder's inequality, the fact that $6^{N_2}Q\subset 72R$ and (ii) and (iv) of Lemma 2.1, we have

$$\begin{aligned}\left|T_\alpha(f\chi_{6^{N_2}Q\setminus 6Q})(x)\right| &\le \left[\int_{6^{N_2}Q\setminus 6Q}\frac{1}{\lambda(x,d(x,z))}\,d\mu(z)\right]^{1-\alpha}\|f\|_{L^{1/\alpha}(\mu)}\\ &\lesssim K_{Q,36R}\|f\|_{L^{1/\alpha}(\mu)}\lesssim K_{Q,R}\|f\|_{L^{1/\alpha}(\mu)}.\end{aligned}$$

This implies that $\mathrm{J}_2\lesssim K_{Q,R}\|f\|_{L^{1/\alpha}(\mu)}$. Similarly, we have

$$\mathrm{J}_1\lesssim K_{Q,6Q}\|f\|_{L^{1/\alpha}(\mu)}\lesssim K_{Q,R}\|f\|_{L^{1/\alpha}(\mu)}.$$

Combining the estimates for J_1, J_2, J_3 and J_4, we obtain (2.25) and hence complete the proof of (II)⇒(III).

(III)⇒(IV) We first show that, for any ball B, bounded function a supported on B and $q_0 := 1/(1-\alpha)$,

$$\int_B |T_\alpha a(x)|^{q_0}\,d\mu(x) \lesssim [\mu(2B)]^{q_0}\|a\|_{L^\infty(\mu)}^{q_0}. \tag{2.26}$$

To prove this, we borrow some ideas from the proof of [25, Lemma 3.1] by considering the following two cases for r_B.

Case (i) $r_B \le \mathrm{diam}(\mathrm{supp}\,\mu)/40$, where $\mathrm{diam}(\mathrm{supp}\,\mu)$ denotes the *diameter of the set* $\mathrm{supp}\,\mu$. By Corollary 2.3 and (III) of Theorem 1.13, we have

$$\int_B |T_\alpha a(x) - m_{\widetilde{B}}(T_\alpha a)|^{q_0}\,d\mu(x) \lesssim \mu(2B)\|a\|_{L^{1/\alpha}(\mu)}^{q_0} \lesssim [\mu(2B)]^{q_0}\|a\|_{L^\infty(\mu)}^{q_0}. \tag{2.27}$$

Thus, by (2.27), to prove (2.26), it suffices to show that

$$|m_{\widetilde{B}}(T_\alpha a)| \lesssim [\mu(2B)]^\alpha \|a\|_{L^\infty(\mu)}. \tag{2.28}$$

We first claim that there exists $j_0 \in \mathbb{N}$ such that

$$\mu(6^{j_0}B \setminus 2B) > 0. \tag{2.29}$$

Indeed, if, for all $j \in \mathbb{N}$, $\mu(6^jB \setminus 2B) = 0$, then we see that $\mu(\mathcal{X} \setminus 2B) = 0$, which implies that $\mathrm{supp}\,\mu \subset \overline{2B}$, the closure of $2B$. This contradicts to that $r_B \le \mathrm{diam}(\mathrm{supp}\,\mu)/40$ and thus (2.29) holds true. Now assume that S is the smallest ball of the form 6^jB such that $\mu(S\setminus 2B) > 0$. We then know that $\mu(6^{-1}S \setminus 2B) = 0$ and $\mu(S \setminus 2B) > 0$. Thus, $\mu(S \setminus (6^{-1}S \cup 2B)) > 0$. By this and [18, Lemma 3.3], we choose $x_0 \in S \setminus (6^{-1}S \cup 2B)$ such that the ball centered at x_0 with the radius $6^{-k}r_S$ for some $k \ge 2$ is doubling. Let B_0 be the biggest ball of this form. Then we see that $B_0 \subset 2S$ and $\mathrm{dist}(B_0, B) \gtrsim r_B$. We now claim that

$$K_{B,2S} \lesssim 1. \tag{2.30}$$

Indeed, if $S = 6B$, then by Lemma 2.1(ii), we have (2.30). If $S \supset 6^2B$, then $(1/12)S \supset 3B$. Notice that, in this case, $\mu(6^{-1}S\setminus 2B) = 0$ implies that $K_{2B,(1/12)S} = 1$. By this, together with (iv) and (ii) of Lemma 2.1, we further have

$$K_{B,2S} \lesssim K_{B,2B} + K_{2B,(1/12)S} + K_{(1/12)S,2S} \lesssim K_{B,2B} + K_{(1/12)S,2S} \lesssim 1.$$

Thus, (2.30) also holds true in this case, which shows (2.30). Moreover, assume that $r_{B_0} = 6^{-k_0}r_S$, where $k_0 \ge 2$, and there exists $N \in \mathbb{N}$ such that $\widetilde{6B_0} = 6^{N+1}B_0$.

By the definition of B_0, we see that $N - k_0 + 1 \geq -1$, hence $r_{6(\widetilde{6B_0})} \geq r_S$ and $2S \subset 24(\widetilde{6B_0})$. Therefore, by (i) through (iv) of Lemma 2.1, we see that

$$K_{B_0,2S} \leq K_{B_0,24(\widetilde{6B_0})} \lesssim K_{B_0,\widetilde{6B_0}} + K_{\widetilde{6B_0},24(\widetilde{6B_0})} \lesssim 1. \tag{2.31}$$

By (2.1), (2.31), (2.30), Lemma 2.1(iii) and Theorem 1.13(III), we know that

$$\begin{aligned}
&|m_{B_0}(T_\alpha a) - m_{\widetilde{B}}(T_\alpha a)| \\
&\leq |m_{B_0}(T_\alpha a) - m_{2S}(T_\alpha a)| + |m_{2S}(T_\alpha a) - m_B(T_\alpha a)| \\
&\quad + |m_B(T_\alpha a) - m_{\widetilde{B}}(T_\alpha a)| \\
&\lesssim (K_{B_0,2S} + K_{B,2S} + K_{B,\widetilde{B}})\|T_\alpha a\|_{\mathrm{RBMO}(\mu)} \\
&\lesssim \|a\|_{L^{1/\alpha}(\mu)} \lesssim [\mu(2B)]^\alpha \|a\|_{L^\infty(\mu)},
\end{aligned} \tag{2.32}$$

Moreover, by (1.7), $\mathrm{dist}(B_0, B) \gtrsim r_B$, (1.2) and (1.3), we conclude that, for all $y \in B_0$,

$$|T_\alpha a(y)| \lesssim \frac{\mu(B)}{[\lambda(c_B, r_B)]^{1-\alpha}} \|a\|_{L^\infty(\mu)} \lesssim [\mu(2B)]^\alpha \|a\|_{L^\infty(\mu)}. \tag{2.33}$$

The estimate (2.28) follows from (2.32) and (2.33), which completes the proof of (2.26) in this case.

Case (ii) $r_B > \mathrm{diam}(\mathrm{supp}\,\mu)/40$. In this case, without loss of generality, we may assume that $r_B \leq 8\mathrm{diam}(\mathrm{supp}\,\mu)$. Then, by Remark 1.2(ii), we see that $B \cap \mathrm{supp}\,\mu$ is covered by finite number balls $\{B_j\}_{j=1}^J$ with radius $r_B/800$, where $J \in \mathbb{N}$ is independent of r_B. For any $j \in \{1, \ldots, J\}$, we define $a_j := \frac{\chi_{B_j}}{\sum_{k=1}^J \chi_{B_k}} a$. Since (2.26) is true if we replace B by $2B_j$ which contains the support of a_j, by (1.7), (2.26), (1.3), (1.2) and the fact that, if $B \cap B_j \neq \emptyset$, then $4B_j \subset 2B$, we have

$$\begin{aligned}
&\int_B |T_\alpha a(x)|^{q_0}\, d\mu(x) \\
&\lesssim \sum_{j=1}^J \int_{B\setminus 2B_j} |T_\alpha a(x)|^{q_0}\, d\mu(x) + \sum_{j=1}^J \int_{2B_j} \cdots \\
&\lesssim \sum_{j=1}^J \int_{B\setminus 2B_j} \left[\int_{B_j} \frac{|a_j(y)|}{[\lambda(x, d(x,y))]^{1-\alpha}}\, d\mu(y)\right]^{q_0} d\mu(x) + \sum_{j=1}^J \|a_j\|_{L^\infty(\mu)}^{q_0} [\mu(4B_j)]^{q_0}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{j=1}^{J} \|a_j\|_{L^\infty(\mu)}^{q_0} \left\{ \int_{B\setminus 2B_j} \left[\int_{B_j} \frac{1}{[\lambda(y, d(x,y))]^{1-\alpha}} \, d\mu(y) \right]^{q_0} d\mu(x) + [\mu(4B_j)]^{q_0} \right\} \\
&\lesssim \sum_{j=1}^{J} \|a_j\|_{L^\infty(\mu)}^{q_0} \left\{ \left[\frac{\mu(B_j)}{(\lambda(c_{B_j}, r_{B_j}))^{1-\alpha}} \right]^{q_0} \mu(B) + [\mu(4B_j)]^{q_0} \right\} \\
&\lesssim \sum_{j=1}^{J} \|a_j\|_{L^\infty(\mu)}^{q_0} \{ [\mu(2B)]^{\alpha q_0} \mu(B) + [\mu(4B_j)]^{q_0} \} \lesssim \|a\|_{L^\infty(\mu)}^{q_0} [\mu(2B)]^{q_0}.
\end{aligned}
$$

Thus, (2.26) also holds true in this case.

Now we turn to prove (IV). By a standard argument (see [21, Theorem 4.1] for the details), it suffices to show that, for any $(\infty, 1)_\lambda$-atomic block b,

$$
\|T_\alpha b\|_{L^{q_0}(\mu)} \lesssim |b|_{H^{1,\infty}_{\mathrm{atb}}(\mu)}. \tag{2.34}
$$

Assume that $\operatorname{supp} b \subset R$ and $b = \sum_{j=1}^{2} \lambda_j a_j$, where, for $j \in \{1, 2\}$, a_j is a function supported in $B_j \subset R$ such that $\|a_j\|_{L^\infty(\mu)} \le [\mu(4B_j)]^{-1} K_{B_j,R}^{-1}$ and $|\lambda_1| + |\lambda_2| \sim |b|_{H^{1,\infty}_{\mathrm{atb}}(\mu)}$. Write

$$
\int_{\mathcal{X}} |T_\alpha b(x)|^{q_0} \, d\mu(x) = \int_{2R} |T_\alpha b(x)|^{q_0} \, d\mu(x) + \int_{\mathcal{X}\setminus 2R} \cdots =: \mathrm{L}_1 + \mathrm{L}_2.
$$

For L_1, we see that

$$
\mathrm{L}_1 \lesssim \sum_{j=1}^{2} |\lambda_j|^{q_0} \int_{2B_j} |T_\alpha a_j(x)|^{q_0} \, d\mu(x) + \sum_{j=1}^{2} |\lambda_j|^{q_0} \int_{2R\setminus 2B_j} \cdots =: \mathrm{L}_{1,1} + \mathrm{L}_{1,2}.
$$

From (2.26), $\|a_j\|_{L^\infty(\mu)} \lesssim [\mu(4B_j)]^{-1} K_{B_j,R}^{-1}$ for $j \in \{1, 2\}$, and Definition 1.11(iii), it follows that

$$
\mathrm{L}_{1,1} \lesssim \sum_{j=1}^{2} |\lambda_j|^{q_0} \|a_j\|_{L^\infty(\mu)}^{q_0} [\mu(4B_j)]^{q_0} \lesssim \sum_{j=1}^{2} |\lambda_j|^{q_0} \lesssim |b|_{H^{1,\infty}_{\mathrm{atb}}(\mu)}^{q_0}.
$$

For $\mathrm{L}_{1,2}$, by (1.7), Minkowski's inequality, (1.2), (1.3), (ii) and (iv) of Lemma 2.1, the fact that $\|a_j\|_{L^\infty(\mu)} \lesssim [\mu(4B_j)]^{-1} K_{B_j,R}^{-1}$ and Definition 1.11(iii), we see that

$$
\begin{aligned}
\mathrm{L}_{1,2} &\lesssim \sum_{j=1}^{2} |\lambda_j|^{q_0} \int_{2R\setminus 2B_j} \left\{ \int_{B_j} \frac{|a_j(y)|}{[\lambda(x, d(x,y))]^{1-\alpha}} \, d\mu(y) \right\}^{q_0} d\mu(x) \\
&\lesssim \sum_{j=1}^{2} |\lambda_j|^{q_0} \left\{ \int_{B_j} |a_j(y)| \left[\int_{2R\setminus 2B_j} \frac{1}{\lambda(x, d(x,y))} \, d\mu(x) \right]^{1/q_0} d\mu(y) \right\}^{q_0}
\end{aligned}
$$

$$\lesssim \sum_{j=1}^{2} |\lambda_j|^{q_0} [\mu(B_j)]^{q_0} \|a_j\|_{L^\infty(\mu)}^{q_0} \int_{2R\setminus 2B_j} \frac{1}{\lambda(c_{B_j}, d(x, c_{B_j}))} \, d\mu(x)$$
$$\lesssim \sum_{j=1}^{2} |\lambda_j|^{q_0} [\mu(B_j)]^{q_0} \|a_j\|_{L^\infty(\mu)}^{q_0} K_{B_j,R} \lesssim \sum_{j=1}^{2} |\lambda_j|^{q_0} \lesssim |b|_{H_{\mathrm{atb}}^{1,\infty}(\mu)}^{q_0}.$$

Therefore, $\mathrm{L}_1 \lesssim |b|_{H_{\mathrm{atb}}^{1,\infty}(\mu)}^{q_0}$.

