TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 6, pp. 2163-2190, December 2013
DOI: 10.11650/tjm.17.2013.3794
This paper is available online at http://journal.taiwanmathsoc.org.tw

EXISTENCE, ASYMPTOTICS AND UNIQUENESS OF TRAVELING WAVES FOR NONLOCAL DIFFUSION SYSTEMS WITH DELAYED NONLOCAL RESPONSE

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Abstract. In this paper, we deal with the existence, asymptotic behavior and uniqueness of traveling waves for nonlocal diffusion systems with delay and global response. We first obtain the existence of traveling wave front by using upper-lower solutions method and Schauder's fixed point theorem for $c > c_*$ and using a limiting argument for $c = c_*$. Secondly, we find a priori asymptotic behavior of (monotone or non-monotone) traveling waves with the help of Ikehara's Theorem by constructing a Laplace transform representation of a solution. Thirdly, we show that the traveling wave front for each given wave speed is unique up to a translation. Last, we apply our results to two models with delayed nonlocal response.

1. INTRODUCTION

The spatial dispersal of cells, organisms or species is clearly central to ecology, and the evolution of dispersal itself is consequently of great importance. The Laplacian reaction-diffusion equation

$$\frac{\partial u(t,x)}{\partial t} = D \triangle u(t,x) + u(t,x)(1-u(t,x)) \tag{1.1}$$

is well known in population dynamics and was investigated by Fisher [12] to model the spatial spread of a mutant in an infinite one-dimensional habitat. Since then, traveling wave fronts for reaction-diffusion systems have attracted much attention in biology,

Received August 5, 2012, accepted June 16, 2013.
Communicated by Chin-Cheng Lin.
2010 *Mathematics Subject Classification*: Primary 35C07, 35R10.
Key words and phrases: Asymptotics, Traveling wave fronts, Nonlocal diffusion, Delay.
[1]This author was Partially supported by National Natural Science Foundation of China (No. 11101282, No. 11271260), by Shanghai university young teacher training program (No. slg11031), by Shanghai Leading Academic Discipline Project (No. XTKX2012) and by Innovation Program of Shanghai Municipal Education Commission (No. 14YZ096).
[2]Supported by National Natural Science Foundation of China and RFDP.

chemistry, epidemiology and physics, see [2, 11, 23]. There are many methods to deal with the existence of traveling wave, for example, the phase space analysis [20] and the Conley index [23]. Due to the practical background and biological realism, delays and nonlocal delays are incorporated into reaction-diffusion equations, see [9, 14-15, 20, 23, 26] and the references therein.

Despite the popularity of Laplacian diffusion models, diffusion has some drawbacks. One important shortcoming for ecological and epidemiological models is that Laplacian diffusion is a local operator where individuals in the population can only influence their immediate neighbors. With diffusion models there is some disconnect between experimentally collected data and a limited number of parameters that are available to fit that data. One method in overcoming these problems with the Laplacian operator is to describe these models concerning with the spatial migration by integral equation. Lee et al. [14] argued that, for processes where the spatial scale for movement is large in comparison with its temporal scale, non-local models using integro-differential equations may allow for better estimation of parameters from data and provide more insight into the biological system. The precise mathematical model is read as

$$u_t(t,x) = \int_{\mathbb{R}} J(x-y)u(t,y)dy - u(t,x) + f(u(t,x)). \tag{1.2}$$

The nonlocal model (1.2) with monostable nonlinearity have been widely investigated by authors (see [1, 4, 7, 8, 9, 22]). More recently, authors in [18, 19] showed the existence of traveling waves for (1.2) with delayed reaction terms satisfying the quasi-monotonicity and the exponential quasi-monotonicity, respectively. Those results in [18, 19] were well applied to the Logistic equation with nonlocal diffusion

$$u_t(t,\, x) = d[J * u(t,x) - u(t,x)] + ru(t-\tau,x)[1-u(t,x)] \tag{1.3}$$

and the Nicholson's blowflies equation with nonlocal diffusion

$$u_t(t,\, x) = d[J * u(t,x) - u(t,x)] - ru(t,x) + rpu(t-\tau,x)e^{-u(t-\tau,x)}, \tag{1.4}$$

where r, p and the delay τ are positive. Authors [26, 28] further investigated two componentwise nonlocal diffusion systems with the weak (exponential) quasi-monotonicity and the partial (exponential) quasi-monotonicity, respectively. The model (1.2) is closely related to traditional reaction-diffusion models. Taking the diffusion kernel

$$J(x) = \delta(x) + \delta''(x), \tag{1.5}$$

where δ is the Dirac delta, (1.2) reduces to the Laplacian reaction diffusion equation (1.1), (see, Medlock et al. [17]).

We notice that the drift of some individuals depends on their present position from all possible positions at previous times. It appears that the first comprehensive attempt to address this phenomena was made for the reaction-diffusion equation by Britton [3]. His idea was that the reaction term with the delay has to involve a weighted spatial averaging over the whole of the infinite domain (for short, the reaction term with delayed nonlocal response). Motivated by these, we also incorporate the nonlocal delayed response into nonlocal diffusion models (1.3) and (1.4), that is,

$$u_t(t,\, x) = d[J * u(t,x) - u(t,x)] + r\int_{\mathbb{R}} k(y)u(t-\tau,\, x-y)dy(1-u(t,\, x)) \tag{1.6}$$

and

$$\begin{aligned} u_t(t,\, x) &= d[J * u(t,x) - u(t,x)] \\ &\quad -ru + rp\int_{\mathbb{R}} k(y)u(t-\tau,\, x-y)dy e^{-\int_{\mathbb{R}} k(y)u(t-\tau,\, x-y)dy}. \end{aligned} \tag{1.7}$$

To our knowledge, the existence, asymptotic behavior and uniqueness of traveling waves for nonlocal diffusion systems (1.6) and (1.7) are not reported. In order to address these results, we first investigate the existence, asymptotic behavior and uniqueness of traveling waves for more general nonlocal diffusion systems with delayed nonlocal response

$$u_t(t,\, x) = d[J * u(t,x) - u(t,x)] + f\Big(u(t,\, x),\, \int_{-\infty}^{\infty} k(y)u(t-\tau,\, x-y)dy\Big) \tag{1.8}$$

where $d > 0,\ \tau \geq 0$ are constants,

$$J * u(t,\, x) = \int_{\mathbb{R}} J(x-y)u(t,\, y)dy \text{ is the standard convolution,}$$

and the functions $J,\, k\ f$ satisfy

(H1) $J \geq 0, \quad J(x) = J(-x),\ \int_{\mathbb{R}} J(y)dy = 1,$ and $\int_{\mathbb{R}} J(y)e^{-\lambda y}dy < \infty,\ \forall\lambda \geq 0.$

(H2) $k \geq 0, \quad k(x) = k(-x),\ \int_{\mathbb{R}} k(y)dy = 1,$ and $\int_{\mathbb{R}} k(y)e^{-\lambda y}dy < \infty,\ \forall\lambda \geq 0.$

(A1) $f \in C^1([0,\, K]^2,\, \mathbb{R}),\ f(0,0) = f(K,\, K) = 0$ and $f(u,u) > 0$ for $u \in (0,\, K)$, and $\partial_2 f(u,\, v) \geq 0$ for $(u,\, v) \in [0,\, K]^2$, where K is a positive constant;

(A2) $\partial_1 f(0,\, 0)u + \partial_2 f(0,\, 0)v \geq f(u,\, v)$ for any $(u,\, v) \in [0,\, K]^2$;

(A3) there exist numbers $L, \kappa > 0$ and $\sigma_1, \sigma_2 \in (0, 1]$ such that

$$|f(u,\, v) - \partial_1 f(0,\, 0)u - \partial_2 f(0,\, 0)v| \leq L(u^{1+\sigma_1} + v^{1+\sigma_2})$$

for any $(u,\, v) \in [0,\, \kappa]^2$.

Remark 1.1. Condition (A1) together with (A2) implies that $\partial_1 f(0,\, 0) + \partial_2 f(0,\, 0) \geq f\Big(\frac{K}{2}, \frac{K}{2}\Big)\frac{2}{K} > 0$.

Remark 1.2. Letting $k(x) = \delta(x)$, (1.8) can be reduced to the following nonlocal diffusion system

$$u_t(t, x) = d[J * u(t,x) - u(t,x)] + f(u(t, x), u(t-\tau, x)).$$

A traveling wave of (1.8) is a solution of special form $u(t, x) = \phi(x+ct)$, where the velocity c and the wave profile ϕ satisfy the following functional differential equation

$$c\phi'(\xi) = d[J * \phi(\xi) - \phi(\xi)] + f\Big(\phi(\xi), \int_{-\infty}^{\infty} k(y)\phi(\xi - c\tau - y)dy\Big) \tag{1.9}$$

with asymptotic boundary conditions

$$\lim_{\xi\to-\infty} \phi(\xi) = 0, \qquad \lim_{\xi\to+\infty} \phi(\xi) = K, \tag{1.10}$$

where 0 and $K > 0$ are two equilibria of (1.9). A traveling wave ϕ is called the traveling wave front if ϕ is monotone.

Now we formulate our main theorems as follows.

Theorem 1.1. [Existence]. *Assume that* (H1)-(H2) *and* (A1)-(A3) *hold. Then there exists a positive constant* c_* *such that for each* $c \geq c_*$, *equation* (1.8) *admits a nondecreasing positive traveling wave front* $u(t, x) = \phi(x+ct)$ *with* $\phi(-\infty) = 0$ *and* $\phi(+\infty) = K$. *Moreover, if* $c > c_*$, *then*

$$\lim_{\xi\to-\infty} \phi(\xi)e^{-\lambda_1\xi} = 1, \quad \lim_{\xi\to-\infty} \phi'(\xi)e^{-\lambda_1\xi} = \lambda_1, \tag{1.11}$$

where $\lambda = \lambda_1 > 0$ *is the smallest root of the equation*

$$\begin{aligned}&\Delta(c, \lambda)\\ &= c\lambda - d\left[\int_{\mathbb{R}} e^{-\lambda y}J(y)dy - 1\right] - \partial_1 f(0, 0) - \partial_2 f(0, 0)e^{-c\tau\lambda}\int_{\mathbb{R}} k(y)e^{-\lambda y}dy = 0.\end{aligned}$$

Theorem 1.2. [Asymptotics]. *Assume that* (H1)-(H2), (A1)-(A3) *hold, and* $\widetilde{\phi}(\xi)$ *is any nonnegative bounded traveling wave of* (1.8) *with* $\widetilde{\phi}(-\infty) = 0$ *and* $\widetilde{\phi} \not\equiv 0$. *Then we have the following conclusions*

(*i*) *For every* $c > c_*$, $\lim_{\xi\to-\infty} \widetilde{\phi}(\xi)e^{-\lambda_1\xi}$ *exists.*

(*ii*) *For* $c = c_*$, *there exists a constant* $\lambda_* > 0$ *such that* $\lim_{\xi\to-\infty} \widetilde{\phi}(\xi)\xi^{-1}e^{-\lambda_*\xi}$ exists.

(*iii*) *For* $0 < c < c_*$, *there is no nonnegative bounded traveling wave with* $\widetilde{\phi}(-\infty) = 0$ *and* $\widetilde{\phi} \not\equiv 0$.

