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JOHN-NIRENBERG INEQUALITIES ON LEBESGUE SPACES WITH VARIABLE EXPONENTS

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Abstract. The John-Nirenberg inequalities on Lebesgue spaces with variable exponents are obtained.

1. Motivation

The well-known John-Nirenberg inequality

$$|\{x \in B : |f - f_B| \geq t\}| \leq C_1 e^{-C_2 t/\|f\|_{BMO}} |B|,$$

where $B \in \mathbb{B} = \{B(x_0, R) : x_0 \in \mathbb{R}^n,\ R > 0\}$ and $f_B = \frac{1}{|B|}\int_B f(x)dx$ denotes the mean value of f over $B \in \mathbb{B}$, is an inequality for BMO functions in terms of Lebesgue measure $|\cdot|$. If we rewrite the above inequality in terms of the $L^1(\mathbb{R}^n)$ norm, we have

$$\|\chi_{\{x\in B:|f-f_B|\geq t\}}\|_{L^1} \leq C_1 e^{-C_2 t/\|f\|_{BMO}} \|\chi_B\|_{L^1}.$$

The above inequality gives us another point of view for the John-Nirenberg inequality. That is, the John-Nirenberg inequality is a norm inequality on $L^1(\mathbb{R}^n)$. Thus, in this paper, we investigate whether we can replace the norm $\|\cdot\|_{L^1}$ by the norms of Lebesgue spaces with variable exponents.

The Lebesgue spaces with variable exponents $L^{p(\cdot)}(\mathbb{R}^n)$ recently gain a lot of attention for researchers interested in the theory of function spaces, differential equations and fluid dynamics. An important result in the study of the Lebesgue spaces with variable exponent is on the boundedness of the Hardy-Littlewood maximal operator M on $L^{p(\cdot)}(\mathbb{R}^n)$. In particular, we find that whenever the exponent function $p(\cdot)$ is locally log-Hölder continuous and satisfies the log-Hölder decay condition, then M is bounded

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on $L^{p(\cdot)}(\mathbb{R}^n)$. Notice that there are also some other exponent functions that guarantee the boundedness of M on $L^{p(\cdot)}(\mathbb{R}^n)$ [3, Chapter 4] and [14, 15, 19, 20].

In this paper, we show that if $p(\cdot)$ is locally log-Hölder continuous and satisfies the log-Hölder decay condition, then we have the John-Nirenberg inequality on $L^{p(\cdot)}(\mathbb{R}^n)$. That is,

$$\|\chi_{\{x\in B:|f-f_B|\geq t\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C_1 e^{-C_2 t/\|f\|_{BMO}}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{1.1}$$

The John-Nirenberg inequalities are also valid for rearrangement-invariant quasi-Banach function spaces [11, Proposition 3.2] if its Boyd's indices locate strictly in between one and infinity.

In the next section, we give some background materials on Lebesgue spaces with variable exponents, recall an estimate on the operator norm of the Hardy-Littlewood maximal operator on Lebesgue spaces with variable exponents and introduce the atom for the Hardy space $H^1(\mathbb{R}^n)$ defined in term of Lebesgue spaces with variable exponents. The main result is stated and proved in Section 3.

2. Preliminarily Results

Let $p : \mathbb{R}^n \to [1, \infty]$ be a Lebesgue measurable function. Write

$$p^- = \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x) \quad \text{and} \quad p^+ = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

The Lebesgue space with the variable exponent $p(\cdot)$ consists of those Lebesgue measurable function $f(x)$ satisfying

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\{\lambda > 0 : \rho_p(f/\lambda) \leq 1\} < \infty$$

where

$$\rho_p(f) = \int_{\mathbb{R}^n\setminus\Omega_\infty^{p(\cdot)}} |f(x)|^{p(x)}dx + \operatorname*{ess\,sup}_{\Omega_\infty^{p(\cdot)}} |f(x)|$$

and $\Omega_\infty^{p(\cdot)} = \{x \in \mathbb{R}^n : p(x) = \infty\}$. The Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space [7, Theorems 3.2.13] and [8]. We call $p(x)$ the exponent function of $L^{p(\cdot)}(\mathbb{R}^n)$. The reader is referred to [7, 13] for some basic properties of $L^{p(\cdot)}(\mathbb{R}^n)$.