On the other hand, from the fact that $\int_{\mathcal{X}} b(y)\, d\mu(y) = 0$, (1.8) and Definition 1.11(iii), we deduce that

$$\begin{aligned}
\mathrm{L}_2 &\le \int_{\mathcal{X}\setminus 2R} \left[\int_R |K_\alpha(x,y) - K_\alpha(x, c_R)||b(y)| \, d\mu(y) \right]^{q_0} d\mu(x) \\
&\lesssim \left[\int_R |b(y)| \, d\mu(y) \right]^{q_0} \sum_{i=1}^{\infty} \int_{2^{i+1}R\setminus 2^i R} \frac{r_R^{\delta q_0}}{\lambda(c_R, d(x, c_R))[d(x, c_R)]^{\delta q_0}} \, d\mu(x) \\
&\lesssim (|\lambda_1| + |\lambda_2|)^{q_0} \sum_{i=1}^{\infty} 2^{-i\delta q_0} \lesssim |b|_{H_{\mathrm{atb}}^{1,\infty}(\mu)}^{q_0},
\end{aligned}$$

which, together with the estimate for L_1, implies (2.34) and hence completes the proof of (III)⇒(IV).

(IV)⇒(V) is obvious, the details being omitted.

(V)⇒(I) We first claim that, for any ball B and $f \in L^1(\mu)$ with bounded support in $(6/5)B$,

$$\frac{1}{\mu(6B)} \int_B |T_\alpha f(y)| \, d\mu(y) \lesssim \|f\|_{L^{1/\alpha}(\mu)}. \tag{2.35}$$

Assume first that $r_B \le \operatorname{diam}(\operatorname{supp}\mu)/40$. We consider the same construction in the proof of (III)⇒(IV). Let B, B_0 and S be the same as there. We know that $B,\ B_0 \subset 2S$, B_0 is doubling, $K_{B,2S} \lesssim 1$, $K_{B_0,2S} \lesssim 1$ and $\operatorname{dist}(B_0, B) \gtrsim r_B$. Let $g = f + C_{B_0}\chi_{B_0}$, where C_{B_0} is a constant such that $\int_{\mathcal{X}} g(x)\, d\mu(x) = 0$. Then g is an $(\infty, 1)_\lambda$-atomic block supported in R. It is easy to show that

$$\|g\|_{H^1(\mu)} \lesssim [\mu(6B)]^{1/q_0} \|f\|_{L^{1/\alpha}(\mu)}, \tag{2.36}$$

where $q_0 := 1/(1-\alpha)$. For $y \in B$, by (1.7), the fact that $\operatorname{dist}(B_0, B) \gtrsim r_B$, (1.3), $\int_{\mathcal{X}} g(x)\, d\mu(x) = 0$, Hölder's inequality and (1.2), we have

$$\begin{aligned}
&|T_\alpha(C_{B_0}\chi_{B_0})(y)| \\
&\lesssim |C_{B_0}| \int_{B_0} \frac{1}{[\lambda(y, d(x,y))]^{1-\alpha}} \, d\mu(x) \lesssim \frac{|C_{B_0}|\mu(B_0)}{[\lambda(c_B, r_B)]^{1-\alpha}} \\
&\lesssim \|f\|_{L^1(\mu)} \frac{1}{[\lambda(c_B, r_B)]^{1-\alpha}} \lesssim \left[\frac{\mu((6/5)B)}{\lambda(c_B, r_B)} \right]^{1-\alpha} \|f\|_{L^{1/\alpha}(\mu)} \lesssim \|f\|_{L^{1/\alpha}(\mu)}.
\end{aligned} \tag{2.37}$$

Denote $\|g\|_{H^1(\mu)}[\mu(B)]^{-1/q_0}$ simply by E. Then by (V) of Theorem 1.13 and (2.36), we conclude that

$$\begin{aligned}\int_B |T_\alpha g(y)|\,d\mu(y) &= \int_0^E \mu(\{y\in B:\ |T_\alpha g(y)|>t\})\,dt + \int_E^\infty \cdots \\ &\lesssim E\mu(B) + \int_E^\infty \|g\|_{H^1(\mu)}^{q_0} t^{-q_0}\,dt \lesssim \mu(6B)\|f\|_{L^{1/\alpha}(\mu)}.\end{aligned} \tag{2.38}$$

The estimates (2.37) and (2.38) imply (2.35) in this case.

If $r_B > \mathrm{diam}(\mathrm{supp}\,\mu)/40$, by an argument similar to that used in the proof of (2.26) in the case of $r_B > \mathrm{diam}(\mathrm{supp}\,\mu)/40$, we can prove that (2.35) also holds true in this case.

Now we turn to prove (I). By Theorem 2.7, we only need to prove that T_α is bounded from $L^{1/\alpha}(\mu)$ into $\mathrm{RBMO}(\mu)$. Repeating the proofs of (2.24) and (2.25) step by step, only needing to replace the $(L^1(\mu), L^{1/(1-\alpha),\infty}(\mu))$-boundedness of T_α by (2.35) when estimating H, we then know that T_α is bounded from $L^{1/\alpha}(\mu)$ into $\mathrm{RBMO}(\mu)$, which completes the proof that (V) implies (I) and hence the proof of Theorem 1.13. ∎

3. Proofs of Theorems 1.15 and 1.19

In order to prove Theorem 1.15, we need a technical lemma which is a variant over non-homogeneous metric measure spaces of [5, Lemma 2].

Lemma 3.1. *Let $\alpha\in(0,1)$, $p\in(1,1/\alpha)$, $\rho\in[5,\infty)$, $r\in(p,1/\alpha)$ and $1/q = 1/r-\alpha$. Then there exists a positive constant C such that, for all $f\in L^r(\mu)$,*

$$\|M_{p,\rho}^{(\alpha)} f\|_{L^q(\mu)} \le C\|f\|_{L^r(\mu)},$$

where

$$M_{p,\rho}^{(\alpha)} f(x) := \sup_{Q\ni x}\left\{\frac{1}{[\mu(\rho Q)]^{1-\alpha p}}\int_Q |f(y)|^p\,d\mu(y)\right\}^{1/p}$$

and the supremum is taken over all balls $Q\ni x$.

Proof. We first prove that

$$\mu\left(\left\{x\in\mathcal{X}:\ M_{p,\rho}^{(\alpha)} f(x)>t\right\}\right) \lesssim \left[\|f\|_{L^p(\mu)}/t\right]^{p/(1-\alpha p)}. \tag{3.1}$$

Let $E := \{x\in\mathcal{X}:\ M_{p,\rho}^{(\alpha)} f(x)>t\}$.

For any $x\in E$, there exists a ball Q_x containing x such that

$$\frac{1}{[\mu(\rho Q_x)]^{1-\alpha p}}\int_{Q_x} |f(y)|^p\,d\mu(y) > t^p. \tag{3.2}$$

By [13, Theorem 1.2] and [18, Lemma 2.5], there exist countable disjoint subsets $\{Q_j\}_j$ of $\{Q_x : x \in E\}$ such that $E \subset \cup_j \rho Q_j$. Let $q := p/(1-\alpha p)$. Then $p/q \le 1$. Hence, by (3.2) and $p/q = 1-\alpha p$, we see that

$$[\mu(E)]^{p/q} \le [\mu(\cup_j \rho Q_j)]^{p/q} \le \sum_j [\mu(\rho Q_j)]^{p/q} \le \sum_j \frac{1}{t^p}\int_{Q_j} |f(y)|^p \, d\mu(y) \le \frac{\|f\|^p_{L^p(\mu)}}{t^p}.$$

Hence $\mu(E) \lesssim t^{-q}\|f\|^q_{L^p(\mu)}$, namely, (3.1) holds true.

Notice that, if $p < s < 1/\alpha$, by using Hölder's inequality, we have $M^{(\alpha)}_{p,\rho} f \le M^{(\alpha)}_{s,\rho} f$. Hence, by the proceeding arguments, we see that $\mu(E) \le [\frac{1}{t}\|f\|_{L^s(\mu)}]^{s/(1-\alpha s)}$, which, together with (3.1) and the Marcinkiewicz interpolation theorem, further implies the desired result and hence completes the proof of Lemma 3.1. ■

Remark 3.2. Let $\alpha \in (0,1)$. By Lemma 3.1, the maximal operators $M^{(\alpha)}_{r,\rho}$ ($r \in (0,\infty)$) and $M^{(\alpha)}_{(\rho)} := M^{(\alpha)}_{1,\rho}$ are bounded from $L^p(\mu)$ to $L^q(\mu)$ for $p \in (r, 1/\alpha)$ and $1/q = 1/p - \alpha$.

Now we introduce the fractional coefficient $\widetilde{K}^{(\alpha)}_{B,S}$ adapted from [5].

Definition 3.3. For any two balls $B := B(c_B, r_B) \subset S$, $\widetilde{K}^{(\alpha)}_{B,S}$ is defined by

$$\widetilde{K}^{(\alpha)}_{B,S} := 1 + \sum_{k=1}^{N_{B,S}} \left[\frac{\mu(6^k B)}{\lambda(c_B, 6^k r_B)}\right]^{1-\alpha},$$

where $\alpha \in [0,1)$ and $N_{B,S}$ is defined as in Remark 1.6.

Now we give out some simple properties of $\widetilde{K}^{(\alpha)}_{B,S}$, which are completely analogous to [5, Lemma 3]. We omit the details; see [8, Lemma 3.5] for the proofs of the case that $\alpha = 0$.

Lemma 3.4. *Let* $\alpha \in [0,1)$.

(i) *For all balls* $B \subset R \subset S$, $\widetilde{K}^{(\alpha)}_{B,R} \le 2\widetilde{K}^{(\alpha)}_{B,S}$.

(ii) *For any* $\rho \in [1,\infty)$, *there exists a positive constant* $C_{(\rho)}$, *depending only on* ρ, *such that, for all balls* $B \subset S$ *with* $r_S \le \rho r_B$, $\widetilde{K}^{(\alpha)}_{B,S} \le C_{(\rho)}$.

(iii) *There exists a positive constant* $C_{(\alpha)}$, *depending on* α, *such that, for all balls* B, $\widetilde{K}^{(\alpha)}_{B,\widetilde{B}} \le C_{(\alpha)}$.

