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INTEGRAL REPRESENTATIONS OF GENERALIZED HARMONIC FUNCTIONS

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Abstract. When generalized harmonic functions belong to the weighted Lebesgue classes, we give the asymptotic behaviors of them at infinity in an n-dimensional cone. Meanwhile, the integral representations of them are also considered, which imply the known representations of classical harmonic functions in the upper half space.

1. Introduction and Results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbf{R}^n (n \geq 2)$ the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, x_n)$, $X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance between two points P and Q in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set $\mathbf{S}$ in $\mathbf{R}^n$ are denoted by $\partial\mathbf{S}$ and $\overline{\mathbf{S}}$, respectively.

We introduce a system of spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \ldots, x_{n-1}, x_n)$ by $x_n = r\cos\theta_1$.

The unit sphere and the upper half unit sphere in $\mathbf{R}^n$ are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}^{n-1}_+$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set Ω, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with Θ and Ω, respectively. For two sets $\Xi \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set $\{(r, \Theta) \in \mathbf{R}^n; r \in \Xi, (1, \Theta) \in \Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Xi \times \Omega$. In particular, the half space $\mathbf{R}_+ \times \mathbf{S}^{n-1}_+ = \{(X, x_n) \in \mathbf{R}^n; x_n > 0\}$ will be denoted by $\mathbf{T}_n$.

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For $P \in \mathbf{R}^n$ and $r > 0$, let $B(P, r)$ denote the open ball with center at P and radius r in $\mathbf{R}^n$. $S_r = \partial B(O, r)$. By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain Ω on $\mathbf{S}^{n-1}$. We call it a cone. Then T_n is a special cone obtained by putting $\Omega = \mathbf{S}_+^{n-1}$. We denote the sets $I \times \Omega$ and $I \times \partial\Omega$ with an interval on $\mathbf{R}$ by $C_n(\Omega; I)$ and $S_n(\Omega; I)$. By $S_n(\Omega; r)$ we denote $C_n(\Omega) \cap S_r$. By $S_n(\Omega)$ we denote $S_n(\Omega; (0, +\infty))$ which is $\partial C_n(\Omega) - \{O\}$.

We denote by dS_r the $(n-1)$-dimensional volume elements induced by the Euclidean metric on S_r and by dw the elements of the Euclidean volume in $\mathbf{R}^n$.

Let $\mathcal{A}_a$ denote the class of nonnegative radial potentials $a(P)$, i.e. $0 \le a(P) = a(r)$, $P = (r, \Theta) \in C_n(\Omega)$, such that $a \in L^b_{loc}(C_n(\Omega))$ with some $b > n/2$ if $n \ge 4$ and with $b = 2$ if $n = 2$ or $n = 3$.

This article is devoted to the stationary Schrödinger equation

$$Sch_a u(P) = -\Delta u(P) + a(P)u(P) = 0 \quad \text{for} \quad P \in C_n(\Omega),$$

where Δ is the Laplace operator and $a \in \mathcal{A}_a$. These solutions called a-harmonic functions or generalized harmonic functions associated with the operator Sch_a. Note that they are classical harmonic functions in the classical case $a = 0$. Under these assumptions the operator Sch_a can be extended in the usual way from the space $C_0^\infty(C_n(\Omega))$ to an essentially self-adjoint operator on $L^2(C_n(\Omega))$ (see [10, 11, 15]). We will denote it Sch_a as well. This last one has a Green function $G(\Omega, a)(P, Q)$. Here $G(\Omega, a)(P, Q)$ is positive on $C_n(\Omega)$ and its inner normal derivative $\partial G(\Omega, a)(P, Q)/\partial n_Q \ge 0$. We denote this derivative by $\mathbb{PI}(\Omega, a)(P, Q)$, which is called the Poisson a-kernel with respect to $C_n(\Omega)$, where $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into $C_n(\Omega)$. We remark that $G(\Omega, 0)(P, Q)$ and $\mathbb{PI}(\Omega, 0)(P, Q)$ are the Green function and Poisson kernel of the Laplacian in $C_n(\Omega)$ respectively.

Let Δ^* be the Laplace-Beltrami operator (spherical part of the Laplace) on $\Omega \subset \mathbf{S}^{n-1}$ and λ_j $(j = 1, 2, 3 \ldots, 0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots)$ be the eigenvalues of the eigenvalue problem for Δ^* on Ω (see, e.g., [16, p. 41])

$$\begin{aligned} \Delta^*\varphi(\Theta) + \lambda\varphi(\Theta) &= 0 \quad \text{in } \Omega, \\ \varphi(\Theta) &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Corresponding eigenfunctions are denoted by φ_{jv} $(1 \le v \le v_j)$, where v_j is the multiplicity of λ_j. We set $\lambda_0 = 0$, norm the eigenfunctions in $L^2(\Omega)$ and $\varphi_1 = \varphi_{11} > 0$. Then there exist two positive constants d_1 and d_2 such that

$$d_1\delta(P) \le \varphi_1(\Theta) \le d_2\delta(P) \tag{1.1}$$

for $P = (1, \Theta) \in \Omega$ (see Courant and Hilbert [3]), where $\delta(P) = \inf_{Q \in \partial C_n(\Omega)} |P - Q|$.

In order to ensure the existences of λ_j $(j = 1, 2, 3 \ldots)$, we put a rather strong assumption on Ω: if $n \ge 3$, then Ω is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $\mathbf{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces.Then $\varphi_{jv} \in C^2(\overline{\Omega})$ $(j = 1, 2, 3, \ldots, 1 \leq v \leq v_j)$ and $\partial\varphi_1/\partial n > 0$ on $\partial\Omega$ (here and below, $\partial/\partial n$ denotes differentiation along the interior normal). Hence well-known estimates (see, e.g., [14, p. 14]) imply the following inequality:

$$\sum_{v=1}^{v_j} \varphi_{jv}(\Theta) \frac{\partial \varphi_{jv}(\Phi)}{\partial n_\Phi} \leq M(n) j^{2n-1}, \tag{1.2}$$

where the symbol $M(n)$ denotes a constant depending only on n.

Let $V_j(r)$ and $W_j(r)$ stand, respectively, for the increasing and non-increasing, as $r \to +\infty$, solutions of the equation

$$-Q''(r) - \frac{n-1}{r} Q'(r) + \left(\frac{\lambda_j}{r^2} + a(r) \right) Q(r) = 0, \quad 0 < r < \infty, \tag{1.3}$$

normalized under the condition $V_j(1) = W_j(1) = 1$.

We shall also consider the class $\mathcal{B}_a$, consisting of the potentials $a \in \mathcal{A}_a$ such that there exists a finite limit $\lim\limits_{r\to\infty} r^2 a(r) = k \in [0, \infty)$, moreover, $r^{-1}|r^2 a(r) - k| \in L(1, \infty)$. If $a \in \mathcal{B}_a$, then generalized harmonic functions are continuous (see [18]).

In the rest of paper, we assume that $a \in \mathcal{B}_a$ and we shall suppress this assumption for simplicity. Further, we use the standard notations $u^+ = \max(u, 0)$, $u^- = -\min(u, 0)$, $[d]$ is the integer part of d and $d = [d] + \{d\}$, where d is a positive real number.