The associate space of $L^{p(\cdot)}(\mathbb{R}^n)$ is given by $L^{p'(\cdot)}(\mathbb{R}^n)$ where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

We call $p'(x)$ the conjugate exponent function $p(x)$.

We have the following Hölder inequality for Lebesgue spaces with variable exponents [3, Theorem 2.26].

Proposition 2.1. *Let* $p : \mathbb{R}^n \to [1, \infty]$ *be a Lebesgue measurable function. Then*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le K_{p(\cdot)} \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

where

$$K_{p(\cdot)} = \big(\frac{1}{p_-} - \frac{1}{p_+} + 1\big)\|\chi_{\Omega_*^{p(\cdot)}}\|_{L^\infty} + \|\chi_{\Omega_\infty^{p(\cdot)}}\|_{L^\infty} + \|\chi_{\Omega_1^{p(\cdot)}}\|_{L^\infty}$$

and

$$\Omega_*^{p(\cdot)} = \{x \in \mathbb{R}^n : 1 < p(x) < \infty\},$$
$$\Omega_1^{p(\cdot)} = \{x \in \mathbb{R}^n : p(x) = 1\}.$$

We now recall the locally log-Hólder continuity condition and the log-Hӧlder decay condition for exponent functions [7, Definitions 4.1.1 and 4.1.4].

Definition 2.1. Let $p : \mathbb{R}^n \to [1, \infty]$ be a Lebesgue measurable function. We say that p is locally log-Hӧlder continuous if there exists $c_1 > 0$ such that

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \le \frac{c_1}{\log(e + |x-y|^{-1})}, \quad \forall x, y \in \mathbb{R}^n. \tag{2.1}$$

We say that p satisfies the log-Hӧlder decay condition if there exist $c_2 > 0$ and $\frac{1}{p_\infty}$ such that

$$\left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| \le \frac{c_2}{\log(e + |x|)}, \quad \forall x \in \mathbb{R}^n. \tag{2.2}$$

We write $p \in \mathcal{P}^{\log}$ if it satisfies (2.1) and (2.2). Moreover, we write $c_{\log}(p(\cdot)) = \max(c_1, c_2)$ and call it the log-Hӧlder constant of $\frac{1}{p}$.

According to Definition 2.1, we have

$$p \in \mathcal{P}^{\log} \Leftrightarrow p' \in \mathcal{P}^{\log}.$$

We state another important feature for $L^{p(\cdot)}(\mathbb{R}^n)$ when $p \in \mathcal{P}^{\log}$.

Proposition 2.2. *If* $p \in \mathcal{P}^{\log}$*, then*

$$\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le N|B|$$

for all $B \in \mathbb{B}$ *where* N *is a constant only depending on* $c_{\log}(p(\cdot))$ *and* n.

Moreover,

$$\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |B|^{\frac{1}{p(x)}}, & \text{if} \quad |B| \le 2^n \text{ and} \quad x \in B, \\ |B|^{\frac{1}{p_\infty}}, & \text{if} \quad |B| \ge 1. \end{cases} \tag{2.3}$$

for every $B \in \mathbb{B}$*. The implicit constants only depend on* $c_{\log}(p(\cdot))$.

The reader is referred to [7, Theorems 4.4.8, 4.5.7 and Corollary 4.5.9] for the proofs of the above results. The statement given in [7, Corollary 4.5.9] is for cubes in $\mathbb{R}^n$. It is easy to see that it is also valid for $B \in \mathbb{B}$.

