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APPROXIMATE CONTROLLABILITY OF FRACTIONAL ORDER NEUTRAL STOCHASTIC INTEGRO-DIFFERENTIAL SYSTEM WITH NONLOCAL CONDITIONS AND INFINITE DELAY

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Abstract. This paper deals with the approximate controllability of fractional order neutral stochastic integro-differential system with nonlocal conditions and infinite delay in Hilbert spaces under the assumptions that the corresponding linear system is approximately controllable. The control function for this system is suitably constructed by using the infinite dimensional controllability operator. With this control function, the sufficient conditions for approximate controllability of the proposed probelm in Hilbert space is established. Further, the results are obtained by using fractional calculus, stochastic analysis techniques, Sadovskii fixed point theorem and similar to the classical linear growth condition and the Lipschitz condition. Finally an example is provided to illustrate the application of the obtained results.

1. INTRODUCTION

Fractional order differential equations have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry, engineering, etc, [10, 12]. Fractional order differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives (see [26] and references therein). There has been a great deal of interest in the solutions of fractional differential equations in analytical and numerical senses. Metzler et al. [18] have discussed in detail about the recent developments in the description of anomalous transport and the random walk's guide

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to anomalous diffusion using the fractional dynamics approach. Lakshmikantham [13] initiated the basic theory for fractional functional differential equations. Benchohra et al. [4] consider the IVP for a particular class of fractional neutral functional differential equations with infinite delay. The theory of fractional differential equations has been extensively studied by many authors [39, 40]. One of the basic qualitative behaviours of a dynamical system is controllability. The controllability problem has been discussed for fractional dynamical systems in [2, 37]. Tai [36] studied the exact controllability of fractional impulsive neutral integro differential systems with a nonlocal Cauchy condition in Banach spaces. However, in order to establish the results, the assumption made in [36] were that the semigroup associated with linear part is compact and the controllability operator is also compact, hence the induced inverse does not exist in the infinite dimensional state space. Thus, the concept of exact controllability is too strong and the approximate controllability is more appropriate for these control systems. Sakthivel et al. [30] proved the approximate controllability by assuming that the C_0-semigroup is compact and the nonlinear function is continuous and uniformly bounded. Recently, Sukavanam et al. [35] have proved some sufficient conditions for the approximate controllability of a fractional order system in which the nonlinear term depends on both state and control variables. Yan [38] proved approximate controllability of partial neutral functional differential systems of fractional order with state-dependent delay by using the Krasnoselskii-Schaefer type fixed point theorem with the fractional power of operators.

The deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore, the study of stochastic problems are more applicable in dynamical system theory. Controllability plays an important role both in deterministic and stochastic system theory. Only few authors have been studied the extensions of deterministic controllability concepts to stochastic control systems. Sakthivel et al. [31] studied existence of pseudo almost automorphic mild solutions to stochastic fractional differential equations by using stochastic analysis theory and fixed point strategy. El-Borai et al. [9] studied semigroup and some fractional stochastic integral equations. Recently Sakthivel et al. [29, 32] established a set of sufficient conditions for obtaining the approximate controllability of fractional stochastic differential systems.

On the other hand, Byszewski et al. [5] introduced nonlocal initial conditions into the initial value problems and argued that the corresponding models more accurately describe the phenomena since more information was taken into account at the onset of the experiment, thereby reducing the ill effects incurred by a single (possibly erroneous) initial measurement. Mophou et al. [19] studied existence of mild solution for some fractional differential equations with nonlocal condition. Chang et al. [6] investigate the fractional order integrodifferential equations with nonlocal conditions in the Riemann-Liouville fractional derivative sense. Zhang et al. [39] studied the existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions and infinite delay. Lin et al. [14] proved the existence results for impulsive neutral stochastic functional integro-differential inclusions with nonlocal initial conditions. Some authors have been studied the existence and controllability of stochastic differential equations with nonlocal conditions (see [3, 22]).

Motivated by the above literature, the aim of the proposed work is to establish sufficient conditions for the approximate controllability of fractional order neutral stochastic integro-differential system with nonlocal conditions and infinite delay of the form

$$\begin{aligned} {}^cD_t^{\alpha}[x(t)-h(t,x_t)] &= A[x(t)-h(t,x_t)]+Bu(t)+f(t,x_t) \\ &\quad +\int_{-\infty}^{t} g(t,s,x_s)dW(s) \quad t\in J=[0,b], \\ x(0)+\mu(x) &= x_0=\phi, \quad \phi\in\mathcal{C}_v, \end{aligned} \tag{1}$$

