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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR NONLOCAL $\overrightarrow{p}(x)$-LAPLACIAN PROBLEM

G. A. Afrouzi* and M. Mirzapour

Abstract. In this paper, we study the nonlocal anisotropic $\overrightarrow{p}(x)$-Laplacian problem of the following form

$$-\sum_{i=1}^{N} M_i\Big(\int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}dx\Big)\partial_{x_i}\Big(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\Big) = f(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

By means of a direct variational approach and the theory of the anisotropic variable exponent Sobolev space, we obtain the existence and multiplicity of weak energy solutions. Moreover, we get much better results with f in a special form.

1. Introduction

The purpose of this paper is to analyze the existence and multiplicity of the nonlocal anisotropic problem

$$(1.1)\qquad -\sum_{i=1}^{N} M_i\Big(\int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}dx\Big)\partial_{x_i}\Big(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\Big) = f(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, for $i \in \{1,\ldots,N\}$, p_i are continuous functions on $\overline{\Omega}$ such that $2 \leq p_i(x) < N$, $M_i : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, satisfying some certain conditions.

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*Corresponding author.

Since the first equation in (1.1) contains an integral over Ω, it is no longer pointwise identity; therefore it is often called nonlocal problem. Problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

presented by Kirchhoff in 1883, see [11]. This equation is an extension of the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibrations. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension.

In the last few decades, one of the topics from the field of partial differential equations that has continuously attracted interest is that concerning the Sobolev space with variable exponents, $W^{1,p(.)}$ (where p is a function depending on X). Naturally, problems involving the $p(.)$-Laplace operator

$$\Delta_{p(x)}u=div(|\nabla u|^{p(x)-2}\nabla u)$$

were intensively studied. Variable Sobolev spaces have been used in the last decades to model various phenomena. Chen, Levine and Rao [3] proposed a frame work for image restoration based on a variable exponent Laplacian. An other application which uses nonhomogeneous Laplace operators is related to the modeling of electrorheological fluids. The first major discovery in electrorheological fluids is due to Willis Winslow in 1949. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the Winslow effect. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for instance in NASA Laboratories. For more information on properties, modelling and the application of variable exponent space to the fluids, we refer to Diening [4], Rajagopal and Ruzicka [14] and Ruzicka [15].

In this paper, the operator involved (1.1) is more general than the $p(.)$-Laplace operator. Thus, the variable exponent Sobolev space $W^{1,p(.)}(\Omega)$ is not adequate to study nonlinear problems of this type. This lead us to seek weak solution for problem (1.1) in a more general variable exponent Sobolev space which was introduced for the first time by Mihăilescu et al [13].

Motivated by the papers [6, 10] and the ideas introduced in [9], the goal of this paper is to study the existence and multiplicity of solutions for problem (1.1).

2. Notations and Preliminaries

We recall in this section some definitions and basic properties of the variable exponent Lebesgue Sobolev space $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. Roughly speaking, anisotropic Lebesgue and Sobolev spaces are functional spaces of Lebesgue's and Sobolev's type in which different space directions have different roles.

Let Ω be a bounded domain of $\mathbb{R}^N$, denote

$$C_+(\overline{\Omega}) = \{h(x) :\ h(x) \in C(\overline{\Omega}),\ h(x) > 1,\ \forall x \in \overline{\Omega}\};$$

for any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \max\{h(x) :\ x \in \overline{\Omega}\},\ \ h^- = \min\{h(x) :\ x \in \overline{\Omega}\};$$

for any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$\begin{aligned} &L^{p(.)}(\Omega) \\ &= \Big\{u :\ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}dx < \infty\Big\}, \end{aligned}$$

endowed with the so-called *Luxemburg norm*

$$|u|_{L^{p(.)}(\Omega)} = |u|_{p(.)} = \inf\Big\{\mu > 0;\ \int_\Omega |\frac{u(x)}{\mu}|^{p(x)}dx \le 1\Big\},$$

and the space $(L^{p(.)}(\Omega), |.|_{p(.)})$ becomes a reflexive Banach space [12].

Proposition 2.1. (see [5, 7]). (**i**) *The space $(L^{p(.)}(\Omega), |.|_{p(.)})$ is a separable, uniformly convex Banach space and its dual space is $L^{q(.)}(\Omega)$, where $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$. For any $u \in L^{p(.)}(\Omega)$ and $v \in L^{q(.)}(\Omega)$, we have*

$$\Big|\int_\Omega uvdx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(.)}|v|_{q(.)} \le 2|u|_{p(.)}|v|_{q(.)}$$

(**ii**) *If $p_1(.),\ p_2(.) \in C_+(\overline{\Omega}),\ p_1(.) \le p_2(.),\ \forall x \in \overline{\Omega}$, then $L^{p_2(.)}(\Omega) \hookrightarrow L^{p_1(.)}(\Omega)$ and the embedding is continuous.*

Proposition 2.2. (see [6]). *If we denote $\rho_{p(.)}(u) = \int_\Omega |u|^{p(x)}dx$, then for $u \in L^{p(.)}(\Omega)$, $(u_n) \subset\in L^{p(.)}(\Omega)$, we have*

(1) $|u|_{p(.)} < 1\ (respectively = 1; > 1) \Longleftrightarrow \rho_{p(.)}(.) < 1\ (respectively = 1; > 1)$,

(2) $for\ u \ne 0,\ |u|_{p(.)} = \lambda \Longleftrightarrow \rho_{p(.)}(\frac{u}{\lambda}) = 1$,

(3) $if\ |u|_{p(.)} > 1,\ then\ |u|_{p(.)}^{p^-} \le \rho_{p(.)}(u) \le |u|_{p(.)}^{p^+}$,

(4) $if\ |u|_{p(.)} < 1,\ then\ |u|_{p(.)}^{p^+} \le \rho_{p(.)}(u) \le |u|_{p(.)}^{p^-}$,

(5) $|u|_{p(.)} \to 0\ (respectively \to \infty) \Longleftrightarrow \rho_{p(.)}(u) \to 0\ (respectively \to \infty)$,

since $p^+ < \infty$.