Let Y be a Banach function space. For any $1 \leq p < \infty$, define Y^p to be the p-convexification of Y. More precisely,

$$Y^p = \{f \in \mathcal{M} : |f|^p \in Y\}$$

where $\mathcal{M}$ is the set of Lebesgue measurable function on $\mathbb{R}^n$. In addition, Y^p is endowed with the norm,

$$\|f\|_{Y^p} = \||f|^p\|_Y^{1/p},$$

and Y^p is a Banach function space with respect to the norm, $\|\cdot\|_{Y^p}$ (in this connection, the reader may consult [16], Volume II, p.53-54).

Let $1 \leq r < \infty$. For the Lebesgue spaces with variable exponents, we have

$$(L^{p(\cdot)}(\mathbb{R}^n))^r = L^{rp(\cdot)}(\mathbb{R}^n).$$

We also have

$$c_{\log}(rp(\cdot)) = \frac{c_{\log}(p(\cdot))}{r} \tag{2.4}$$

$$K_{rp(\cdot)} \leq K_{p(\cdot)}. \tag{2.5}$$

Thus, we can assume that the family of exponent functions $rp(\cdot)$, $1 \leq r < \infty$, satisfies (2.1) and (2.2) with the same constant $c_{\log}(p(\cdot))$ Furthermore, in view of (2.4), we are allowed to assume that the constants appeared in Proposition 2.2 are independent of r.

We now state the boundedness result of the Hardy-Littlewood maximal operator on $L^{p(\cdot)}(\mathbb{R}^n)$.

Theorem 2.3. *Let $p : \mathbb{R}^n \to [1, \infty]$ be a Lebesgue measurable function. If $p \in \mathcal{P}^{\log}$ and $p^- > 1$, then there exits $K > 0$ only depending on the dimension n and $c_{\log}(p(\cdot))$ such that*

$$\|\mathrm{M} f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq K(p^-)'\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

For any $r \geq 1$, if M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, it is also bounded on $L^{rp(\cdot)}(\mathbb{R}^n)$ because of $(\mathrm{M} f)^r \leq \mathrm{M}(|f|^r)$. In addition, according to [5, Theorem 8.1], if $p^- > 1$ and $p^+ < \infty$, then M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if M is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$.

Therefore, it is legitimate to study the boundedness of M on the family $(L^{rp(\cdot)}(\mathbb{R}^n))'$, $1 \leq r < \infty$. We are especially interested on the operator norms of M on this family.

In view of (2.4), we find that for any $1 \leq r < \infty$, $rp(\cdot)$ also satisfies (2.1) and (2.2) with $c_1 = c_2 = c_{\log}(p(\cdot))$. Therefore, we have a slightly refinement of the above result on the family of Lebesgue spaces with variable exponents $(L^{rp(\cdot)}(\mathbb{R}^n))'$, $1 \leq r < \infty$.

Corollary 2.4. *Let $1 \leq r < \infty$ and $p : \mathbb{R}^n \to [1, \infty]$ be a Lebesgue measurable function. If $p \in \mathcal{P}^{\log}$ and $p^+ < \infty$, then there exits $K > 0$ only depending on the dimension n and $c_{\log}(p(\cdot))$ such that*

$$\|\mathrm{M} f\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} \leq Krp^+ \|f\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} \tag{2.6}$$

for all $f \in (L^{rp(\cdot)}(\mathbb{R}^n))'$.

Proof. The conjugate exponent function of $rp(x)$ is given by

$$q(x) = (rp(x))' = \frac{rp(x)}{rp(x) - 1}.$$

Therefore,

$$q^- = \frac{rp^+}{rp^+ - 1} > 1 \quad \text{and} \quad (q^-)' = rp^+.$$

Hence, our result follows. ■

We introduce the definition of the atom associated with any given Banach function space. It plays the same role as the L^p atom for the atomic decomposition on Hardy spaces. Indeed, in [10, 12], we obtain the atomic decomposition and characterization of Hardy spaces by using the following atoms.