Denote

$$\iota^{\pm}_{j,k} = \frac{2 - n \pm \sqrt{(n-2)^2 + 4(k + \lambda_j)}}{2} \ (j = 0, 1, 2, 3 \ldots).$$

It is known (see [7]) that in the case under consideration the solutions to equation (1.3) have the asymptotics

$$V_j(r) \sim d_3 r^{\iota^{+}_{j,k}}, \ W_j(r) \sim d_4 r^{\iota^{-}_{j,k}}, \ \text{as } r \to \infty, \tag{1.4}$$

where d_3 and d_4 are some positive constants.

Remark 1. $\iota^{+}_{j,0} = j$ $(j = 0, 1, 2, 3, \ldots)$ in the case $\Omega = \mathbf{S}^{n-1}_{+}$.

It is known that the following expansion for the Green function $G(\Omega, a)(P, Q)$ (see [5, Ch. 11], [9], [10]) holds:

$$G(\Omega, a)(P, Q) = \sum_{j=1}^{\infty} \frac{1}{\chi'(1)} V_j(\min(r, t)) W_j(\max(r, t)) \left(\sum_{v=1}^{v_j} \varphi_{jv}(\Theta) \varphi_{jv}(\Phi) \right), \tag{1.5}$$

where $P = (r, \Theta)$, $Q = (t, \Phi)$, $r \neq t$ and $\chi'(s) = w\left(W_1(r), V_1(r)\right)|_{r=s}$ is their Wronskian. The series converges uniformly if either $r \leq st$ or $t \leq sr$ $(0 < s < 1)$. In the case $a = 0$, this expansion coincides with the well-known result by Lelong-Ferrand (see [12]). The expansion (1.5) can also be rewritten in terms of the Gegenbauer polynomials.

For a nonnegative integer m and two points $P = (r, \Theta), Q = (t, \Phi) \in C_n(\Omega)$, we put

$$K(\Omega, a, m)(P, Q) = \begin{cases} 0 & \text{if} \quad 0 < t < 1, \\ \widetilde{K}(\Omega, a, m)(P, Q) & \text{if} \quad 1 \leq t < \infty, \end{cases}$$

where

$$\widetilde{K}(\Omega, a, m)(P, Q) = \sum_{j=1}^{m} \frac{1}{\chi'(1)} V_j(r) W_j(t) \left(\sum_{v=1}^{v_j} \varphi_{jv}(\Theta) \varphi_{jv}(\Phi) \right).$$

To obtain Poisson a-integral representations of generalized harmonic functions in a cone, we use the following modified kernel function defined by

$$G(\Omega, a, m)(P, Q) = G(\Omega, a)(P, Q) - K(\Omega, a, m)(P, Q)$$

for two points $P = (r, \Theta), Q = (t, \Phi) \in C_n(\Omega)$.

Put

$$U(\Omega, a, m; u)(P) = \int_{S_n(\Omega)} \mathbb{PI}(\Omega, a, m)(P, Q) u(Q) d\sigma_Q,$$

where

$$\mathbb{PI}(\Omega, a, m)(P, Q) = \frac{\partial G(\Omega, a, m)(P, Q)}{\partial n_Q}, \; \mathbb{PI}(\Omega, a, 0)(P, Q) = \mathbb{PI}(\Omega, a)(P, Q),$$

$u(Q)$ is a continuous function on $\partial C_n(\Omega)$ and $d\sigma_Q$ is the surface area element on $S_n(\Omega)$.

Remark 2. The kernel function $\mathbb{PI}(\mathbf{S}_+^{n-1}, 0, m)(P, Q)$ coincides with ones in Finkelstein-Scheinberg [6], Kheyfits [9], Siegel-Talvila [17] and Deng [4] (see [10]).

If γ is a real number and $\gamma \geq 0$ ($resp.$ $\gamma < 0$), we assume in addition that $1 \leq p < \infty$,

$$\iota^+_{[\gamma],k} + \{\gamma\} > (-\iota^+_{1,k} - n + 2)p + n - 1,$$

$$(resp. \; -\iota^+_{[-\gamma],k} - \{-\gamma\} > (-\iota^+_{1,k} - n + 2)p + n - 1,)$$

in case $p > 1$

$$\frac{\iota^+_{[\gamma],k} + \{\gamma\} - n + 1}{p} < \iota^+_{m+1,k} < \frac{\iota^+_{[\gamma],k} + \{\gamma\} - n + 1}{p} + 1;$$

$$\left(resp. \quad \frac{-\iota^+_{[-\gamma],k}-\{-\gamma\}-n+1}{p} < \iota^+_{m+1,k} < \frac{-\iota^+_{[-\gamma],k}-\{-\gamma\}-n+1}{p}+1; \right)$$

and in case $p = 1$

$$\iota^+_{[\gamma],k} + \{\gamma\} - n + 1 \le \iota^+_{m+1,k} < \iota^+_{[\gamma],k} + \{\gamma\} - n + 2.$$

$$\left(resp. \quad -\iota^+_{[-\gamma],k} - \{-\gamma\} - n + 1 \le \iota^+_{m+1,k} < -\iota^+_{[-\gamma],k} - \{-\gamma\} - n + 2. \right)$$

If these conditions all hold, we write $\gamma \in \mathcal{C}(k, p, m, n)$ $(resp.\ \gamma \in \mathcal{D}(k, p, m, n))$.

Let $\gamma \in \mathcal{C}(k, p, m, n)$ $(resp.\ \gamma \in \mathcal{D}(k, p, m, n))$ and u be a continuous function on $\partial C_n(\Omega)$ satisfying

$$\int_{S_n(\Omega)} \frac{|u(t,\Phi)|^p}{1+t^{\iota^+_{[\gamma],k}+\{\gamma\}}} d\sigma_Q < \infty. \tag{1.6}$$
$$\left(resp. \int_{S_n(\Omega)} |u(t,\Phi)|^p (1+t^{\iota^+_{[-\gamma],k}+\{-\gamma\}}) d\sigma_Q < \infty. \right)$$

Siegel-Talvila (cf. [17, Corollary 2.1]) proved the following result.

Theorem A. *If u is a continuous function on ∂T_n satisfying*

$$\int_{\partial T_n} \frac{|u(t,\Phi)|}{1+t^{n+m}} dQ < \infty,$$

then the function $U(\mathbf{S}^{n-1}_+, 0, m; u)(P)$ satisfies

$$U(\mathbf{S}^{n-1}_+, 0, m; u) \in C^2(T_n) \cap C^0(\overline{T_n}),$$
$$\Delta U(\mathbf{S}^{n-1}_+, 0, m; u) = 0 \text{ in } T_n,$$
$$U(\mathbf{S}^{n-1}_+, 0, m; u) = u \text{ on } \partial T_n,$$
$$\lim_{r\to\infty, P=(r,\Theta)\in T_n} U(\mathbf{S}^{n-1}_+, 0, m; u)(P) = o(r^{m+1}\cos^{1-n}\theta_1).$$

First of all we start with the following result.