The variable exponent Sobolev space $W^{1,p(.)}(\Omega)$ is defined by

$$W^{1,p(.)}(\Omega) = \{u \in L^{p(.)}(\Omega) : \ |\nabla u| \in L^{p(.)}(\Omega)\},$$

and it can equipped with the norm

$$||u|| = |u|_{p(.)} + |\nabla u(x)|_{p(.)}.$$

As shown by Zhikov [18, 19] the smooth functions are in general not dense in $W^{1,p(.)}(\Omega)$, but if the exponent variable p in $C_+(\overline{\Omega})$ is logarithmic Hölder continuous, that is,

$$|p(x) - p(y)| \leq \frac{-M}{\log(|x-y|)} \ \text{ for all } \ x, y \in \Omega \ \text{ such that } \ |x-y| \leq \frac{1}{2},$$

then the smooth functions are dense in $W^{1,p(.)}(\Omega)$.
The Sobolev space with zero boundary values $W_0^{1,p(.)}(\Omega)$, defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $||.||$. Of course also the norms $||u|| = |\nabla u|_{p(.)}$ and $||u|| = \sum_{i=1}^N |\partial_{x_i} u|_{p(.)}$ are equivalent norms in $W_0^{1,p(.)}(\Omega)$. Note that when $s \in C_+(\overline{\Omega})$ and $s(x) < p^*(x)$ for all $x \in \overline{\Omega}$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = \infty$ if $p(x) \geq N$, then the embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{s(.)}(\Omega)$ is compact and continuous.

Finally, we introduce a natural generalization of the variable exponent Sobolev space $W_0^{1,p(.)}(\Omega)$ that will enable us to study with sufficient accuracy problem (1.1). For this purpose, let us denote by $\vec{p} : \overline{\Omega} \to \mathbb{R}^N$ the vectorial function $\vec{p} = (p_1(.), p_2(.), ..., p_N(.))$. We define $X = W_0^{1,\vec{p}(x)}(\Omega)$, the *anisotropic variable exponent space*, as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$||u|| = \sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)}.$$

It was proved that $W_0^{1,\vec{p}(x)}(\Omega)$ is a reflexive Banach space for any $\vec{p}(x) \in \mathbb{R}^N$ with $p_i^- > 1$ for all $i \in \{1, \ldots, N\}$ and the $\vec{p}(x)$-Laplacian operator $-\Delta_{\vec{p}(x)} : W_0^{1,\vec{p}(x)}(\Omega) \to (W_0^{1,\vec{p}(x)}(\Omega))^*$

$$-\Delta_{\vec{p}(x)} u = \sum_{i=1}^N \partial_{x_i} \left(|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \right)$$

is strictly monotone homeomorphism [2].

In order to facilitate the manipulation of the space $W_0^{1,\vec{p}(x)}(\Omega)$ we introduce $\vec{P}_+,\ \vec{P}_- \in \mathbb{R}^N$ and $P_+^+,\ P_-^+,\ P_+^-,\ P_-^- \in \mathbb{R}^+$ as

$$\vec{P}_+ = (p_1^+, p_2^+, \dots, p_N^+), \quad \vec{P}_- = (p_1^-, p_2^-, \dots, p_N^-),$$

$$P_+^+ = \max\{p_1^+, p_2^+, \dots, p_N^+\}, \quad P_-^+ = \max\{p_1^-, p_2^-, \dots, p_N^-\},$$

$$P_+^- = \min\{p_1^+, p_2^+, \dots, p_N^+\}, \quad P_-^- = \min\{p_1^-, p_2^-, \dots, p_N^-\}.$$

Throughout this paper, we assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1, \tag{2.1}$$

and define $P_-^* \in \mathbb{R}^+$ and $P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^* = \frac{N}{\displaystyle\sum_{i=1}^{N} \frac{1}{p_i^-} - 1}, \qquad P_{-,\infty} = \max\{P_-^+, P_-^*\}.$$

In addition, for the Caratheodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, we consider the antiderivative $F : \Omega \times \mathbb{R} \to \mathbb{R}$,

$$F(x,s) = \int_0^s f(x,t)dt.$$

With the previous notation, the functions $M_i,\ f$ satisfy the conditions:

($\mathbf{M_0}$) For each $i = 1, \dots, N,\ M_i : (0,+\infty) \to (0,+\infty)$ is continuous and $M_i \in L^1(0,t)$ for any $t > 0$.

($\mathbf{F_0}$) For every $(x,t) \in \Omega \times \mathbb{R}$

$$|f(x,t)| \le \sum_{i=1}^{m} b_i(x)|t|^{q_i(x)-1},$$

where $b_i(x) \ge 0,\ b_i(x) \ne 0,\ b_i \in L^{r_i}(\Omega) \cap L^\infty(\Omega),\ r_i,\ q_i \in C_+(\overline{\Omega}),\ P_+^+ < q_i(x) < P_-^*$, and there are $s_i \in C_+(\overline{\Omega})$, such that $P_+^+ < s_i(x) < P_-^*,\ \frac{1}{r_i(x)} + \frac{q_i(x)}{s_i(x)} = 1$.

Proposition 2.3. (see [13].) *Let $\Omega \subset \mathbb{R}^N$ $(N \ge 3)$ is a bounded domain with smooth boundary. Assume relation (2.1) is satisfied. For any $q \in C(\overline{\Omega})$ verifying*

$$1 < q(x) < P_{-,\infty}, \quad \forall x \in \overline{\Omega},$$

then the embedding

$$W_0^{1,\vec{p}(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$$

is continuous and compact.

It should be noticed that from the condition $(\mathbf{F_0})$, we have $P_{-,\infty} = \max\{P_-^+, P_-^*\} = P_-^*$. Define for $i = 1, \dots, N$,

$$
\begin{aligned}
\widehat{M_i}(t) &= \int_0^t M_i(s)ds, \qquad \forall t \geq 0,\\
I_i(u) &= \int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx,\\
J_i(u) &= \widehat{M_i}(I_i(u)) = \widehat{M_i}\Big(\int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx\Big), \qquad \forall u \in X,\\
J(u) &= \sum_{i=1}^N J_i(u), \qquad \forall u \in X,\\
\phi(u) &= \int_\Omega F(x,u)dx, \qquad \forall u \in X,\\
E(u) &= J(u) - \phi(u), \qquad \forall u \in X.
\end{aligned}
$$

Proposition 2.4. (see [9]). *Let $(\mathbf{F_0})$ and $(\mathbf{M_0})$ hold. Then for $i \in \{1, \dots, N\}$ the following statements hold:*

$(\mathbf{1})$ $\widehat{M_i} \in C^0([0,\infty)) \cap C^1((0,\infty))$, $\widehat{M_i}(0) = 0$, $\widehat{M_i}'(t) = M_i(t) > 0$ *for any* $t > 0$, $\widehat{M_i}$ *is strictly increasing on* $[0,\infty)$.