(iv) *There exists a positive constant* c, *depending on* C_λ *and* α, *such that, for all balls* $B \subset R \subset S$, $\widetilde{K}^{(\alpha)}_{B,S} \le \widetilde{K}^{(\alpha)}_{B,R} + c\widetilde{K}^{(\alpha)}_{R,S}$.

(v) *There exists a positive constant* $\widetilde{c}$, *depending on* C_λ *and* α, *such that, for all balls* $B \subset R \subset S$, $\widetilde{K}^{(\alpha)}_{R,S} \le \widetilde{c}\widetilde{K}^{(\alpha)}_{B,S}$.

Now we introduce the sharp maximal operator $\widetilde{M}^{\#,\alpha}$ associated with $\widetilde{K}^{(\alpha)}_{B,S}$.

Definition 3.5. Let $\alpha \in [0,1)$. For all $f \in L^1_{\rm loc}(\mu)$ and $x \in \mathcal{X}$, the *sharp maximal function* $\widetilde{M}^{\#,\alpha} f$ of f is defined by

$$\widetilde{M}^{\#,\alpha} f(x) := \sup_{B \ni x} \frac{1}{\mu(6B)} \int_B |f(y) - m_{\widetilde{B}} f| \, d\mu(y) + \sup_{(Q,R) \in \Delta_x} \frac{|m_Q f - m_R f|}{\widetilde{K}^{(\alpha)}_{Q,R}},$$

where $\Delta_x := \{(Q,R) : \ x \in Q \subset R \text{ and } Q,\ R \text{ are doubling balls}\}$.

Similar to [2, Theorem 4.2], we have the following lemma.

Lemma 3.6. *Let $f \in L^1_{\rm loc}(\mu)$ satisfy that $\int_{\mathcal{X}} f(x)\, d\mu(x) = 0$ when $\|\mu\| := \mu(\mathcal{X}) < \infty$. Assume that, for some $p \in (1,\infty)$, $\inf\{1, Nf\} \in L^p(\mu)$. Then there exists a positive constant C, independent of f, such that $\|Nf\|_{L^p(\mu)} \le C \|\widetilde{M}^{\#,\alpha} f\|_{L^p(\mu)}$.*

The following two lemmas are completely analogous to [5, Lemmas 5 and 6], the details being omitted.

Lemma 3.7. *For any $\alpha \in [0,1)$, there exists some positive constant P_α (big enough), depending only on C_λ in* (1.2) *and α, such that, if $m \in \mathbb{N}$, $B_1 \subset \cdots \subset B_m$ are concentric balls with $\widetilde{K}^{(\alpha)}_{B_i, B_{i+1}} > P_\alpha$ for $i \in \{1, \ldots, m-1\}$, then there exists a positive constant C, depending only on C_λ and α, such that $\sum_{i=1}^{m-1} \widetilde{K}^{(\alpha)}_{B_i, B_{i+1}} \le C \widetilde{K}^{(\alpha)}_{B_1, B_m}$.*

Lemma 3.8. *For any $\alpha \in [0,1)$, there exists some positive constant $\widetilde{P_\alpha}$ (large enough), depending on C_λ, β_6 as in* (1.2) *with $\eta = 6$ and α, such that, if $x \in \mathcal{X}$ is some fixed point and $\{f_B\}_{B \ni x}$ is a collection of numbers such that $|f_B - f_S| \le \widetilde{K}^{(\alpha)}_{B,S} C_x$ for all doubling balls $B \subset S$ with $x \in B$ satisfying $\widetilde{K}^{(\alpha)}_{B,S} \le \widetilde{P_\alpha}$, then there exists a positive constant C, depending on C_λ, β_6, α and $\widetilde{P_\alpha}$, such that $|f_B - f_S| \le C_4 \widetilde{K}^{(\alpha)}_{B,S} C_x$ for all doubling balls $B \subset S$ with $x \in B$, where C_x is a positive constant, depending on x, and C_4 a positive constant depending only on C_λ, β_6 and α.*

The following theorem is adapted from [5, Theorem 1].

Theorem 3.9. *Let $b \in \mathrm{RBMO}(\mu)$ and T_α for $\alpha \in (0,1)$ be as in* (1.9) *with kernel K_α satisfying* (1.7) *and* (1.8), *which is bounded from $L^p(\mu)$ into $L^q(\mu)$ for all $p \in (1, 1/\alpha)$ and $1/q = 1/p - \alpha$. Then the commutator $[b, T_\alpha]$ satisfies that there exists a positive constant C such that, for all $f \in L^p(\mu)$, $\|[b, T_\alpha] f\|_{L^q(\mu)} \le C \|b\|_{\mathrm{RBMO}(\mu)} \|f\|_{L^p(\mu)}$.*

Proof. The case that $\mu(\mathcal{X}) < \infty$ can be proved by a way similar to the proof of [8, Theorem 3.10]. Thus, without loss of generality, we may assume that $\mu(\mathcal{X}) = \infty$. Let $p \in (1, 1/\alpha)$. We first claim that, for all $r \in (1,\infty)$, $f \in L^p(\mu)$ and $x \in \mathcal{X}$,

$$\widetilde{M}^{\#,\alpha}([b,T_\alpha]f)(x) \lesssim \|b\|_{\mathrm{RBMO}(\mu)}\left\{M^{(\alpha)}_{r,5}f(x)+M_{r,6}(T_\alpha f)(x)+T_\alpha(|f|)(x)\right\}. \tag{3.3}$$

Once (3.3) is proved, taking $1<r<p<1/\alpha$, by Lemma 2.5(ii), Lemma 3.6, an argument similar to that used in the proof of [8, Theorem 3.10], and Remark 3.2, we conclude that

$$\begin{aligned}\|[b,T_\alpha]f\|_{L^q(\mu)} &\le \|N([b,T_\alpha]f)\|_{L^q(\mu)} \lesssim \|\widetilde{M}^{\#,\alpha}([b,T_\alpha]f)\|_{L^q(\mu)}\\ &\lesssim \|b\|_{\mathrm{RBMO}(\mu)}\left\{\|M^{(\alpha)}_{r,5}f\|_{L^q(\mu)}+\|M_{r,6}(T_\alpha f)\|_{L^q(\mu)}+\|T_\alpha f\|_{L^q(\mu)}\right\}\\ &\lesssim \|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{L^p(\mu)},\end{aligned}$$

which is just the desired conclusion.

To show (3.3), by Definition 1.9, there exists a family of numbers, $\{b_Q\}_Q$, such that, for any ball Q,

$$\int_Q |b(y)-b_Q|\,d\mu(y) \lesssim \mu(6Q)\|b\|_{\mathrm{RBMO}(\mu)}$$

and, for all balls Q, R with $Q\subset R$, $|b_Q-b_R|\lesssim K_{Q,R}\|b\|_{\mathrm{RBMO}(\mu)}$. For any ball Q, let

$$h_Q := m_Q(T_\alpha([b-b_Q]f\chi_{\mathcal{X}\setminus(6/5)Q})).$$

Next we show that, for all x and Q with $Q\ni x$,

$$\frac{1}{\mu(6Q)}\int_Q |[b,T_\alpha]f(y)-h_Q|\,d\mu(y) \lesssim \|b\|_{\mathrm{RBMO}(\mu)}\left\{M^{(\alpha)}_{p,5}f(x)+M_{p,6}(T_\alpha f)(x)\right\} \tag{3.4}$$

and, for all balls Q, R with $Q\subset R$ and $Q\ni x$,

$$|h_Q-h_R| \lesssim \|b\|_{\mathrm{RBMO}(\mu)}\left\{M^{(\alpha)}_{p,5}f(x)+T_\alpha(|f|)(x)\right\}K_{Q,R}\widetilde{K}^{(\alpha)}_{Q,R}. \tag{3.5}$$

To prove (3.4), for a fixed ball Q and x with $x\in Q$, we write $[b,T_\alpha]f$ as

$$[b,T_\alpha]f=[b-b_Q]T_\alpha f - T_\alpha([b-b_Q]f_1)-T_\alpha([b-b_Q]f_2), \tag{3.6}$$

where $f_1 := f\chi_{(6/5)Q}$ and $f_2 := f-f_1$.

Let us first estimate the term $[b-b_Q]T_\alpha f$. By Hölder's inequality and [18, Corollary 6.3], we see that

$$\begin{aligned}&\frac{1}{\mu(6Q)}\int_Q |[b(y)-b_Q]T_\alpha f(y)|\,d\mu(y)\\ &\le \left[\frac{1}{\mu(6Q)}\int_Q |b(y)-b_Q|^{p'}\,d\mu(y)\right]^{1/p'}\left[\frac{1}{\mu(6Q)}\int_Q |T_\alpha f(y)|^p\,d\mu(y)\right]^{1/p}\\ &\lesssim \|b\|_{\mathrm{RBMO}(\mu)}M_{p,6}(T_\alpha f)(x),\end{aligned} \tag{3.7}$$

which is desired.

To estimate $T_\alpha([b-b_Q]f_1)$, take $s:=\sqrt{p}$ and $1/r:=1/s-\alpha$. From Hölder's inequality, the $(L^s(\mu),L^r(\mu))$-boundedness of T_α and [18, Corollary 6.3], it follows that

$$\begin{aligned}
&\frac{1}{\mu(6Q)}\int_Q |T_\alpha([b-b_Q]f_1)(y)|\,d\mu(y)\\
&\le \frac{[\mu(Q)]^{1-1/r}}{\mu(6Q)}\|T_\alpha([b-b_Q]f_1)\|_{L^r(\mu)} \lesssim \frac{[\mu(Q)]^{1-1/r}}{\mu(6Q)}\|(b-b_Q)f_1\|_{L^s(\mu)}\\
&\lesssim \frac{1}{[\mu(6Q)]^{1/r}}\left\{\int_{(6/5)Q}|b(y)-b_Q|^{ss'}\,d\mu(y)\right\}^{\frac{1}{ss'}}\left[\int_{(6/5)Q}|f(y)|^p\,d\mu(y)\right]^{\frac{1}{p}}\\
&\lesssim \|b\|_{\mathrm{RBMO}(\mu)}M^{(\alpha)}_{p,5}f(x),
\end{aligned}\tag{3.8}$$

which is desired.

By (3.6), (3.7) and (3.8), to obtain (3.4), we still need to estimate the difference $|T_\alpha([b-b_Q]f_2)-h_Q|$ by writing that, for all $y_1,\ y_2\in Q$,

$$\begin{aligned}
&|T_\alpha([b-b_Q]f_2)(y_1)-T_\alpha([b-b_Q]f_2)(y_2)|\\
&\lesssim \int_{6Q\setminus(6/5)Q}|K_\alpha(y_1,z)-K_\alpha(y_2,z)||b(z)-b_Q||f(z)|\,d\mu(z)\,d\mu(z)+\int_{\mathcal{X}\setminus 6Q}\cdots\\
&=: \mathrm{I}_1+\mathrm{I}_2.
\end{aligned}$$

Let c_Q and r_Q be the center and the radius of Q, respectively. To estimate I_1, from (1.7) and Hölder's inequality, together with (1.2) and (1.3), it follows that

$$\begin{aligned}
\mathrm{I}_1 &\lesssim \int_{6Q\setminus(6/5)Q}\left(\frac{1}{[\lambda(y_1,d(y_1,z))]^{1-\alpha}}+\frac{1}{[\lambda(y_2,d(y_2,z))]^{1-\alpha}}\right)|f(z)||b(z)-b_Q|\,d\mu(z)\\
&\lesssim \left[\frac{1}{\mu(30Q)}\int_{6Q}|b(z)-b_Q|^{p'}\,d\mu(z)\right]^{1/p'}\left\{\frac{1}{[\mu(30Q)]^{1-\alpha p}}\int_{6Q}|f(z)|^p\,d\mu(z)\right\}^{1/p}\\
&\lesssim \|b\|_{\mathrm{RBMO}(\mu)}M^\alpha_{p,5}f(x),
\end{aligned}$$

which is desired.