Theorem 1. *If $\gamma \in \mathcal{C}(k, p, m, n)$ $(resp.\ \gamma \in \mathcal{D}(k, p, m, n))$ and u is a continuous function on $\partial C_n(\Omega)$ satisfying (1.6), then the function $U(\Omega, a, m; u)(P)$ satisfies*

$$U(\Omega, a, m; u) \in C^2(C_n(\Omega)) \cap C^0(\overline{C_n(\Omega)}),$$
$$Sch_a U(\Omega, a, m; u) = 0 \text{ in } C_n(\Omega),$$
$$U(\Omega, a, m; u) = u \text{ on } \partial C_n(\Omega)$$

$$\lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)} r^{\frac{-\iota^+_{[\gamma],k}-\{\gamma\}+n-1}{p}}\varphi_1^{n-1}(\Theta)U(\Omega,a,m;u)(P)=0.$$
$$\left(resp. \quad \lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)} r^{\frac{\iota^+_{[-\gamma],k}+\{-\gamma\}+n-1}{p}}\varphi_1^{n-1}(\Theta)U(\Omega,a,m;u)(P)=0.\right)$$

Remark 3. Mizuta-Shimomura (see [13, Theorem 1 with $\lambda = n$]) treated the case $\Omega = \mathbf{S}_+^{n-1}$ and $a = 0$.

If we put $p = 1$, $\zeta = n$ and $\iota^+_{[\gamma],k} + \{\gamma\} = \iota^+_{m+1,k} + n - 1$ in Theorem 1, by (1.4) we obtain

Corollary 2. *If u is a continuous function on $\partial C_n(\Omega)$ satisfying*

$$\int_{S_n(\Omega)} \frac{|u(t,\Phi)|}{1+V_{m+1}(t)t^{n-1}}d\sigma_Q < \infty, \tag{1.7}$$

then the function $U(\Omega,a,m;u)(P)$ is a generalized harmonic function of $P \in \partial C_n(\Omega)$ and

$$\lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)} r^{-\iota^+_{m+1,k}}\varphi_1^{n-1}(\Theta)U(\Omega,a,m;u)(P)=0.$$

By the boundedness of $\varphi_1(\Theta)$, we immediately have

Corollary 3. Under the assumptions of Corollary 2, we have

$$\lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)} r^{-\iota^+_{m+1,k}}\int_\Omega |U(\Omega,a,m;u)(P)|\varphi_1(\Theta)dS_1 = 0. \tag{1.8}$$

For real numbers $\beta \geq 1$, we denote $\mathcal{C}(\Omega,\beta,a)$ the class of all measurable functions $f(t,\Phi)$ $(Q=(t,\Phi)=(Y,y_n)\in C_n(\Omega))$ satisfying the following inequality

$$\int_{C_n(\Omega)} \frac{|f(t,\Phi)|\varphi_1}{1+V_{[\beta]}(t)t^{n+\{\beta\}}}dw < \infty \tag{1.9}$$

and the class $\mathcal{D}(\Omega,\beta,a)$, consists of all measurable functions $g(t,\Phi)$ $(Q=(t,\Phi)=(Y,y_n)\in S_n(\Omega))$ satisfying

$$\int_{S_n(\Omega)} \frac{|g(t,\Phi)|V_1(t)W_1(t)}{1+\chi'(t)V_{[\beta]}(t)t^{n+\{\beta\}-1}}\frac{\partial\varphi_1}{\partial n}d\sigma_Q < \infty. \tag{1.10}$$

We will also consider the class of all continuous functions $u(t,\Phi)$ $((t,\Phi)\in\overline{C_n(\Omega)})$ generalized harmonic in $C_n(\Omega)$ with $u^+(t,\Phi)\in\mathcal{C}(\Omega,\beta,a)$ $((t,\Phi)\in C_n(\Omega))$ and $u^+(t,\Phi)\in\mathcal{D}(\Omega,\beta,a)$ $((t,\Phi)\in S_n(\Omega))$ is denoted by $\mathcal{E}(\Omega,\beta,a)$.

Remark 4. Notice that $\chi'(t)t = \tau_{1,k}V_1(t)W_1(t)$. If $a = 0$, (1.9) and (1.10) are equivalent to

$$\int_{C_n(\Omega)} \frac{|f(t,\Phi)|\varphi_1}{1+t^{n+\iota^+_{[\beta],0}+\{\beta\}}}dw < \infty \tag{1.11}$$

and

$$\int_{S_n(\Omega)} \frac{|g(t,\Phi)|}{1+t^{n+\iota^+_{[\beta],0}+\{\beta\}-2}}\frac{\partial\varphi_1}{\partial n}d\sigma_Q < \infty \tag{1.12}$$

respectively from (1.4). We suppose in addition that $\Omega = \mathbf{S}_+^{n-1}$ and $\alpha = \beta - 1$ in (1.11)-(1.12), by Remark 1 we have

$$\int_{T_n} \frac{y_n|f(Y,y_n)|}{1+t^{n+\alpha+2}}dQ < \infty \text{ and } \int_{\partial T_n} \frac{|g(Y,0)|}{1+t^{n+\alpha}}dY < \infty,$$

which yield that $\mathcal{E}(\mathbf{S}_+^{n-1}, \alpha+1, 0)$ is equivalent to $(CH)_\alpha$ in the notation of [4].

Let us recall the classical case $a = 0$. If $u(P) \le 0$ is classical harmonic in T_n, continuous on $\overline{T_n}$ and $u \in \mathcal{E}(\mathbf{S}_+^{n-1}, 1, 0)$, then there exists a constant $d_5 \le 0$ such that (see [8, 19])

$$u(P) = d_5 x_n + \int_{\partial T_n} \mathbb{PI}(\mathbf{S}_+^{n-1}, 0)(P,Q)u(Q)dQ, \tag{1.13}$$

where $P = (X, x_n) \in T_n$, $\mathbb{PI}(\mathbf{S}_+^{n-1}, 0)(P,Q) = 2w_n^{-1}x_n|P-Q|^{-n}$ is the classical harmonic Poisson kernel for T_n and w_n is the area of the unit sphere in $\mathbf{R}^n$.

Deng (see [4]) has constructed a similar representation to (1.13) for $u \in \mathcal{E}(\mathbf{S}_+^{n-1}, \beta, 0)$, which is the integral with a modified classical Poisson kernel derived by subtracting of some special harmonic polynomials from $\mathbb{PI}(\mathbf{S}_+^{n-1}, 0)(P,Q)$. We will construct an integral representation of a generalized harmonic function as a modified Poisson a-integral corresponding to the operator Sch_a in a cone.

Next, we state our main results as follows.

Theorem 2. *If $u \in \mathcal{E}(\Omega, \beta, a)$, then $u \in \mathcal{D}(\Omega, \beta, a)$.*

Theorem 3. *If $u \in \mathcal{E}(\Omega, \beta, a)$, m is an integer such that $V_m(t) < V_{[\beta]}(t) + t^{\{\beta\}} \le V_{m+1}(t)$ $(t \ge 1)$, then the following properties hold:*

(I) If $\beta = 1$, then the integral

$$\int_{S_n(\Omega)} \mathbb{PI}(\Omega, a, 0)(P,Q)u(Q)d\sigma_Q,$$

is absolutely convergent, it represents a generalized harmonic function $U(\Omega, a, 0; u)(P)$ on $C_n(\Omega)$ and can be continuously extended to $\overline{C_n(\Omega)}$ such that $U(\Omega, a, 0; u)(P) = u(P)$ for $P = (r, \Theta) \in S_n(\Omega)$ and there exists a constant d_6 such that $u(P) = d_6 V_1(r)\varphi_1(\Theta) + U(\Omega, a, 0; u)(P)$ for $P = (r, \Theta) \in C_n(\Omega)$.