$(\mathbf{2})$ J_i, ϕ, $E \in C^0(X)$, $J_i(0) = \phi(0) = E(0) = 0$. $J_i \in C^1(X\backslash\{0\})$, $\phi \in C^1(X)$, $E \in C^1(X\backslash\{0\})$. *For every $u \in X\backslash\{0\}$ and $v \in X$, it holds that*

$$E'(u)v = \sum_{i=1}^N M_i\Big(\int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx\Big)\int_\Omega |\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u \partial_{x_i} v dx - \int_\Omega f(x,u)v dx.$$

Thus $u \in X\backslash\{0\}$ is a weak solution of (1.1) if and only if u is a nontrivial critical point of E.

$(\mathbf{3})$ *The functional $J_i : X \to \mathbb{R}$ is sequentially weakly lower semi-continuous, $\phi : X \to \mathbb{R}$ is sequentially weakly continuous, and thus E is sequentially weakly lower semi-continuous.*

$(\mathbf{4})$ *The mapping $\phi' : X \to X^*$ is sequentially weakly-strongly continuous.*

Proposition 2.5 (See [9]). *Let $(\mathbf{F_0})$ and $(\mathbf{M_0})$ hold. Then the mapping J' and $E' : X\backslash\{0\} \to X^*$ are of type (S_+), namely,*

$$u_n \rightharpoonup u \quad \textit{and} \quad \limsup_{n\to+\infty} J'(u_n)(u_n - u) \leq 0 \quad \textit{implies } u_n \to u.$$

Corollary 2.6. *Let $(\mathbf{F_0})$ and $(\mathbf{M_0})$ hold. Then for any $c \neq 0$, every bounded $(PS)_c$ sequence for E, i.e. a bounded sequence $\{u_n\} \subset X\backslash\{0\}$ such that $E(u_n) \to c$ and $E'(u_n) \to 0$, has a strongly convergent subsequence and such u is a nonzero solution of (1.1).*

Proof. Let $\{u_n\} \subset X\backslash\{0\}$ be bounded $(PS)_c$ sequence for E with $c \neq 0$. Then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightharpoonup u$ in X. The condition $E'(u_n) \to 0$ implies that $E'(u_{n_k})(u_{n_k} - u) \to 0$. Since E' is of type (S_+), we have $u_{n_k} \to u$ in X. If, in addition, $E(u_n) \to c \neq 0$, then, by the continuity of E at u, $E(u) = c \neq 0 = E(0)$. Thus $u \neq 0$, and by the continuity of E' at u, $E'(u) = \lim_{n_k\to\infty} E'(u_{n_k}) = 0$. ■

Remark 2.7. By Corollary (2.6), to verify that E satisfies $(PS)_c$ with $c \neq 0$, it is sufficient to prove that every $(PS)_c$ sequence with $c \neq 0$ is bounded.

Remark 2.8. Under assumption $(\mathbf{M_0})$, the function M_i may be singular at 0 and in this case the energy functional E may be non-differentiable at 0. It is obvious that, under assumptions $(\mathbf{F_0})$ and $(\mathbf{M_0})$, if in addition, for each $i = 1, \dots, N$, M_i is continuous at 0, then $E \in C^1(X)$ and $E : X \to X^*$ is of type (S_+).

In the sequel, we use $c,\ c',\ C,\ C',\ M$, to denote the general nonnegative or positive constant (the exact value may change from line to line).

3. Solutions with Negative Energy

Theorem 3.1. *Let* $(\mathbf{F_0})$ *and* $(\mathbf{M_0})$ *and the following conditions hold:*

$(\mathbf{M_1})$ *For each* $i = 1, \dots, N$, *there are positive constants* $\gamma_i,\ M$ *and* C *such that* $\widehat{M_i}(t) \geq Ct^{\gamma_i}$ *for* $t \geq M$.

$(\mathbf{H_1})$ $q^+ < \gamma_i P_-^-$ *for* $i = 1, \dots, N$.

Then the functional E *is coercive, that is,* $E(u) \to \infty$ *as* $||u|| \to \infty$, *and* E *attains its infimum in* X *at some* $u_0 \in X$. *Therefore,* u_0 *is a solution of (1.1) if* E *is differentiable at* u_0, *and in particular, if* $u_0 \neq 0$.

Proof. Set $\epsilon = \min\{\gamma_i P_-^- - q^+ \ : \ i = 1, \dots, N\}$. Then by $(\mathbf{H_1})$, $\epsilon > 0$. For $||u||$ large enough, by $(\mathbf{M_1})$, we have that

$$J_i(u) = \widehat{M}\Big(\int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx\Big) \geq \widehat{M}\Big(\frac{1}{P_+^+} |\partial_{x_i} u|_{p_i(x)}^{P_-^-}\Big)$$
$$\geq C(|\partial_{x_i} u|_{p_i(x)})^{\gamma_i P_-^-} \geq C(|\partial_{x_i} u|_{p_i(x)})^{q^+ + \epsilon},$$

and hence,

$$J(u) = \sum_{i=1}^N J_i(u) \geq \sum_{i=1}^N C(|\partial_{x_i} u|_{p_i(x)})^{q^+ + \epsilon} \geq C||u||^{q^+ + \epsilon}.$$

For simplicity, in $(\mathbf{F_0})$ we assume that $m = 1$, $b_1 = b$, $s_1 = s$ and $r_1 = r$. we have

$$\begin{aligned}|\phi(u)| = \Big|\int_\Omega F(x,u)dx\Big| &\le \int_\Omega |F(x,u)|dx \le \int_\Omega \frac{b(x)}{q(x)}|u|^{q(x)}dx\\ &\le \frac{2}{q^-}|b|_{r(x)}\Big||u|^{q(x)}\Big|_{\frac{s(x)}{q(x)}} \le \frac{2}{q^-}|b|_{r(x)}(|u|_{s(x)})^{q^+}\\ &\le c||u||^{q^+}.\end{aligned}$$