Theorem 1.3. [Uniqueness]. *Assume that (H1)-(H2) and (A1)-(A3) hold. For* $c \geq c_*$, *let* φ, ψ *be two travelling wave fronts of* (1.9) *and* (1.10) *with wave speed* c. *Then* ϕ *is a translation of* ψ; *more precisely, there exists* $\bar{\xi} \in \mathbb{R}$ *such that* $\phi(\xi) = \psi(\xi + \bar{\xi})$.

This paper is organized as follows. Section 2 is devoted to the existence of traveling wave front for the nonlocal diffusion system with delayed nonlocal response by using the super-sub solution method and the Schauder's fixed point theorem for $c > c_*$ and using a limiting argument for $c = c_*$. In Section 3, we find a priori asymptotic behavior of traveling waves with the help of Ikehara's Theorem by constructing a Laplace transform presentation of a solution for a class of the nonlocal diffusion system. In Section 4, the traveling wave front obtained in Section 1 is unique up to a translation by using the technique in [5, 6]. Last Section, we apply our results to another version of the classical Logistic model and the Nicholson's blowflies model with delayed nonlocal response.

2. Existence of Traveling Wave Fronts

Let

$$C_{[0,\,K]}(\mathbb{R},\,\mathbb{R}) = \{\phi \in (\mathbb{R},\,\mathbb{R})\,|0 \le \phi(\xi) \le K, \xi \in \mathbb{R}\}\,.$$

Define the operator $T : C_{[0,\,K]}(\mathbb{R},\,\mathbb{R}) \to C_{[0,\,K]}(\mathbb{R},\,\mathbb{R})$ by

$$T(\phi)(\xi) = \frac{1}{c} e^{-\frac{\beta}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{\beta}{c}y} H(\phi)(y)dy, \tag{2.1}$$

where

$$H[\phi](\xi) = dJ{*}\phi(\xi) + (\beta - d)\phi(\xi) + f\Big(\phi(\xi),\, \int_{\mathbb{R}} k(y)\phi(\xi - c\tau - y)dy\Big), \quad \xi \in \mathbb{R}. \tag{2.2}$$

By (H1)-(H2) and (A1), T is well defined. It is easy to see that a fixed point ϕ of T or a solution of the equation

$$\phi(\xi) = T(\phi)(\xi), \quad \xi \in \mathbb{R} \tag{2.3}$$

is a traveling wave solution of (1.8).

Since $\partial_2 f(u,\,v) \ge 0$ for $(u,\,v) \in [0,\,K]^2$, it is easy to see that the function f satisfies the following quasimonotone condition.

Lemma 2.1. *Assume that (H2) and (A1) hold. Then there is a positive constant* $\beta > \max_{(u,\,v)\in[0,\,K]^2} |\partial_1 f(u,\,v)| + d$ *such that*

$$f\Big(\phi_1(\xi),\, \int_{\mathbb{R}} k(y)\phi_1(\xi - c\tau - y)dy\Big) - f\Big(\phi_2(\xi),\, \int_{\mathbb{R}} k(y)\phi_2(\xi - c\tau - y)dy\Big) + (\beta - d)(\phi_1(\xi) - \phi_2(\xi)) \ge 0$$

where $\phi_1,\, \phi_2 \in C(\mathbb{R},\,\mathbb{R})$ *with* $0 \le \phi_2(\xi) \le \phi_1(\xi) \le K$ *for* $\xi \in \mathbb{R}$.

Now we introduce the concept of upper and lower solutions of the integral equation (2.3).

Definition 2.1. A continuous bounded function ϕ is called an upper solution of (2.3) if

$$T(\phi)(\xi) \leq \phi(\xi), \quad \text{for all } \xi \in \mathbb{R}. \tag{2.4}$$

A lower solution of (2.3) is defined in a similar way by reversing the inequality in (2.4).

Define a function

$$\begin{aligned} &\Delta(c,\, \lambda) \\ &= c\lambda - d\left[\int_{\mathbb{R}} e^{-\lambda y} J(y)dy - 1\right] - \partial_1 f(0,\, 0) - \partial_2 f(0,\, 0) e^{-c\tau\lambda} \int_{\mathbb{R}} k(y) e^{-\lambda y} dy. \end{aligned} \tag{2.5}$$

It is easily seen that the following lemma holds.

Lemma 2.2. *Assume that* (H1)-(H2) *and* (A1)-(A3) *hold. Then there exists a unique* $c_* > 0$ *such that*

(*i*) *if* $c \geq c_*$, *then there exist two positive numbers* λ_1 *and* λ_2 *with* $\lambda_1 \leq \lambda_2$ *such that*

$$\Delta(c,\, \lambda_1) = \Delta(c,\, \lambda_2) = 0;$$

(*ii*) *if* $c < c_*$, *then* $\Delta(c,\, \lambda) < 0$ *for all* $\lambda \geq 0$;

(*iii*) *if* $c = c_*$, *then* $\lambda_1 = \lambda_2 = \lambda_*$, *and if* $c > c_*$, *then* $\lambda_1 < \lambda_* < \lambda_2$ *and*

$$\Delta(c,\, .) > 0 \quad \text{in } (\lambda_1,\, \lambda_2)), \quad \Delta(c,\, .) < 0 \quad \text{in } \mathbb{R} \setminus [\lambda_1,\, \lambda_2].$$

For the above constants $c > c_*$ and λ_1, λ_2 given in Lemma 2.2, we define the following continuous functions:

$$\overline{\phi}(\xi) := \min\{K,\, e^{\lambda_1 \xi}\}, \quad \xi \in \mathbb{R}, \tag{2.6}$$

and

$$\underline{\phi}(\xi) := \max\{0,\, e^{\lambda_1 \xi} - q e^{\gamma \lambda_1 \xi}\}, \quad \xi \in \mathbb{R}, \tag{2.7}$$

where $\gamma \in (1, \min\{1 + \sigma_1, 1 + \sigma_2, \frac{\lambda_2}{\lambda_1}\})$. Clearly, for sufficiently large q, we have $0 \leq \underline{\phi}(\xi) \leq \overline{\phi}(\xi) \leq K$ and $\underline{\phi}(\xi) \not\equiv 0$ for $\xi \in \mathbb{R}$.

Lemma 2.3. *Assume that* (H1)-(H2) *and* (A1)-(A3) *hold. Then for* $c > c_*$, $\overline{\phi}(\xi)$ *and* $\underline{\phi}(\xi)$ *are an upper solution and a lower solution of* (2.3)*, respectively.*

Proof. Since $0 \le \overline{\phi}(\xi) \le K$ for $\xi \in \mathbb{R}$, it follows from Lemma 2.1 that

$$\begin{aligned} H(\overline{\phi})(\xi) &= d\int_{\mathbb{R}} J(y)\overline{\phi}(\xi - y)dy + (\beta - d)\overline{\phi}(\xi) + f\Big(\overline{\phi}(\xi),\, \int_{\mathbb{R}} k(y)\overline{\phi}(\xi - c\tau - y)dy\Big) \\ &\le dK + (\beta - d)K + f(K, K) = \beta K \end{aligned}$$

and

$$T(\overline{\phi})(\xi) = \frac{1}{c}e^{-\frac{\beta}{c}\xi}\int_{-\infty}^{\xi} e^{\frac{\beta}{c}y}H(\overline{\phi})(y)dy \le \frac{1}{c}e^{-\frac{\beta}{c}\xi}\int_{-\infty}^{\xi} e^{\frac{\beta}{c}y}\beta K dy = K. \tag{2.8}$$

On the other hand, noting that $f(u, v) \le \partial_1 f(0, 0)u + \partial_2 f(0, 0)v$ for $u, v \in [0, K]$ and $0 \le \overline{\phi}(\xi) \le e^{\lambda_1 \xi}$ for $\xi \in \mathbb{R}$, it follows from $\Delta(c, \lambda_1) = 0$ and $\beta > \max_{(u,\, v)\in[0,\, K]^2} |\partial_1 f(u,\, v)| + d$ that

$$\begin{aligned} H(\overline{\phi})(\xi) \le\, & d\int_{\mathbb{R}} J(y)\overline{\phi}(\xi - y)dy + (\beta - d)\overline{\phi}(\xi) + \partial_1 f(0, 0)\overline{\phi}(\xi) \\ & + \partial_2 f(0, 0)\int_{\mathbb{R}} k(y)\overline{\phi}(\xi - c\tau - y)dy \\ \le\, & e^{\lambda_1\xi}\Big[d\int_{\mathbb{R}} J(y)e^{-\lambda_1 y}dy + (\beta - d + \partial_1 f(0, 0)) \\ & + \partial_2 f(0, 0)\int_{\mathbb{R}} k(y)e^{-\lambda_1(c\tau + y)}dy\Big] \\ =\, & e^{\lambda_1\xi}(c\lambda_1 + \beta) \end{aligned}$$

and

$$\begin{aligned} & T(\overline{\phi})(\xi) \\ & = \frac{1}{c}e^{-\frac{\beta}{c}\xi}\int_{-\infty}^{\xi} e^{\frac{\beta}{c}y}H(\overline{\phi})(y)dy \le \frac{1}{c}e^{-\frac{\beta}{c}\xi}\int_{-\infty}^{\xi} e^{\frac{\beta}{c}y}e^{\lambda_1 y}(c\lambda_1 + \beta)dy = e^{\lambda_1\xi}. \end{aligned} \tag{2.9}$$

According to the definition of $\overline{\phi}(\xi)$ and (2.8) and (2.9), it is obvious to see that

$$T(\overline{\phi})(\xi) \le \overline{\phi}(\xi), \quad \text{for all } \xi \in \mathbb{R}.$$

Thus, $\overline{\phi}(\xi)$ is an upper solution of (2.3).

Letting $\xi_0 = -\frac{\ln q}{\lambda_1(\gamma - 1)}$, then we have

$$\underline{\phi}(\xi) = 0 \quad \text{for } \xi \ge \xi_0 \quad \text{and} \quad \underline{\phi}(\xi) = e^{\lambda_1\xi} - qe^{\gamma\lambda_1\xi} \quad \text{for } \xi \le \xi_0.$$

Obviously, it follows from Lemma 2.1 that

$$T(\underline{\phi})(\xi) \ge 0 \quad \text{for } \xi \in \mathbb{R}. \tag{2.10}$$

By (A3), we have

$$f(u,\,v) \geq \partial_1 f(0,\,0)u + \partial_2 f(0,\,0)v - L(u^{1+\sigma_1} + v^{1+\sigma_2}) \text{ for any}(u,\,v) \in [0,\,\kappa]^2. \tag{2.11}$$

It is easily seen that there exists $Q_1(\gamma) \gg 1$ such that

$$e^{\lambda_1 \xi} - q e^{\gamma\lambda_1 \xi} \leq \kappa \quad \text{for } q \geq Q_1(\gamma).$$

Therefore, $0 \leq \underline{\phi}(\xi) \leq \kappa$. Since $\xi_0 < 0$ and $1 + \sigma_i > \gamma$, $i = 1, 2$, it is easy to see that

$$\overline{\phi}(\xi) \geq \underline{\phi}(\xi) \geq e^{\lambda_1 \xi} - q e^{\gamma\lambda_1 \xi} \quad \text{for } \xi \in \mathbb{R} \tag{2.12}$$

and

$$[\underline{\phi}(\xi)]^{1+\sigma_i} \leq e^{\gamma\lambda_1 \xi} \quad \text{for } \xi \in \mathbb{R},\, i = 1, 2. \tag{2.13}$$