(*II*) *If* $\beta > 1$, *then*

(*i*) *The integral*

$$\int_{S_n(\Omega)} \mathbb{PI}(\Omega, a, m)(P,Q)u(Q)d\sigma_Q,$$

is absolutely convergent, it represents a generalized harmonic function $U(\Omega, a, m; u)(P)$ *on* $C_n(\Omega)$ *and can be continuously extended to* $\overline{C_n(\Omega)}$ *such that* $U(\Omega, a, m; u)(P) = u(P)$ *for* $P = (r, \Theta) \in S_n(\Omega)$.

(*ii*) *There exists a generalized harmonic polynomial*

$$h(P) = \sum_{j=0}^{m} \left(\sum_{v=1}^{v_j} d_{jv} \varphi_{jv}(\Theta) \right) V_j(r)$$

vanishing continuously on $\partial C_n(\Omega)$ *such that* $u(P) = U(\Omega, a, m; u)(P) + h(P)$ *for* $P = (r, \Theta) \in C_n(\Omega)$, *where* d_{jv} *are constants.*

The following results generalize Deng's result (see [4]) to the conical case.

Corollary 4. *If* $u \in \mathcal{E}(\Omega, \beta, 0)$ *(see Remark 4 for* $\mathcal{E}(\Omega, \beta, 0)$*), then* $u \in \mathcal{D}(\Omega, \beta, 0)$.

Corollary 5. *If* $u \in \mathcal{E}(\Omega, \beta, 0)$, m *is an integer such that* $\iota^+_{m,0} < \iota^+_{[\beta],0} + \{\beta\} \leq \iota^+_{m+1,0}$, *then the following properties hold:*

(*I*) *If* $\beta = 1$, *then the integral*

$$\int_{S_n(\Omega)} \mathbb{PI}(\Omega, 0, 0)(P,Q)u(Q)d\sigma_Q,$$

is absolutely convergent, it represents a harmonic function $U(\Omega, 0, 0; u)(P)$ *on* $C_n(\Omega)$ *and can be continuously extended to* $\overline{C_n(\Omega)}$ *such that* $U(\Omega, 0, 0; u)(P) = u(P)$ *for* $P = (r, \Theta) \in S_n(\Omega)$ *and there exists a constant* d_7 *such that* $U(P) = d_7 r^{\iota^+_{1,0}} \varphi_1(\Theta) + U(\Omega, 0, 0; u)(P)$ *for* $P = (r, \Theta) \in C_n(\Omega)$.

(*II*) *If* $\beta > 1$, *then*

(*i*) *The integral*

$$\int_{S_n(\Omega)} \mathbb{PI}(\Omega, 0, m)(P,Q)u(Q)d\sigma_Q,$$

is absolutely convergent, it represents a harmonic function $U(\Omega, 0, m; u)(P)$ *on* $C_n(\Omega)$ *and can be continuously extended to* $\overline{C_n(\Omega)}$ *such that* $U(\Omega, 0, m; u)(P) = u(P)$ *for* $P = (r, \Theta) \in S_n(\Omega)$.

(*ii*) *There exists a harmonic polynomial*

$$h(P) = \sum_{j=0}^{m} \left(\sum_{v=1}^{v_j} d'_{jv} \varphi_{jv}(\Theta) \right) r^{\iota^+_{j,0}}$$

vanishing continuously on $\partial C_n(\Omega)$ *such that* $u(P) = U(\Omega, 0, m; u)(P) + h(P)$ *for* $P = (r, \Theta) \in C_n(\Omega)$, *where* d'_{jv} *are constants.*

2. LEMMAS

Throughout this paper, let M denote various constants independent of the variables in questions, which may be different from line to line.

Lemma 1.

(i) $\mathbb{PI}(\Omega,a)(P,Q)\le Mr^{\iota^{-}_{1,k}}t^{\iota^{+}_{1,k}-1}\varphi_1(\Theta)$

(ii) *(resp.* $\mathbb{PI}(\Omega,a)(P,Q)\le Mr^{\iota^{+}_{1,k}}t^{\iota^{-}_{1,k}-1}\varphi_1(\Theta))$

for any $P=(r,\Theta)\in C_n(\Omega)$ *and any* $Q=(t,\Phi)\in S_n(\Omega)$ *satisfying* $0<\frac{t}{r}\le\frac{4}{5}$ $(resp.\ 0<\frac{r}{t}\le\frac{4}{5})$;

(iii) $\mathbb{PI}(\Omega,0)(P,Q)\le M\frac{\varphi_1(\Theta)}{t^{n-1}}+M\frac{r\varphi_1(\Theta)}{|P-Q|^n}$

for any $P=(r,\Theta)\in C_n(\Omega)$ *and any* $Q=(t,\Phi)\in S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))$.

Proof. (i) and (ii) are obtained by A. Kheyfits (see [5, Ch. 11]). (iii) follows from V. S. Azarin (see [2, Lemma 4 and Remark]).

Lemma 2. (see [10]). *For a non-negative integer* m*, we have*

$$|\mathbb{PI}(\Omega,a,m)(P,Q)|\le M(n,m,s)V_{m+1}(r)\frac{W_{m+1}(t)}{t}\varphi_1(\Theta)\frac{\partial\varphi_1(\Phi)}{\partial n_\Phi}\tag{2.1}$$

for any $P=(r,\Theta)\in C_n(\Omega)$ *and* $Q=(t,\Phi)\in S_n(\Omega)$ *satisfying* $r\le st$ $(0<s<1)$*, where* $M(n,m,s)$ *is a constant dependent of* n*,* m *and* s*.*

The following Lemma plays an important role in our discussions, which is due to B. Ya. Levin and A. Kheyfits (see [5, p. 356]).