where $0<\alpha<1$; ${}^cD_t^{\alpha}$ denotes the Caputo fractional derivative operator of order α. Here, $x(\cdot)$ takes value in the Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. The operator A generates a strongly continuous semigroup of bounded linear operators $T(t)$ in H (see [24]). The control function $u(\cdot)$ is takes the values in $L_2^{\mathfrak{F}}(J,U)$, a Banach space of admissible control functions, for a separable Hilbert space U. B is a bounded linear operator from U into H. Let K be another separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_K$ and the norm $\|\cdot\|_K$. Suppose $\{W(t)\}_{t\geq 0}$ is a given K valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q\geq 0$. We are also employing the same notation $\|\cdot\|$ for the norm of $L(K,H)$, where $L(K,H)$ denotes the space of all bounded linear operators from K into H, simply $L(H)$ if $K=H$. The histories $x_t:(-\infty,0]\to\mathcal{C}_v$ defined by $x_t=\{x(t+\theta),\theta\in(-\infty,0]\}$ belong to the phase space $\mathcal{C}_v$ which is defined in Section 2. $h:J\times\mathcal{C}_v\to H$, $f:J\times\mathcal{C}_v\to H$ and $g:J\times J_1\times\mathcal{C}_v\to L_Q(K,H)$, are appropriate functions, where $J_1=(-\infty,b]$ and $L_Q(K,H)$ denotes the space of all Q- Hilbert Schmidt operators from K into H. Cui et al. [7] proved the existence result for fractional neutral stochastic integro-differential equations with infinite delay by using the Sadovskii's fixed point theorem. Under this assumptions, let g be a strongly measurable mapping such that $\int_{-\infty}^{t}\|g(t,s,x_s)\|_Q^2ds<\infty$. $\mu:C(J,H)\to H$ is bounded and the initial data x_0 is an $\mathfrak{F}_0$- adapted H- valued random variable independent of Wiener process W.

To the best of our knowledge, there is no work reported on the approximate controllability of fractional order neutral stochastic integro-differential system with nonlocal conditions and infinite delay in Hilbert spaces. The paper is organized as follows: Section 2 contains preliminaries such as definitions of fractional calculus and lemmas. In section 3 we discuss the main result of this paper. In section 4. an example is given to illustrate our results.

2. Preliminaries

For more details of this section, the reader may refer to [1, 8, 20, 23, 25, 27, 34] and the references therein. Throughout the paper $(H,\|\cdot\|)$ and $(K,\|\cdot\|_K)$ denote real separable Hilbert spaces. Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space equipped with a normal filtration $\{\mathfrak{F}_t, t \in J\}$ satisfying the usual conditions(i.e., right continuous and $\mathfrak{F}_0$ containing all P- null sets of $\mathfrak{F}$). An H-valued random variable is an $\mathfrak{F}-$ measurable function $x(t) : \Omega \to H$, and the collection of random variables $S = \{x(t, \omega) : \Omega \to H|_{t\in J}\}$ is called a stochastic process. Generally, we just write $x(t)$ instead of $x(t, \omega)$ and $x(t) : J \to H$ in the space of S. Let $\{\zeta_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of K. Suppose that $\{W(t); t \geq 0\}$ is a K- valued Wiener process with finite trace nuclear covariance operator $Q \geq 0$, denote $Tr(Q) = \sum_{i=1}^{\infty} \lambda_i < \infty$, which satisfies $Q\zeta_i = \lambda_i\zeta_i$. So, actually, $W(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\beta_i(t)\zeta_i$, where $\{\beta_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. We assume that $\mathfrak{F}_t = \sigma\{W(s) : 0 \leq s \leq t\}$ is the σ- algebra generated by W and $\mathfrak{F}_b = \mathfrak{F}$. Let $\chi \in L(K, H)$ and define

$$\|\chi\|_Q^2 = \text{Tr}(\chi Q \chi^*) = \sum_{i=1}^{\infty} \|\sqrt{\lambda_i}\chi\zeta_i\|^2.$$

If $\|\chi\|_Q < \infty$, then χ is called a Q-Hilbert Schmidt operator. Let $L_Q(K, H)$ denote the space of all Q-Hilbert Schmidt operators $\chi : K \to H$. The completion $L_Q(K, H)$ of $L(K, H)$ with respect to the topology induces by the norm $\|\cdot\|_Q$ where $\|\chi\|_Q^2 = \langle \chi, \chi \rangle$ is a Hilbert space with the above norm topology. The collection of all strongly measurable, square integrable H valued random variables, denoted by $L_2(\Omega, \mathfrak{F}, P; H) \equiv L_2(\Omega; H)$, is a Banach space equipped with norm $\|x(\cdot)\|_{L_2} = (E\|x(\cdot; \omega)\|_H^2)^{1/2}$, where the expectation, E is defined by $E(h_1) = \int_\Omega h_1(\omega)dP$. Let $J_1 = (-\infty, b]$ and $C(J_1, L_2(\Omega; H))$ be the Banach space of all continuous maps from J_1 into $L_2(\Omega; H)$ satisfying the condition $\sup_{t\in J_1} E\|x(t)\|^2 < \infty$.
Now, we present the abstract phase space $\mathcal{C}_v$. Assume that $v : (-\infty, 0] \to (0, +\infty)$ is a continuous function satisfying $l = \int_{-\infty}^0 v(t)dt < +\infty$. The Banach space $\Big(\mathcal{C}_v, \|\cdot\|_{\mathcal{C}_v}\Big)$ induced by the function v is defined as follows

$$\mathcal{C}_v = \begin{cases} \varphi : (-\infty, 0] \to H, \text{ for any } \quad a > 0, E(|\varphi(\theta)|^2)^{1/2} \\ \qquad\qquad\qquad\qquad \text{is a boundedand measurable function} \\ \text{on} \quad [-a, 0] \quad \text{and} \displaystyle\int_{-\infty}^0 v(s) sup_{s\leq\theta\leq 0} E(|\varphi(\theta)|^2)^{1/2} ds < +\infty\}. \end{cases}.$$

If $\mathcal{C}_v$ is endowed with the norm $\|\varphi\|_{\mathcal{C}_v} = \int_{-\infty}^0 v(s) sup_{s\leq\theta\leq 0} E(|\varphi(\theta)|^2)^{1/2} ds, \quad \varphi \in \mathcal{C}_v$. Denote by $C((-\infty, b], H)$ the space of all continuous H- valued stochastic process $\{\xi(t), \quad t \in (-\infty, b]\}$. Let $\mathcal{C}_b = \{x; \quad x \in C((-\infty, b], H), x_0 = \phi \in \mathcal{C}_v\}$.