Thus, according to (2.11)-(2.13), we can obtain

$$\begin{aligned}\left[\int_{\mathbb{R}} k(y)\underline{\phi}(\xi - c\tau - y)dy\right]^{1+\sigma_2} &= \left\{\int_{\mathbb{R}} k(y)\left[\left(\underline{\phi}(\xi - c\tau - y)\right)^{1+\sigma_2}\right]^{\frac{1}{1+\sigma_2}} dy\right\}^{1+\sigma_2} \\ &\leq e^{\gamma\lambda_1 \xi}\left[\int_{\mathbb{R}} k(y) e^{-\frac{\gamma\lambda_1}{1+\sigma_2}(c\tau + y)} dy\right]^{1+\sigma_2}\end{aligned}$$

and

$$\begin{aligned}&H(\underline{\phi})(\xi) \\ &= dJ * \underline{\phi}(\xi) + (\beta - d)\underline{\phi}(\xi) + f\Big(\underline{\phi}(\xi),\, \int_{\mathbb{R}} k(y)\underline{\phi}(\xi - c\tau - y)dy\Big) \\ &\geq dJ * \underline{\phi}(\xi) + (\beta - d)\underline{\phi}(\xi) + \partial_1 f(0,\,0)\underline{\phi}(\xi) + \partial_2 f(0,\,0)\int_{\mathbb{R}} k(y)\underline{\phi}(\xi - c\tau - y)dy \\ &\quad - L[\underline{\phi}(\xi)]^{1+\sigma_1} - L\left[\int_{\mathbb{R}} k(y)\underline{\phi}(\xi - c\tau - y)dy\right]^{1+\sigma_2} \\ &\geq d\int_{\mathbb{R}} J(y)\Big(e^{\lambda_1(\xi - y)} - q e^{\gamma\lambda_1(\xi - y)}\Big)dy + (\beta - d + \partial_1 f(0,\,0))\Big(e^{\lambda_1 \xi} - q e^{\gamma\lambda_1 \xi}\Big) \\ &\quad + \partial_2 f(0,\,0)\int_{\mathbb{R}} k(y)\Big(e^{\lambda_1(\xi - c\tau - y)} - q e^{\gamma\lambda_1(\xi - c\tau - y)}\Big)dy \\ &\quad - L e^{\gamma\lambda_1 \xi} - L e^{\gamma\lambda_1 \xi}\left[\int_{\mathbb{R}} k(y) e^{-\frac{\gamma\lambda_1}{1+\sigma_2}(c\tau + y)} dy\right]^{1+\sigma_2} \\ &= (c\lambda_1 + \beta)e^{\lambda_1 \xi} - (c\gamma\lambda_1 + \beta)q e^{\gamma\lambda_1 \xi} \\ &\quad + e^{\gamma\lambda_1 \xi}\left\{\Delta(c, \gamma\lambda_1)q - L - L\left[\int_{\mathbb{R}} k(y) e^{-\frac{\gamma\lambda_1}{1+\sigma_2}(c\tau + y)} dy\right]^{1+\sigma_2}\right\}.\end{aligned}$$

Therefore, choosing $q \geq \max\{Q_1(\gamma), Q_2(\gamma)\}$, where

$$Q_2(\gamma) := \frac{L + L\left[\int_{\mathbb{R}} k(y) e^{-\frac{\gamma\lambda_1}{1+\sigma_2}(c\tau+y)} dy\right]^{1+\sigma_2}}{\Delta(c, \gamma\lambda_1)},$$

we have

$$H[\underline{\phi}](\xi) \geq (c\lambda_1 + \beta)e^{\lambda_1 \xi} - (c\gamma\lambda_1 + \beta)q e^{\gamma\lambda_1 \xi}. \tag{2.14}$$

It follows from (2.14) that

$$\begin{aligned} &T(\underline{\phi})(\xi) \\ &\geq \frac{1}{c} e^{-\frac{\beta}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{\beta}{c}y} \Big[(c\lambda_1+\beta)e^{\lambda_1 y} - (c\gamma\lambda_1+\beta)q e^{\gamma\lambda_1 y}\Big] dy = e^{\lambda_1\xi} - q e^{\gamma\lambda_1\xi}. \end{aligned} \tag{2.15}$$

According to (2.10), (2.15) and the definition of $\underline{\phi}(\xi)$, we have

$$T(\underline{\phi})(\xi) \geq \underline{\phi}(\xi), \quad \text{for all } \xi \in \mathbb{R}.$$

Thus, $\underline{\phi}(\xi)$ is a lower solution of (2.3). This completes the proof. ■

In the following, we introduce the exponential decay norm. For $0 < \lambda < \lambda_1$, define

$$B_\lambda(\mathbb{R}, \mathbb{R}) = \{\phi : \ \phi(\xi) \in C(\mathbb{R}, \mathbb{R}) \text{ and} \sup_{\xi\in\mathbb{R}} |\phi(\xi)| e^{-\lambda\xi} < \infty\}.$$

It is easy to check that $B_\lambda(\mathbb{R}, \mathbb{R})$ is a Banach space equipped with the norm $\| \cdot \|_\lambda$ defined by $\| \phi \|_\lambda = \sup_{\xi\in\mathbb{R}} |\phi(\xi)| e^{-\lambda\xi}$ for $\phi \in B_\lambda(\mathbb{R}, \mathbb{R})$.

Let $\overline{\phi}(\xi)$ and $\underline{\phi}(\xi)$ be given above and define the set Γ by

$$\Gamma := \left\{ \phi \in C_{[0,K]}(\mathbb{R}, \mathbb{R}) \;\middle|\; \begin{array}{l} \text{(i) } \phi(\xi) \text{ is nondecreasing on } \mathbb{R}; \\ \text{(ii) } \underline{\phi}(\xi) \leq \phi(\xi) \leq \overline{\phi}(\xi) \text{ for all } \xi \in \mathbb{R}. \end{array} \right\}$$

It is obvious that Γ is nonempty, convex and compact in $B_\lambda(\mathbb{R}, \mathbb{R})$. For the operator T defined by (2.1), we have the following lemma.

Lemma 2.4. *Assume that* (H1)-(H2) *and* (A1)-(A3) *hold. Then we have*

(1) $T(\Gamma) \subset \Gamma$*;*

(2) $T : \Gamma \to \Gamma$ *is completely continuous with respect to the norm* $\| \cdot \|_{B_\lambda}$ *in* B_λ.

Proof. According to Lemma 2.1 and Lemma 2.3, it is easily seen that for any $\phi \in \Gamma$,

$$\underline{\phi}(\xi) \leq T(\underline{\phi})(\xi) \leq T(\phi)(\xi) \leq T(\overline{\phi})(\xi) \leq \overline{\phi}(\xi),$$

and $T(\phi)(\xi)$ is nondecreasing on $\mathbb{R}$. Thus, $T(\Gamma)\subset\Gamma$.

Since $f\in C^1([0,\,K]^2,\mathbb{R})$, there exists $M>0$ such that $|f(u_1,v_1)-f(u_2,v_2)|\le M(|u_1-u_2|+|v_1-v_2|)$ for $u_i,v_i\in[0,K], i=1,2$. Thus, for $\phi,\ \psi\in\Gamma$, we obtain

$$\begin{aligned}
&|H(\phi)(\xi)-H(\psi)(\xi)|\\
\le\,& d\int_{\mathbb{R}}J(y)|\phi(\xi-y)-\psi(\xi-y)|dy+(\beta-d+M)|\phi(\xi)-\psi(\xi)|\\
&+M\int_{\mathbb{R}}k(y)|\phi(\xi-c\tau-y)-\psi(\xi-c\tau-y)|dy\\
\le\,& e^{\lambda\xi}\left[d\int_{\mathbb{R}}J(y)e^{-\lambda y}dy+\beta-d+M+M\int_{\mathbb{R}}k(y)e^{-\lambda(y+c\tau)}dy\right]\|\phi-\psi\|_{B_\lambda}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\|T(\phi)-T(\psi)\|_{B_\lambda}\\
=\,&\sup_{\xi\in\mathbb{R}}|T(\phi)(\xi)-T(\psi)(\xi)|e^{-\lambda\xi}\\
\le\,&\sup_{\xi\in\mathbb{R}}e^{\frac{-(c\lambda+\beta)\xi}{c}}\frac{1}{c}\int_{-\infty}^{\xi}e^{\frac{(c\lambda+\beta)y}{c}}|H(\phi)(y)-H(\psi)(y)|e^{-\lambda y}dy\\
\le\,&\frac{1}{c\lambda+\beta}\left[d\int_{\mathbb{R}}J(y)e^{-\lambda y}dy+\beta-d+M+M\int_{\mathbb{R}}k(y)e^{-\lambda(y+c\tau)}dy\right]\|\phi-\psi\|_{B_\lambda},
\end{aligned}$$

which implies that $T:\Gamma\to\Gamma$ is continuous. On the other hand, for any $\phi\in\Gamma$, $\xi\in\mathbb{R}$, we have

$$H(\phi)(\xi)\le\beta K$$

and for $\xi_1\ge\xi_2,\ \xi_1,\xi_2\in\mathbb{R}$,

$$\begin{aligned}
&|T(\phi)(\xi_1)-T(\phi)(\xi_2)|\\
\le\,&\frac{\beta}{c}K\Big[e^{-\frac{\beta}{c}\xi_1}\int_{-\infty}^{\xi_1}e^{\frac{\beta}{c}y}dy-e^{-\frac{\beta}{c}\xi_2}\int_{-\infty}^{\xi_2}e^{\frac{\beta}{c}y}dy\Big]\\
\le\,&\frac{\beta}{c}K\Big[e^{-\frac{\beta}{c}\xi_1}\int_{\xi_2}^{\xi_1}e^{\frac{\beta}{c}y}dy+\Big|e^{-\frac{\beta}{c}\xi_2}-e^{-\frac{\beta}{c}\xi_1}\Big|\int_{-\infty}^{\xi_2}e^{\frac{\beta}{c}y}dy\Big]\\
=\,&2K\Big[1-e^{-\frac{\beta}{c}(\xi_1-\xi_2)}\Big].
\end{aligned}$$

which imply that $\{T(\phi)(\xi):\ \phi\in\Gamma\}$ is uniformly bounded and equicontinuous in $\xi\in\mathbb{R}$. Thus, by Arzela-Ascoli theorem, for any given sequence $\{\psi_n\}_{n\in\mathbb{N}^+}$ in $T(\Gamma)$, there exist $n_k\to\infty$ and $\psi\in C(\mathbb{R},\mathbb{R})$ such that $\lim_{k\to\infty}\psi_{n_k}(\xi)=\psi(\xi)$ uniformly for ξ in any compact subset of $\mathbb{R}$. Since $\underline{\phi}(\xi)\le\psi_{n_k}(\xi)\le\overline{\phi}(\xi)$ for any $\xi\in\mathbb{R}$, we have $\underline{\phi}(\xi)\le\psi(\xi)\le\overline{\phi}(\xi)$ for any $\xi\in\mathbb{R}$, and therefore $\psi(\xi)\in\Gamma$. Note that

$$\lim_{\xi\to\pm\infty}(\overline{\phi}(\xi)-\underline{\phi}(\xi))e^{-\lambda\xi}=0.$$

Thus, for any $\epsilon > 0$, we can find $M_0 > 0$ such that

$$|\psi_{n_k}(\xi) - \psi(\xi)|e^{-\lambda\xi} \leq (\overline{\phi}(\xi) - \underline{\phi}(\xi))e^{-\lambda\xi} < \epsilon \quad \text{for any } |\xi| > M_0.$$