Lemma 3. *If* $R>r>0$ *and* $u(t,\Phi)$ *is a generalized harmonic function on a domain containing* $C_n(\Omega;(r,R))$*, then*

$$\begin{aligned}&\int_{S_n(\Omega;R)}\frac{\chi'(R)}{V_1(R)}u(R,\Phi)\varphi_1(\Phi)dS_R\\&+\int_{S_n(\Omega;(r,R))}u(t,\Phi)\frac{\partial\varphi_1}{\partial n}\Psi(t)d\sigma_Q+d_6(r)+d_7(r)\frac{W_1(R)}{V_1(R)}=0,\end{aligned}\tag{2.2}$$

where

$$\begin{aligned}\Psi(t)&=W_1(t)-\frac{W_1(R)}{V_1(R)}V_1(t),\\d_8(r)&=\int_{S_n(\Omega;r)}u(r,\Phi)\varphi_1(\Phi)W_1'(r)-W_1(r)\varphi_1(\Phi)\frac{\partial u}{\partial n}dS_r,\\d_9(r)&=\int_{S_n(\Omega;r)}V_1(r)\varphi_1(\Phi)\frac{\partial u}{\partial n}-u(r,\Phi)\varphi_1(\Phi)V_1'(r)dS_r.\end{aligned}$$

Lemma 4. (see [11, Theorem 1]). *If m is an nonnegative integer and $u(r,\Theta)$ is a generalized harmonic function on $C_n(\Omega)$ satisfying*

$$\int_{\Omega} u^{+}(r,\Theta)dS_1 = O(r^{\iota^{+}_{m,k}}),\ \textit{as}\ r\to\infty, \tag{2.3}$$

then

$$u(r,\Theta)=\sum_{j=0}^{m}\left(\sum_{v=1}^{v_j} d_{jv}\varphi_{jv}(\Theta)\right)V_j(r),$$

where d_{jv} are constants.

Corollary 5. *Obviously, the conclusion of Lemma 4 holds true if (2.3) is replaced by*

$$\liminf_{r\to\infty,(r,\Theta)\in C_n(\Omega)} r^{-\iota^{+}_{m+1,k}}\int_{\Omega} u^{+}(r,\Theta)\varphi_1(\Theta)dS_1=0.$$

3. Proof of Theorem 1

We only prove the case $p>1$ and $\gamma\geq 0$, the remaining cases can be proved similarly.

For any fixed $P=(r,\Theta)\in C_n(\Omega)$, take a number satisfying $R>\max(1,\frac{r}{s})$ $(0<s<\frac{4}{5})$. If $\iota^{+}_{m+1,k}>\frac{\iota^{+}_{[\gamma],k}+\{\gamma\}-n+1}{p}$ and $\frac{1}{p}+\frac{1}{q}=1$, then $(-\iota^{+}_{m+1,k}-n+1+\frac{\iota^{+}_{[\gamma],k}+\{\gamma\}}{p})q+n-1<0$.

Then

$$\begin{aligned}
&\int_{S_n(\Omega;(R,\infty))}|\mathbb{PI}(\Omega,a,m)(P,Q)||u(Q)|d\sigma_Q\\
&\leq V_{m+1}(r)\varphi_1(\Theta)\int_{S_n(\Omega;(R,\infty))}\frac{|u(Q)|}{V_{m+1}(t)t^{n-1}}d\sigma_Q\\
&\leq Mr^{\iota^{+}_{m+1,k}}\varphi_1(\Theta)\Bigg(\int_{S_n(\Omega;(R,\infty))}\frac{|u(Q)|^p}{t^{\iota^{+}_{[\gamma],k}+\{\gamma\}}}d\sigma_Q\Bigg)^{\frac{1}{p}}\\
&\quad\times\Bigg(\int_{S_n(\Omega;(\frac{r}{s},\infty))}t^{(-\iota^{+}_{m+1,k}-n+1+\frac{\iota^{+}_{[\gamma],k}+\{\gamma\}}{p})q}d\sigma_Q\Bigg)^{\frac{1}{q}}\\
&\leq Mr^{\frac{\iota^{+}_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta)\\
&<\infty.
\end{aligned}$$

from (1.4), (1.6), Lemma 2 and Hölder's inequality.

Then $U(\Omega,a,m;u)(P)$ is finite for any $P\in C_n(\Omega)$. Since $\mathbb{PI}(\Omega,a,m)(P,Q)$ is a generalized harmonic function of $P\in C_n(\Omega)$ for any $Q\in S_n(\Omega)$, $U(\Omega,a,m;u)(P)$ is also a generalized harmonic function of $P\in C_n(\Omega)$.

Now we study the boundary behavior of $U(\Omega, a, m; u)(P)$. Let $Q' = (t', \Phi') \in \partial C_n(\Omega)$ be any fixed point and l be any positive number satisfying $l > \max(t'+1, \frac{4}{5}R)$.

Set $\chi_{S(l)}$ is the characteristic function of $S(l) = \{Q = (t, \Phi) \in \partial C_n(\Omega), t \leq l\}$ and write

$$U(\Omega, a, m; u)(P) = U'(P) - U''(P) + U'''(P),$$

where

$$U'(P) = \int_{S_n(\Omega;(0,\frac{5}{4}l])} \mathbb{PI}(\Omega, a)(P, Q)u(Q)d\sigma_Q,$$

$$U''(P) = \int_{S_n(\Omega;(1,\frac{5}{4}l])} \frac{\partial K(\Omega, a, m)(P, Q)}{\partial n_Q} u(Q)d\sigma_Q$$

and

$$U'''(P) = \int_{S_n(\Omega;(\frac{5}{4}l,\infty))} \mathbb{PI}(\Omega, a, m)(P, Q)u(Q)d\sigma_Q.$$

Notice that $U'(P)$ is the Poisson a-integral of $u(Q)\chi_{S(\frac{5}{4}l)}$, we have $\lim_{P\to Q', P\in C_n(\Omega)} U'(P) = u(Q')$. Since $\lim_{\Theta\to\Phi'} \varphi_{jv}(\Theta) = 0$ $(j = 1, 2, 3\ldots; 1 \leq v \leq v_j)$ as $P = (r, \Theta) \to Q' = (t', \Phi') \in S_n(\Omega)$, we have $\lim_{P\to Q', P\in C_n(\Omega)} U''(P) = 0$ from the definition of the kernel function $K(\Omega, a, m)(P, Q)$. $U'''(P) = O(r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta))$ and therefore tends to zero.

So the function $U(\Omega, a, m; u)(P)$ can be continuously extended to $\overline{C_n(\Omega)}$ such that

$$\lim_{P\to Q', P\in C_n(\Omega)} U(\Omega, a, m; u)(P) = u(Q')$$

for any $Q' = (t', \Phi') \in \partial C_n(\Omega)$ from the arbitrariness of l.

For any $\epsilon > 0$, there exists $R_\epsilon > 1$ such that

$$\int_{S_n(\Omega;(R_\epsilon,\infty))} \frac{|u(Q)|^p}{1 + t^{\iota^+_{[\gamma],k}+\{\gamma\}}} d\sigma_Q < \epsilon. \tag{3.1}$$

The relation $G(\Omega, a)(P, Q) \leq G(\Omega, 0)(P, Q)$ implies this inequality (see [1])

$$\mathbb{PI}(\Omega, a)(P, Q) \leq \mathbb{PI}(\Omega, 0)(P, Q). \tag{3.2}$$

For $0 < s < \frac{4}{5}$ and any fixed point $P = (r, \Theta) \in C_n(\Omega)$ satisfying $r > \frac{5}{4}R_\epsilon$, let $I_1 = S_n(\Omega; (0, 1))$, $I_2 = S_n(\Omega; [1, R_\epsilon])$, $I_3 = S_n(\Omega; (R_\epsilon, \frac{4}{5}r])$, $I_4 = S_n(\Omega; (\frac{4}{5}r, \frac{5}{4}r))$, $I_5 = S_n(\Omega; [\frac{5}{4}r, \frac{r}{s}))$, $I_6 = S_n(\Omega; [1, \frac{r}{s}))$ and $I_7 = S_n(\Omega; [\frac{r}{s}, \infty))$, we write

$$U(\Omega, a, m; u)(P) \leq \sum_{i=1}^{7} U_{\Omega,a,i}(P),$$

where

$$U_{\Omega,a,i}(P) = \int_{I_i} |\mathbb{PI}(\Omega,a)(P,Q)||u(Q)|d\sigma_Q \ (i=1,2,3,4,5),$$
$$U_{\Omega,a,6}(P) = \int_{I_6} |\mathbb{PI}(\Omega,a,m)(P,Q)||u(Q)|d\sigma_Q,$$
$$U_{\Omega,a,7}(P) = \int_{I_7} |\frac{\partial \widetilde{K}(\Omega,a,m)(P,Q)}{\partial n_Q}||u(Q)|d\sigma_Q.$$