Thus,

$$E(u) = J(u) - \phi(u) \ge C||u||^{q^+ + \epsilon} - c||u||^{q^+} \to +\infty,$$

that is, E is coercive. Since E is sequentially weakly lower semi-continuous and X is reflexive, E attains its infimum in X at some $u_0 \in X$. In this case where E is differentiable at u_0, u_0 is a solution of (1.1). ■

Theorem 3.2. *Let* $(\mathbf{F_0})$, $(\mathbf{M_0})$ *and* $(\mathbf{H_1})$ *and the following conditions hold:*

- $(\mathbf{M_1})$ *For each i=1, …,N, there exists* $\alpha_i > 0$ *such that* $\limsup_{t\to 0^+} \frac{\widehat{M}_i(t)}{t^{\alpha_i}} < +\infty$.
- $(\mathbf{F_1})$ *There exists a positive constant* $\delta > 0$ *such that* $f(x,t) \ge b_0(x)t^{q_0(x)-1}$ *for* $x \in \Omega$ *and* $0 < t \le \delta$, *where* $b_0 \ge 0$, $b_0(x) \in C(\Omega, \mathbb{R})$, $b_0 \ne 0$, $q_0(x) \in C_+(\overline{\Omega})$, $q_0^+ < P_-^-$.
- $(\mathbf{H_2})$ $q_0^+ < \alpha_i P_-^-$ *for* $i = 1, \ldots, N$.

Then (1.1) has at least one nontrivial solution which is a global minimizer of the functional E.

Proof. Setting $\epsilon_1 = \min\{\alpha_i P_-^- - q_0^+ \ : \ i = 1, \ldots, N\}$, then by $(\mathbf{H_2})$, $\epsilon_1 > 0$. From Theorem 3.1 we know that E has a minimizer u_0. It is clear that $F(x,0) = 0$ and consequently $E(0) = 0$. As $b_0 \ge 0$ and $b_0 \ne 0$, we can find an open set $\Omega_0 \subset \Omega$ such that $b_0(x) > 0$ for $x \in \Omega$. Take $\omega \in C_0^\infty(\Omega)\backslash\{0\}$. Then, by $(\mathbf{F_1})$, $(\mathbf{M_2})$ and $(\mathbf{H_2})$, for sufficiently small $\lambda > 0$, we have that

$$\begin{aligned}J_i(\lambda\omega) = \widehat{M}_i\Big(\int_\Omega \frac{\lambda^{p_i(x)}|\partial_{x_i}\omega|^{p_i(x)}}{p_i(x)}dx\Big) &\le c\Big(\int_\Omega \frac{\lambda^{p_i(x)}|\partial_{x_i}\omega|^{p_i(x)}}{p_i(x)}dx\Big)^{\alpha_i}\\ \le c\lambda^{\alpha_i P_-^-}\Big(\int_\Omega \frac{|\partial_{x_i}\omega|^{p_i(x)}}{p_i(x)}dx\Big)^{\alpha_i} &\le c\lambda^{\alpha_i P_-^-} \le c\lambda^{q_0^+ + \epsilon_1}.\end{aligned}$$

Thus for sufficiently small $\lambda > 0$,

$$\begin{aligned}E(\lambda\omega) = J(\lambda\omega) - \phi(\lambda\omega) &= \sum_{i=1}^N J_i(\lambda\omega) - \int_\Omega F(x,\lambda\omega)dx\\ &\le c\lambda^{q_0^+ + \epsilon_1} - C\lambda^{q_0^+} < 0.\end{aligned}$$

Hence $E(u) < 0$ which shows $u_0 \neq 0$. ∎

Since X is a reflexive and separable Banach space, then X^* is too. There exist (see [17]) $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\text{span}\ \{e_j\ : j = 1, 2, ...\}}, \quad X^* = \overline{\text{span}\ \{e_j^*\ : j = 1, 2, ...\}},$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases}$$

where $\langle ., . \rangle$ denote the duality product between X and X^*. We define

$$X_j = \text{span}\ \{e_j\}, \quad Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \tag{3.1}$$

Lemma 3.3. (see [6]). *Assume that $\psi : X \to \mathbb{R}$ is weakly-strongly continuous and $\psi(0) = 0,\ \nu > 0$ is a given number. Set*

$$\beta_k = \sup_{u \in Z_k, ||u|| \leq \nu} |\psi(u)|,$$

then $\beta_k \to 0$ as $k \to \infty$.

Theorem 3.4. *Let all the hypotheses of Theorem 3.2 hold, and let, in addition, f satisfy the following condition:*

($\mathbf{f_2}$) *$f(x, -t) = -f(x, t)$ for $x \in \Omega$ and $t \in \mathbb{R}$.*

Then (1.1) has a sequence of solutions $\{\pm u_k\}$ such that $E(\pm u_k) < 0$, and $E(\pm u_k) \to 0$ as $k \to \infty$.

Proof. Denote by $\kappa(A)$ the genus of A. Denote

$$\sum = \{A \subset X \backslash \{0\}\ :\ A \quad \text{is compact and} \quad A = -A\},$$

$$\sum_k = \{A \in \sum\ :\ \kappa(A) \geq k\},$$

$$c_k = \inf_{A \in \sum_k} \sup_{u \in A} E(u), \ \ k = 1, 2, \ldots,$$

we have $-\infty < c_1 \leq c_2 \leq \cdots \leq c_k \leq c_{k+1} \ldots.$
For any k, we can choose a k-dimensional linear subspace E_k of $W_0^{k,p(.)}(\Omega)$ such that $E_k \subset C_0^\infty(\Omega)$. As the norms on E_k are equivalent to each other, there exists

$\rho_k \in (0,1)$ such that $u \in E_k$ with $||u|| \leq \rho_k$ implies $|u|_{L^\infty} < \delta$. Set $S_{\rho_k}^{(k)} = \{u \in E_k \ : \ ||u|| = \rho_k\}$. Since $S_{\rho_k}^{(k)}$ is compact, we can find a positive constant d_k such that

$$\int_\Omega \frac{b_0(x)}{q_0(x)} |u|^{q_0(x)} dx \geq d_k, \qquad \forall u \in S_{\rho_k}^{(k)}.$$

For $u \in S_{\rho_k}^{(k)}$ and $t \in (0,1)$, we have

$$E(tu) \leq \frac{t^{\alpha_i P_-^-}}{P_-^-} \rho_k^{P_-^-} - t^{q_0^+} d_k.$$

By $(\mathbf{H_2})$, we can find $t_k \in (0,1)$ and $\epsilon_k > 0$ such that $E(t_k u) \leq -\epsilon_k < 0$ for every $u \in S_{\rho_k}^{(k)}$, which implies $E(u_k) \leq -\epsilon_k < 0$ for every $u \in S_{t_k\rho_k}^{(k)}$. Since $\kappa(S_{t_k\rho_k}^{(k)}) = k$, we get the conclusion $c_k \leq -\epsilon_k < 0$.