For any $y_1,y_2\in Q$, by (1.8), (1.3), (1.2), Hölder's inequality and [18, Corollary 6.3], we know that

$$\begin{aligned}
\mathrm{I}_2 &\lesssim \int_{\mathcal{X}\setminus 6Q}\frac{[d(y_1,y_2)]^\delta}{[d(y_1,z)]^\delta[\lambda(y_1,d(y_1,z))]^{1-\alpha}}|b(z)-b_Q||f(z)|\,d\mu(z)\\
&\lesssim \sum_{k=1}^\infty\int_{2^k(6Q)\setminus 2^{k-1}(6Q)}\frac{(2r_Q)^\delta}{[2^{k-1}\times 6r_Q]^\delta}\frac{1}{[\lambda(c_Q,2^{k-1}\times 6r_Q)]^{1-\alpha}}|b(z)-b_Q||f(z)|\,d\mu(z)
\end{aligned}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{-k\delta} \frac{1}{[\mu(2^k \times 30Q)]^{1-\alpha}} \left[\int_{2^k(6Q)} |b(z) - b_{2^k(6Q)}||f(z)|\,d\mu(z) \right.$$
$$\left. + k\|b\|_{\mathrm{RBMO}(\mu)} \int_{2^k(6Q)} |f(z)|\,d\mu(z) \right]$$
$$\lesssim \sum_{k=1}^{\infty} 2^{-k\delta} \left(\left[\frac{1}{\mu(2^k \times 30Q)} \int_{2^k(6Q)} |b(z) - b_{2^k(6Q)}|^{p'}\,d\mu(z) \right]^{\frac{1}{p'}} \right.$$
$$\times \left\{ \frac{1}{[\mu(2^k \times 30Q)]^{1-\alpha p}} \int_{2^k(6Q)} |f(z)|^p\,d\mu(z) \right\}^{1/p}$$
$$\left. + k\|b\|_{\mathrm{RBMO}(\mu)} \left\{ \frac{1}{[\mu(2^k \times 30Q)]^{1-\alpha p}} \int_{2^k(6Q)} |f(z)|^p\,d\mu(z) \right\}^{1/p} \right)$$
$$\lesssim \sum_{k=1}^{\infty} (k+1) 2^{-k\delta} \|b\|_{\mathrm{RBMO}(\mu)} M_{p,5}^{(\alpha)} f(x) \lesssim \|b\|_{\mathrm{RBMO}(\mu)} M_{p,5}^{(\alpha)} f(x),$$

where we used the fact that

$$|b_Q - b_{2^k(6/5)Q}| \lesssim K_{Q,2^k(6Q)} \|b\|_{\mathrm{RBMO}(\mu)} \lesssim k\|b\|_{\mathrm{RBMO}(\mu)}.$$

Combining the estimates for I_1 and I_2, we see that, for all $y \in Q$,

$$|T_\alpha([b - b_Q]f_2)(y) - h_Q| \lesssim \|b\|_{\mathrm{RBMO}(\mu)} M_{p,5}^{(\alpha)} f(x).$$

Thus,

$$\frac{1}{\mu(6Q)} \int_Q |T_\alpha([b - b_Q]f_2)(y) - h_Q|\,d\mu(y) \lesssim \|b\|_{\mathrm{RBMO}(\mu)} M_{p,5}^{(\alpha)} f(x),$$

which, together with (3.6), (3.7) and (3.8), implies (3.4).

Now we show the regularity condition (3.5) for the numbers $\{h_Q\}_Q$. Consider two balls $Q \subset R$ with $x \in Q$ and let $N := N_{Q,R} + 1$. Write $|h_Q - h_R|$ as

$$|m_Q(T_\alpha([b - b_Q]f\chi_{\mathcal{X}\setminus(6/5)Q})) - m_R(T_\alpha([b - b_Q]f\chi_{\mathcal{X}\setminus(6/5)R}))|$$
$$\le |m_Q(T_\alpha([b - b_Q]f\chi_{6Q\setminus(6/5)Q}))| + |m_Q(T_\alpha([b_Q - b_R]f\chi_{\mathcal{X}\setminus 6Q}))|$$
$$+ |m_Q(T_\alpha([b - b_R]f\chi_{6^N Q\setminus 6Q}))| + |m_Q(T_\alpha([b - b_R]f\chi_{\mathcal{X}\setminus 6^N Q}))$$
$$- m_R(T_\alpha([b - b_R]f\chi_{\mathcal{X}\setminus 6^N Q}))| + |m_R(T_\alpha([b - b_R]f\chi_{6^N Q\setminus(6/5)R}))|$$
$$=: \mathrm{U}_1 + \mathrm{U}_2 + \mathrm{U}_3 + \mathrm{U}_4 + \mathrm{U}_5.$$

Following the proof of [5, Theorem 1], it is easy to see that

$$\mathrm{U}_1 + \mathrm{U}_4 + \mathrm{U}_5 \lesssim \|b\|_{\mathrm{RBMO}(\mu)} M_{p,5}^{(\alpha)} f(x)$$

and $\mathrm{U}_2 \lesssim K_{Q,R}\|b\|_{\mathrm{RBMO}(\mu)}[T_\alpha(|f|)(x) + M^{(\alpha)}_{p,5}f(x)]$.

Now we turn to the estimate for U_3. For $y \in Q$, by (1.7) and Hölder's inequality, we conclude that

$$\begin{aligned}
&|T_\alpha([b-b_R]f\chi_{6^N Q\setminus 6Q})(y)|\\
&\lesssim \sum_{k=1}^{N-1} \frac{1}{[\lambda(x_Q,6^k r_Q)]^{1-\alpha}} \int_{6^{k+1}Q\setminus 6^k Q} |b(y)-b_R||f(y)|\,d\mu(y)\\
&\lesssim \sum_{k=1}^{N-1} \frac{1}{[\lambda(x_Q,6^k r_Q)]^{1-\alpha}} \left[\int_{6^{k+1}Q} |b(y)-b_R|^{p'}\,d\mu(y)\right]^{1/p'} \left[\int_{6^{k+1}Q} |f(y)|^p\,d\mu(y)\right]^{1/p}.
\end{aligned}$$

Notice that, by Minkowski's inequality and Lemma 2.1(i), we see that

$$\begin{aligned}
&\left[\int_{6^{k+1}Q} |b(y)-b_R|^{p'}\,d\mu(y)\right]^{1/p'}\\
&\le \left[\int_{6^{k+1}Q} |b(y)-b_{6^{k+1}Q}|^{p'}\,d\mu(y)\right]^{1/p'} + \left[\mu(6^{k+1}Q)\right]^{1/p'} |b_{6^{k+1}Q}-b_R|\\
&\lesssim K_{Q,R}\|b\|_{\mathrm{RBMO}(\mu)} \left[\mu(5\times 6^{k+1}Q)\right]^{1/p'}.
\end{aligned}$$

Thus, by (1.7), (1.3) and (1.2), we conclude that

$$\begin{aligned}
&|T_\alpha([b-b_R]f\chi_{6^N Q\setminus 6Q})(y)|\\
&\lesssim K_{Q,R}\|b\|_{\mathrm{RBMO}(\mu)} \sum_{k=1}^{N-1} \frac{[\mu(5\times 6^{k+1}Q)]^{1-1/p}}{[\lambda(x_Q,6^k r_Q)]^{1-\alpha}} \left[\int_{6^{k+1}Q} |f(y)|^p\,d\mu(y)\right]^{1/p}\\
&\lesssim K_{Q,R}\|b\|_{\mathrm{RBMO}(\mu)} \sum_{k=1}^{N_{Q,R}} \left[\frac{\mu(6^{k+2}Q)}{\lambda(x_Q,6^k r_Q)}\right]^{1-\alpha}\\
&\quad\times \left\{\frac{1}{[\mu(5\times 6^{k+1}Q)]^{1-\alpha p}} \int_{6^{k+1}Q} |f(y)|^p\,d\mu(y)\right\}^{1/p}\\
&\lesssim K_{Q,R}\widetilde{K}^{(\alpha)}_{Q,R}\|b\|_{\mathrm{RBMO}(\mu)} M^{(\alpha)}_{p,5}f(x).
\end{aligned}$$

Taking the mean over Q, we obtain $\mathrm{U}_3 \lesssim K_{Q,R}\widetilde{K}^{(\alpha)}_{Q,R}\|b\|_{\mathrm{RBMO}(\mu)} M^{(\alpha)}_{p,5}f(x)$, which, together with the estimates U_1, U_2, U_4 and U_5, further implies (3.5).

By (3.4), if Q is a doubling ball and $x \in Q$, we have

$$|m_Q([b,T_\alpha]f) - h_Q| \lesssim \|b\|_{\mathrm{RBMO}(\mu)} \left[M^{(\alpha)}_{p,5}f(x) + M_{p,6}(T_\alpha f)(x)\right]. \tag{3.9}$$

Since, for any ball Q with $x \in Q$, $K_{Q,\widetilde{Q}} \le C$ and $\widetilde{K}^{(\alpha)}_{Q,\widetilde{Q}} \le C$, by (3.4), (3.5) and (3.9), we see that

$$\begin{aligned}
&\frac{1}{\mu(6Q)}\int_Q |[b,T_\alpha]f(y)-m_{\widetilde{Q}}([b,T_\alpha]f)|\,d\mu(y)\\
&\le \frac{1}{\mu(6Q)}\int_Q |[b,T_\alpha]f(y)-h_Q|\,d\mu(y)+|h_Q-h_{\widetilde{Q}}|+|h_{\widetilde{Q}}-m_{\widetilde{Q}}([b,T_\alpha]f)|\\
&\lesssim \|b\|_{\mathrm{RBMO}(\mu)}\left\{M_{p,5}^{(\alpha)}f(x)+M_{p,6}(T_\alpha f)(x)+T_\alpha(|f|)(x)\right\}.
\end{aligned} \tag{3.10}$$

On the other hand, for all doubling balls $Q\subset R$ with $x\in Q$ such that $\widetilde{K}_{Q,R}^{(\alpha)}\le \widetilde{P_\alpha}$, where $\widetilde{P_\alpha}$ is the constant as in Lemma 3.8, by (3.5), we have

$$|h_Q-h_R|\lesssim K_{Q,R}\|b\|_{\mathrm{RBMO}(\mu)}\left[M_{p,5}^{(\alpha)}f(x)+T_\alpha(|f|)(x)\right]\widetilde{P_\alpha}.$$

Hence, by Lemma 3.8, we know that, for all doubling balls $Q\subset R$ with $x\in Q$,

$$|h_Q-h_R|\lesssim \widetilde{K}_{Q,R}^{(\alpha)}\|b\|_{\mathrm{RBMO}(\mu)}\left[M_{p,5}^{(\alpha)}f(x)+T_\alpha(|f|)(x)\right]$$

and, using (3.9), we further obtain

$$\begin{aligned}
&|m_Q([b,T_\alpha]f)-m_R([b,T_\alpha]f)|\\
&\lesssim \widetilde{K}_{Q,R}^{(\alpha)}\|b\|_{\mathrm{RBMO}(\mu)}\left\{M_{p,5}^{(\alpha)}f(x)+M_{p,6}(T_\alpha f)(x)+T_\alpha(|f|)(x)\right\},
\end{aligned}$$

which, together with (3.10), induces (3.3) and hence completes the proof of Theorem 3.9. ∎

To prove Theorem 1.15, we need to recall some notation from [14]. Let C_i^k be as in Section 1. For any sequence $\vec{b}:=(b_1,\dots,b_k)$ of functions and all i-tuples $\sigma:=\{\sigma(1),\dots,\sigma(i)\}\in C_i^k$, let $\vec{b}_\sigma:=(b_{\sigma(1)},\dots,b_{\sigma(i)})$ and

$$\|\vec{b}_\sigma\|_{\mathrm{RBMO}(\mu)}:=\prod_{j=1}^{i}\|b_{\sigma(j)}\|_{\mathrm{RBMO}(\mu)}.$$

For any $\sigma\in C_i^k$ and $z\in\mathcal{X}$, let

$$\left[m_{\widetilde{B}}(\vec{b})-\vec{b}(z)\right]_\sigma:=\prod_{j=1}^{i}\left[m_{\widetilde{B}}(b_{\sigma(j)})-b_{\sigma(j)}(z)\right]$$

and $T_{\alpha,\vec{b}_\sigma}:=[b_{\sigma(i)},[b_{\sigma(i-1)},\cdots,[b_{\sigma(1)},T_\alpha]\cdots]]$. In particular, when $\sigma:=\{1,\dots,k\}$, $T_{\alpha,\vec{b}_\sigma}$ coincides with $T_{\alpha,\vec{b}}$ as in (1.11).