If $\iota^+_{[\gamma],k}+\{\gamma\} > (-\iota^+_{1,k}-n+2)p+n-1$, then $(\iota^+_{1,k}-1+\frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q+n-1>0$. By (1.6), (3.1), Lemma 1 (i) and Hölder's inequality, we have the following growth estimates

$$\begin{aligned} &U_{\Omega,a,2}(P)\\ &\leq Mr^{\iota^-_{1,k}}\varphi_1(\Theta)\int_{I_2} t^{\iota^+_{1,k}-1}|u(Q)|d\sigma_Q\\ &\leq Mr^{\iota^-_{1,k}}\varphi_1(\Theta)\Big(\int_{I_2}\frac{|u(Q)|^p}{t^{\iota^+_{[\gamma],k}+\{\gamma\}}}d\sigma_Q\Big)^{\frac{1}{p}}\Big(\int_{I_2} t^{(\iota^+_{1,k}-1+\frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q}d\sigma_Q\Big)^{\frac{1}{q}}\\ &\leq Mr^{\iota^-_{1,k}}R_\epsilon^{\iota^+_{1,k}+n-2+\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta). \end{aligned} \tag{3.3}$$

$$U_{\Omega,a,1}(P) \leq Mr^{\iota^-_{1,k}}\varphi_1(\Theta). \tag{3.4}$$

$$U_{\Omega,a,3}(P) \leq M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta). \tag{3.5}$$

If $\iota^+_{m,k} > \frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}$, then $(\iota^-_{1,k}-1+\frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q+n-1<0$. We obtain by (3.1), Lemma 1 (ii) and Hölder's inequality

$$\begin{aligned} &U_{\Omega,a,5}(P)\\ &\leq Mr^{\iota^+_{1,k}}\varphi_1(\Theta)\int_{S_n(\Omega;[\frac{5}{4}r,\infty))} t^{\iota^-_{1,k}-1}|u(Q)|d\sigma_Q\\ &\leq Mr^{\iota^+_{1,k}}\varphi_1(\Theta)\Big(\int_{S_n(\Omega;[\frac{5}{4}r,\infty))}\frac{|u(Q)|^p}{t^{\iota^+_{[\gamma],k}+\{\gamma\}}}d\sigma_Q\Big)^{\frac{1}{p}}\\ &\qquad\Big(\int_{S_n(\Omega;[\frac{5}{4}r,\infty))} t^{(\iota^-_{1,k}-1+\frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q}d\sigma_Q\Big)^{\frac{1}{q}}\\ &\leq M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta). \end{aligned} \tag{3.6}$$

By (3.2) and Lemma 1 (iii), we consider the inequality

$$U_{\Omega,a,4}(P) \leq U_{\Omega,0,4}(P) \leq U'_{\Omega,0,4}(P) + U''_{\Omega,0,4}(P),$$

where

$$U'_{\Omega,0,4}(P) = M\varphi_1(\Theta)\int_{I_4} t^{1-n}|u(Q)|d\sigma_Q,$$

$$U''_{\Omega,0,4}(P) = Mr\varphi_1(\Theta)\int_{I_4} \frac{|u(Q)|}{|P-Q|^n}d\sigma_Q.$$

We first have

$$(3.7)\qquad \begin{aligned} &U'_{\Omega,0,4}(P)\\ &= M\varphi_1(\Theta)\int_{I_4} t^{\iota^+_{1,k}+\iota^-_{1,k}-1}|u(Q)|d\sigma_Q\\ &\leq Mr^{\iota^+_{1,k}}\varphi_1(\Theta)\int_{S_n(\Omega;(\frac{4}{5}r,\infty))} t^{\iota^-_{1,k}-1}|u(Q)|d\sigma_Q\\ &\leq M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta), \end{aligned}$$

which is similar to the estimate of $U_{\Omega,a,5}(P)$.

Next, we shall estimate $U''_{\Omega,0,4}(P)$.

Take a sufficiently small positive number d_{10} such that $I_4 \subset B(P, \frac{1}{2}r)$ for any $P = (r,\Theta) \in \Pi(d_{10})$, where

$$\Pi(d_{10}) = \{P = (r,\Theta) \in C_n(\Omega);\ \inf_{z\in\partial\Omega} |(1,\Theta)-(1,z)| < d_{10},\ 0 < r < \infty\}.$$

and divide $C_n(\Omega)$ into two sets $\Pi(d_{10})$ and $C_n(\Omega) - \Pi(d_{10})$.

If $P = (r,\Theta) \in C_n(\Omega) - \Pi(d_{10})$, then there exists a positive d'_{10} such that $|P-Q| \geq d'_{10}r$ for any $Q \in S_n(\Omega)$, and hence

$$(3.8)\qquad \begin{aligned} U''_{\Omega,0,4}(P) &\leq M\varphi_1(\Theta)\int_{I_4} t^{1-n}|u(Q)|d\sigma_Q\\ &\leq M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta), \end{aligned}$$

which is similar to the estimate of $U'_{\Omega,0,4}(P)$.

We shall consider the case $P = (r,\Theta) \in \Pi(d_{10})$. Now put

$$H_i(P) = \{Q \in I_4;\ 2^{i-1}\delta(P) \leq |P-Q| < 2^i\delta(P)\}.$$

Since $S_n(\Omega) \cap \{Q \in \mathbf{R}^n : |P-Q| < \delta(P)\} = \varnothing$, we have

$$U''_{\Omega,0,4}(P) = M \sum_{i=1}^{i(P)} \int_{H_i(P)} r\varphi_1(\Theta) \frac{|u(Q)|}{|P-Q|^n} d\sigma_Q,$$

where $i(P)$ is a positive integer satisfying $2^{i(P)-1}\delta(P) \leq \frac{r}{2} < 2^{i(P)}\delta(P)$.