Since $\lim_{k\to\infty}(\psi_{n_k}(\xi) - \psi(\xi))e^{-\lambda\xi} = 0$ uniformly for $\xi \in [-M_0, M_0]$, there exists $K' > 0$ such that for $k \geq K'$,

$$|\psi_{n_k}(\xi) - \psi(\xi)|e^{-\lambda\xi} < \epsilon \quad \text{for any } |\xi| \leq M_0.$$

It follows that $\|\psi_{n_k} - \psi\|_{B_\lambda} \to 0$ as $k \to \infty$. Thus, we can see that $T : \Gamma \to \Gamma$ is compact with respect to the norm $\|\cdot\|_{B_\lambda}$ in B_λ. This completes the proof. ■

Proof of Theorem 1.1. For $c > c_*$, it follows from Lemma 3.3 and the Schauder's fixed point theorem that T has a fixed point $\phi(\xi)$ in Γ. Since $\phi(\xi)$ is nondecreasing and bounded, $l =: \lim_{\xi\to\infty}\phi(\xi) > 0$ exists. By L. Hopital's rule and (A1), we can obtain $l = K$. Since $\max\{0, e^{\lambda_1\xi} - qe^{\gamma\lambda_1\xi}\} \leq \phi(\xi) \leq \min\{K, e^{\lambda_1\xi}\}, \quad \xi \in \mathbb{R}$, it follows that $\lim_{\xi\to-\infty}\phi(\xi) = 0$ and

$$\lim_{\xi\to-\infty}|\phi(\xi)e^{-\lambda_1\xi} - 1| \leq \lim_{\xi\to-\infty} qe^{(\gamma-1)\lambda_1\xi} = 0$$

which implies that

$$\lim_{\xi\to-\infty}\phi(\xi)e^{-\lambda_1\xi} = 1. \tag{2.16}$$

According to (2.11), (2.16) and $\lim_{\xi\to-\infty}\phi(\xi) = 0$, it is easily seen that

$$\begin{aligned}
&\lim_{\xi\to-\infty}\Big|f\Big(\phi(\xi), \int_{\mathbb{R}} k(y)\phi(\xi - c\tau - y)dy\Big) - \partial_1 f(0,0)\phi(\xi) \\
&\quad - \partial_2 f(0,0)\int_{\mathbb{R}} k(y)\phi(\xi - c\tau - y)dy\Big|e^{-\lambda_1\xi} \\
&\leq \lim_{\xi\to-\infty} L\Big\{[\phi(\xi)]^{1+\sigma_1} + \Big(\int_{\mathbb{R}} k(y)\phi(\xi - c\tau - y)dy\Big)^{1+\sigma_2}\Big\}e^{-\lambda_1\xi} = 0
\end{aligned}$$

and

$$\lim_{\xi\to-\infty} e^{-\lambda_1\xi}\int_{\mathbb{R}}\phi(\xi - y)J(y)dy = \int_{\mathbb{R}} e^{-\lambda_1 y}J(y)dy.$$

Therefore,

$$\begin{aligned}
&\lim_{\xi\to-\infty}\phi'(\xi)e^{-\lambda_1\xi} \\
&= \frac{1}{c}\lim_{\xi\to-\infty}\left\{d[J * \phi(\xi) - \phi(\xi)] + f\left(\phi(\xi), \int_{\mathbb{R}} k(y)\phi(\xi - c\tau - y)dy\right)\right\}e^{-\lambda_1\xi} \\
&= \frac{1}{c}\left\{d\left[\int_{\mathbb{R}} e^{-\lambda_1 y}J(y)dy - 1\right] + \partial_1 f(0,0) + \partial_2 f(0,0)\int_{\mathbb{R}} k(y)e^{-(c\tau+y)\lambda_1}dy\right\} \\
&= \lambda_1.
\end{aligned}$$

For $c = c_*$, it could be obtained by a limiting argument similar to that of [24, 29]. We omit the details.

For $c \geq c_*$, we can obtain $\phi(\xi) > 0$ for $\xi \in \mathbb{R}$. Indeed, note that $0 \leq \phi(\xi) \leq K$ for $\xi \in \mathbb{R}$. If $\phi(\xi_0) = 0$, $\phi(\xi) = 0$ for $\xi < \xi_0$ since $\phi(\xi)$ is nondecreasing. By (1.9), for $\xi < \xi_0$, we have $c\phi'(\xi) = dJ * \phi(\xi) > 0$, which is a contradiction. This completes the proof of Theorem 1.1. ∎

3. Asymptotic Behavior of Traveling Waves

In this section, we will find a priori estimate and asymptotic behavior of any nonnegative traveling wave solution with the help of Ikehara's Theorem.

We recall a version of Ikehara's Theorem.

Lemma 3.1. ([4], Proposition 2.3). *Let $l(\lambda) = \int_0^{+\infty} u(x)e^{-\lambda x}dx$, with u being a positive decreasing function. Assume that $l(\lambda)$ has the representation*

$$l(\lambda) = \frac{h(\lambda)}{(\lambda + \alpha)^{k+1}},$$

where $k > -1$ and h is analytic in the strip $-\alpha \leq \Re\lambda < 0$. Then

$$\lim_{x\to\infty} \frac{u(x)}{x^k e^{-\alpha x}} = \frac{h(-\alpha)}{\Gamma(\alpha + 1)}.$$

In what follows, we assume that $\widetilde{\phi}(x + ct)$ is any nonnegative bounded traveling wave of (1.8) with $\widetilde{\phi}(-\infty) = 0$ and $\widetilde{\phi} \not\equiv 0$. Then we can obtain the following results.

Lemma 3.2. *Assume that (H1)-(H2) and (A1) hold. Then the function $\widetilde{\phi}(\xi)$ is strictly positive for $\xi \in \mathbb{R}$.*

Proof. Suppose on the contrary that there exists $\xi_1 \in \mathbb{R}$ such that $\widetilde{\phi}(\xi_1) = 0$. Since $\widetilde{\phi}$ is a nonnegative bounded traveling wave with $\widetilde{\phi}(-\infty) = 0$ and $\widetilde{\phi} \not\equiv 0$, $\xi_0 := \sup\{\xi \in \mathbb{R} \,|\, \widetilde{\phi}(\xi) = 0\}$ is well defined and $\widetilde{\phi}(\xi_0) = \widetilde{\phi}'(\xi_0) = 0$. Thus,

$$\begin{aligned} 0 = c\widetilde{\phi}'(\xi_0) &= d[J * \widetilde{\phi}(\xi_0) - \widetilde{\phi}(\xi_0)] + f\Big(\widetilde{\phi}(\xi_0), \int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi_0 - c\tau - y)dy\Big) \\ &= d\int_{\mathbb{R}} J(y)\widetilde{\phi}(\xi_0 - y)dy + f\Big(0, \int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi_0 - c\tau - y)dy\Big) \\ &\geq d\int_{\mathbb{R}} J(y)\widetilde{\phi}(\xi_0 - y)dy + f(0,0) = d\int_{\mathbb{R}} J(y)\widetilde{\phi}(\xi_0 - y)dy. \end{aligned}$$

which implies that $\widetilde{\phi}(\xi_0 - y) = 0$ a.e. on $\mathbb{R}$. This contradict the definition of ξ_0. This completes the proof. ∎

Lemma 3.3. *Assume that (H1)-(H2) and (A1)-(A3) hold. Then* $\int_{-\infty}^{\xi} \widetilde{\phi}(\theta)d\theta < \infty$ *for any* $\xi \in \mathbb{R}$.

Proof. Let $\varpi_1 = \partial_1 f(0,0) + \partial_2 f(0,0) > 0$ and $\varpi_2 = \partial_2 f(0,0) - \partial_1 f(0,0)$. Since $\lim_{\xi\to-\infty} \widetilde{\phi}(\xi) = 0$, there exists $\xi' < 0$ such that for any $\xi < \xi'$,

$$\begin{aligned}&\frac{\varpi_1}{4}\left(\widetilde{\phi}(\xi) + \int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi - c\tau - y)dy\right)\\ &> L\left([\widetilde{\phi}(\xi)]^{1+\sigma_1} + \left[\int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi - c\tau - y)dy\right]^{1+\sigma_2}\right).\end{aligned}$$

Then for any $\xi < \xi'$, we have

$$\begin{aligned}
&c\widetilde{\phi}'(\xi)\\
&= d(J * \widetilde{\phi}(\xi) - \widetilde{\phi}(\xi)) + f\Big(\widetilde{\phi}(\xi), \int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi - c\tau - y)dy\Big)\\
&\geq d(J * \widetilde{\phi}(\xi) - \widetilde{\phi}(\xi)) + \partial_1 f(0,0)\widetilde{\phi}(\xi) + \partial_2 f(0,0)\int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi - c\tau - y)dy\\
&\quad -L[\widetilde{\phi}(\xi)]^{1+\sigma_1} - L\left[\int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi - c\tau - y)dy\right]^{1+\sigma_2}\\
&= d(J * \widetilde{\phi}(\xi) - \widetilde{\phi}(\xi)) - L\left([\widetilde{\phi}(\xi)]^{1+\sigma_1} + \left[\int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi - c\tau - y)dy\right]^{1+\sigma_2}\right)\\
&\quad +\frac{\varpi_1}{4}\widetilde{\phi}(\xi) + \frac{\varpi_2}{2}\left(\int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi - c\tau - y)dy - \widetilde{\phi}(\xi)\right)\\
&\quad +\frac{\varpi_1}{4}\int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi - c\tau - y)dy + \frac{\varpi_1}{4}\left(\int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi - c\tau - y)dy + \widetilde{\phi}(\xi)\right)\\
&\geq d(J * \widetilde{\phi}(\xi) - \widetilde{\phi}(\xi)) + \frac{\varpi_1}{4}\widetilde{\phi}(\xi) + \frac{\varpi_2}{2}\left(\int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi - c\tau - y)dy - \widetilde{\phi}(\xi)\right)\\
&\quad +\frac{\varpi_1}{4}\int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi - c\tau - y)dy.
\end{aligned} \tag{3.1}$$

According to Fubini's theorem and Lebesgue's dominated convergent theorem, we have

$$\begin{aligned}
\int_{\eta}^{\xi}(J * \widetilde{\phi}(\theta) - \widetilde{\phi}(\theta))d\theta &= \int_{\eta}^{\xi}\left(\int_{\mathbb{R}} J(\theta)(\widetilde{\phi}(\theta - \vartheta) - \widetilde{\phi}(\vartheta))d\theta\right)d\vartheta\\
&= -\int_{\eta}^{\xi}\left(\int_{\mathbb{R}} J(\theta)\theta\int_0^1 \widetilde{\phi}'(\vartheta - t\theta)dtd\vartheta\right)d\theta\\
&= -\int_{\mathbb{R}} J(\theta)\theta\int_0^1 (\widetilde{\phi}(\xi - t\theta) - \widetilde{\phi}(\eta - t\theta))dtd\theta\\
&\to -\int_{\mathbb{R}} J(\theta)\theta\int_0^1 \widetilde{\phi}(\xi - t\theta)dtd\theta
\end{aligned} \tag{3.2}$$

as $\eta \to -\infty$, and

$$\begin{aligned}
&\int_\eta^\xi \left(\int_{\mathbb{R}} k(y)\widetilde{\phi}(\theta - c\tau - y)dy - \widetilde{\phi}(\theta)\right) d\theta \\
&= \int_\eta^\xi \left(\int_{\mathbb{R}} k(y)[\widetilde{\phi}(\theta - c\tau - y) - \widetilde{\phi}(\theta)]dy\right) d\theta \\
&= -\int_\eta^\xi \left(\int_{\mathbb{R}} k(y)(c\tau + y)\int_0^1 \widetilde{\phi}'(\theta - (c\tau + y)t)dtdy\right) d\theta \qquad (3.3)\\
&= -\int_{\mathbb{R}} k(y)(c\tau + y)\left(\int_0^1 [\widetilde{\phi}(\xi - (c\tau + y)t) - \widetilde{\phi}(\eta - (c\tau + y)t)]dt\right) dy \\
&\to -\int_{\mathbb{R}} \left(k(y)(c\tau + y)\int_0^1 \widetilde{\phi}(\xi - (c\tau + y)t)dt\right) dy
\end{aligned}$$