Set $\|\cdot\|_b$ be a seminorm defined by

$$\|x\|_b = \|x_0\|_{\mathcal{C}_v} + \sup_{s\in[0,t]} \big(E|x(s)|^2\big)^{1/2}, \quad x \in \mathcal{C}_b.$$

In addition to the familiar Young, Holder and Minkowski inequalities, the inequality of the form $(\sum_{i=1}^{n} a_i)^m \leq n^{m-1}\sum_{i=1}^{n} a_i^m$, where a_i is a nonnegative constants $(i = 1, 2, \ldots, n)$ and $m, n \in \mathbb{N}$, is helpful in establishing various estimates.

Definition 2.1. The fractional integral of order α with the lower limit 0 for a function f is defined as

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds, \quad t > 0, \quad \alpha > 0,$$

provided the right hand side is pointwise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Caputo derivative of order α with the lower limit 0 for a function f can be written as

$${}^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}}ds = I^{n-\alpha}f^n(t), \quad t > 0, \quad 0 \leq n-1 < \alpha < n.$$

The Caputo derivative of a constant equal to zero. If f is an abstract function with values in H, then the integrals which appear in the above definitions are taken in Bochner's sense (see [17]).

Definition 2.3. [7, 39] An H- valued stochastic process $\{x(t), t \in (-\infty, b]\}$ is a mild solution of the system (1) if $x(0) + \mu(x) = x_0 = \phi \in \mathcal{C}_v$, and for each $u \in L_2^{\mathfrak{F}}(J, U)$ the process x satisfies the following integral equation

$$\begin{aligned} x(t) &= \widehat{T}_{\alpha}(t)[\phi(0) - \mu(x) - h(0,\phi)] + h(t, x_t) \\ &\quad + \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s)Bu(s)ds + \int_0^t (t-s)^{\alpha-1} \\ &\quad \times T_{\alpha}(t-s)f(s,x_s)ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s)\Big[\int_{-\infty}^{s} g(s,\tau,x_{\tau})dW(\tau)\Big]ds, \end{aligned} \tag{2}$$

Where $\widehat{T}_{\alpha}(t)x = \int_0^{\infty}\eta_{\alpha}(\theta)T(t^{\alpha}\theta)xd\theta$, $T_{\alpha}(t)x = \alpha\int_0^{\infty}\theta\eta_{\alpha}(\theta)T(t^{\alpha}\theta)xd\theta$, $\eta_{\alpha}(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\overline{w}_{\alpha}(\theta^{-\frac{1}{\alpha}}) \geq 0$, $\overline{w}_{\alpha}(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}sin(n\pi\alpha)$, $\theta \in (0,\infty)$, η_{α} is a probability density function defined on $(0,\infty)$, that is $\eta_{\alpha}(\theta) \geq 0$, $\theta \in (0,\infty)$ and $\int_0^{\infty}\eta_{\alpha}(\theta)d\theta = 1$.

Lemma 2.4. (see [40]). *The operators $\widehat{T}_{\alpha}(t)$ and $T_{\alpha}(t)$ have the following properties:*

(a) *For any fixed $t \geq 0$, the operator $\widehat{T}_{\alpha}(t)$ and $T_{\alpha}(t)$ are linear and bounded operators, i.e., for any $x \in H$, $\|\widehat{T}_{\alpha}(t)x\| \leq M\|x\|$ and $\|T_{\alpha}(t)x\| \leq \frac{M\alpha}{\Gamma(1+\alpha)}\|x\|$.*

(b) *$\{\widehat{T}_\alpha(t), t \geq 0\}$ and $\{T_\alpha(t), t \geq 0\}$ are strongly continuous.*
(c) *For every $t > 0$, $\widehat{T}_\alpha(t)$ and $T_\alpha(t)$ are also compact operators.*

It is convenient to introduce the relevant operators and the basic controllability condition

(1) The operator $L_0^b \in \mathcal{L}(L_2^{\mathfrak{F}}(J, H), L_2(\Omega, \mathfrak{F}_b, H))$ is defined by

$$L_0^b u = \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) Bu(s) ds,$$

clearly the adjoint $(L_0^b)^* : L_2(\Omega, \mathfrak{F}_b, H) \to L_2^{\mathfrak{F}}(J, H)$ is defined by

$$[(L_0^b)^* z](t) = B^* T_\alpha^*(b-t) E\{z|\mathfrak{F}_t\}.$$

(2) The controllability operator Π_0^b associated with the linear stochastic system of (1) is defined by

$$\Pi_0^b\{\cdot\} = L_0^b (L_0^b)^* \{\cdot\} = \int_0^b (b-s)^{2(\alpha-1)} T_\alpha(b-s) BB^* T_\alpha^*(b-s) E\{\cdot|\mathfrak{F}_s\} ds$$

which belongs to $\mathcal{L}(L_2(\mathfrak{F}_b, H), L_2(\mathfrak{F}_b, H))$ and the controllability operator $\psi_t^b \in \mathcal{L}(H, H)$ is

$$\psi_t^b = \int_t^b (b-s)^{2(\alpha-1)} T_\alpha(b-s) BB^* T_\alpha^*(b-s) ds, \quad 0 \leq s \leq t.$$

Let $x(t; \phi, u)$ denotes state value of the system (1) at time t corresponding to the control $u \in L_2^{\mathfrak{F}}(J, U)$. In particular, the state of system (1) at $t = b, x(b; \phi, u)$ is called the terminal state with control u. $\mathfrak{R}(b; \phi, u) = \{x(b; \phi, u), \quad u \in L_2^{\mathfrak{F}}(J, U)\}$ is called the reachable set of the system (1).