From (1.1) we see that $r\varphi_1(\Theta) \leq M\delta(P)$ $(P = (r,\Theta) \in C_n(\Omega))$. Similar to the estimate of $U'_{\Omega,0,4}(P)$, we obtain

$$\begin{aligned}
&\int_{H_i(P)} r\varphi_1(\Theta) \frac{|u(Q)|}{|P-Q|^n} d\sigma_Q \\
&\leq \int_{H_i(P)} r\varphi_1(\Theta) \frac{|u(Q)|}{(2^{i-1}\delta(P))^n} d\sigma_Q \\
&\leq M 2^{(1-i)n} \varphi_1^{1-n}(\Theta) \int_{H_i(P)} t^{1-n} |u(Q)| d\sigma_Q \\
&\leq M\epsilon r^{\frac{\iota^+_{[\gamma],k} + \{\gamma\} - n + 1}{p}} \varphi_1^{1-n}(\Theta)
\end{aligned}$$

for $i = 0, 1, 2, \ldots, i(P)$.

So

$$U''_{\Omega,0,4}(P) \leq M\epsilon r^{\frac{\iota^+_{[\gamma],k} + \{\gamma\} - n + 1}{p}} \varphi_1^{1-n}(\Theta). \tag{3.9}$$

We only consider $U_{\Omega,a,6}(P)$ in the case $m \geq 1$, since $U_{\Omega,a,6}(P) \equiv 0$ for $m = 0$. By the definition of $\widetilde{K}(\Omega, a, m)$, (1.2) and Lemma 2, we see

$$U_{\Omega,a,6}(P) \leq \frac{M}{\chi'(1)} \sum_{j=0}^{m} j^{2n-1} q_j(r),$$

where

$$q_j(r) = V_j(r)\varphi_1(\Theta) \int_{I_6} \frac{W_j(t)|u(Q)|}{t} d\sigma_Q.$$

To estimate $q_j(r)$, we write

$$q_j(r) \leq q'_j(r) + q''_j(r),$$

where

$$\begin{aligned}
q'_j(r) &= V_j(r)\varphi_1(\Theta) \int_{I_2} \frac{W_j(t)|u(Q)|}{t} d\sigma_Q, \\
q''_j(r) &= V_j(r)\varphi_1(\Theta) \int_{S_n(\Omega;(R_\epsilon, \frac{r}{s}))} \frac{W_j(t)|u(Q)|}{t} d\sigma_Q.
\end{aligned}$$

If $\iota^+_{m+1,k} < \frac{\iota^+_{[\gamma],k} + \{\gamma\} - n + 1}{p} + 1$, then $(-\iota^+_{m+1,k} - n + 2 + \frac{\iota^+_{[\gamma],k} + \{\gamma\}}{p})q + n - 1 > 0$. Notice that

$$V_j(r) \frac{V_{m+1}(t)}{V_j(t)t} \leq M \frac{V_{m+1}(r)}{r} \leq M r^{\iota^+_{m+1,k} - 1} \ (t \geq 1, R_\epsilon < \frac{r}{s}).$$

Thus, by (1.4), (1.6) and Hölder's inequality we conclude

$$\begin{aligned}
q'_j(r) &= V_j(r)\varphi_1(\Theta)\int_{I_2}\frac{|u(Q)|}{V_j(t)t^{n-1}}d\sigma_Q\\
&\le MV_j(r)\varphi_1(\Theta)\int_{I_2}\frac{V_{m+1}(t)}{t^{\iota^+_{m+1,k}}}\frac{|u(Q)|}{V_j(t)t^{n-1}}d\sigma_Q\\
&\le r^{\iota^+_{m+1,k}-1}\varphi_1(\Theta)\Big(\int_{I_2}\frac{|u(Q)|^p}{t^{\iota^+_{[\gamma],k}+\{\gamma\}}}d\sigma_Q\Big)^{\frac{1}{p}}\Big(\int_{I_2}t^{(-\iota^+_{m+1,k}-n+2+\frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q}d\sigma_Q\Big)^{\frac{1}{q}}\\
&\le Mr^{\iota^+_{m+1,k}-1}R_\epsilon^{-\iota^+_{m+1,k}+1+\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta).
\end{aligned}$$

Analogous to the estimate of $q'_j(r)$, we have

$$q''_j(r)\le M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta).$$

Thus we can conclude that

$$q_j(r)\le M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta),$$

which yields

$$U_{\Omega,a,6}(P)\le M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta). \tag{3.10}$$

If $\iota^+_{m+1,k}>\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}$, then $(-\iota^+_{m+1,k}-n+1+\frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q+n-1<0$. By (3.1), Lemma 2 and Hölder's inequality we have

$$\begin{aligned}
U_{\Omega,0,7}(P) &\le MV_{m+1}(r)\varphi_1(\Theta)\int_{I_7}\frac{|u(Q)|}{V_{m+1}(t)t^{n-1}}d\sigma_Q\\
&\le MV_{m+1}(r)\varphi_1(\Theta)\Big(\int_{I_7}\frac{|u(Q)|^p}{t^{\iota^+_{[\gamma],k}+\{\gamma\}}}d\sigma_Q\Big)^{\frac{1}{p}}\\
&\quad\Big(\int_{I_7}t^{(-\iota^+_{m+1,k}-n+1+\frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q}d\sigma_Q\Big)^{\frac{1}{q}}\\
&\le M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta).
\end{aligned}\tag{3.11}$$

Combining (3.3)-(3.11), we obtain that if R_ϵ is sufficiently large and ϵ is sufficiently small, then $U(\Omega,a,m;u)(P)=o(r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1^{1-n}(\Theta))$ as $r\to\infty$, where $P=(r,\Theta)\in C_n(\Omega;(R_\epsilon,+\infty))$. Then we complete the proof of Theorem 1.

4. Proof of Theorem 2

We apply the formula (2.2) with $R > r = 1$ to $u = u^+ - u^-$ in $C_n(\Omega; (1, R))$.

$$\begin{aligned} &m_+(R) + \int_{S_n(\Omega;(1,R))} u^+ \Psi(t) \frac{\partial \varphi_1}{\partial n} d\sigma_Q + d_6 + \frac{W_1(R)}{V_1(R)} d_7 \\ &= m_-(R) + \int_{S_n(\Omega;(1,R))} u^- \Psi(t) \frac{\partial \varphi_1}{\partial n} d\sigma_Q, \end{aligned} \tag{4.1}$$

where

$$m_\pm(R) = \int_{S_n(\Omega;R)} \frac{\chi'(R)}{V_1(R)} u^\pm \varphi_1 dS_R,$$

$$\begin{aligned} d_6 &= \int_{S_n(\Omega;1)} u\varphi_1 W_1'(1) - W_1(1)\varphi_1 \frac{\partial u}{\partial n} dS_1\,, \\ d_7 &= \int_{S_n(\Omega;1)} V_1(1)\varphi_1 \frac{\partial u}{\partial n} - u\varphi_1 V_1'(1) dS_1. \end{aligned}$$

Since $u \in \mathcal{E}(\Omega, \beta, a)$, we obtain by (1.9)

$$\begin{aligned} &\int_1^\infty \frac{m_+(R)V_1(R)}{\chi'(R)V_{[\beta]}(R)R^{n+\{\beta\}}} dR \\ &= \int_{C_n(\Omega;(1,\infty))} \frac{u^+\varphi_1}{V_{[\beta]}(t)t^{n+\{\beta\}}} dw \le 2\int_{C_n(\Omega)} \frac{u^+\varphi_1}{1+V_{[\beta]}(t)t^{n+\{\beta\}}} dw < \infty. \end{aligned} \tag{4.2}$$