$\eta \to -\infty$. Integrating (3.1) from $-\infty$ to ξ, according to (3.2) and (3.3), then for any $\xi < \xi'$,

$$\begin{aligned}
&c\widetilde{\phi}(\xi)+\frac{\omega_2}{2}\int_{\mathbb{R}} \left(k(y)(c\tau + y)\int_0^1 \widetilde{\phi}(\xi-(c\tau+y)t)dt\right) dy+d\int_{\mathbb{R}} J(\theta)\theta\int_0^1 \widetilde{\phi}(\xi-t\theta)dtd\theta \\
&\geq \frac{\varpi_1}{4}\int_{-\infty}^\xi \widetilde{\phi}(\theta)d\theta + \frac{\varpi_1}{4}\int_{-\infty}^\xi \left(\int_{\mathbb{R}} k(y)\widetilde{\phi}(\theta - c\tau - y)dy\right) d\theta \\
&\geq \frac{\varpi_1}{4}\int_{-\infty}^\xi \widetilde{\phi}(\theta)d\theta.
\end{aligned}$$

Thus, we can obtain that $\int_{-\infty}^\xi \widetilde{\phi}(\theta)d\theta<\infty$ for any $\xi\in\mathbb{R}$. This completes the proof. ■

Lemma 3.4. *Assume that (H1)-(H2) and (A1)-(A3) hold. Then there exists a positive constant ϱ such that $\widetilde{\phi}(\xi) = O(e^{\varrho\xi})$ as $\xi\to-\infty$. Moreover,* $\sup_{\xi\in\mathbb{R}} \widetilde{\phi}(\xi)e^{-\varrho\xi}<\infty$.

Proof. Letting $V(\xi) = \int_{-\infty}^\xi \widetilde{\phi}(\theta)d\theta$, it is easily seen that $V(\xi) > 0$ is nondecreasing and $\lim_{\xi\to-\infty} V(\xi)=0$. First, we claim that $\int_{-\infty}^\xi V(\theta)d\theta < +\infty$ for any $\xi \in \mathbb{R}$.

If $\omega_2 \geq 0$, for any $\xi < \xi'$, ξ' given in Lemma 3.3, integrating (3.1) from $-\infty$ to ξ, we have

$$\begin{aligned}
c\widetilde{\phi}(\xi) &\geq d(J * V(\xi) - V(\xi)) + \frac{\varpi_1}{4}V(\xi) \\
&\quad+\frac{\varpi_2}{2}(k * V(\xi - c\tau) - V(\xi)) + \frac{\varpi_1}{4}k * V(\xi - c\tau).
\end{aligned} \qquad (3.4)$$

Integrating (3.4) from $-\infty$ to ξ, it follows that

$$cV(\xi) \geq d\int_{-\infty}^{\xi}(J*V(\theta)-V(\theta))d\theta+\frac{\varpi_1}{4}\int_{-\infty}^{\xi}V(\theta)d\theta \tag{3.5}$$
$$+\frac{\varpi_2}{2}\int_{-\infty}^{\xi}(k*V(\theta-c\tau)-V(\theta))d\theta+\frac{\varpi_1}{4}\int_{-\infty}^{\xi}k*V(\theta-c\tau)d\theta.$$

Letting $Q(\xi)=\int_{-\infty}^{\xi}J(\theta)d\theta$, it is easily seen that $Q(-\xi)=1-Q(\xi)$. Since $V(\xi)$ is nondecreasing, it follows that

$$\int_{-\infty}^{\xi}J*V(\theta)=\int_{-\infty}^{\xi}V(\theta+\xi)Q(\theta)d\theta$$

and

$$\begin{aligned}\int_{0}^{+\infty}V(\xi+\theta)Q(\theta)d\theta &\geq \int_{0}^{+\infty}V(\xi-\theta)Q(\theta)d\theta\\ &=\int_{-\infty}^{0}V(\xi+\theta)Q(-\theta)d\theta\\ &=\int_{-\infty}^{0}V(\xi+\theta)(1-Q(\theta))d\theta\end{aligned}$$

which implies that

$$\int_{-\infty}^{\xi}V(\theta+\xi)Q(\theta)d\theta\geq\int_{-\infty}^{0}V(\xi+\theta)d\theta=\int_{-\infty}^{\xi}V(\theta)d\theta.$$

Thus, we have

$$\int_{-\infty}^{\xi}[J*V(\theta)-V(\theta)]d\theta\geq 0. \tag{3.6}$$

Similarly, we can also obtain

$$\int_{-\infty}^{\xi}[k*V(\theta-c\tau)-V(\theta-c\tau)]d\theta\geq 0. \tag{3.7}$$

By (3.5)-(3.7), we obtain

$$\begin{aligned}cV(\xi)&\geq\frac{\varpi_1}{4}\int_{-\infty}^{\xi}V(\theta)d\theta+\frac{\varpi_2}{2}\int_{-\infty}^{\xi}[V(\theta-c\tau)-V(\theta)]d\theta\\ &=\frac{\varpi_1}{4}\int_{-\infty}^{\xi}V(\theta)d\theta-\frac{\varpi_2 c\tau}{2}\int_{0}^{1}V(\xi-c\tau t)dt,\end{aligned} \tag{3.8}$$

which implies that

$$cV(\xi)\geq\frac{\varpi_1}{4}\int_{-\infty}^{\xi}V(\theta)d\theta-\frac{\varpi_2 c\tau}{2}V(\xi) \tag{3.9}$$

since $V(\xi)$ is nondecreasing. By (3.9), $V(\xi)$ is integrable on $(-\infty, \xi']$. Thus, if $\varpi_2 \geq 0$, then $\int_{-\infty}^{\xi} V(\theta)d\theta < +\infty$ for any $\xi \in \mathbb{R}$.

If $\omega_2 < 0$, then $\partial_1 f(0, 0) > \partial_2 f(0, 0) \geq 0$. Then for any $\xi \leq \xi'$ (sufficiently large $-\xi' > 0$), we have

$$\begin{aligned} c\widetilde{\phi}'(\xi) &= d(J * \widetilde{\phi}(\xi) - \widetilde{\phi}(\xi)) + f\Big(\widetilde{\phi}(\xi), \int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi - c\tau - y)dy\Big) \\ &\geq d(J * \widetilde{\phi}(\xi) - \widetilde{\phi}(\xi)) + f(\widetilde{\phi}(\xi), 0) \\ &\geq d(J * \widetilde{\phi}(\xi) - \widetilde{\phi}(\xi)) + \frac{1}{2}\partial_1 f(0, 0)\widetilde{\phi}(\xi). \end{aligned} \tag{3.10}$$

Thus, it follow from (3.10) that

$$c\widetilde{\phi}(\xi) \geq d(J * V(\xi) - V(\xi)) + \frac{1}{2}\partial_1 f(0, 0)V(\xi). \tag{3.11}$$

Integrating(3.11) from $-\infty$ to ξ, we obtain

$$\begin{aligned} cV(\xi) &\geq d\int_{-\infty}^{\xi} [J * V(\theta) - V(\theta)]d\theta + \frac{1}{2}\partial_1 f(0, 0)\int_{-\infty}^{\xi} V(\theta)d\theta \\ &\geq \frac{1}{2}\partial_1 f(0, 0)\int_{-\infty}^{\xi} V(\theta)d\theta. \end{aligned} \tag{3.12}$$

Therefore, if $\varpi_2 < 0$, then $\int_{-\infty}^{\xi} V(\theta)d\theta < +\infty$ also holds for any $\xi \in \mathbb{R}$.

Furthermore, we can verify that $V(\xi) = O(e^{\varrho\xi})$ as $\xi \to -\infty$.

In fact, for any $r > \frac{2\varpi_2 c\tau + 4c}{\varpi_1} > 0$ and $\varpi_2 \geq 0$, it is clear that

$$\Big(\frac{\varpi_2 c\tau}{2} + c\Big)V(\xi) \geq \frac{\varpi_1}{4}\int_{\xi-r}^{\xi} V(\theta)d\theta - \geq \frac{r\varpi_1}{4}V(\xi - r).$$

For any $r > \frac{(2+c\tau)c}{\varpi_1} > 0$ and $\varpi_2 < 0$, we can obtain

$$cV(\xi) \geq \frac{1}{2}\partial_1 f(0, 0)\int_{\xi-r}^{\xi} V(\theta)d\theta \geq \frac{r\partial_1 f(0, 0)}{2}V(\xi - r).$$

Thus, there exists $r_0 > 0$ and some ρ with $0 < \rho < 1$ such that

$$V(\xi - r_0) \leq \rho V(\xi).$$

Let $g(\xi) = V(\xi)e^{-\varrho\xi}$, where $\varrho = \frac{1}{r_0}\ln\frac{1}{\rho} < \mu$. Then

$$g(\xi - r_0) = V(\xi - r_0)e^{-\varrho(\xi - r_0)} = \frac{1}{\rho}V(\xi - r_0)e^{-\varrho\xi} \leq V(\xi)e^{-\varrho\xi} = g(\xi). \tag{3.13}$$

Noting that $g(\xi)$ is bounded for all $\xi \in [\xi' - r_0, \xi']$, then (3.13) implies that $g(\xi)$ is bounded for all $\xi \leq \xi'$, that is, $V(\xi) = O(e^{\varrho\xi})$ as $\xi \to -\infty$.

Since $V(\xi) = O(e^{\varrho\xi})$ as $\xi \to -\infty$, by (H1)-(H2), then $\int_{\mathbb{R}} J(\theta)V(\xi - \theta)d\theta = O(e^{\varrho\xi})$ as $\xi \to -\infty$ and $\int_{\mathbb{R}} k(\theta)V(\xi - \theta - c\tau)d\theta = O(e^{\varrho\xi})$ as $\xi \to -\infty$. Integrating the first equality of (3.4) from $-\infty$ to ξ and by $\lim_{\xi\to-\infty} \widetilde{\phi}(\xi) = 0$, it follows that

$$\begin{aligned} c\widetilde{\phi}(\xi) =& d\int_{\mathbb{R}} J(\theta)V(\xi-\theta)d\theta - dV(\xi) + \int_{-\infty}^{\xi} f\Big(\widetilde{\phi}(\theta), \int_{\mathbb{R}} k(\theta)V(\xi-\theta-c\tau)d\theta\Big)d\theta \\ \leq& d\int_{\mathbb{R}} J(\theta)V(\xi-\theta)d\theta - dV(\xi) \\ &+ \int_{-\infty}^{\xi} \Big[(\partial_1 f(0,0)\widetilde{\phi}(\theta) + \partial_2 f(0,0)\int_{\mathbb{R}} k(\vartheta)\widetilde{\phi}(\theta-\vartheta-c\tau)d\vartheta\Big]d\theta \\ =& d\int_{\mathbb{R}} J(\theta)V(\xi-\theta)d\theta - dV(\xi) \\ &+ \partial_1 f(0,0)V(\xi) + \partial_2 f(0,0)\int_{\mathbb{R}} k(\vartheta)V(\xi-\vartheta-c\tau)d\vartheta \end{aligned}$$

which implies that $\widetilde{\phi}(\xi) = O(e^{\varrho\xi})$ as $\xi \to -\infty$. Since both $\widetilde{\phi}(\xi)$ and $e^{-\varrho\xi}$ are bounded if $\xi > \xi'$, then $\sup_{\xi\in\mathbb{R}} \widetilde{\phi}(\xi)e^{-\varrho\xi} < \infty$. This completes the proof. ■

Remark 3.1. Lemma 3.4 implies that $\int_{\mathbb{R}} \widetilde{\phi}(\theta)e^{-\lambda\theta}d\theta < \infty$ for any $0 < \Re\lambda < \varrho$.