Now we are ready to prove Theorem 1.15.

Proof of Theorem 1.15. By Lemma 2.4, it suffices to prove that $T_{\alpha,\vec{b}}$ is bounded from $L^p(\mu)$ into $L^q(\mu)$ for all $p \in (1, 1/\alpha)$ and $1/q = 1/p - \alpha$. We show this by induction on k.

By Theorem 3.9, the conclusion is valid for $k = 1$. Now assume that $k \geq 2$ is an integer and, for any $i \in \{1, \ldots, k-1\}$ and any subset $\sigma = \{\sigma(1), \ldots, \sigma(i)\}$ of $\{1, \ldots, k-1\}$, $T_{\alpha,\vec{b}_\sigma}$ is bounded from $L^p(\mu)$ to $L^q(\mu)$ for the same p, q as those such that T_α is bounded from $L^p(\mu)$ to $L^q(\mu)$.

The case that $\mu(\mathcal{X}) < \infty$ can be proved by a way similar to that used in the proof of [8, Theorem 3.10], the details being omitted. Thus, without loss of generality, we may assume that $\mu(\mathcal{X}) = \infty$. We first claim that, for any $r \in (1, \infty)$, $f \in L^p(\mu)$ and $x \in \mathcal{X}$,

$$\begin{aligned}\widetilde{M}^{\#,\,\alpha}(T_{\alpha,\vec{b}}f)(x) \lesssim \|\vec{b}\|_{\mathrm{RBMO}(\mu)} &\left[M_{r,6}T_\alpha f(x) + M_{r,5}^{(\alpha)} f(x)\right] \\ &+ \sum_{i=1}^{k-1} \sum_{\sigma \in C_i^k} \|\vec{b}_\sigma\|_{\mathrm{RBMO}(\mu)} M_{r,6}(T_{\alpha,\vec{b}_{\sigma'}}f)(x). \end{aligned} \tag{3.11}$$

Once (3.11) is proved, by Lemmas 2.5 and 2.6, an argument similar to that used in the proof of Theorem 3.9, and Remark 3.2, we conclude that, for all $p \in (1, 1/\alpha)$, $1/q = 1/p - \alpha$ and $f \in L^p(\mu)$,

$$\begin{aligned}\|T_{\alpha,\vec{b}}f\|_{L^q(\mu)} &\leq \|N(T_{\alpha,\vec{b}}f)\|_{L^q(\mu)} \lesssim \left\|\widetilde{M}^{\#,\alpha}(T_{\alpha,\vec{b}}f)\right\|_{L^q(\mu)} \\ &\lesssim \|\vec{b}\|_{\mathrm{RBMO}(\mu)} \left[\|M_{r,6}(T_\alpha f)\|_{L^q(\mu)} + \|M_{r,5}^{(\alpha)}(f)\|_{L^q(\mu)}\right] \\ &\quad + \sum_{i=1}^{k-1} \sum_{\sigma \in C_i^k} \|\vec{b}_\sigma\|_{\mathrm{RBMO}(\mu)} \|M_{r,6}(T_{\alpha,\vec{b}_{\sigma'}}f)\|_{L^q(\mu)} \\ &\lesssim \|\vec{b}\|_{\mathrm{RBMO}(\mu)} \left[\|T_\alpha f\|_{L^q(\mu)} + \|f\|_{L^p(\mu)} + \sum_{i=1}^{k-1} \sum_{\sigma \in C_i^k} \|T_{\alpha,\vec{b}_{\sigma'}}f\|_{L^q(\mu)}\right] \\ &\lesssim \|\vec{b}\|_{\mathrm{RBMO}(\mu)} \|f\|_{L^p(\mu)},\end{aligned}$$

which is desired.

As in the proof of [14, Theorem 2], to prove (3.11), it suffices to show that, for all x and B with $B \ni x$,

$$\begin{aligned}&\frac{1}{\mu(6B)} \int_B |T_{\alpha,\vec{b}}f(y) - h_B| \, d\mu(y) \\ &\lesssim \|\vec{b}\|_{\mathrm{RBMO}(\mu)} \left[M_{r,6}(T_\alpha f)(x) + M_{r,5}^{(\alpha)} f(x)\right] \\ &\quad + \sum_{i=1}^{k-1} \sum_{\sigma \in C_i^k} \|\vec{b}_\sigma\|_{\mathrm{RBMO}(\mu)} M_{r,6}(T_{\alpha,\vec{b}_{\sigma'}}f)(x)\end{aligned} \tag{3.12}$$

and, for an arbitrary ball Q, a doubling ball R with $Q \subset R$ and $x \in Q$,

$$|h_Q - h_R| \lesssim \left[\widetilde{K}_{Q,R}\right]^k \widetilde{K}_{Q,R}^{(\alpha)} \left\{ \|\vec{b}\|_{\mathrm{RBMO}(\mu)} \{M_{r,6} T_\alpha f(x) + M_{r,5}^{(\alpha)} f(x)\} + \sum_{i=1}^{k-1} \sum_{\sigma \in C_i^k} \|\vec{b}_\sigma\|_{\mathrm{RBMO}(\mu)} M_{r,6}(T_{\alpha,\vec{b}_{\sigma'}} f)(x) \right\}, \tag{3.13}$$

where

$$h_Q := m_Q \left(T_\alpha \left(\prod_{i=1}^k [m_{\widetilde{Q}}(b_i) - b_i] f \chi_{\mathcal{X} \setminus \frac{6}{5}Q} \right) \right)$$

and

$$h_R := m_R \left(T_\alpha \left(\prod_{i=1}^k [m_R(b_i) - b_i] f \chi_{\mathcal{X} \setminus \frac{6}{5}R} \right) \right).$$

Let us first prove (3.12). With the aid of the formula that, for all y, $z \in \mathcal{X}$,

$$\prod_{i=1}^k [m_{\widetilde{Q}}(b_i) - b_i(z)] = \sum_{i=0}^k \sum_{\sigma \in C_i^k} [b(y) - b(z)]_{\sigma'} [m_{\widetilde{Q}}(b) - b(y)]_\sigma,$$

where, if $i = 0$, then $\sigma' = \{1, \ldots, k\}$ and $\sigma = \emptyset$, $[m_{\widetilde{Q}}(b) - b(y)]_\emptyset = 1$, it is easy to see that, for all $y \in \mathcal{X}$,

$$T_{\alpha,\vec{b}} f(y) = T_\alpha \left(\prod_{i=1}^k [m_{\widetilde{Q}}(b_i) - b_i] f \right)(y) - \sum_{i=1}^k \sum_{\sigma \in C_i^k} [m_{\widetilde{Q}}(b) - b(y)]_\sigma T_{\alpha,\vec{b}_{\sigma'}} f(y),$$

where, if $i = k$, $T_{\alpha,\vec{b}_{\sigma'}} f(y) := T_\alpha(f)(y)$. Therefore, for all balls $Q \ni x$, we have

$$\begin{aligned}
&\frac{1}{\mu(6Q)} \int_Q \left| T_{\alpha,\vec{b}} f(y) - h_Q \right| d\mu(y) \\
&\le \frac{1}{\mu(6Q)} \int_Q \left| T_\alpha \left(\prod_{i=1}^k [m_{\widetilde{Q}}(b_i) - b_i] f \chi_{\frac{6}{5}Q} \right)(y) \right| d\mu(y) \\
&\quad + \sum_{i=1}^k \sum_{\sigma \in C_i^k} \frac{1}{\mu(6Q)} \int_Q \left| \left[m_{\widetilde{Q}}(b) - b(y) \right]_\sigma \right| \left| T_{\alpha,\vec{b}_{\sigma'}} f(y) \right| d\mu(y) \\
&\quad + \frac{1}{\mu(6Q)} \int_Q \left| T_\alpha \left(\prod_{i=1}^k \left[m_{\widetilde{Q}}(b_i) - b_i \right] f \chi_{\mathcal{X} \setminus \frac{6}{5}Q} \right)(y) - h_Q \right| d\mu(y) \\
&=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.
\end{aligned}$$

Take $1/s^2 = 1/r - \alpha$. Using the boundedness of T_α from $L^{s/(1+s\alpha)}(\mu)$ into $L^s(\mu)$ for $s \in (1,\infty)$ and some arguments similar to those used in the proofs of [14, Theorem 1.1] and [8, Theorem 1.9], we conclude that, for all $x \in \mathcal{X}$, $\mathrm{I}_1 \lesssim \|\vec{b}\|_{\mathrm{RBMO}(\mu)} M^{(\alpha)}_{r,5} f(x)$,

$$\mathrm{I}_2 \lesssim \sum_{i=1}^{k} \sum_{\sigma\in C_i^k} \|\vec{b}_\sigma\|_{\mathrm{RBMO}(\mu)} M_{r,6}\left(T_{\alpha,\vec{b}_{\sigma'}} f\right)(x)$$

and $\mathrm{I}_3 \lesssim \|\vec{b}_\sigma\|_{\mathrm{RBMO}(\mu)} M^{(\alpha)}_{r,5} f(x)$, which imply (3.12).

Now we turn to prove (3.13). Let Q be an arbitrary ball and R a doubling ball in $\mathcal{X}$ such that $x \in Q \subset R$. Denote $N_{Q,R}+1$ simply by N. Write

$$\begin{aligned}
&|h_Q - h_R|\\
&\le \left| m_R\left[T_\alpha\left(\prod_{i=1}^{k}\left[m_{\widetilde{Q}}(b_i) - b_i\right] f\chi_{\mathcal{X}\setminus 6^N Q}\right)\right]\right.\\
&\quad \left. - m_Q\left[T_\alpha\left(\prod_{i=1}^{k}\left[m_{\widetilde{Q}}(b_i) - b_i\right] f\chi_{\mathcal{X}\setminus 6^N Q}\right)\right]\right|\\
&\quad + \left| m_R\left[T_\alpha\left(\prod_{i=1}^{k}\left[m_{\widetilde{Q}}(b_i) - b_i\right] f\chi_{\mathcal{X}\setminus 6^N Q}\right)\right]\right.\\
&\quad \left. - m_R\left[T_\alpha\left(\prod_{i=1}^{k}\left[m_R(b_i) - b_i\right] f\chi_{\mathcal{X}\setminus 6^N Q}\right)\right]\right|\\
&\quad + \left| m_Q\left[T_\alpha\left(\prod_{i=1}^{k}\left[m_{\widetilde{Q}}(b_i) - b_i\right] f\chi_{6^N Q\setminus \frac{6}{5}Q}\right)\right]\right|\\
&\quad + \left| m_R\left[T_\alpha\left(\prod_{i=1}^{k}\left[m_R(b_i) - b_i\right] f\chi_{6^N Q\setminus \frac{6}{5}R}\right)\right]\right| =: \mathrm{L}_1 + \mathrm{L}_2 + \mathrm{L}_3 + \mathrm{L}_4.
\end{aligned}$$

An estimate similar to that for I_3, together with $K_{Q,R} \lesssim \widetilde{K}_{Q,R}$, we see that, for all $x \in \mathcal{X}$, $\mathrm{L}_1 \lesssim [\widetilde{K}_{Q,R}]^k \|\vec{b}\|_{RBMO(\mu)} M^{(\alpha)}_{r,5} f(x)$.