Definition 2.2. Let X be a Banach function space on $\mathbb{R}^n$. We call a function $A(x)$ a $(1, X)$-atom if there exists a $B(x_0, R) \in \mathbb{B}$, $x_0 \in \mathbb{R}^n$ and $R > 0$, such that

$$\mathrm{supp} A \subset 3B = B(x_0, 3R), \tag{2.7}$$

$$\int_{\mathbb{R}^n} A(x) dx = 0, \tag{2.8}$$

$$\|A\|_X \leq \|\chi_B\|_X |B|^{-1}. \tag{2.9}$$

We call B the ball associated with the $(1, X)$-atom, $A(x)$. We denote the set of $(1, X)$-atoms as $\mathcal{A}_{1,X}$.

For the detail studies of atoms on variable Lebesgue spaces and variable Hardy spaces, the reader is referred to [4, 17].

The following lemma gives an estimate on the H^1 norm of the above atom. For completeness, we recall the definition of the Hardy space H^1 [21, Chapter III]. Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ denote the classes of Schwartz functions and tempered distributions, respectively.

Definition 2.3. Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \Phi(x) dx \neq 0$. The Hardy space H^1 consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H^1} = \|\mathrm{M}_\Phi f\|_{L^1(\mathbb{R}^n)} < \infty$$

where

$$\mathrm{M}_\phi f(x) = \sup_{t>0} |(f * \Phi_t)(x)|$$

and $\Phi_t(x) = t^{-n}\Phi(x/t)$.

The proof of the following lemma is based on the well-known result from the atomic decomposition of Hardy space. Since the independence of the constant C on r in (2.10) is the crucial ingredient of the proof for our main result, therefore, for completeness, we provide the proof.

Lemma 2.5. *Let $1 \leq r < \infty$. If $p \in \mathcal{P}^{\log}$ and $p^+ < \infty$, then, for any $(1, (L^{rp(\cdot)}(\mathbb{R}^n))')$-atom A and $\epsilon > 0$, we have $A \in H^1(\mathbb{R}^n)$ with*

$$\|A\|_{H^1} \leq Crp^+ \tag{2.10}$$

for some constant $C > 0$ independent of A and r.

Proof. Without loss of generality, we assume that supp$A \subseteq B = B(0, 3h)$, $h > 0$.

Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy supp$\Phi \in B(0, 1)$ and $\int_{\mathbb{R}^n} \Phi(x)dx \neq 0$. For any locally integrable function, f, we consider the mapping

$$\mathrm{M}_\Phi(f) = \sup_{t>0} |f * \Phi_t|,$$

where $\Phi_t(x) = t^{-n}\Phi(x/t)$, $t > 0$. As $p \in \mathcal{P}^{\log}$ we have $p^+ < \infty$ and $\mathrm{M}_\Phi(f) < C\mathrm{M}(f)$ for some constant $C > 0$. By applying Corollary 2.4, there exists a constant $C > 0$ independent of f and r such that

$$\|\mathrm{M}_\Phi(f)\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} \leq Crp^+ \|f\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'}, \quad \forall f \in (L^{rp(\cdot)}(\mathbb{R}^n))'. \tag{2.11}$$

We consider

$$\begin{aligned} \|A\|_{H^1(\mathbb{R}^n)} &= \|\mathrm{M}_\Phi(A)\|_{L^1} \leq 2(\|\chi_{2B}\mathrm{M}_\Phi(A)\|_{L^1} + \|(1-\chi_{2B})\mathrm{M}_\Phi(A)\|_{L^1}) \\ &= I + II. \end{aligned}$$