By the genus theory, each c_k is a critical value of E, hence there is a sequence of solutions $\{\pm u_k \ : \ k = 1, 2, \ldots\}$ of problem (1.1) such that $E(\pm u_k) = c_k < 0$.

At last, we will prove $c_k \to 0$ as $k \to \infty$. Since E is coercive, then there exists a constant $\eta > 0$ such that $E(u) > 0$ when $||u|| \geq \eta$. For any $A \in \sum_k$, let Y_k and Z_k be the subspace of X as mentioned above. According to the properties of genus, we know $A \cap Z_k \neq \emptyset$. Set

$$\beta_k = \sup_{u \in Z_k, \ ||u|| \leq \eta} |\phi(u)|,$$

we know $\beta_k \to 0$ as $k \to \infty$. When $u \in Z_k$ and $||u|| \leq \eta$, we have $E(u) \geq -\beta_k$ and then $c_k \geq -\beta_k$, which concludes $c_k \to 0$ as $k \to \infty$. ∎

4. Solutions with Positive nergy

In this section we will find the Mountain Pass critical points of the energy functional E associated to problem (1.1).

Lemma 4.1. *Let* $(\mathbf{M_0})$, $(\mathbf{F_1})$ *and the following conditions be satisfied:*

$(\mathbf{M_1})'$ *The condition* $(\mathbf{M_1})$ *holds and* $\gamma_i P_-^- > 1$ *for* $i = 1, \ldots, N$.

$(\mathbf{M_3})$ *For each* $i = 1, \ldots, N$, *there exist* $\lambda_i > 0$ *and* $M > 0$ *such that* $\lambda_i \widehat{M_i}(t) \geq M_i(t)t$ *for* $t \geq M$.

$(\mathbf{F_3})$ *There exist* $\mu > 0$ *and* $M > 0$ *such that* $0 \leq \mu F(x,t) \leq f(x,t)t$ *for* $|t| \geq M$ *and* $x \in \Omega$.

$(\mathbf{H_3})$ $\lambda_i P_+^+ < \mu$ *for* $i = 1, \ldots, N$.

Then E *satisfies condition* $(PS)_c$ *for any* $c \neq 0$.

Proof. By $(\mathbf{M_3})$, for each $i = 1, \dots, N$, and for sufficiently large $|\partial_{x_i} u|_{p_i(.)}$,

$$\begin{aligned}
\lambda_i P_+^+ J_i(u) &= \lambda_i P_+^+ \widehat{M}_i\Big(\int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}dx\Big)\\
&\geq P_+^+ M_i\Big(\int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}dx\Big)\int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}dx\\
&\geq M_i\Big(\int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}dx\Big)\int_\Omega |\partial_{x_i} u|^{p_i(x)}dx = J_i'(u)u.
\end{aligned}$$

In [8] it was proved that, $(\mathbf{M_1})'$ and $(\mathbf{F_3})$ imply that, given any $\epsilon \in (0, \mu)$, there exists C_ϵ such that

$$\phi'(u)u - (\mu - \epsilon)\phi(u) \geq -C_\epsilon \text{ for } u \in X.$$

Now let $\{u_n\} \subset X\backslash\{0\}$, $E(u_n) \to c \neq 0$ and $E'(u_n) \to 0$. By $(\mathbf{H_3})$, there exists $\epsilon > 0$ small enough such that $\lambda_i P_+^+ < (\mu - \epsilon)$ for $i = 1, \dots, N$. Setting $d = \min\{\gamma_i P_-^- \ ; \ i = 1, \dots, N\}$ and $e = (\mu - \epsilon) - \lambda_i P_+^+$, then $d > 1$ and $e > 0$. Since $\{u_n\}$ is a $(PS)_c$ sequence, for sufficiently large n, we have

$$\begin{aligned}
(\mu - \epsilon)c + 1 + ||u_n|| &\geq (\mu - \epsilon)E(u_n) - E'(u_n)u_n\\
&\geq (\mu-\epsilon)\sum_{i=1}^N J_i(u_n) - \sum_{i=1}^N J_i'(u_n)u_n + \phi'(u_n)u_n - (\mu-\epsilon)\phi(u_n)\\
&\geq \sum_{i=1}^N \Big((\mu - \epsilon) - \lambda_i P_+^+\Big) J_i(u_n) - c - C_\epsilon\\
&\geq eJ(u_n) - c - C_\epsilon\\
&\geq c'||u_n||^d - C.
\end{aligned}$$

This shows that $\{||u_n||\}$ is bounded because $d > 1$. By Corollary 2.6, E satisfies condition $(PS)_c$ for any $c \neq 0$. ■

Lemma 4.2. *Under the hypotheses of Lemma 4.1, for any* $\omega \in X\backslash\{0\}$, $E(s\omega) \to -\infty$ *as* $s \to +\infty$.