By some arguments similar to those used in the proofs of [14, Theorem 1.1] and [8, Theorem 1.9], we easily see that, for all $x \in \mathcal{X}$,

$$\begin{aligned}
\mathrm{L}_2 \lesssim \left[\widetilde{K}_{Q,R}\right]^k &\left\{\sum_{i=1}^{k-1} \sum_{\sigma\in C_i^k} \|\vec{b}_{\sigma'}\|_{\mathrm{RBMO}(\mu)} M_{r,6}\left(T_{\alpha,\vec{b}} f(x)\right)\right.\\
&\left. + \|\vec{b}\|_{\mathrm{RBMO}(\mu)} M_{r,6}(T_\alpha f)(x) + \|\vec{b}\|_{\mathrm{RBMO}(\mu)} M^{(\alpha)}_{r,5} f(x)\right\},
\end{aligned}$$

$\mathrm{L}_3 \lesssim [\widetilde{K}_{Q,R}]^k \widetilde{K}_{Q,R}^{(\alpha)} \|\vec{b}\|_{\mathrm{RBMO}(\mu)} M_{r,5}^{(\alpha)} f(x)$ and $\mathrm{L}_4 \lesssim \|\vec{b}\|_{\mathrm{RBMO}(\mu)} M_{r,5}^{(\alpha)} f(x)$.

Combining the estimates for L_1, L_2, L_3 and L_4, we then obtain (3.13) and hence complete the proof of Theorem 1.15. ■

Now we are ready to prove Theorem 1.19. In what follows, for any $k \in \mathbb{N}$ and $i \in \{1, \ldots, k\}$, let C_i^k be as in the introduction. For all sequences of numbers, $r := (r_1, \ldots, r_k)$, and i-tuples $\sigma := \{\sigma(1), \ldots, \sigma(i)\} \in C_i^k$, let $\vec{b}$ and $\vec{b}_\sigma$ be as in Theorem 1.15,

$$\|\vec{b}_\sigma\|_{\mathrm{Osc}_{\exp L^{r_\sigma}}(\mu)} := \prod_{j=1}^{i} \|b_{\sigma(j)}\|_{\mathrm{Osc}_{\exp L^{r_{\sigma(j)}}}(\mu)}$$

and, in particular,

$$\|\vec{b}\|_{\mathrm{Osc}_{\exp L^{r}}(\mu)} := \prod_{j=1}^{k} \|b_j\|_{\mathrm{Osc}_{\exp L^{r_j}}(\mu)}.$$

Then we prove Theorem 1.19.

Proof of Theorem 1.19. Without loss of generality, by homogeneity, we may assume that $\|f\|_{L^1(\mu)} = 1$ and $\|b_i\|_{\mathrm{Osc}_{\exp L^{r_i}}(\mu)} = 1$ for all $i \in \{1, \ldots, k\}$.

We prove the theorem by two steps: $k = 1$ and $k > 1$.

Step (i) $k = 1$. It is easy to see that the conclusion of Theorem 1.19 automatically holds true if $t \le \beta_6 \|f\|_{L^1(\mu)}/\mu(\mathcal{X})$ when $\mu(\mathcal{X}) < \infty$. Thus, we only need to deal with the case that $t > \beta_6 \|f\|_{L^1(\mu)}/\mu(\mathcal{X})$. For any given bounded function f with bounded support, $q_0 := 1/(1-\alpha)$ and any $t > \beta_6 \|f\|_{L^1(\mu)}/\mu(\mathcal{X})$, applying Lemma 2.6 to f with t replaced by t^{q_0}, and letting R_j be as in Lemma 2.6(iii), we see that $f = g + h$, where $g := f\chi_{\mathcal{X}\setminus \cup_j 6B_j} + \sum_j \varphi_j$ and $h := \sum_j (\omega_j f - \varphi_j) =: \sum_j h_j$. Let $p_1 \in (1, 1/\alpha)$ and $1/q_1 := 1/p_1 - \alpha$. By (2.7), we easily know that $\|g\|_{L^\infty(\mu)} \lesssim t^{q_0}$. From this, the boundedness of T_α from $L^{p_1}(\mu)$ to $L^{q_1}(\mu)$ and (2.19), it follows that

$$\begin{aligned}\mu(\{x \in \mathcal{X} : \ |T_{\alpha,b} g(x)| > t\}) &\lesssim t^{-q_1} \int_{\mathcal{X}} |T_{\alpha,b} g(y)|^{q_1} \, d\mu(y) \lesssim t^{-q_1} \|g\|_{L^{p_1}(\mu)}^{q_1} \\ &\lesssim t^{-q_1} t^{q_0(p_1-1)q_1/p_1} \|f\|_{L^1(\mu)}^{q_1/p_1} \lesssim t^{-q_0},\end{aligned}$$

where $T_{\alpha,b} := T_{\alpha,b_1}$. On the other hand, by (2.3) with $p = 1$ and t replaced by t^{q_0}, and the fact that the sequence of balls, $\{B_j\}_j$, is pairwise disjoint, we see that $\mu\left(\cup_j 6^2 B_j\right) \lesssim t^{-q_0} \int_{\mathcal{X}} |f(y)| \, d\mu(y) \lesssim t^{-q_0}$, and hence the proof of Step (i) can be reduced to proving

$$\begin{aligned}&\mu\left(\left\{x \in \mathcal{X} \setminus \left(\bigcup_j 6^2 B_j\right) : \ |T_{\alpha,b} h(x)| > t\right\}\right) \\ &\quad \lesssim \left[\|\Phi_{1/r}(t^{-1}|f|)\|_{L^1(\mu)} + \Phi_{1/r}(t^{-1}\|f\|_{L^1(\mu)})\right]^{q_0}.\end{aligned} \tag{3.14}$$

For each fixed j and all $x \in \mathcal{X}$, let $b_j(x) := b(x) - m_{\widetilde{B}_j}(b)$ and write

$$T_{\alpha,b}h(x) = \sum_j b_j(x) T_\alpha h_j(x) - \sum_j T_\alpha(b_j h_j)(x) =: \mathrm{I}(x) + \mathrm{II}(x).$$

For the term $\mathrm{II}(x)$, by the boundedness of T_α from $L^1(\mu)$ to $L^{q_0,\infty}(\mu)$, we conclude that

$$\begin{aligned}
&\mu\left(\{x \in \mathcal{X} : |\mathrm{II}(x)| > t\}\right)\\
&\lesssim t^{-q_0}\left[\sum_j \int_{\mathcal{X}} |b_j(y) h_j(y)|\, d\mu(y)\right]^{q_0}\\
&\lesssim t^{-q_0}\left[\sum_j \int_{\mathcal{X}} |b(y) - m_{\widetilde{B}_j}(b)||f(y)|\omega_j(y)\, d\mu(y)\right]^{q_0}\\
&\quad + t^{-q_0}\left[\sum_j \|\varphi_j\|_{L^\infty(\mu)} \int_{R_j} |b(y) - m_{\widetilde{B}_j}(b)|\, d\mu(y)\right]^{q_0} =: \mathrm{U} + \mathrm{V}.
\end{aligned}$$

By Lemma 2.6(iii), we easily know that R_j is also $(6, \beta_6)$-doubling and $R_j = \widetilde{R}_j$. Thus, from Lemmas 2.2 and 2.1, an argument similar to that used in the proof of [14, Theorem 1.2], (2.5) and the fact that $\{6B_j\}_j$ is a sequence of finite overlapping balls, we deduce that

$$\mathrm{V} \lesssim t^{-q_0}\left[\sum_j \|\varphi_j\|_{L^\infty(\mu)} \mu(R_j)\right]^{q_0} \lesssim t^{-q_0}\left[\int_{\mathcal{X}} |f(y)|\, d\mu(y)\right]^{q_0}. \tag{3.15}$$

On the other hand, by the generalized Hölder inequality ([8, Lemma 4.1]), Lemma 2.2 and an argument similar to that used in the proof of [14, Theorem 1.2], we have

$$\mathrm{U} \lesssim \left[\|\Phi_{1/r}(t^{-1}|f|)\|_{L^1(\mu)} + \Phi_{1/r}(t^{-1}\|f\|_{L^1(\mu)})\right]^{q_0}. \tag{3.16}$$

Combining (3.15) and (3.16), we know that

$$\mu\left(\{x \in \mathcal{X} : |\mathrm{II}(x)| > t\}\right) \lesssim \left[\|\Phi_{1/r}(t^{-1}|f|)\|_{L^1(\mu)} + \Phi_{1/r}(t^{-1}\|f\|_{L^1(\mu)})\right]^{q_0}, \tag{3.17}$$

which is desired.

Now we turn our attention to $\mathrm{I}(x)$. Let x_j be the center of B_j. Let θ be a bounded function with $\|\theta\|_{L^{q_0'}(\mu)} \le 1$ and the support contained in $\mathcal{X} \setminus (\cup_j 6^2 B_j)$. By the vanishing moment of h_j and (1.8), we see that

$$
\begin{aligned}
&\int_{\mathcal{X}\setminus(\cup_j 6^2 B_j)} |\mathrm{I}(x)\theta(x)|\, d\mu(x)\\
&\lesssim \sum_j \int_{\mathcal{X}\setminus 2R_j} |b_j(x)\theta(x)| \left| \int_{\mathcal{X}} h_j(y)[K_\alpha(x,y)-K_\alpha(x,x_j)]\, d\mu(y)\right| d\mu(x)\\
&\quad + \sum_j \int_{2R_j\setminus 6^2 B_j} |b_j(x)\theta(x)||T_\alpha h_j(x)|\, d\mu(x)\\
&\lesssim \sum_j r_{R_j}^\delta \int_{\mathcal{X}} |h_j(y)|\, d\mu(y) \int_{\mathcal{X}\setminus 2R_j} \frac{|b_j(x)\theta(x)|}{[d(x,x_j)]^\delta[\lambda(x_j,d(x,x_j))]^{1-\alpha}}\, d\mu(x)\\
&\quad + \sum_j \int_{2R_j\setminus 6^2 B_j} |b_j(x)\theta(x)||T_\alpha(\omega_j f)(x)|\, d\mu(x)\\
&\quad + \sum_j \int_{2R_j} |b_j(x)\theta(x)||T_\alpha(\varphi_j)(x)|\, d\mu(x) =: \mathrm{G}+\mathrm{H}+\mathrm{J}.
\end{aligned}
$$

From (1.2), Hölder's inequality, Corollary 2.3, (2.1), (i) through (iv) of Lemma 2.1, we deduce that

$$
\begin{aligned}
&\int_{\mathcal{X}\setminus 2R_j} \frac{|b_j(x)\theta(x)|}{[d(x,x_j)]^\delta[\lambda(x_j,d(x,x_j))]^{1-\alpha}}\, d\mu(x)\\
&\lesssim \sum_{k=1}^\infty \left(2^k r_{R_j}\right)^{-\delta} \frac{1}{[\lambda(x_j,2^k r_{R_j})]^{1-\alpha}} \int_{2^{k+1}R_j} |b(x)-m_{\widetilde{2^{k+1}R_j}}(b)||\theta(x)|\, d\mu(x)\\
&\quad + \sum_{k=1}^\infty \left(2^k r_{R_j}\right)^{-\delta} \frac{1}{[\lambda(x_j,2^k r_{R_j})]^{1-\alpha}} |m_{\widetilde{B_j}}(b)-m_{\widetilde{2^{k+1}R_j}}(b)| \int_{2^{k+1}R_j} |\theta(x)|\, d\mu(x)\\
&\lesssim \sum_{k=1}^\infty \left(2^k r_{R_j}\right)^{-\delta} \left[\frac{\mu(2^{k+2}R_j)}{\lambda(x_j,6^k r_{R_j})}\right]^{1-\alpha}\\
&\quad + \sum_{k=1}^\infty K_{\widetilde{B_j},\widetilde{2^{k+1}R_j}} \left(2^k r_{R_j}\right)^{-\delta} \left[\frac{\mu(6^{k+1}R_j)}{\lambda(x_j,6^k r_{R_j})}\right]^{1-\alpha}\\
&\lesssim r_{R_j}^{-\delta},
\end{aligned}
$$

where we used the fact that

$$
K_{\widetilde{B_j},\widetilde{2^{k+1}R_j}} \lesssim K_{\widetilde{B_j},R_j} + K_{R_j,2^{k+1}R_j} + K_{2^{k+1}R_j,\widetilde{2^{k+1}R_j}} \lesssim K_{R_j,2^{k+1}R_j} \lesssim k.
$$

Since $\|h_j\|_{L^1(\mu)} \lesssim \int_{\mathcal{X}} |f(y)|\omega_j(y)\, d\mu(y)$, we further see that $\mathrm{G} \lesssim \|f\|_{L^1(\mu)}$.