Definition 2.5. The system (1) is approximately controllable on J if $\overline{\mathfrak{R}(b; \phi, u)} = L_2(\Omega, \mathfrak{F}, H)$, where $\overline{\mathfrak{R}(b; \phi, u)}$ is the closure of the reachable set.

Lemma 2.6. *For any $\overline{x}_b \in L_2(\Omega, \mathfrak{F}, P; H)$ there exists $\gamma(s) \in L_2^{\mathfrak{F}}(\Omega, L_2(J; L_Q(K, H)))$ such that $\overline{x}_b = E\overline{x}_b + \int_0^b \gamma(s) dW(s)$.*

We define the control function in the following form

$$\begin{aligned} & u^\lambda(t, x) \\ &= B^*(b-t)^{\alpha-1} T_\alpha^*(b-t) \Big[(\lambda I + \psi_0^b)^{-1} \\ & \qquad \Big(E\overline{x}_b - \widehat{T}_\alpha(b)(\phi(0) - \mu(x) - h(0, \phi)) - h(b, x_b) \Big) \Big] \\ & \quad - B^*(b-t)^{\alpha-1} T_\alpha^*(b-t) \int_0^t (\lambda I + \psi_s^b)^{-1} (b-s)^{\alpha-1} T_\alpha(b-s) f(s, x_s) ds \end{aligned} \tag{3}$$

$$-B^*(b-t)^{\alpha-1}T^*_\alpha(b-t)\int_0^t(\lambda I+\psi^b_s)^{-1}(b-s)^{\alpha-1}T_\alpha(b-s)$$
$$\Big[\int_{-\infty}^s g(s,\tau,x_\tau)dW(\tau)\Big]ds$$
$$+B^*(b-t)^{\alpha-1}T^*_\alpha(b-t)\int_0^t(\lambda I+\psi^b_s)^{-1}\gamma(s)dW(s).$$

Lemma 2.7. *Assume that $x\in\mathcal{C}_b$, then for all $t\in J$, $x_t\in\mathcal{C}_v$. Moreover,*

$$l\big(E|x(t)|^2\big)^{1/2}\le\|x_t\|_{\mathcal{C}_v}\le l\sup_{s\in[0,t]}\big(E|x(s)|^2\big)^{1/2}+\|x_0\|_{\mathcal{C}_v}.$$

To prove our main results, we list the following basic assumptions of this paper.

(H_1) The function $h,f:J\times\mathcal{C}_v\to H$ are continuous, and there exist positive constants M_h, M_f such that

$$E\|h(t,x)-h(t,y)\|^2_H\le M_h\|x-y\|^2_{\mathcal{C}_v},$$
$$E\|h(t,x)\|^2_H\le M_h(1+\|x\|^2_{\mathcal{C}_v}),$$
$$E\|f(t,x)-f(t,y)\|^2_H\le M_f\|x-y\|^2_{\mathcal{C}_v},$$
$$E\|f(t,x)\|^2_H\le M_f(1+\|x\|^2_{\mathcal{C}_v}),\quad\text{for every}\quad x,y\in\mathcal{C}_v,t\in J.$$

(H_2) μ is continuous and there exists some positive constants M_μ such that

$$E\|\mu(x)-\mu(y)\|^2_H\le M_\mu\|x-y\|^2_{\mathcal{C}_v},$$
$$E\|\mu(x)\|^2_H\le M_\mu(1+\|x\|^2_{\mathcal{C}_v}),\quad\text{for every}\quad x,y\in\mathcal{C}_v.$$

(H_3) For each $\varphi\in\mathcal{C}_v$, $k(t)=\lim_{a\to\infty}\int_{-a}^0 g(t,s,\varphi)dW(s)$ exists and is continuous. Further, there exists a positive constant M_k such that $E\|k(t)\|^2_H\le M_k$.

(H_4) The function $g:J\times J_1\times\mathcal{C}_v\to L_Q(K,H)$ satisfies the following

(i) for each $x\in\mathcal{C}_v$, $g(\cdot,\cdot,x):J\times J\to L_Q(K,H)$ is strongly measurable, and for each $(t,s)\in J\times J$, $g(t,s,\cdot):\mathcal{C}_v\to L_Q(K,H)$ is continuous and there exists a constant $L_g>0$ such that

$$\int_0^t E\|g(t,s,x)-g(t,s,y)\|^2_Q ds\le L_g\|x-y\|^2_{\mathcal{C}_v},\quad\text{for every}\quad x,y\in\mathcal{C}_v.$$

(ii) there is a positive integrable function $m\in L^1([0,b])$ and a continuous non-decreasing function $M_g:[0,\infty)\to(0,\infty)$ such that for every $(t,s,x)\in J\times J\times\mathcal{C}_v$, we have

$$\int_0^t E\|g(t,s,x)\|^2_Q ds\ \le\ m(t)M_g(\|x\|^2_{\mathcal{C}_v}),\ \lim_{r\to\infty}inf\frac{M_g(r)}{r}ds=\Delta<\infty.$$