We use (2.1) to $L^{rp(\cdot)}(\mathbb{R}^n)$ to estimate I. By using (2.5), we find that

$$\begin{aligned} I &\leq K_{rp(\cdot)} \|\mathrm{M}_\Phi(A)\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} \|\chi_{2B}\|_{L^{rp(\cdot)}(\mathbb{R}^n)} \\ &\leq K_{p(\cdot)} \|\mathrm{M}_\Phi(A)\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} \|\chi_{2B}\|_{L^{rp(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Proposition 2.2, Definition (2.2) and inequality (2.11) ensure that

$$\begin{aligned} I &\leq Crp^+ \|A\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} \|\chi_B\|_{L^{rp(\cdot)}(\mathbb{R}^n)} \\ &\leq Crp^+ \|\chi_B\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} \|\chi_B\|_{L^{rp(\cdot)}(\mathbb{R}^n)} |B|^{-1} \leq Crp^+ \end{aligned} \tag{2.12}$$

for some $C > 0$ independent of r and A because the constants in Proposition 2.2 are independent of r.

We now consider II. As $x \notin 2B$, $\Phi \in \mathcal{S}(\mathbb{R}^n)$ and supp$\Phi \in B(0,1)$, we use the vanishing moment condition for A, and find that, for any $N > 0$,

$$\begin{aligned}|(A * \Phi_t)(x)| &= \left| \int_{3B} A(y)(\Phi_t(x-y) - \Phi_t(x))dy \right| \\ &\le t^{-n} \int_{3B} |A(y)| \frac{C_N |y/t|}{(1+|x/t|)^N} dy \\ &\le \frac{C_N t^{-(1+n)}}{(1+t^{-1}|x|)^N} \int_{3B} |A(y)||y| dy,\end{aligned}$$

where C_N depends on n and N only.

As the center of B is the origin, we have $|y| \le C|B|^{1/n}$, $\forall y \in 3B$, for some $C > 0$. Thus, using the Hölder inequality on $L^{rp(\cdot)}(\mathbb{R}^n)$, Proposition 2.2 asserts that

$$\begin{aligned}|(A * \Phi_t)(x)| &\le \frac{C_N t^{-(1+n)} |B|^{1/n}}{(1+t^{-1}|x|)^N} \|A\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} \|\chi_B\|_{L^{rp(\cdot)}(\mathbb{R}^n)} \\ &\le C_N \frac{t^{-(1+n)} |B|^{1/n}}{(1+t^{-1}|x|)^N}.\end{aligned}$$

By taking $N > 1 + n$, we have

$$\sup_{t>0} |(A * \Phi_t)(x)| \le C_N \frac{|B|^{1/n}}{|x|^{2(1+n)}}. \tag{2.13}$$

Let $h = 2^a$ where $a \in \mathbb{Z}$. Applying L^1 norm on both sides of (2.13), we find that

$$II \le C 2^a \left(\sum_{j=a}^{\infty} \frac{2^{jn}}{2^{j(1+n)}} \right) \le C \tag{2.14}$$

for some constant $C > 0$ independent of A and r. Thus, (2.12) and (2.14) prove (2.10). ■

3. Main Results

We first have a supporting lemma for our main result. In fact, it is a special case of a more general result on the characterization of BMO by r.-i. Banach function spaces. The reader is referred to [10, 11, 12] for further details.

Lemma 3.1. *Let $1 \le r < \infty$ and $p \in \mathcal{P}^{\log}$. We have*

$$\frac{\|(f-f_B)\chi_B\|_{L^{rp(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|^{1/r}_{L^{p(\cdot)}(\mathbb{R}^n)}} \le Crp^+\|f\|_{BMO}, \quad \forall f \in BMO \tag{3.1}$$

where $C>0$ is independent of f and r.