On the other hand, applying Hölder's inequality, Corollary 2.3, (2.1), (iv), (i) and (iii) of Lemma 2.1, the boundedness of T_α from $L^{p_1}(\mu)$ to $L^{q_1}(\mu)$ with $p_1 \in (p_0, 1/\alpha)$ and $1/q_1 = 1/p_1 - \alpha$, (2.7), and the fact that $\{6Q_j\}_j$ is a sequence of finite overlapping balls, we obtain

$$\begin{aligned}
\mathrm{J} &\leq \sum_j \int_{2R_j} \left[|b(x) - m_{\widetilde{2R_j}}(b)| + |m_{\widetilde{B_j}}(b) - m_{\widetilde{2R_j}}(b)|\right] |T_\alpha(\varphi_j)(x)\theta(x)|\,d\mu(x) \\
&\leq \|\theta\|_{L^{q_0'}(\mu)} \sum_j \left\{ \left[\int_{2R_j} |b(x) - m_{\widetilde{2R_j}}(b)|^{q_0} |T_\alpha\varphi_j(x)|^{q_0}\,d\mu(x)\right]^{1/q_0} \right. \\
&\quad \left. + \left[\int_{2R_j} |T_\alpha\varphi_j(x)|^{q_0}\,d\mu(x)\right]^{1/q_0} \left|m_{\widetilde{B_j}}(b) - m_{\widetilde{2R_j}}(b)\right| \right\} \\
&\lesssim \sum_j \left\{ \|T_\alpha\varphi_j\|_{L^{q_1}(\mu)} \left[\int_{2R_j} |b(x) - m_{\widetilde{2R_j}}(b)|^{q_0(q_1/q_0)'}\,d\mu(x)\right]^{1/q_0 - 1/q_1} \right. \\
&\quad \left. + [\mu(4R_j)]^{1/q_0-1/q_1} |m_{\widetilde{B_j}}(b) - m_{\widetilde{2R_j}}(b)| \right\} \lesssim \sum_j [\mu(4R_j)]^{1/q_0-1/q_1} \|\varphi_j\|_{L^{p_1}(\mu)} \\
&\lesssim \sum_j [\mu(4R_j)]^{1/q_0-1/q_1} \|\varphi_j\|_{L^\infty(\mu)} [\mu(R_j)]^{1/p_1} \lesssim \int_{\mathcal{X}} |f(x)|\,d\mu(x),
\end{aligned}$$

where we used the fact that

$$|m_{\widetilde{B_j}}(b) - m_{\widetilde{2R_j}}(b)| \leq |m_{\widetilde{B_j}}(b) - m_{R_j}(b)| + |m_{R_j}(b) - m_{\widetilde{2R_j}}(b)| \lesssim 1.$$

To estimate H, by (1.7), (1.2) and (1.3), we see that, for all $x \in 2R_j \setminus 6^2 B_j$,

$$|T_\alpha(\omega_j f)(x)| \lesssim \frac{1}{[\lambda(x_j, d(x, x_j))]^{1-\alpha}} \int_{6B_j} |f(y)|\omega_j(y)\,d\mu(y),$$

which further implies that

$$\begin{aligned}
\mathrm{H} &\lesssim \sum_j \left\{ \int_{2R_j \setminus R_j} \frac{|b_j(x)\theta(x)|}{[\lambda(x_j, d(x, x_j))]^{1-\alpha}}\,d\mu(x) + \int_{R_j \setminus 6^2 B_j} \cdots \right\} \int_{\mathcal{X}} |f(y)|\omega_j(y)\,d\mu(y) \\
&\lesssim \sum_j \left\{ \frac{1}{[\lambda(x_j, r_{R_j})]^{1-\alpha}} \left[\int_{\mathcal{X}} |b_j(x)|^{q_0}\,d\mu(x)\right]^{1/q_0} + \sum_{k=0}^{N-1} \left[\frac{\mu((3\times 6^2)^{k+2} B_j)}{\lambda(x_j, (3\times 6^2)^k r_{B_j})}\right]^{1-\alpha} \right. \\
&\quad \left. + \sum_{k=0}^{N-1} \left[\frac{\mu((3\times 6^2)^{k+1} B_j)}{\lambda(x_j, (3\times 6^2)^k r_{B_j})}\right]^{1-\alpha} |m_{\widetilde{B_j}}(b) - m_{\widetilde{(3\times 6^2)^{k+1} B_j}}(b)| \right\} \\
&\quad \times \int_{\mathcal{X}} |f(y)|\omega_j(y)\,d\mu(y),
\end{aligned}$$

where $N \in \mathbb{N}$ satisfies that $R_j = (3\times 6^2)^N B_j$. Obviously, for each $k \in \{0, \ldots, N-1\}$, $(3\times 6^2)^k B_j \subset R_j$ and hence

$$|m_{\widetilde{B_j}}(b) - m_{\widetilde{(3\times 6^2)^{k+1} B_j}}(b)| \lesssim K_{B_j, (3\times 6^2)^{k+1} B_j} \lesssim K_{B_j, R_j} \lesssim 1.$$

Consequently, by the fact that R_j is the smallest $(3\times 6^2, C_\lambda^{(3\times 6^2)+1})$-doubling ball of the family $\{(3\times 6^2)^k B_j\}_{k\in\mathbb{N}}$ and an argument similar to that used in the proof of Lemma 3.4(iii), we see that

$$\begin{aligned}\mathrm{H} &\lesssim \sum_j \left(1+\sum_{k=0}^{N-1}\left[\frac{\mu((3\times 6^2)^k B_j)}{\lambda(x_j,(3\times 6^2)^k r_{B_j})}\right]^{1-\alpha}\right)\int_{\mathcal{X}} |f(y)|\omega_j(y)\,d\mu(y)\\ &\lesssim \int_{\mathcal{X}} |f(y)|\,d\mu(y).\end{aligned}$$

Combining the estimates for G, H and J, we then conclude that

$$\int_{\mathcal{X}\setminus(\cup_j 6^2 B_j)} |\mathrm{I}(x)\theta(x)|\,d\mu(x) \lesssim \|f\|_{L^1(\mu)}.$$

Thus, we have

$$\begin{aligned}&\mu\left(\left\{x\in\mathcal{X}\setminus\left(\bigcup_j 6^2 B_j\right) : |\mathrm{I}(x)|>t\right\}\right)\\ &\lesssim t^{-q_0}\int_{\mathcal{X}\setminus(\cup_j 6^2 B_j)} |\mathrm{I}(x)|^{q_0}\,d\mu(x) \lesssim \left[t^{-1}\int_{\mathcal{X}\setminus(\cup_j 6^2 B_j)} |f(x)|\,d\mu(x)\right]^{q_0},\end{aligned}$$

which, together with (3.17), implies (3.14) and hence completes the proof of Theorem 1.19 in the case that $k=1$.

Step (ii) $k>1$. The proof of this case is completely analogous to that of [14, Theorem 1.2], the details being omitted, which completes the proof of Theorem 1.19. ∎

4. Some Applications

In this section, we apply all the results of Theorems 1.13, 1.15 and 1.19 to a specific example of fractional integrals to obtain some interesting conclusions.

We first need the following notion.

Definition 4.1. Let $\epsilon\in(0,\infty)$. A dominating function λ is said to satisfy the ϵ-*weak reverse doubling condition* if, for all $r\in(0,2\operatorname{diam}(\mathcal{X}))$ and $a\in(1,2\operatorname{diam}(\mathcal{X})/r)$, there exists a number $C(a)\in[1,\infty)$, depending only on a and $\mathcal{X}$, such that, for all $x\in\mathcal{X}$,

$$\lambda(x,ar)\ge C(a)\lambda(x,r) \tag{4.1}$$

and, moreover,

$$\sum_{k=1}^{\infty}\frac{1}{[C(a^k)]^{\epsilon}}<\infty. \tag{4.2}$$

Remark 4.2.

(i) We remark that the 1-weak reverse doubling condition is just the weak reverse doubling condition introduced in [9, Definition 3.1]. Moreover, it is easy to see that, if $\epsilon_1 < \epsilon_2$ and λ satisfies the ϵ_1-weak reverse doubling condition, then λ also satisfies the ϵ_2-weak reverse doubling condition.

(ii) Assume that $\mathrm{diam}(\mathcal{X}) = \infty$. Let $a = 2^k$ and $r = 2^{-k}$ in (4.1). Then, by (4.2), we see that, for any fixed $x \in \mathcal{X}$,

$$\lim_{k\to\infty} \lambda(x, 2^{-k}) \le \lim_{k\to\infty} \frac{1}{C(2^k)}\lambda(x, 1) = 0.$$

Thus, by the fact that $r \to \lambda(x, r)$ is non-decreasing for any fixed $x \in \mathcal{X}$, we further know that $\lim_{r\to 0} \lambda(x, r) = 0$.

On the other hand, by (4.2), we see that $\lim_{k\to\infty} C(2^k) = \infty$. Letting $a = 2^k$ and $r = 1$ in (4.1), by an argument similar to that used for the case $r \to 0$, we know that, for any fixed $x \in \mathcal{X}$, $\lim_{r\to\infty} \lambda(x, r) = \infty$.

(iii) By Remark 1.4(i), the dominating function in the Euclidean space $\mathbb{R}^d$ with a Radon measure μ as in (1.1) is $\lambda(x, r) := C_0 r^{\kappa}$, which satisfies the ϵ-weak reverse doubling condition for any $\epsilon \in (0, \infty)$.

(iv) If $(\mathcal{X}, d, \mu)$ is an RD-*space*, namely, a space of homogeneous type in the sense of Coifman and Weiss with a measure μ satisfying both the doubling and the reverse doubling conditions, then $\lambda(x, r) := \mu(B(x, r))$ is the dominating function satisfying the ϵ-weak reverse doubling condition for any $\epsilon \in (0, \infty)$. It is known that a connected space of homogeneous type in the sense of Coifman and Weiss is always an RD-space (see [47, p. 65] and [9, Remark 3.4(ii)]).

(v) We remark that the ϵ-weak reverse doubling condition is much weaker than the assumption introduced by Bui and Duong in [2, Subsection 7.3]: there exists $m \in (0, \infty)$ such that, for all $x \in \mathcal{X}$ and a, $r \in (0, \infty)$, $\lambda(x, ar) = a^m \lambda(x, r)$.

Before we give an example, we first establish a technical lemma adapted from [10, Lemma 2.1]. It turns out that the integral kernel $1/[\lambda(y, d(x, y))]^{1-\alpha}$ for $\alpha \in (0, 1)$ is locally integrable.