(H_5) Let $R(\lambda, \psi_t^b) = (\lambda I + \psi_t^b)^{-1}$ for $\lambda > 0$. For each $0 \le t \le b$, the operator $\lambda(\lambda I + \psi_t^b)^{-1} \to 0$ in the strong operator topology as $\lambda \to 0^+$. Observe that the linear fractional deterministic control system

$$(4)\qquad \begin{aligned} {}^cD_t^\alpha x(t) &= Ax(t) + Bu(t), \quad t \in [0, b], \\ x(0) &= x_0, \end{aligned}$$

corresponding to equation (1) is approximately controllable on $[t, b]$ iff the operator $\lambda(\lambda I + \psi_t^b)^{-1} \to 0$ strongly as $\lambda \to 0^+$. The approximate controllability for linear fractional deterministic control system (4) is a natural generalization of approximate controllability of linear first order control system (Theorem 2 in [15, 16]).

Theorem 2.8. (see [28]). *Assume that Φ is a condensing operator on a Banach space $\mathbb{H}$. i.e., Φ is continuous and takes bounded sets into bounded sets, and $\alpha(\Phi(\mathbb{B})) \le \alpha(\mathbb{B})$ for every bounded set of $\mathbb{B}$ of $\mathbb{H}$ with $\alpha(\mathbb{B}) > 0$. If $\Phi(\mathbb{D}) \subset \mathbb{D}$ for a convex, closed and bounded set of $\mathbb{D}$ of $\mathbb{H}$, then Φ has a fixed point in $\mathbb{D}$ (where $\alpha(\cdot)$ denotes the Kuratowski Measure of non compactness).*

3. Approximate Controllability

Theorem 3.1. *Assume that the hypotheses (H_1)-(H_4) hold then for each $0 < \lambda \le 1$, the operator D has a fixed point in $\mathcal{C}_b$ provided that*

$$\begin{aligned} &20M^2M_\mu l^2 + 20M_h l^2 + 100\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{N_1 b}{\lambda^2} \\ &\quad\Big(M^2M_\mu l^2 + M_h l^2 + \Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^{2\alpha}}{\alpha^2}M_f l^2 + 2\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2 \\ &\frac{b^{2\alpha}}{\alpha^2}l^2 Tr(Q)\Delta\sup_{t\in J} m(s)\Big) + 20\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^{2\alpha}}{\alpha^2}M_f l^2 \\ &\quad + 40\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^{2\alpha}}{\alpha^2}l^2 Tr(Q)\Delta\sup_{t\in J} m(s) < 1 \end{aligned}$$

and

$$\begin{aligned} &\Big((M^2M_\mu + M_h)\Big(3 + \frac{12bN_1}{\lambda^2}\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\Big) \\ &+ \frac{12N_1}{\lambda^2}\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^4\frac{b^{2\alpha+1}}{\alpha^2}(M_f + 2Tr(Q)L_g)\Big) < 1. \end{aligned}$$

Proof. Define the mapping $D : \mathcal{C}_b \to \mathcal{C}_b$ as

$$(Dx)(t) = \begin{cases} \phi(t), & t \in (-\infty, 0] \\ \widehat{T}_\alpha(t)[\phi(0) - \mu(x) - h(0,\phi)] + h(t, x_t) & \\ \quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) Bu(s) ds & \\ \quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x_s) ds & \\ \quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) [\int_{-\infty}^s g(s,\tau,x_\tau) dW(\tau)] ds, & t \in J. \end{cases}.$$

Now, we are able to show that D has a fixed point in the space $\mathcal{C}_b$, which is the mild solution for the system (1). Let $x(t) = z(t) + \widehat{\rho}(t), \quad -\infty < t \le b$, where $\widehat{\rho}(t)$ is defined by

$$\widehat{\rho}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \widehat{T}_\alpha(t)\phi(0), & t \in J. \end{cases}.$$

It is evident that z satisfies $z_0 = 0, \quad t \in (-\infty, 0]$ and

$$\begin{aligned} z(t) \;=\; & \widehat{T}_\alpha(t)\Big[-\mu(z+\widehat{\rho}) - h(0,\phi)\Big] + h(t, z_t + \widehat{\rho}_t) \\ & + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) Bu(s) ds + \int_0^t (t-s)^{\alpha-1} \\ & \times T_\alpha(t-s) f(s, z_s + \widehat{\rho}_s) ds \\ & + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \Big[\int_{-\infty}^s g(s,\tau,z_\tau + \widehat{\rho}_\tau) dW(\tau)\Big] ds. \end{aligned}$$

Define the Banach space $\big(\mathcal{C}_b^0, \|\cdot\|_b\big)$ induced by $\mathcal{C}_b$, $\mathcal{C}_b^0 = \{z \in \mathcal{C}_b, \quad z_0 = 0 \in \mathcal{C}_v\}$ with norm

$$\|z\|_b = \|z_0\|_{\mathcal{C}_v} + \sup_{s\in[0,b]} \big(E|z(s)|^2\big)^{1/2} = \sup_{s\in[0,b]} \big(E|z(s)|^2\big)^{1/2}.$$