From (1.10), we conclude that

$$\begin{aligned} &\int_1^\infty \frac{V_1(R)}{\chi'(R)V_{[\beta]}(R)R^{n+\{\beta\}}} \int_{S_n(\Omega;(1,R))} u^+\Psi(t) \frac{\partial \varphi_1}{\partial n} d\sigma_Q dR \\ &= \int_{S_n(\Omega;(1,\infty))} u^+ V_1(t) \int_t^\infty \frac{V_1(R)}{\chi'(R)V_{[\beta]}(R)R^{n+\{\beta\}}} \\ &\qquad \left(\frac{W_1(t)}{V_1(t)} - \frac{W_1(R)}{V_1(R)}\right) dR \frac{\partial \varphi_1}{\partial n} d\sigma_Q \\ &\le M \int_{S_n(\Omega;(1,\infty))} \frac{u^+ V_1(t) W_1(t)}{\chi'(t)V_{[\beta]}(t)t^{n+\{\beta\}-1}} \frac{\partial \varphi_1}{\partial n} d\sigma_Q \\ &\le M \int_{S_n(\Omega)} \frac{u^+ V_1(t) W_1(t)}{1+\chi'(t)V_{[\beta]}(t)t^{n+\{\beta\}-1}} \frac{\partial \varphi_1}{\partial n} d\sigma_Q \\ &< \infty. \end{aligned} \tag{4.3}$$

Combining (4.1), (4.2) and (4.3), we obtain

$$\begin{aligned}
&\int_1^\infty \frac{V_1(R)}{\chi'(R)V_{[\beta]}(R)R^{n+\frac{\{\beta\}}{2}}} \int_{S_n(\Omega;(1,R))} u^-\Psi(t)\frac{\partial\varphi_1}{\partial n}d\sigma_Q dR \\
\leq &\int_1^\infty \frac{m_+(R)V_1(R)}{\chi'(R)V_{[\beta]}(R)R^{n+\frac{\{\beta\}}{2}}}dR \\
&+\int_1^\infty \frac{V_1(R)}{\chi'(R)V_{[\beta]}(R)R^{n+\frac{\{\beta\}}{2}}} \int_{S_n(\Omega;(1,R))} u^+\Psi(t)\frac{\partial\varphi_1}{\partial n}d\sigma_Q dR \\
&+\int_1^\infty \frac{1}{\chi'(R)V_{[\beta]}(R)R^{n+\frac{\{\beta\}}{2}}}\left(V_1(R)d_6+W_1(R)d_7\right)dR \\
< \ &\infty.
\end{aligned}$$

Set

$$\begin{aligned}
&\mathcal{H}(\beta) \\
= &\lim_{t\to\infty}\frac{\chi'(t)V_{[\beta]}(t)t^{n+\{\beta\}-1}}{W_1(t)}\int_t^\infty \frac{V_1(R)}{\chi'(R)V_{[\beta]}(R)R^{n+\frac{\{\beta\}}{2}}}\left(\frac{W_1(t)}{V_1(t)}-\frac{W_1(R)}{V_1(R)}\right)dR \\
= &\lim_{t\to\infty} t^{\iota^+_{[\beta],k}+\iota^+_{1,k}+n+\{\beta\}-2}\int_t^\infty \frac{1}{R^{\iota^+_{[\beta],k}-\iota^+_{1,k}+\frac{\{\beta\}}{2}+1}}\left(\frac{1}{t^{\chi_{1,k}}}-\frac{1}{R^{\chi_{1,k}}}\right)dR,
\end{aligned}$$

where $\chi_{1,k}=\iota^+_{1,k}-\iota^-_{1,k}$.

By the L'hospital's rule, we have

$$\mathcal{H}(\beta)=\begin{cases} \frac{\chi_{1,k}}{(\iota^+_{[\beta],k}-\iota^+_{1,k})(\iota^+_{[\beta],k}+\iota^+_{1,k}+n-2)} & \text{if} \quad \{\beta\}=0, \\ +\infty & \text{if} \quad \{\beta\}\neq 0, \end{cases}$$

which yields that there exists a positive constant M such that for any $t\geq 1$,

$$\int_t^\infty \frac{V_1(R)}{\chi'(R)V_{[\beta]}(R)R^{n+\frac{\{\beta\}}{2}}}\Psi(t)dR \geq \frac{MV_1(t)W_1(t)}{\chi'(t)V_{[\beta]}(t)t^{n+\{\beta\}-1}}.$$

Then

$$\begin{aligned}
&M\int_{S_n(\Omega;(1,\infty))}\frac{u^-V_1(t)W_1(t)}{\chi'(t)V_{[\beta]}(t)t^{n+\{\beta\}-1}}\frac{\partial\varphi_1}{\partial n}d\sigma_Q \\
\leq &\int_{S_n(\Omega;(1,\infty))} u^-\int_t^\infty \frac{V_1(R)}{\chi'(R)V_{[\beta]}(R)R^{n+\frac{\{\beta\}}{2}}}\Psi(t)dR\frac{\partial\varphi_1}{\partial n}d\sigma_Q \\
< \ &\infty,
\end{aligned}$$

which shows that $u\in\mathcal{D}(\Omega,\beta,a)$ from $|u|=u^++u^-$. Then Theorem 2 is proved.

5. Proof of Theorem 3

To prove (II). Notice that $V_m(t) < V_{[\beta]}(t)t^{\{\beta\}} \leq V_{m+1}(t)$ $(t \geq 1)$ and condition (1.10) is stronger than (1.7). So the proofs of (i) are similar to them as in Theorem 1. Here we omit them.

Finally we consider the function $u(P) - U(\Omega, a, m; u)(P)$, which is generalized harmonic in $C_n(\Omega)$ and vanishes continuously on $\partial C_n(\Omega)$.

Since

$$0 \leq (u(P) - U(\Omega, a, m; u)(P))^+ \leq u^+(P) + (U(\Omega, a, m; u))^-(P) \tag{5.1}$$

for any $P \in C_n(\Omega)$.

Further, (1.4) and (1.9) give that

$$\liminf_{r\to\infty,(r,\Theta)\in C_n(\Omega)} r^{-\iota^+_{m+1,k}} \int_\Omega u^+(r,\Theta)\varphi_1(\Theta)dS_1 = 0. \tag{5.2}$$

By virtue of (1.8), (5.1), (5.2) and Corollary 5, the conclusion (ii) holds.

If $u \in \mathcal{E}(\Omega, 1, a)$, then $u \in \mathcal{E}(\Omega, \beta, a)$ for each $\beta > 1$ and there exists a constant d_9 such that

$$u(P) = d_{11}V_1(r)\varphi_1(\Theta) + U(\Omega, a, 1; u)(P)$$

for all $P \in C_n(\Omega)$. So if we take $d_6 = d_{11} - \int_{S_n(\Omega;[1,\infty))} P(\Omega, a, 1)(0, Q)u(Q)d\sigma_Q$, we see that $u(P) = d_6V_1(r)\varphi_1(\Theta) + U(\Omega, a, 0; u)(P)$ holds for all $P \in C_n(\Omega)$. We complete the proof of Theorem 3.

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