Proof. Setting $\tau = \min\{\mu - \lambda_i P_+^+ \ : \ i = 1, \dots, N\}$, then by $(\mathbf{H_3})$, $\tau > 0$. Let $\omega \in X\backslash\{0\}$ be given. From $(\mathbf{M_3})$, for each $i = 1, \dots, N$, and sufficiently large $t > 0$ we have

$$\widehat{M}_i(t) \leq C_i t^{\lambda_i}$$

and then it follows that for s large enough

$$J_i(s\omega) \leq d_1 s^{\lambda_i P_+^+} \leq d_1 s^{\mu - \tau},$$

where d_1 is a positive constant depending on ω. Thus for s large enough we have

$$J(s\omega) \leq N d_1 s^{\mu-\tau}.$$

From $(\mathbf{F_3})$ for $x \in \Omega$ and $t \in \mathbb{R}$ we have

$$F(x,t) \geq C|t|^{\mu} - c,$$

which implies that for s large enough

$$\phi(s\omega) = \int_{\Omega} F(x, s\omega)dx \geq d_2 s^{\mu},$$

where d_2 is a positive constant depending on ω. Hence for s large enough, we have

$$E(s\omega) \leq d_1 s^{\mu-\tau} - d_2 s^{\mu},$$

and consequently, $E(s\omega) \to -\infty$ as $s \to +\infty$. ■

Lemma 4.3. *Let* $(\mathbf{F_0})$, $(\mathbf{M_0})$ *and the following conditions be satisfied:*

$(\mathbf{M_4})$ *For each* $i = 1, \ldots, N$, *there exists* $\beta_i > 0$ *such that* $\liminf_{t\to 0^+} \frac{\widehat{M}_i(t)}{t^{\beta_i}} > 0$.

$(\mathbf{F_4})$ *There exists* $r_1(x) \in C^0(\overline{\Omega})$ *such that* $P_+^+ < r_1(x) < P_-^*(x)$ *for* $x \in \overline{\Omega}$ *and* $\liminf_{t\to 0} \frac{|F(x,t)|}{|t|^{r_1(x)}} < +\infty$ *uniformly in* $x \in \Omega$.

$(\mathbf{H_4})$ $\beta_i P_+^+ < r_1^-$ *for* $i = 1, \ldots, N$.

Then there exist positive constants ρ *and* δ *such that* $E(u) \geq \delta$ *for* $||u|| = \rho$.

Proof. Setting $\epsilon = \min\{r_1^- - \beta_i P_+^+ \ : \ i = 1, \ldots, N\}$, then by $(\mathbf{H_4})$, $\epsilon > 0$. It follows from $(\mathbf{M_4})$ that for $||u||$ small enough

$$\begin{aligned} J_i(u) &= \widehat{M}\Big(\int_{\Omega} \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx\Big) \geq \widehat{M}\Big(\frac{1}{P_+^+} |\partial_{x_i} u|_{p_i(x)}^{P_+^+}\Big) \\ &\geq C(|\partial_{x_i} u|_{p_i(x)})^{\beta_i P_+^+} \geq C(|\partial_{x_i} u|_{p_i(x)})^{r_1^- - \epsilon}, \end{aligned}$$

and hence,

$$J(u) = \sum_{i=1}^{N} J_i(u) \geq \sum_{i=1}^{N} C(|\partial_{x_i} u|_{p_i(x)})^{r_1^- - \epsilon} \geq C||u||^{r_1^- - \epsilon}.$$

It follows from $(\mathbf{F_0})$ and $(\mathbf{F_4})$ that for sufficiently small $||u||$,

$$|\phi(u)| \leq C'||u||^{r_1^-}.$$

Thus, for sufficiently small $||u||$, $E(u) \geq C||u||^{r_1^- - \epsilon} - C'||u||^{r_1^-}$. From this we obtain the assertion of Lemma 4.3. ■

By the famous Mountain pass lemma [1], from Lemmas 4.1-4.3 we have the following:

Theorem 4.4. *Let all hypotheses of Lemmas 4.1-4.3 hold. Then* (1.1) *has a non-trivial solution with positive energy.*

5. The Case of Concave-convex Nonlinearity

In this section, we will obtain much better results with f in a special form. We have the following theorem:

Theorem 5.1. *Let* $f(x,t) = a(x)|u|^{\alpha(x)-2}u + b(x)|u|^{q(x)-2}u$*, where*

$$\alpha,\ q \in C_+(\overline{\Omega}),\quad 1 < \alpha^- \le \alpha^+ < P_-^- \le P_+^+ < q^-,\quad P_+^+ < q(x) < P_-^*(x),$$

$$a(x) > 0,\quad a \in L^\infty(\overline{\Omega}) \cap L^{r_1(\cdot)}(\Omega),\quad \frac{1}{r_1(x)} + \frac{\alpha(x)}{s_1(x)} = 1,$$

$$b(x) > 0,\quad b \in L^\infty(\overline{\Omega}) \cap L^{r_2(\cdot)}(\Omega),\quad \frac{1}{r_2(x)} + \frac{\alpha(x)}{s_2(x)} = 1,$$

$$p(x) \le s_1(x) \le P_-^*(x),\quad p(x) \le s_2(x) \le P_-^*(x).$$

Then, we have

- $(\mathbf{i})$ *If* $(\mathbf{M_0})$, $(\mathbf{M_1})'$, $(\mathbf{M_3})$, $(\mathbf{H_3})$ *hold and we also assume that* $\alpha^+ < \gamma_i P_-^- < q^+$ *and* $1 < \lambda_i P_+^+ < q^-$*, then problem (1.1) has solutions* $\{\pm u_k\}_{k=1}^\infty$ *such that* $E(\pm u_k) \to +\infty$ *as* $k \to +\infty$.
- $(\mathbf{ii})$ *If* $(\mathbf{M_0})$, $(\mathbf{M_1})'$, $(\mathbf{M_3})$, $(\mathbf{M_4})$ *hold and also assume that* $\alpha^- < \beta_i P_+^+$ *and* $\alpha^+ < \lambda_i P_-^-$*, then problem (1.1) has solutions* $\{\pm v_k\}_{k=1}^\infty$ *such that* $E(\pm v_k) < 0,\ E(\pm v_k) \to 0$ *as* $k \to \infty$.

We will use the following Fountain theorem and the Dual Fountain theorem to prove Theorem 5.1.

Lemma 5.2. (Fountain Theorem, see [16]). *Let*

$(\mathbf{A1})$ $E \in C^1(X, \mathbb{R})$ *be an even functional, where* $(X, ||.||)$ *is a separable and reflexive Banach space, the subspaces* $X_k,\ Y_k$ *and* Z_k *are defined by* (3.1).
If for each $k \in \mathbb{N}$*, there exist* $\rho_k > r_k > 0$ *such that*

$(\mathbf{A2})$ $\inf\{E(u)\ :\ u \in Z_k,\ ||u|| = r_k\} \to +\infty$ *as* $k \to +\infty$.