Set $B_q = \{z \in \mathcal{C}_b^0, \quad \|z\|_b^2 \le q\}$ for some $q > 0$. Then B_q, for each q, is a bounded, closed convex set in $\mathcal{C}_b^0$. For $z \in B_q$, by Lemma 2.7, we have

$$\begin{aligned} \|z_t + \widehat{\rho}_t\|_{\mathcal{C}_v}^2 \;\le\; & 2\big(\|z_t\|_{\mathcal{C}_v}^2 + \|\widehat{\rho}_t\|_{\mathcal{C}_v}^2\big), \\ \le\; & 4\big(l^2 \sup_{s\in[0,t]} E\|z(s)\|^2 + \|z_0\|_{\mathcal{C}_v}^2 + l^2 \sup_{s\in[0,t]} E\|\widehat{\rho}(s)\|^2 + \|\widehat{\rho}_0\|_{\mathcal{C}_v}^2\big), \\ \le\; & 4l^2(q + M^2 E\|\phi(0)\|_H^2) + 4\|\phi\|_{\mathcal{C}_v}^2. \end{aligned}$$

For each positive number q, B_q is clearly a bounded closed convex set in $\mathcal{C}_b^0$. We claim that there exists a positive number q such that $D(B_q) \subset B_q$. If this is not true, then for each positive integer q, there exist $z^q \in B_q$ and $t \in (-\infty, b]$ such that $E\|D(z^q)(t)\|_H^2 > q$

$$
\begin{aligned}
q \leq\ & E\|\widehat{T}_\alpha(t)\Big[-\mu(z^q+\widehat{\rho})-h(0,\phi)\Big]+h(t,z_t^q+\widehat{\rho}_t)\\
&+\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)B\Big\{B^*(b-s)^{\alpha-1}\\
&\times T_\alpha^*(b-s)\Big[(\lambda I+\psi_0^b)^{-1}(E\overline{x}_b+\widehat{T}_\alpha(b)[\mu(z^q+\widehat{\rho})+h(0,\phi)]-h(b,z_b^q+\widehat{\rho}_b))\\
&+\int_0^t(\lambda I+\psi_s^b)^{-1}\gamma(s)dW(s)\Big]-B^*(b-s)^{\alpha-1}T_\alpha^*(b-s)\\
&\quad\int_0^t(\lambda I+\psi_s^b)^{-1}(b-s)^{\alpha-1}T_\alpha(b-s)\\
&\times f(s,z_s^q+\widehat{\rho}_s)ds-B^*(b-s)^{\alpha-1}T_\alpha^*(b-s)\int_0^t(\lambda I+\psi_s^b)^{-1}(b-s)^{\alpha-1}T_\alpha(b-s)\\
&\times[\int_{-\infty}^s g(s,\tau,z_\tau^q+\widehat{\rho}_\tau)dW(s)]ds\Big\}ds+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s,z_s^q+\widehat{\rho}_s)ds\\
&+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)[\int_{-\infty}^s g(s,\tau,z_\tau^q+\widehat{\rho}_\tau)dW(\tau)]ds\|_H^2,\\
\leq\ & 5M^2M_\mu(1+\|z^q+\widehat{\rho}\|_{\mathcal{C}_v}^2)+5M^2M_h(1+\|\phi\|_{\mathcal{C}_v}^2)+5M_h(1+\|z_t^q+\widehat{\rho}_t\|_{\mathcal{C}_v}^2)+25\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\\
&\times\frac{N_1 b}{\lambda^2}(\|\overline{x}_b\|^2+M^2M_\mu(1+\|z^q+\widehat{\rho}\|_{\mathcal{C}_v}^2)+M^2M_h(1+\|\phi\|_{C_v}^2)+M_h(1+\|z_t^q+\widehat{\rho}_t\|_{C_v}^2))\\
&+25\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^4\frac{N_1}{\lambda^2}\frac{b^{\alpha+1}}{\alpha}\int_0^t(b-s)^{\alpha-1}E\|f(s,z_s^q+\phi_s)\|_H^2ds+25\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^4\frac{N_1}{\lambda^2}\frac{b^{\alpha+1}}{\alpha}\\
&\times\int_0^t(b-s)^{\alpha-1}E\|\int_{-\infty}^s g(s,\tau,z_\tau^q+\phi_\tau)dW(\tau)\|_Q^2ds+5\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^\alpha}{\alpha}\int_0^t(t-s)^{\alpha-1}\\
&\times E\|f(s,z_s^q+\phi_s)\|_H^2ds+5\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^\alpha}{\alpha}\int_0^t(t-s)^{\alpha-1}\\
&\times E\|\int_{-\infty}^s g(s,\tau,z_\tau^q+\phi_\tau)dW(\tau)\|_Q^2ds,\\
\leq\ & 5M^2M_\mu(1+4l^2(q+M^2E\|\phi(0)\|^2)+4\|\phi\|_{\mathcal{C}_v}^2)+5M^2M_h(1+\|\phi\|_{\mathcal{C}_v}^2)+5M_h(1+4l^2(q\\
&+M^2E\|\phi(0)\|^2)+4\|\phi\|_{\mathcal{C}_v}^2)+25\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{N_1 b}{\lambda^2}\Big[\|\overline{x}_b\|^2+M^2M_\mu(1+4l^2(q+M^2E\|\phi(0)\|^2)\\
&+4\|\phi\|_{\mathcal{C}_v}^2)+M^2M_h(1+\|\phi\|_{\mathcal{C}_v}^2)+M_h(1+4l^2(q+M^2E\|\phi(0)\|^2)+4\|\phi\|_{\mathcal{C}_v}^2)+\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\\
&\times\frac{b^{2\alpha}}{\alpha^2}M_f(1+4l^2(q+M^2E\|\phi(0)\|^2)+4\|\phi\|_{\mathcal{C}_v}^2)+\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^{2\alpha}}{\alpha^2}\{2M_k+2Tr(Q)M_g\\
&\times(4l^2(q+M^2E\|\phi(0)\|^2)+4\|\phi\|_{\mathcal{C}_v}^2)\sup_{s\in J}m(s)\}\Big]\\
&+5\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^{2\alpha}}{\alpha^2}M_f(1+4l^2(q+M^2E\|\phi(0)\|^2)
\end{aligned}
$$