Proof. For any $f \in BMO$ and $B \in \mathbb{B}$, by [3, Theorem 2.34] we have a $h \in (L^{rp(\cdot)}(\mathbb{R}^n))'$ satisfying $\|h\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} \le 1$, supp$h \subseteq B$ and

$$\|(f-f_B)\chi_B\|_{L^{rp(\cdot)}(\mathbb{R}^n)} \le \frac{1}{k_{rp(\cdot)}}\left|\int_B h(x)(f(x)-f_B)dx\right|$$

where

$$\frac{1}{k_{rp(\cdot)}} = \|\chi_{\Omega_*^{rp(\cdot)}}\|_{L^\infty} + \|\chi_{\Omega_\infty^{rp(\cdot)}}\|_{L^\infty} + \|\chi_{\Omega_1^{rp(\cdot)}}\|_{L^\infty}.$$

It is obvious that there exists a $\tilde{B} \in \mathbb{B}$ such that $|B| = |\tilde{B}|$, $B \cap \tilde{B} = \emptyset$ and $\mathrm{dist}(B,\tilde{B}) = 0$. Define A by

$$A(x) = \begin{cases} h(x), & x \in B; \\ -\dfrac{1}{|B|}\displaystyle\int_B h(y)dy, & x \in \tilde{B}. \end{cases}$$

Thus, A fulfills conditions (2.7) and (2.8). Recall that $3B = B(x_B, 3R_B)$ where x_B and R_B denote the center and the radius of B, respectively. Obviously, $B, \tilde{B} \subset 3B$. Let $a \in B \cap 3B$ and $b \in \tilde{B} \cap 3B$.

When $|B| \le 2^n$, in view of (2.3) and the fact that $1 \le (rp(a))', (rp(b))' \le \infty$, we have

$$\begin{aligned} \|\chi_B\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} &\approx |B|^{\frac{1}{(rp(a))'}} \approx |3B|^{\frac{1}{(rp(a))'}} \approx |3B|^{\frac{1}{(rp(b))'}} \approx |\tilde{B}|^{\frac{1}{(rp(b))'}} \\ &\approx \|\chi_{\tilde{B}}\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'}. \end{aligned} \tag{3.2}$$

Similarly, when $|B| > 2^n$, we have

$$\|\chi_B\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} \approx |B|^{\frac{1}{(rp_\infty)'}} \approx \|\chi_{\tilde{B}}\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'}. \tag{3.3}$$

Similar to the constants given in (2.3), the implicit constants in the above estimates are independent of r.

Moreover, by Proposition 2.2, we obtain

$$\begin{aligned} &\|A\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} \\ &\le \|h\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} + \left|\frac{1}{|B|}\int_B h(y)dy\right| \|\chi_{\tilde{B}}\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} \\ &\le \|h\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} + C\frac{1}{|B|}\|h\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'}\|\chi_B\|_{L^{rp(\cdot)}(\mathbb{R}^n)}\|\chi_{\tilde{B}}\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} \\ &\le C\|h\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'} \le C \end{aligned}$$

in view of (3.2) and (3.3). Thus, A is a constant multiple of a $(1,(L^{rp(\cdot)}(\mathbb{R}^n))')$-atom.

According to Lemma 2.5, we assure that A belongs to $H^1(\mathbb{R}^n)$ with

$$\|A\|_{H^1} \le Crp^+ \frac{|B|}{\|\chi_B\|_{(L^{rp(\cdot)}(\mathbb{R}^n))'}} \le Crp^+ \|\chi_B\|^{1/r}_{L^{p(\cdot)}(\mathbb{R}^n)}$$

for some constant $C > 0$ independent of A and r.

As the dual space of $H^1(\mathbb{R}^n)$ is BMO, we find that

$$\begin{aligned}
\frac{\|(f-f_B)\chi_B\|_{L^{rp(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|^{1/r}_{L^{p(\cdot)}(\mathbb{R}^n)}} &\le \frac{C}{\|\chi_B\|^{1/r}_{L^{p(\cdot)}(\mathbb{R}^n)}} \left| \int_B h(x)(f(x)-f_B)dx \right| \\
&= \frac{C}{\|\chi_B\|^{1/r}_{L^{p(\cdot)}(\mathbb{R}^n)}} \left| \int_{\mathbb{R}^n} A(x)(f(x)-f_B)\chi_B(x)dx \right| \\
&\le \frac{2\|A\|_{H^1}\|f\|_{BMO}}{\|\chi_B\|^{1/r}_{L^{p(\cdot)}(\mathbb{R}^n)}} \le Crp^+\|f\|_{BMO}. \qquad \blacksquare
\end{aligned}$$

We now ready to state and prove our main result, the John-Nirenberg inequality on Lebesgue spaces with variable exponents.