$(\mathbf{A3})$ $\max\{E(u)\ :\ u \in Y_k,\ ||u|| = \rho_k\} \le 0$.

$(\mathbf{A4})$ E *satisfies the (PS) condition for every* $c > 0$.

Then E *has an unbounded sequence of critical values tending to* $+\infty$.

Lemma 5.3. (Dual Fountain Theorem, see [16]). *Assume* $(\mathbf{A1})$ *is satisfied and there is* $k_0 > 0$ *so that, for each* $k \ge k_0$*, there exist* $\rho_k > r_k > 0$ *such that*

$(\mathbf{B1})$ $a_k = \inf\{E(u)\ :\ u \in Z_k,\ ||u|| = \rho_k\} \geq 0.$

$(\mathbf{B2})$ $b_k = \max\{E(u)\ :\ u \in Y_k,\ ||u|| = r_k\} < 0.$

$(\mathbf{B3})$ $d_k = \inf\{E(u)\ :\ u \in Z_k,\ ||u|| \leq \rho_k\}\ \to 0$ *as* $k \to +\infty.$

$(\mathbf{B4})$ E *satisfies the* $(PS)_c^*$ *condition for every* $c \in [d_{k_0}, 0).$

Then E *has a sequence of negative critical values converging to* $0.$

Definition 5.4. We say that E satisfies the $(PS)_c^*$ condition (with respect to (Y_n)), if any sequence $\{u_{n_j}\} \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$, $E(u_{n_j}) \to c$ and $(E|_{Y_{n_j}})'(u_{n_j}) \to 0$, contains a subsequence converging to a critical point of E.

Lemma 5.5. *Assume that the conditions in Theorem 5.1 hold, then* J *satisfies the* $(PS)_c^*$ *condition.*

Proof. Suppose $\{u_{n_j}\} \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$, $E(u_{n_j}) \to c$ and $(E|_{Y_{n_j}})'(u_{n_j}) \to 0$. Similar to the method in Lemma 4.1, we have that

$$\begin{aligned}
&(\mu-\epsilon)c + 1 + ||u_{n_j}|| \\
&\geq (\mu-\epsilon)E(u_{n_j}) - E'(u_{n_j})u_{n_j} \\
&\geq (\mu-\epsilon)\sum_{i=1}^{N} J_i(u_{n_j}) - \sum_{i=1}^{N} J_i'(u_{n_j})u_{n_j} + \phi'(u_{n_j})u_{n_j} - (\mu-\epsilon)\phi(u_{n_j}) \\
&\geq \sum_{i=1}^{N}\Big((\mu-\epsilon) - \lambda_i P_+^+\Big)J_i(u_{n_j}) - c - C_\epsilon \\
&\geq eJ(u_{n_j}) - c - C_\epsilon \\
&\geq c'||u_{n_j}||^d - C,
\end{aligned}$$

hence, we can get that $\{||u_{n_j}||\}$ is bounded. Going if necessary to a subsequence, we can assume $u_{n_j} \rightharpoonup u$ in X. As $X = \overline{\cup_{n_j} Y_{n_j}}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. Hence

$$\begin{aligned}
\lim_{n_j\to+\infty}\langle E'(u_{n_j}), u_{n_j} - u\rangle &= \lim_{n_j\to+\infty}\langle E'(u_{n_j}), u_{n_j} - v_{n_j}\rangle + \lim_{n_j\to+\infty}\langle E'(u_{n_j}), v_{n_j} - u\rangle \\
&= \lim_{n_j\to+\infty}\Big\langle (E|_{Y_{n_j}})'(u_{n_j}), u_{n_j} - v_{n_j}\Big\rangle \\
&= 0.
\end{aligned}$$

As E' is of (S_+) type, we conclude $u_{n_j} \to u$, furthermore we have $E'(u_{n_j}) \to E'(u)$. Let us prove $E'(u) = 0$ below. Taking $\omega_k \in Y_k$, notice that when $n_j \geq k$ we have

$$\begin{aligned}
\langle E'(u), \omega_k\rangle &= \langle E'(u) - E'(u_{n_j}), \omega_k\rangle + \langle E'(u_{n_j}), \omega_k\rangle \\
&= \langle E'(u) - E'(u_{n_j}), \omega_k\rangle + \Big\langle (E|_{Y_{n_j}})'(u_{n_j}), \omega_k\Big\rangle.
\end{aligned}$$

Going to the limit on the right side of the above equation reaches

$$\langle E'(u), \omega_k\rangle = 0, \quad \forall \omega_k \in Y_k,$$

so $E'(u) = 0$, this show that E satisfies the $(PS)_c^*$ condition for every $c \in \mathbb{R}$. ∎

5.1. Proof of Theorem 5.1

($\mathbf{i}$) We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that $(\mathbf{A_2})$ and $(\mathbf{A_3})$ are satisfied.

$(\mathbf{A_2})$ For $k = 1, 2, \dots,$ denote

$$\theta_k = \sup_{v\in Z_k, ||v||\le 1} \int_\Omega \frac{a(x)}{\alpha(x)}|v|^{\alpha(x)}dx, \quad \beta_k = \sup_{v\in Z_k, ||v||\le 1} \int_\Omega \frac{b(x)}{q(x)}|v|^{q(x)}dx,$$

then $\theta_k > 0,\ \beta_k > 0$ and $\theta_k \to 0,\ \beta_k \to 0$, as $k \to \infty$. When $u \in Z_k,\ ||u|| \ge M$,

$$E(u) \ge \frac{1}{P_+^+}||u||^d - \theta_k||u||^{\alpha^+} - \beta_k||u||^{q^+},$$

where d is defined in Lemma 4.1. For sufficiently large k, we have $\theta_k < \frac{1}{2P_+^+}$. As $\alpha^+ < \gamma_i P_-^-$ for $i = 1, \dots, N$, it follows $\alpha^+ < d$, we get

$$E(u) \ge \frac{1}{2P_+^+}||u||^d - \beta_k||u||^{q^+}.$$

At this stage, we fix r_k as follows:

$$r_k = (2P_+^+\beta_k q^+)^{\frac{1}{d-q^+}} \to +\infty \quad \text{as} \quad k \to +\infty.$$