$$+4\|\phi\|^2_{\mathcal{C}_v})+5\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^{2\alpha}}{\alpha^2}\{2M_k+2Tr(Q)M_g(4l^2(q+M^2E\|\phi(0)\|^2)$$
$$+4\|\phi\|^2_{\mathcal{C}_v})\sup_{s\in J}m(s)\},$$

dividing on both sides by q and taking $q\to\infty$, we get

$$20M^2M_\mu l^2+20M_hl^2+100\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{N_1b}{\lambda^2}\Big(M^2M_\mu l^2+M_hl^2$$
$$+\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^{2\alpha}}{\alpha^2}M_fl^2+2\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2$$
$$\frac{b^{2\alpha}}{\alpha^2}l^2Tr(Q)\Delta\sup_{t\in J}m(s)\Big)+20\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^{2\alpha}}{\alpha^2}M_fl^2$$
$$+40\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^{2\alpha}}{\alpha^2}l^2Tr(Q)\Delta\sup_{t\in J}m(s)\geq 1,$$

where $N_1=\|(t-s)^{2(\alpha-1)}T_\alpha(t-s)BB^*T^*_\alpha(t-s)\|^2$. This is contradiction to our assumption. Hence, for the some positive integer q, $D(B_q)\subset B_q$.

Next we show that the operator $D=D_1+D_2$ is condensing, the operators D_1 and D_2 are defined on B_q by, respectively

$$(D_1z)(t)=\widehat{T}_\alpha(t)\Big[-\mu(z+\widehat{\rho})-h(0,\phi)\Big]+h(t,z_t+\widehat{\rho}_t)$$
$$+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)Bu(s)ds,$$
$$(D_2z)(t)=\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s,z_s+\widehat{\rho}_s)ds+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)$$
$$\times\Big[\int_{-\infty}^s g(s,\tau,z_\tau+\widehat{\rho}_\tau)dW(\tau)\Big]ds.$$

In order to use Theorem 2.8, we will verify that D_1 is a contraction while D_2 is completely continuous. For better readability, we break the proof into a sequence of steps.

Step 1. D_1 is a contraction on B_q.
Let $t\in J$ and $z_1,z_2\in B_q$, we have

$$E\|(D_1z_1)(t)-(D_1z_2)(t)\|^2_H$$
$$\leq 3E\|\widehat{T}_\alpha(t)(\mu(z_1+\widehat{\rho})-\mu(z_2+\widehat{\rho}))\|^2_H+3E\|h(t,z_{1,t}+\widehat{\rho}_t)-h(t,z_{2,t}+\widehat{\rho}_t)\|^2_H$$
$$+3E\|\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)B\Big\{B^*(b-s)^{\alpha-1}T^*_\alpha(b-s)(\lambda I+\psi^b_0)^{-1}$$
$$\Big[\widehat{T}_\alpha(t)(\mu(z_1+\widehat{\rho})-\mu(z_2+\widehat{\rho}))$$

$$
\begin{aligned}
&+(h(t, z_{1,t}+\widehat{\rho}_t) - h(t, z_{2,t}+\widehat{\rho}_t))\Big] \\
&-B^*(b-s)^{\alpha-1}T^*_\alpha(b-s)\int_0^t (\lambda I+\psi_s^b)^{-1}(b-s)^{\alpha-1}T_\alpha(b-s) \\
&\times(f(s, z_{1,s}+\widehat{\rho}_s) - f(s, z_{2,s}+\widehat{\rho}_s))ds \\
&-B^*(b-s)^{\alpha-1}T^*_\alpha(b-s)\int_0^t (\lambda I+\psi_s^b)^{-1}(b-s)^{\alpha-1} \\
&\times T_\alpha(b-s)\Big(\int_0^s g(s,\tau, z_{1,\tau}+\widehat{\rho}_\tau)dW(\tau) - \int_0^s g(s,\tau, z_{2,\tau}+\widehat{\rho}_\tau)dW(\tau)\Big)ds\Big\}ds\|^2_H, \\
\leq\ & 3M^2M_\mu\|z_1-z_2\|^2_{\mathcal{C}_v}+3M_h\|z_{1,t}-z_{2,t}\|^2_{\mathcal{C}_v}+\frac{12bN_1}{\lambda^2}\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\Big[M^2M_\mu\|z_1-z_2\|^2_{\mathcal{C}_v} \\
&+M_h\|z_{1,t}-z_{2,t}\|^2_{\mathcal{C}_v}+\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^\alpha}{\alpha}\int_0^t (b-s)^{\alpha-1}M_f\|z_{1,s}-z_{2,s}\|^2_{\mathcal{C}_v}ds \\
&+2Tr(Q)\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^\alpha}{\alpha}\int_0^t (b-s)^{\alpha-1}L_g\|z_{1,s}-z_{2,s}\|^2_{\mathcal{C}_v}ds\Big], \\
\leq\ & \Big((M^2M_\mu+M_h)\Big(3+\frac{12bN_1}{\lambda^2}\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\Big) \\
&+\frac{12N_1}{\lambda^2}\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^4\frac{b^{2\alpha+1}}{\alpha^2}(M_f+2Tr(Q)L_g)\Big) \\
&\times\sup_{0\leq s\leq t} E\|z_1(s)-z_2(s)\|^2.
\end{aligned}
$$

Therefore

$$
\Big((M^2M_\mu+M_h)\Big(3+\frac{12bN_1}{\lambda^2}\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\Big) +\frac{12N_1}{\lambda^2}\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^4\frac{b^{2\alpha+1}}{\alpha^2}(M_f+2Tr(Q)L_g)\Big)<1.
$$

Thus, D_1 is a contraction mapping.