Theorem 3.2. *Let $p \in \mathcal{P}^{\log}$ and $p^+ < \infty$. There exist constants $C_1, C_2 > 0$ such that for any $f \in BMO$, for any $B \in \mathbb{B}$ and any $t > 0$,*

$$\|\chi_{\{x\in B:|f(x)-f_B|\ge t\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C_1 e^{-\frac{C_2 t}{\|f\|_{BMO}}} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Proof. Lemma 3.1 gives

$$\frac{\||f-f_B|^r\chi_B\|^{1/r}_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|^{1/r}_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C_1 rp^+ \|f\|_{BMO}$$

when $r \ge 1$ and C_1 is independent of r and f. That is,

$$\||f-f_B|^r\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le (C_0 r)^r \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

where $C_0 = C_1 p^+ \|f\|_{BMO}$.

Using Chebysheff's inequality, we assert that

$$\|\chi_{\{x\in B:|f(x)-f_B|\ge t\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le (C_0 r)^r t^{-r} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

When $t \geq 2C_0$, we take $r = t/2C_0 \geq 1$ and find that

$$\|\chi_{\{x\in B:|f(x)-f_B|\geq t\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq (\frac{1}{2})^r \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} = e^{-C_2 t/\|f\|_{BMO}}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

where $C_2 = (2C_1 p^+)^{-1} \ln 2$.

Finally, when $t \leq 2C_0$, we have $e^{-C_2 t/\|f\|_{BMO}} \geq 1/4$ and, hence,

$$\|\chi_{\{x\in B:|f(x)-f_B|\geq t\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 4e^{-C_2 t/\|f\|_{BMO}}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \qquad \blacksquare$$

We have an application of the above result.

Corollary 3.3. *Let* $p \in \mathcal{P}^{\log}$ *and* $p^+ < \infty$. *For any* $f \in BMO$ *and* $0 < \mu < C_2/\|f\|_{BMO}$, *we obtain*

$$\sup_{B\in\mathbb{B}} \frac{\|e^{\mu|f-f_B|}\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} < \infty. \tag{3.4}$$

The proof of the above corollary is a simple modification of the well known result for the BMO function in the exponential class [21, p.146]

$$\frac{1}{|B|}\int_B e^{\mu|f-f_B|}dx \leq C. \tag{3.5}$$

For brevity, we skip the details.

Applying the Hölder inequality on $L^{p(\cdot)}(\mathbb{R}^n)$, we find that

$$\begin{aligned}\int_B e^{\mu|f-f_B|}dx &\leq C\|e^{\mu|f-f_B|}\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{(L^{p(\cdot)}(\mathbb{R}^n))'} \\ &\leq C\frac{|B|\|e^{\mu|f-f_B|}\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.\end{aligned}$$

Hence, (3.5) can be derived by (3.4). Therefore, Corollary 3.3 is a generalization of (3.5).

Roughly speaking, the proof of Theorem 3.2 shows that the validity of the John-Nirenberg inequality on Lebesgue spaces with variable exponents relies on the results from (2.3) and (2.6). Therefore, the John-Nirenberg inequality on Lebesgue spaces with variable exponents can be further extended to some exponent functions that does not satisfy (2.1) and (2.2), for instance, the example from Nekvinda [18, 20]. As mentioned in [7, Remarks 4.2.8 and 4.3.10], the inequality (2.6) is also valid for the exponent function given in [18]. Moreover, there are replacement for the estimates in (2.3) in terms of the harmonic mean of $p(\cdot)$, see [7, Theorem 4.5.7]. For brevity, we leave the details to the reader.

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