Proof of Theorem 1.2. For any λ with $0 < \Re\lambda < \varrho$ and using Remark 3.1, we can now define a two-sided Laplace transform of $\widetilde{\phi}$ by

$$\mathbb{L}(\lambda) \equiv \int_{\mathbb{R}} e^{-\lambda\theta}\widetilde{\phi}(\theta)d\theta.$$

Note that

$$\begin{aligned} &\int_{\mathbb{R}} e^{-\lambda\theta} J * \widetilde{\phi}(\theta)d\theta \\ &= \int_{\mathbb{R}} e^{-\lambda\vartheta}J(\vartheta)\int_{\mathbb{R}} \widetilde{\phi}(\theta-\vartheta)e^{-\lambda(\theta-\vartheta)}d\theta d\vartheta = \mathbb{L}(\lambda)\int_{\mathbb{R}} e^{-\lambda\vartheta}J(\vartheta)d\vartheta \end{aligned}$$

and

$$\int_{\mathbb{R}} e^{-\lambda\theta} k * \widetilde{\phi}(\theta - c\tau)d\theta = \mathbb{L}(\lambda)e^{-c\tau\lambda}\int_{\mathbb{R}} e^{-\lambda\vartheta}k(\vartheta)d\vartheta$$

Since the first equality of (3.1) can be written as

$$\begin{aligned}(3.14)\qquad & d(J*\widetilde{\phi}(\xi)-\widetilde{\phi}(\xi))-c\widetilde{\phi}'(\xi)+\partial_1 f(0,0)\widetilde{\phi}(\xi)\\ &+\partial_2 f(0,0)\int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi-c\tau-y)dy\\ &=\partial_1 f(0,0)\widetilde{\phi}(\xi)+\partial_2 f(0,0)\int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi-c\tau-y)dy\\ &-f\Big(\widetilde{\phi}(\xi),\int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi-c\tau-y)dy\Big)\\ &=: R(\widetilde{\phi})(\xi),\end{aligned}$$

we have

$$\Delta(\lambda,c)\mathbb{L}(\lambda)=\int_{-\infty}^{\infty} e^{-\lambda\theta}R(\widetilde{\phi})(\theta)d\theta. \tag{3.15}$$

It is easily seen that the left-hand side of (3.15) is analytic for $\Re\lambda\in(0,\varrho)$. According to (A3), for any $\overline{u}>0$, there exists $\overline{L}>0$ such that

$$f(u,v)\geq \partial_1 f(0,0)u+\partial_2 f(0,0)v-\overline{L}\left(u^{1+\sigma_1}+v^{1+\sigma_2}\right),\ \forall\, u,v\in[0,\overline{u}],$$

where

$$\overline{L}=:\max\Big\{L,\delta^{-(1+\min\{\sigma_1,\sigma_2\})}\max_{u,v\in[0,\overline{u}]}\{\partial_1 f(0,0)u+\partial_2 f(0,0)v-f(u,v)\}\Big\}.$$

Thus,

$$-\overline{L}\left([\widetilde{\phi}(\xi)]^{1+\sigma_1}+\Big[\int_{\mathbb{R}} k(y)\widetilde{\phi}(\xi-c\tau-y)dy\Big]^{1+\sigma_2}\right)\leq R(\widetilde{\phi})(\xi)\leq 0.$$

Choosing $\nu>0$ such that $\nu<\min\{\sigma_1,\sigma_2\}\varrho$. Then for any $\lambda\in(0,\varrho+\nu)$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\lambda\theta}[\widetilde{\phi}(\theta)]^{1+\sigma_1}d\theta &=\int_{\mathbb{R}} e^{-(\lambda-\nu)\theta}\widetilde{\phi}(\theta)\Big(\widetilde{\phi}(\theta)e^{-\frac{\nu\theta}{\sigma_1}}\Big)^{\sigma_1}d\theta\\ &\leq \mathbb{L}(\lambda-\nu)\Big(\sup_{\xi\in\mathbb{R}}\widetilde{\phi}(\xi)e^{-\frac{\nu\xi}{\sigma_1}}\Big)^{\sigma_1}<+\infty\end{aligned}$$

and

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{-\lambda\theta}\Big[\int_{\mathbb{R}} k(y)\widetilde{\phi}(\theta-c\tau-y)dy\Big]^{1+\sigma_2}d\theta\\ =&\int_{-\infty}^{\infty} e^{-(\lambda-\nu)(\theta-c\tau)}\int_{\mathbb{R}} k(y)\widetilde{\phi}(\theta-c\tau-y)dy e^{-\lambda c\tau}e^{-\nu(\theta-c\tau)}\\ &\Big[\int_{\mathbb{R}} k(y)\widetilde{\phi}(\theta-c\tau-y)dy\Big]^{\sigma_2}d\theta\\ =&\int_{-\infty}^{\infty}\Big\{e^{-(\lambda-\nu)(\theta-c\tau)}\int_{\mathbb{R}} k(y)\widetilde{\phi}(\theta-c\tau-y)dy e^{-\lambda c\tau}\end{aligned}$$

$$\begin{aligned}
&\times \Big[\int_{\mathbb{R}} k(y)e^{-\frac{\nu}{\sigma_2}(\theta-c\tau-y)}\widetilde{\phi}(\theta-c\tau-y)e^{-\frac{\nu}{\sigma_2}y}dy\Big]^{\sigma_2}\Big\}d\theta\\
\le& \Big[\int_{\mathbb{R}} k(y)e^{-\frac{\nu}{\sigma_2}y}dy\Big]^{\sigma_2}\Big(\sup_{\xi\in\mathbb{R}}\widetilde{\phi}(\xi)e^{-\frac{\nu\xi}{\sigma_2}}\Big)^{\sigma_2}\int_{-\infty}^{\infty} e^{-(\lambda-\nu)(\theta-c\tau)}\\
&\int_{\mathbb{R}} k(y)\widetilde{\phi}(\theta-c\tau-y)dye^{-\lambda c\tau}d\theta\\
=& e^{-\lambda c\tau}\Big[\int_{\mathbb{R}} k(y)e^{-\frac{\nu}{\sigma_2}y}dy\Big]^{\sigma_2}\Big(\sup_{\xi\in\mathbb{R}}\widetilde{\phi}(\xi)e^{-\frac{\nu\xi}{\sigma_2}}\Big)^{\sigma_2}\\
&\times\int_{\mathbb{R}} k(y)\Big(\int_{-\infty}^{\infty} e^{-(\lambda-\nu)(\theta-c\tau-y)}\widetilde{\phi}(\theta-c\tau-y)e^{-(\lambda-\nu)y}dy\Big)d\theta\\
=& \mathbb{L}(\lambda-\nu)e^{-\lambda c\tau}\Big[\int_{\mathbb{R}} k(y)e^{-\frac{\nu}{\sigma_2}y}dy\Big]^{\sigma_2}\Big(\sup_{\xi\in\mathbb{R}}\widetilde{\phi}(\xi)e^{-\frac{\nu\xi}{\sigma_2}}\Big)^{\sigma_2}\int_{\mathbb{R}} k(y)e^{-(\lambda-\nu)y}dy<+\infty.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\Big|\int_{-\infty}^{\infty} e^{-\lambda\theta}R(\widetilde{\phi})(\theta)d\theta\Big|\\
\le&\overline{L}\int_{-\infty}^{\infty} e^{-\lambda\theta}\Big|[\widetilde{\phi}(\theta)]^{1+\sigma_1}+\Big[\int_{\mathbb{R}} k(y)\widetilde{\phi}(\theta-c\tau-y)dy\Big]^{1+\sigma_2}\Big|d\theta<+\infty
\end{aligned}$$

We now use a property of Laplace transform ([25], p. 58). Since $\widetilde{\phi}(\xi)>0$, there exists a real κ such that $\mathbb{L}(\lambda)$ is analytic for $0<\Re\lambda<\kappa$ and $\mathbb{L}(\lambda)$ has a singularity at $\lambda=\kappa$. Hence, $c\ge c_*$, $\mathbb{L}(\lambda)$ is analytic for $0<\Re\lambda<\lambda_1$ and $\mathbb{L}(\lambda)$ has a singularity at $\lambda=\lambda_1$.

We first prove (iii) of Theorem 1.2. We argue by contradiction, that is, for $0<c<c_*$, there is a nonnegative bounded traveling wave with $\widetilde{\phi}(-\infty)=0$ and $\widetilde{\phi}\not\equiv 0$. Since $\Delta(c,\lambda)$ has no real zeros, $\mathbb{L}(\lambda)$ is analytic for all λ with $\Re\lambda>0$. Using (3.15), it follows that

$$\int_{-\infty}^{\infty} e^{-\lambda\theta}[\Delta(\lambda,c)\widetilde{\phi}(\theta)-R(\widetilde{\phi})(\theta)]d\theta=0$$

which implies a contradiction since $\Delta(\lambda,c)\to+\infty$ as $\lambda\to+\infty$.

Next we prove (i) and (ii) of Theorem 1.2. From now on, we study the case $c\ge c_*$. In order to apply Lemma 3.1, we rewrite (3.15) as

$$\int_{-\infty}^{0} e^{-\lambda\theta}\widetilde{\phi}(\theta)d\theta=\frac{\int_{-\infty}^{\infty}e^{-\lambda\theta}R(\widetilde{\phi})(\theta)d\theta}{\Delta(\lambda,c)}-\int_{0}^{\infty} e^{-\lambda\theta}\widetilde{\phi}(\theta)d\theta. \tag{3.16}$$

Note that $\int_0^\infty\widetilde{\phi}(\theta)e^{-\lambda\theta}d\theta$ is analytic for $\Re\lambda>0$. Also, $\Delta(\lambda,c)=0$ does not have any zero with $\Re\lambda=\lambda_1$ other than $\lambda=\lambda_1$. In fact, letting $\lambda=\lambda_1+i\beta$, then $\Delta(\lambda,c)=0$

implies that

$$(3.17)\quad \begin{aligned} c\lambda_1 &= d\left[\int_{\mathbb{R}} e^{-\lambda_1 y} J(y)\cos\beta y dy - 1\right] + \partial_1 f(0,0) \\ &+\partial_2 f(0,0)\int_{\mathbb{R}} e^{-\lambda_1(y+c\tau)} k(y)\cos\beta(c\tau+y)dy \end{aligned}$$

and

$$(3.18)\quad c\beta = d\int_{\mathbb{R}} e^{-\lambda y}\sin\beta y J(y)dy + \partial_2 f(0,0)\int_{\mathbb{R}} e^{-\lambda_1(y+c\tau)} k(y)\sin\beta(c\tau+y)dy.$$

According to (3.17) and $\Delta(\lambda_1, c) = 0$, we can obtain

$$(3.19)\quad d\int_{\mathbb{R}} e^{-\lambda y}\sin^2\frac{\beta y}{2} J(y)dy + \partial_2 f(0,0)\int_{\mathbb{R}} e^{-\lambda(y+c\tau)}\sin^2\frac{\beta(c\tau+y)}{2} k(y)dy = 0.$$

If $\partial_2 f(0,0) = 0$, (3.19) can imply $\sin\frac{\beta y}{2} = 0$ and it is easily seen that $\beta = 0$ by (3.18). If $\partial_2 f(0,0) > 0$, according to (3.19), then we have $\sin\frac{\beta y}{2} = 0$ and $\sin\frac{\beta(c\tau+y)}{2} = 0$ which imply that $\beta = 0$ by (3.18).