Consequently, if $||u|| = r_k$ then

$$E(u) \ge \Big(1 - \frac{1}{q^+}\Big)\frac{r_k^d}{2P_+^+} - C \to +\infty \quad \text{as} \quad k \to +\infty.$$

$(\mathbf{A_3})$ From $(\mathbf{M_3})$, it is easy to obtain that for t large enough $\widehat{M_i}(t) \le Ct^{\lambda_i}$. For $k = 1, 2, \dots,$ denote

$$e_k = \inf_{v\in Y_k, ||v||=1} \int_\Omega \frac{b(x)}{q(x)}|v|^{q(x)}dx.$$

Then $e_k > 0$. Setting $d' = \max\{\lambda_i P_+^+ \ :\ i = 1, \dots, N\}$, then $1 < d' < q^-$. For any $v \in Y_k$ with $||v|| = 1$ and t large enough, since dim$Y_k < \infty$, all norms are equivalent in Y_k, we have

$$\begin{aligned} E(tv) &\le \frac{c}{P_-^-}\sum_{i=1}^N t^{\lambda_i P_+^+} - e_k t^{q^-} \\ &\le N\frac{c}{P_-^-}t^{d'} - e_k t^{q^-}. \end{aligned}$$

As $d^{'} < q^{-}$, there exists $\rho_k > r_k$ such that $t = \rho_k$ concludes $E(tv) \leq 0$ and then

$$\max_{u \in Y_k, ||u|| = \rho_k} E(u) \leq 0,$$

so $(\mathbf{A_3})$ is satisfied.

$(\mathbf{ii})$ We use the Dual Fountain theorem to prove conclusion $(\mathbf{ii})$, and now it remains for us to prove that there exist $\rho_k > r_k > 0$ such that if k is large enough $(\mathbf{B_1})$, $(\mathbf{B_2})$ and $(\mathbf{B_3})$ are satisfied.

$(\mathbf{B_1})$ Let θ_k and β_k be defined as above. Setting $d^{''} = \max\{\beta_i P_+^+ \ : \ i = 1, \ldots, N\}$, then $\alpha^- < d^{''}$. When $v \in Z_k, \ ||v|| = 1$ and t small enough, we have

$$\begin{aligned} E(tv) &\geq \frac{1}{P_+^+} \sum_{i=1}^{N} t^{\beta_i P_+^+} - \theta_k t^{\alpha^-} - \beta_k t^{q^-} \\ &\geq \frac{N}{P_+^+} t^{d^{''}} - \theta_k t^{\alpha^-} - \beta_k t^{P_+^+}. \end{aligned}$$

For sufficiently large k we have $\beta_k < \frac{1}{2P_+^+}$, thus

$$E(tv) \geq \frac{N}{P_+^+} t^{d^{''}} - \theta_k t^{\alpha^-}.$$

Choose $\rho_k = \left(\frac{2P_+^+ \theta_k}{N}\right)^{\frac{1}{d^{''} - \alpha^-}}$, then for sufficiently large k, $\rho_k < 1$. When $t = \rho_k, \ v \in Z_k$ with $||v|| = 1$, we have

$$E(tv) \geq \left(\frac{2P_+^+}{N}\right)^{\frac{\alpha^-}{d^{''} - \alpha^-}} \theta_k^{\frac{\alpha^-}{d^{''} - \alpha^-}} - \left(\frac{2P_+^+}{N}\right)^{\frac{\alpha^-}{d^{''} - \alpha^-}} \theta_k^{\frac{\alpha^-}{d^{''} - \alpha^-}} = 0.$$

Since $d^{''} > \alpha^-, \ \theta_k \to 0$, we know that $\rho_k \to 0$ as $k \to +\infty$, so $(\mathbf{B_1})$ is satisfied.

$(\mathbf{B_2})$ For $k = 1, 2, \ldots$, denote

$$\delta_k = \inf_{v \in Y_k, ||v|| = 1} \int_{\Omega} \frac{a(x)}{\alpha(x)} |v|^{\alpha(x)} dx,$$

then $\delta_k > 0$. Setting $d_0 = \min\{\lambda_i P_-^- \ : \ i = 1, \ldots, N\}$, then $\alpha^+ < d_0$. Using $(\mathbf{M_3})$, for $v \in Y_k, \ ||v|| = 1$ and t small enough, we have

$$\begin{aligned} E(tv) &\leq \frac{1}{P_-^-} \sum_{i=1}^{N} t^{\lambda_i P_-^-} - \delta_k t^{\alpha^+} \\ &\leq \frac{N}{P_-^-} t^{d_0} - \theta_k t^{\alpha^+}. \end{aligned}$$

Since $\dim Y_k = k$, condition $\alpha^+ < d_0$ implies that there exists a $r_k \in (0, \rho_k)$ such that $E(u) < 0$ when $||u|| = r_k$. Hence $b_k = \max\{E(u) \ : \ u \in Y_k, \ ||u|| = r_k\} < 0$, hence

$(\mathbf{B_2})$ is satisfied.
$(\mathbf{B_3})$ From the proof above and $Y_k \cap Z_k \neq \emptyset$, we have

$$\begin{aligned} d_k &= \inf\{E(u) \ : \ u \in Z_k, \ ||u||_{\vec{p}(x)} \leq \rho_k\} \leq b_k \\ &= \max\{E(u) \ : \ u \in Y_k, \ ||u||_{\vec{p}(x)} = r_k\} < 0. \end{aligned}$$

For $v \in Z_k, \ ||v|| = 1$ and $u = tv$ small enough, we have

$$\begin{aligned} E(u) = E(tv) &\geq \frac{N}{2P_+^+} t^{d''} - \theta_k t^{\alpha^-} \\ &\geq -\theta_k t^{\alpha^-} \geq -\theta_k \rho_k^{\alpha^-} \geq -\theta_k, \end{aligned}$$

hence $d_k \to 0$, so $(\mathbf{B_3})$ is satisfied.

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G. A. Afrouzi and M. Mirzapour
Department of Mathematics
Faculty of Mathematical Sciences
University of Mazandaran
Babolsar, Iran
E-mail: afrouzi@umz.ac.ir
mirzapour@stu.umz.ac.ir