Lemma 4.3. *Let $\alpha \in (0, 1)$ and λ satisfy the α-weak reverse doubling condition. Then there exists a positive constant C, depending on α, such that, for all $x \in \mathcal{X}$ and $r \in (0, 2\,\mathrm{diam}(\mathcal{X}))$,*

$$\int_{B(x,r)} \frac{1}{[\lambda(y, d(x, y))]^{1-\alpha}}\, d\mu(y) \le C[\lambda(x, r)]^{\alpha}.$$

Proof. From (1.3), (1.2), (4.1) and (4.2), we deduce that

$$\begin{aligned}
&\int_{B(x,r)} \frac{1}{[\lambda(y, d(x,y))]^{1-\alpha}}\, d\mu(y) \\
&\lesssim \int_{B(x,r)} \frac{1}{[\lambda(x, d(x,y))]^{1-\alpha}}\, d\mu(y) \lesssim \sum_{j=0}^{\infty} \frac{\mu(B(x, 2^{-j}r))}{[\lambda(x, 2^{-j-1}r)]^{1-\alpha}} \\
&\lesssim \sum_{j=0}^{\infty} \frac{\lambda(x, 2^{-j}r)}{[\lambda(x, 2^{-j-1}r)]^{1-\alpha}} \lesssim \sum_{j=0}^{\infty} [\lambda(x, 2^{-j-1}r)]^{\alpha} \\
&\lesssim \sum_{j=1}^{\infty} \frac{1}{[C(2^j)]^{\alpha}} [\lambda(x,r)]^{\alpha} \lesssim [\lambda(x,r)]^{\alpha},
\end{aligned}$$

which completes the proof of Lemma 4.3. ∎

For all $\alpha \in (0,1)$, $f \in L_b^{\infty}(\mu)$ and $x \in \mathcal{X}$, the *fractional integral* $I_\alpha f(x)$ is defined by

$$I_\alpha f(x) := \int_{\mathcal{X}} \frac{f(y)}{[\lambda(y, d(x,y))]^{1-\alpha}}\, d\mu(y). \tag{4.3}$$

Notice that, if $(\mathcal{X}, d, \mu) := (\mathbb{R}^d, |\cdot|, \mu)$, $\lambda(x,r) := C_0 r^{\kappa}$ with $\kappa \in (0, d]$ and the measure μ is as in (1.1), then I_α is just the classical fractional integral in the non-doubling space $(\mathbb{R}^d, |\cdot|, \mu)$.

We now show that the kernel of I_α satisfies all the assumptions of this article. By (1.3), we know that the integral kernel $K_\alpha(x,y) := \frac{1}{[\lambda(y, d(x,y))]^{1-\alpha}}$ satisfies (1.7). By Remark 1.4(iii), without loss of generality, we may assume that λ satisfies that there exist ϵ, $\widetilde{C} \in (0,\infty)$ such that, for all $x \in \mathcal{X}$, $r \in (0,\infty)$ and $t \in [0, r]$,

$$|\lambda(x, r+t) - \lambda(x,r)| \le \widetilde{C} \frac{t^{\epsilon}}{r^{\epsilon}} \lambda(x,r). \tag{4.4}$$

Remark 4.4. By (4.4), we see that, for a fixed $x \in \mathcal{X}$, $r \to \lambda(x,r)$ is continuous on $(0,\infty)$.

Now we show that the integral kernel K_α of I_α also satisfies (1.8).

Proposition 4.5. *Assume that λ satisfies* (4.4). *Then the integral kernel K_α of I_α in* (4.3) *satisfies* (1.8).

Proof. For all x, $\widetilde{x}$, $y \in \mathcal{X}$ with $d(x,y) \ge 2d(x, \widetilde{x})$, we consider the following two cases.

Case (i) $d(x,y) \le d(\widetilde{x},y)$. Let $t = d(\widetilde{x},y) - d(x,y)$ and $r = d(x,y)$. Then, by $0 \le t \le d(x,\widetilde{x}) \le \frac{1}{2}d(x,y) \le d(x,y) = r$ and (4.4), we see that

$$\begin{aligned}&|\lambda(y,d(\widetilde{x},y)) - \lambda(y,d(x,y))|\\&\lesssim \frac{[d(\widetilde{x},y)-d(x,y)]^{\epsilon}}{[d(x,y)]^{\epsilon}}\lambda(y,d(x,y)) \lesssim \left[\frac{d(x,\widetilde{x})}{d(x,y)}\right]^{\epsilon}\lambda(y,d(x,y)).\end{aligned}$$

From this, $d(x,y) \le d(\widetilde{x},y)$, Definition 1.3 and (1.3), we further deduce that

$$\begin{aligned}&|K_\alpha(x,y) - K_\alpha(\widetilde{x},y)|\\&\le \left|\frac{1}{\lambda(y,d(x,y))} - \frac{1}{\lambda(y,d(\widetilde{x},y))}\right|^{1-\alpha} = \frac{|\lambda(y,d(\widetilde{x},y)) - \lambda(y,d(x,y))|^{1-\alpha}}{[\lambda(y,d(\widetilde{x},y))\lambda(y,d(x,y))]^{1-\alpha}}\\&\lesssim \frac{[d(x,\widetilde{x})]^{\epsilon(1-\alpha)}}{[d(x,y)]^{\epsilon(1-\alpha)}[\lambda(y,d(\widetilde{x},y))]^{1-\alpha}} \lesssim \frac{[d(x,\widetilde{x})]^{\epsilon(1-\alpha)}}{[d(x,y)]^{\epsilon(1-\alpha)}[\lambda(x,d(x,y))]^{1-\alpha}}.\end{aligned}$$

This finishes the proof of (1.8) in this case.

Case (ii) $d(\widetilde{x},y) \le d(x,y)$. In this case, since $d(x,y) \ge 2d(x,\widetilde{x})$, it follows that

$$d(x,\widetilde{x}) \le \frac{1}{2}d(x,y) \le \frac{1}{2}[d(x,\widetilde{x}) + d(\widetilde{x},y)],$$

and hence $d(x,\widetilde{x}) \le d(\widetilde{x},y)$. Then, by an argument similar to that used in the proof of Case (i), we see that

$$|K_\alpha(x,y) - K_\alpha(\widetilde{x},y)| \lesssim \frac{[d(x,\widetilde{x})]^{\epsilon(1-\alpha)}}{[d(\widetilde{x},y)]^{\epsilon(1-\alpha)}[\lambda(x,d(x,y))]^{1-\alpha}},$$

which, together with $d(x,y) \le d(x,\widetilde{x}) + d(\widetilde{x},y) \le 2d(\widetilde{x},y)$, further implies that (1.8) holds true in this case. This finishes the proof of Proposition 4.5. ■

To consider the boundedness of I_α on Lebesgue spaces, we need the following Welland inequality in the present setting, which is a variant of [11, Theorem 6.4].

Lemma 4.6. *Assume that* $\operatorname{diam}(\mathcal{X}) = \infty$. *Let* $\alpha \in (0,1)$, $\epsilon \in (0, \min\{\alpha, 1-\alpha\})$ *and* λ *satisfy the* ϵ*-weak reverse doubling condition. Then there exists a positive constant* C*, independent of* f *and* x*, such that, for all* $x \in \mathcal{X}$ *and* $f \in L_b^\infty(\mu)$,

$$|I_\alpha f(x)| \le C\left[M_{1,6}^{(\alpha+\epsilon)}f(x)M_{1,6}^{(\alpha-\epsilon)}f(x)\right]^{1/2},$$

where $M_{1,6}^{(\alpha)}$ *for* $\alpha \in (0,1)$ *is defined as in Lemma 3.1.*

Proof. Without loss of generality, we may assume that the right-hand side of the desired inequality is finite. Let $s \in (0,\infty)$. We write

$$|I_\alpha f(x)| \le \int_{B(x,s)} \frac{|f(y)|}{[\lambda(y,d(x,y))]^{1-\alpha}}\,d\mu(y) + \int_{\mathcal{X}\setminus B(x,s)} \cdots =: \mathrm{I} + \mathrm{II}.$$

By (1.3), (1.2), (4.1) and (4.2), we see that

$$\begin{aligned}
\mathrm{I} &\lesssim \int_{B(x,s)} \frac{|f(y)|}{[\lambda(x,d(x,y))]^{1-\alpha}}\,d\mu(y) \lesssim \sum_{j=0}^{\infty} \frac{1}{[\lambda(x,2^{-j-1}s)]^{1-\alpha}} \int_{B(x,2^{-j}s)} |f(y)|\,d\mu(y) \\
&\sim \sum_{j=0}^{\infty} \frac{[\lambda(x,2^{-j-1}s)]^{\epsilon}}{[\lambda(x,2^{-j-1}s)]^{1-\alpha+\epsilon}} \int_{B(x,2^{-j}s)} |f(y)|\,d\mu(y) \\
&\lesssim [\lambda(x,s)]^{\epsilon} \sum_{j=1}^{\infty} \frac{1}{[C(2^j)]^{\epsilon}} M_{1,6}^{(\alpha-\epsilon)} f(x) \lesssim [\lambda(x,s)]^{\epsilon} M_{1,6}^{(\alpha-\epsilon)} f(x).
\end{aligned}$$

Similarly, we also see that $\mathrm{II} \lesssim [\lambda(x,s)]^{-\epsilon} M_{1,6}^{(\alpha+\epsilon)} f(x)$. Thus,

$$|\mathrm{I}_\alpha f(x)| \lesssim [\lambda(x,s)]^{\epsilon} M_{1,6}^{(\alpha-\epsilon)} f(x) + [\lambda(x,s)]^{-\epsilon} M_{1,6}^{(\alpha+\epsilon)} f(x).$$

By Remark 4.2(ii) and Remark 4.4, we can choose $s \in (0,\infty)$ such that

$$[\lambda(x,s)]^{\epsilon} := \left[\frac{M_{1,6}^{(\alpha+\epsilon)} f(x)}{M_{1,6}^{(\alpha-\epsilon)} f(x)} \right]^{1/2}.$$

Then we obtain the desired conclusion and hence complete the proof of Lemma 4.6. ∎

Now we are ready to state the main theorem of this section.

Theorem 4.7. *Assume that* $\operatorname{diam}(\mathcal{X}) = \infty$*. Let* $\alpha \in (0,1)$, $p \in (1, 1/\alpha)$ *and* $1/q = 1/p - \alpha$*. If* λ *satisfies the* ϵ*-weak reverse doubling condition for some* $\epsilon \in (0, \min\{\alpha, 1-\alpha, 1/q\})$*, then* I_α *is bounded from* $L^p(\mu)$ *into* $L^q(\mu)$*.*

Proof. Let $\frac{1}{q_\epsilon^+} := \frac{1}{q} - \epsilon$, $\frac{1}{q_\epsilon^-} := \frac{1}{q} + \epsilon$, $q^+ := 2\frac{q_\epsilon^+}{q}$ and $q^- := 2\frac{q_\epsilon^-}{q}$. Then we have $1 < p < q_\epsilon^- < q < q_\epsilon^+ < \infty$, $1 < q^- < q^+ < \infty$ and $1/q^+ + 1/q^- = 1$. From Lemma 4.6, Hölder's inequality and Lemma 3.1, it follows that

$$\begin{aligned}
\|I_\alpha f\|_{L^q(\mu)} &\lesssim \left\| \left[M_{1,6}^{(\alpha+\epsilon)} f \right]^{q/2} \right\|_{L^{q^+}(\mu)}^{1/q} \left\| \left[M_{1,6}^{(\alpha-\epsilon)} f \right]^{q/2} \right\|_{L^{q^-}(\mu)}^{1/q} \\
&\sim \|M_{1,6}^{(\alpha+\epsilon)} f\|_{L^{q_\epsilon^+}(\mu)}^{1/2} \|M_{1,6}^{(\alpha-\epsilon)} f\|_{L^{q_\epsilon^-}(\mu)}^{1/2} \lesssim \|f\|_{L^p(\mu)}^{1/2} \|f\|_{L^p(\mu)}^{1/2} \sim \|f\|_{L^p(\mu)},
\end{aligned}$$

which completes the proof of Theorem 4.7. ∎

From Theorems 4.7, 1.13, 1.15 and 1.19, we immediately deduce the following interesting conclusions, the details being omitted.

Corollary 4.8. *Under the same assumption as that of Theorem 4.7, all the conclusions of Theorems 1.13, 1.15 and 1.19 hold true, if T_α therein is replaced by I_α as in* (4.3).

ACKNOWLEDGMENTS

The first author would like to express his deep thanks to Professors Yan Meng, Haibo Lin and Dongyong Yang for some helpful discussions on the subject of this paper. The authors would also like to thank the referee for her/his careful reading and several helpful comments on this article.

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