Assume that $\widetilde{\phi}(\xi)$ is increasing for large $-\xi > 0$, then we can choose a translation of $\widetilde{\phi}$ that is increasing for $\xi < 0$. Letting $u(\xi) = \widetilde{\phi}(-\xi)$ and

$$\begin{aligned} \mathbf{T}(u)(\xi) &= \partial_1 f(0,0)u(\xi) + \partial_2 f(0,0)\int_{\mathbb{R}} k(y)u(\xi+c\tau+y)dy \\ &-f\Big(u(\xi), \int_{\mathbb{R}} k(y)u(\xi+c\tau+y)dy\Big), \end{aligned}$$

it is clear that $u(\xi)$ is decreasing $\xi > 0$ and

$$\begin{aligned} \int_0^{+\infty} e^{\lambda\theta}u(\theta)d\theta &= \frac{\int_{-\infty}^{\infty} e^{\lambda\theta}\mathbf{T}(u)(\theta)d\theta}{\Delta(\lambda, c)} - \int_0^{\infty} e^{-\lambda\theta}\widetilde{\phi}(\theta)d\theta \\ &=: \frac{h(\lambda)}{(\lambda-\lambda_1)^{i+1}}, \end{aligned}$$

where $i = 0$ for $c > c_*$, and $i = 1$ for $c = c_*$, and

$$h(\lambda) = \frac{(\lambda-\lambda_1)^{i+1}\int_{-\infty}^{\infty} e^{\lambda\theta}\mathbf{T}(u)(\theta)d\theta}{\Delta(\lambda, c)} - (\lambda-\lambda_1)^{i+1}\int_0^{\infty} e^{-\lambda\theta}\widetilde{\phi}(\theta)d\theta.$$

By Lemma 2.2, $\lim_{\lambda\to\lambda_1} h(\lambda)$ exists. Therefore, $h(\lambda)$ is analytic for all $0 < \Re\lambda \le \lambda_1$. Then Lemma 3.1 implies that

$$\lim_{\xi\to+\infty}\frac{u(\xi)}{\xi^k e^{-\lambda_1\xi}} \text{ exists, i.e., } \lim_{\xi\to-\infty}\frac{\widetilde{\phi}(\xi)}{|\xi|^k e^{\lambda_1\xi}} \text{ exists,}$$

that is,

$$\lim_{\xi\to-\infty}\frac{\widetilde{\phi}(\xi)}{e^{\lambda_1\xi}} \text{ exists for } \ c>c_*, \ \lim_{\xi\to-\infty}\frac{\widetilde{\phi}(\xi)}{|\xi|e^{\lambda_*\xi}} \text{ exists for } \ c=c_*.$$

Now we assume that $\widetilde{\phi}(\xi)$ is not monotone for large $-\xi>0$. letting

$$p=\frac{1+M+d}{c} \quad \text{and } \overline{U}(\xi)=\widetilde{\phi}(\xi)e^{p\xi},$$

where $M=\max\limits_{(u,v)\in[0,K]^2}\{|\partial_1 f(u,v)|\}$, then for large enough $-\xi>0$, we have

$$c\overline{U}'(\xi)=dJ*\widetilde{\phi}(\xi)e^{p\xi}+[(1+M)\widetilde{\phi}(\xi)+f\Big(\widetilde{\phi}(\xi),\int_{\mathbb{R}}k(y)\widetilde{\phi}(\xi-c\tau-y)dy\Big)]e^{p\xi}>0.$$

Then we can choose a translation of $\overline{U}$ which is increasing for $\xi<0$. Letting $\bar{u}(\xi)=\overline{U}(-\xi)$, it is clear that $\bar{u}(\xi)$ is decreasing $\xi>0$. Let $\overline{\mathbb{L}}(\lambda)=\int_{\mathbb{R}}e^{-\lambda\xi}\overline{U}(\xi)d\xi$. Noting that $\overline{\mathbb{L}}(\lambda)=\mathbb{L}(\lambda-p)$ and repeating the above argument, we have

$$\lim_{\xi\to+\infty}\frac{\overline{U}(\xi)}{\xi^i e^{-(p+\lambda_1)\xi}}=\lim_{\xi\to-\infty}\frac{\widetilde{\phi}(\xi)}{|\xi|^i e^{\lambda_1\xi}} \text{ exists.}$$

Thus, it follows that

$$\lim_{\xi\to-\infty}\frac{\widetilde{\phi}(\xi)}{e^{\lambda_1\xi}} \text{ exists for } \ c>c_*, \ \lim_{\xi\to-\infty}\frac{\widetilde{\phi}(\xi)}{|\xi|e^{\lambda_*\xi}} \text{ exists for } \ c=c_*.$$

This completes the proof of Theorem 1.2. ■

4. Uniqueness of the Traveling Wave Front

In this section, we will prove that the traveling wave front obtained in Theorem 1.1 is unique up to a translation by using the technique in [5, 6].

Lemma 4.1. *Assume that (H1)-(H2) and (A1)-(A3) hold. Then there exists $\rho_0\in(0,K)$ such that for any solution (c,ϕ) of* (1.9) *and* (1.10) *and any $\rho\in(0,\rho_0)$,*

$$f((1+\rho)\phi(\xi),(1+\rho)k*\phi(\xi-c\tau))-(1+\rho)f(\phi(\xi),k*\phi(\xi-c\tau))<0$$

for all ξ satisfying $\phi(\xi)>K-\rho_0$.

Proof. Since $\phi(+\infty)=K$, there exist $\rho_0\in(0,K)$ and large enough $M_0>0$ such that $\phi(\xi)>K-\rho_0$ for $\xi>M_0$. On the other hand, when $\xi\to+\infty$, it follows from the Taylor expansion that

$$f(\phi(\xi), k*\phi(\xi - c\tau)) = (\phi(\xi) - K)\partial_1 f(K,K) + (k*\phi(\xi - c\tau) - K)\partial_2 f(K,K)$$
$$+ o(|\phi(\xi) - K|) + o(|k*\phi(\xi - c\tau) - K|),$$
$$\partial_1 f(\phi(\xi), k*\phi(\xi - c\tau)) = \partial_1 f(K,K) + O(|\phi(\xi) - K|) + O(|k*\phi(\xi - c\tau) - K|),$$
$$\partial_2 f(\phi(\xi), k*\phi(\xi - c\tau)) = \partial_2 f(K,K) + O(|\phi(\xi) - K|) + O(|k*\phi(\xi - c\tau) - K|).$$

Thus, we have

$$\frac{d}{d\rho}[f((1+\rho)\phi(\xi), (1+\rho)k*\phi(\xi - c\tau)) - (1+\rho)f(\phi(\xi), k*\phi(\xi - c\tau))]\Big|_{\rho=0}$$
$$= \phi(\xi)\partial_1 f(\phi(\xi), k*\phi(\xi - c\tau)) + k*\phi(\xi - c\tau)\partial_2 f(\phi(\xi), k*\phi(\xi - c\tau))$$
$$- f(\phi(\xi), k*\phi(\xi - c\tau))$$
$$= \partial_1 f(K,K) + \partial_2 f(K,K) + O(|\phi(\xi) - K|) + O(|k*\phi(\xi - c\tau) - K|).$$

Since $\partial_1 f(K,K) + \partial_2 f(K,K) < 0$, we may choose $\rho_0 > 0$ small enough such that

$$\frac{d}{d\rho}[f((1+\rho)\phi(\xi), (1+\rho)k*\phi(\xi - c\tau)) - (1+\rho)f(\phi(\xi), k*\phi(\xi - c\tau))]\Big|_{\rho=0} < 0$$

for all ξ satisfying $\phi(\xi) > K - \rho_0$. This completes the proof. ∎

For any fixed solution (c, ϕ) of (1.9) and (1.10), we define

$$\kappa = \kappa(\phi) := \sup\left\{\frac{\phi(\xi)}{\phi'(\xi)}\Big|\phi(\xi) \le K - \rho_0\right\}.$$

Then $0 < \kappa < +\infty$ since $\lim\limits_{\xi\to-\infty} \frac{\phi'(\xi)}{\phi(\xi)} = \lambda_1$.

Lemma 4.2. *Assume that (H1)-(H2) and (A1)-(A) hold. Let (c, ϕ_1) and (c, ϕ_2) be two solutions of* (1.9) *and* (1.10)*, there exists $\rho \in (0, \rho_0]$ such that $(1+\rho)\phi_1(\xi - \kappa\rho) \ge \phi_2(\xi)$ for $\xi \in \mathbb{R}$. Then $\phi_1(\xi) \ge \phi_2(\xi)$ for $\xi \in \mathbb{R}$.*

Proof. Let $w(\rho, \xi) := (1+\rho)\phi_1(\xi - \kappa\rho) - \phi_2(\xi)$ and

$$\rho^* := \inf\{\rho > 0 | w(\rho, \cdot) \ge 0 \text{ on } \mathbb{R}\}.$$

By the continuity, $w(\rho^*, \cdot) \ge 0$ on $\mathbb{R}$. Next we prove that $\rho^* = 0$. Suppose on the contrary that $q^* \in (0, \rho_0]$. By the definition of κ, we have

$$\frac{d}{d\rho}w(\rho, \xi) = \phi_1(\xi - \kappa\rho) - (1+\rho)\kappa\phi_1'(\xi - \kappa\rho) < 0 \tag{4.1}$$

for all ξ satisfying $\phi(\xi) \leq K - \rho_0$. According to the definition of ρ^* and $w(\rho^*, \cdot) \geq 0$ on $\mathbb{R}$, (4.1) implies that there exists ξ_0 satisfying $\phi_1(\xi_0 - \kappa\rho^*) > K - \rho_0$ such that $w(\rho^*, \xi_0) = w_\xi(\rho^*, \xi_0) = 0$, $J * \phi_2(\xi_0) \leq J * \phi_1(\xi_0^*)$ and $k * \phi_2(\xi_0 - c\tau) \leq (1+\rho^*)k * \phi_1(\xi_0^* - c\tau)$, where $\xi_0^* = \xi_0 - \kappa\rho^*$. Then, it follows that

$$\begin{aligned} 0 =& c\phi_2'(\xi_0) - d(J * \phi_2(\xi_0) - \phi_2(\xi_0)) - f(\phi_2(\xi_0), k * \phi_2(\xi_0 - c\tau)) \\ \geq& (1+\rho^*)[c\phi_1'(\xi_0^*) - d(J * \phi_1(\xi_0^*) - \phi_1(\xi_0^*))] \\ &- f((1+\rho^*)\phi_1(\xi_0^*), (1+\rho^*)k * \phi_1(\xi_0^* - c\tau)) \\ =& (1+\rho^*)f(\phi_1(\xi_0^*), k * \phi_1(\xi_0^* - c\tau)) - f((1+\rho^*)\phi_1(\xi_0^*), (1+\rho^*)k * \phi_1(\xi_0^* - c\tau)) \end{aligned}$$

which contradicts Lemma 4.1. Hence $\rho^* = 0$ and $\phi_1(\xi) \geq \phi_2(\xi)$ for $\xi \in \mathbb{R}$. This completes the proof. ■

Lemma 4.3. *Assume that* (H1)-(H2) *and* (A1)-(A3) *hold. Let* (c, ϕ_1) *and* (c, ϕ_2) *be two solutions of* (1.9) *and* (1.10) *satisfying* $\phi_2 \leq \phi_1$. *Then either* $\phi_2 \equiv \phi_1$ *or* $\phi_2 < \phi_1$ *on* $\mathbb{R}$.

Proof. Suppose that there exists ξ_0 such that $\phi_1(\xi_0) = \phi_2(\xi_0)$. Since ϕ_1 and ϕ_2 are the solutions of (1.9) and (1.10), we have $T(\phi_1)(\xi_0) = \phi_1(\xi_0)$ and $T(\phi_2)(\xi_0) = \phi_2(\xi_0)$. Thus

$$0 = \phi_1(\xi_0) - \phi_2(\xi_0) = \frac{1}{c}e^{-\frac{\beta}{c}\xi_0}\int_{-\infty}^{\xi_0} e^{\frac{\beta}{c}y}[H(\phi_1)(y) - H(\phi_2)(y)]dy. \tag{4.2}$$

According to Lemma 2.1, it follows that $H(\phi_1)(\xi) \geq H(\phi_2)(\xi)$ for all $\xi \in \mathbb{R}$. Therefore, (4.2) implies that $H(\phi_1)(y) = H(\phi_2)(y)$ for all $y \leq \xi_0$, i.e.,

$$\begin{aligned} &dJ * \phi_1(y) + (\beta - d)\phi_1(y) + f\Big(\phi_1(y), \int_{\mathbb{R}} k(s)\phi_1(y - c\tau - s)dy\Big) \\ &= dJ * \phi_2(y) + (\beta - d)\phi_2(y) + f\Big(\phi_2(y), \int_{\mathbb{R}} k(s)\phi_2(y - c\tau - s)dy\Big), \end{aligned}$$

which implies that

$$0 \geq d\int_{\mathbb{R}} J(s)[\phi_1(y-s) - \phi_2(y-s)]ds$$

by Lemma 2.1. Hence $\phi_1(\xi) = \phi_2(\xi)$ for all $\xi \in \mathbb{R}$. This completes the proof. ■

Proof of Theorem 1.3. Let (c, ϕ_1) and (c, ϕ_2) be two solutions of (1.9) and (1.10). By translation, we may assume that $\phi_1(0) = \phi_2(0) = \frac{K}{2}$. By Theorem 1.2, $\lim_{\xi\to-\infty} \frac{\phi_2(\xi)}{\phi_1(\xi)}$ exists. Hence we may assume that $\lim_{\xi\to-\infty} \frac{\phi_2(\xi)}{\phi_1(\xi)} \leq 1$ (otherwise, we may consider $\lim_{\xi\to-\infty} \frac{\phi_1(\xi)}{\phi_2(\xi)}$). Then $\lim_{\xi\to-\infty} \frac{\phi_2(\xi - z)}{\phi_1(\xi)} < 1$ for all $z > 0$.

For any fixed number $z > 0$, there exists $\xi_1 > 0$ such that $\phi_1(\xi) > \phi_2(\xi - z)$ on $(-\infty, -\xi_1]$. Thus, there exists large enough $z_0 > 0$ such that $(1+\rho_0)\phi_1(\xi - \kappa\rho_0) \geq \phi_2(\xi - z_0)$ for all $\xi \in \mathbb{R}$. Applying Lemma 4.2, we have $\phi_1(\xi) \geq \phi_2(\xi - z_0)$ for all $\xi \in \mathbb{R}$. We may define

$$z^* := \inf\{z > 0 | \phi_1(\xi) \geq \phi_2(\xi - z), \text{ for all } \xi \in \mathbb{R}\}.$$

We claim that $z^* = 0$. Indeed, suppose on the contradiction that $z^* > 0$. According to $\lim\limits_{\xi \to -\infty} \frac{\phi_2(\xi - z^*)}{\phi_1(\xi - \frac{z^*}{2})} < 1$, there exists ξ_2 such that

$$\phi_1(\xi - \frac{z^*}{2}) \geq \phi_2(\xi - z^*) \quad \text{for all } \xi \in (-\infty, -\xi_2]. \tag{4.3}$$

According to $\phi_1(+\infty) = K$ and $\phi_1'(+\infty) = 0$, there exists $\xi_3 \gg 1$ such that

$$\frac{d}{d\rho}[(1+\rho)\phi_1(\xi - 2\kappa\rho)] = \phi_1 - 2\kappa(1+\rho)\phi_1' > 0$$

for all $\rho \in [0, 1]$ and $\xi \in (-\infty, -\xi_3]$. Thus, for all $\rho \in [0, 1]$ and $\xi \in (-\infty, -\xi_3]$, we have

$$(1+\rho)\phi_1(\xi - 2\kappa\rho) \geq \phi(\xi) \geq \phi_2(\xi - z^*). \tag{4.4}$$

Now we consider $\xi \in [-\xi_2, \xi_3]$. Since $\phi_1(\cdot) \geq \phi_2(\cdot - z^*)$ in $\mathbb{R}$ and $\phi_1(z^*) > \phi_2(0)$, by Lemma 4.3, we have $\phi_1(\cdot) > \phi_2(\cdot - z^*)$ in $\mathbb{R}$. Thus, we can choose $0 < \epsilon < \min\{\rho_0, \frac{z^*}{4\kappa}\}$ such that

$$\phi_1(\cdot - 2\kappa\epsilon) \geq \phi_2(\cdot - z^*) \quad \text{on} \quad [-\xi_2, \xi_3]. \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), it follows that $(1+\epsilon)\phi_1(\cdot - 2\kappa\epsilon) \geq \phi_2(\cdot - z^*)$ in $\mathbb{R}$. According to Lemma 4.2, we have

$$\phi_1(\xi - \kappa\epsilon) \geq \phi_2(\xi - z^*), \ \forall \xi \in \mathbb{R}.$$

which contradicts the definition of z^*. Therefore, $z^* = 0$. Since $\phi_1(0) = \phi_2(0) = \frac{K}{2}$, by Lemma 4.3, $\phi_1 \equiv \phi_2$ on $\mathbb{R}$. This completes the proof of Theorem 3. ■

5. Applications

We first investigate the existence, asymptotics and uniqueness of traveling waves for (1.6). Since $f(u, v) = r(1-u)v$ satisfies assumptions (A1)-(A3), we can obtain

Theorem 5.1. *Assume that* (*H1*)-(*H2*) *hold. Then there exists a positive constant* c_* *such that for each* $c \geq c_*$, (1.6) *admits a unique (up to translation) nondecreasing positive traveling wave front* $u(t, x) = \phi(x + ct)$ *connecting 0 and 1. Moreover, if* $c > c_*$, *then*

$$\lim_{\xi \to -\infty} \phi(\xi) e^{-\lambda_1 \xi} = 1, \quad \lim_{\xi \to -\infty} \phi'(\xi) e^{-\lambda_1 \xi} = \lambda_1,$$

where $\lambda_1 > 0$ *is the smallest root of the equation*

$$\Delta(c, \lambda) = c\lambda - d\left[\int_{\mathbb{R}} e^{-\lambda y} J(y)dy - 1\right] - re^{-c\tau\lambda} \int_{\mathbb{R}} e^{-\lambda y} k(y)dy = 0.$$

Theorem 5.2. *Assume that* (*H1*)-(*H2*) *hold and* $\widetilde{\phi}(\xi)$ *is a nonnegative bounded traveling wave of* (1.6) *with* $\widetilde{\phi}(-\infty) = 0$ *and* $\widetilde{\phi} \not\equiv 0$. *Then we have the following conclusions*

(i) *For every* $c > c_*$, $\lim_{\xi \to -\infty} \widetilde{\phi}(\xi) e^{-\lambda_1 \xi}$ *exists.*

(ii) *For* $c = c_*$, *there exists a constant* $\lambda_* > 0$ *such that* $\lim_{\xi \to -\infty} \widetilde{\phi}(\xi) \xi^{-1} e^{-\lambda_* \xi}$ exists.

(iii) *For* $0 < c < c_*$, *there is no nonnegative bounded traveling wave with* $\widetilde{\phi}(-\infty) = 0$ *and* $\widetilde{\phi} \not\equiv 0$.

Next, consider the following diffusive Nicholson's blowflies equation (1.7), where $r > 0$ and $\tau \geq 0$. When $1 < p \leq e$, $f(u, v) = -ru + rpve^{-v}$ satisfies (A1) and (A2). Therefore, we have the following results.

Theorem 5.3. *Assume that* (*H1*)-(*H2*) *hold. Then there exists a positive constant* c_* *such that for each* $c \geq c_*$, (1.7) *admits a unique (up to translation) nondecreasing positive traveling wave front* $u(t, x) = \phi(x + ct)$ *connecting 0 and* $\ln p$. *Moreover, if* $c > c_*$, *then*

$$\lim_{\xi \to -\infty} \phi(\xi) e^{-\lambda_1 \xi} = 1, \quad \lim_{\xi \to -\infty} \phi'(\xi) e^{-\lambda_1 \xi} = \lambda_1,$$

where $\lambda_1 > 0$ *is the smallest root of the equation*

$$\Delta(c, \lambda) = c\lambda - d\left[\int_{\mathbb{R}} e^{-\lambda y} J(y)dy - 1\right] + r - rpe^{-c\tau\lambda} \int_{\mathbb{R}} e^{-\lambda y} k(y)dy = 0.$$

Theorem 5.4. *Assume that* (*H1*)-(*H2*) *hold and* $\widetilde{\phi}(\xi)$ *is a nonnegative bounded traveling wave of* (1.7) *with* $\widetilde{\phi}(-\infty) = 0$ *and* $\widetilde{\phi} \not\equiv 0$. *Then we have the following conclusions*

(i) *For every* $c > c_*$, $\lim_{\xi \to -\infty} \widetilde{\phi}(\xi) e^{-\lambda_1 \xi}$ *exists.*

(ii) *For* $c = c_*$, *there exists a constant* $\lambda_* > 0$ *such that* $\lim_{\xi \to -\infty} \widetilde{\phi}(\xi) \xi^{-1} e^{-\lambda_* \xi}$ exists.

(*iii*) *For* $0 < c < c_*$, *there is no nonnegative bounded traveling wave with* $\widetilde{\phi}(-\infty) = 0$ *and* $\widetilde{\phi} \not\equiv 0$.

Remark 5.3. Letting $k(x) = \delta(x)$, (1.6) and (1.7) can be reduced to nonlocal diffusion systems (1.3) and (1.4). Thus, our results improve and complement the previous works.

ACKNOWLEDGMENTS

The authors thank the referee for his valuable comments and suggestions that help the improvement of the manuscript.

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