Step 2. D_2 maps bounded sets to bounded set in B_q.
In fact, if $z\in B_q$, from Lemma 2.7, it follows that

$$
\|z_t+\widehat{\rho}_t\|^2_{\mathcal{C}_v}\leq 4l^2(q+M^2E\|\phi(0)\|^2_H)+4\|\phi\|^2_{\mathcal{C}_v}=q^*, \quad \text{for all} \quad t\in J.
$$

$$
\begin{aligned}
&E\|D_2z(t)\|^2_H \\
\leq\ & 2E\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,z_s+\widehat{\rho}_s)ds\|^2_H+2E\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s) \\
&\times[\int_{-\infty}^s g(s,\tau,z_\tau+\widehat{\rho}_\tau)dW(\tau)]ds\|^2_Q, \\
\leq\ & 2\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}M_f(1+\|z_s+\widehat{\rho}_s\|^2_{\mathcal{C}_v})ds+2\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^\alpha}{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^t (t-s)^{\alpha-1}(2M_k+2Tr(Q)m(s)M_g(\|z_s+\widehat{\rho}_s\|)^2_{\mathcal{C}_v})ds,\\
&\leq 2\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^{2\alpha}}{\alpha^2}M_f(1+q^*)+4\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{b^{2\alpha}}{\alpha^2}(M_k+Tr(Q)M_g(q^*)\sup_{t\in J}m(s)),\\
&\leq q^{**},
\end{aligned}
$$

which is shows the desired result of the claim.

Step 3. The set of functions $\{D_2z, \quad z\in B_q\}$ is an equicontinuous on J.

Let $0<\epsilon<t<b$ and $\delta>0$ such that $\|T_\alpha(s)-T_\alpha(s^*)\|<\epsilon$, for every $s,s^*\in J$ with $\|s-s^*\|\leq\delta$. Let $0<t_1<t_2\leq b$, for each $z\in B_q$, we have

$$
\begin{aligned}
&E\|D_2z(t_2)-D_2z(t_1)\|^2_H\\
\leq\ & \|\int_0^{t_2}(t_2-s)^{\alpha-1}T_\alpha(t_2-s)f(s,z_s+\widehat{\rho}_s)ds+\int_0^{t_2}(t_2-s)^{\alpha-1}T_\alpha(t_2-s)\\
&\times[\int_{-\infty}^s g(s,\tau,z_\tau+\widehat{\rho}_\tau)dW(\tau)]ds-\int_0^{t_1}(t_1-s)^{\alpha-1}T_\alpha(t_1-s)f(s,z_s+\widehat{\rho}_s)ds\\
&-\int_0^{t_1}(t_1-s)^{\alpha-1}T_\alpha(t_1-s)[\int_{-\infty}^s g(s,\tau,z_\tau+\widehat{\rho}_\tau)dW(\tau)]ds\|^2_H,\\
\leq\ & 6E\|\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]T_\alpha(t_2-s)f(s,z_s+\widehat{\rho}_s)ds\|^2_H\\
&+6E\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}T_\alpha(t_2-s)f(s,z_s+\widehat{\rho}_s)ds\|^2_H+6E\|\int_0^{t_1}(t_1-s)^{\alpha-1}\\
&\times[T_\alpha(t_2-s)-T_\alpha(t_1-s)]f(s,z_s+\widehat{\rho}_s)ds\|^2_H+6E\|\int_0^{t_1}[(t_2-s)^{\alpha-1}\\
&-(t_1-s)^{\alpha-1}]T_\alpha(t_2-s)[\int_{-\infty}^s g(s,\tau,z_\tau+\widehat{\rho}_\tau)dW(\tau)]ds\|^2_Q\\
&+6E\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}T_\alpha(t_2-s)[\int_{-\infty}^s g(s,\tau,z_\tau+\widehat{\rho}_\tau)dW(\tau)]ds\|^2_Q\\
&+6E\|\int_0^{t_1}(t_1-s)^{\alpha-1}[T_\alpha(t_2-s)-T_\alpha(t_1-s)][\int_{-\infty}^s g(s,\tau,z_\tau+\widehat{\rho}_\tau)dW(\tau)]ds\|^2_Q,\\
\leq\ & 6\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\int_0^{t_1}|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}|^2M_f(1+q^*)ds\\
&+6\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\int_{t_1}^{t_2}|(t_2-s)^{\alpha-1}|^2M_f(1+q^*)ds\\
&+6\epsilon^2\int_0^{t_1}|(t_1-s)^{\alpha-1}|^2M_f(1+q^*)ds+6\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\int_0^{t_1}|(t_2-s)^{\alpha-1}
\end{aligned}
$$

$$-(t_1-s)^{\alpha-1}|^2(2M_k+2Tr(Q)m(s)M_g(q^*))ds$$
$$+6\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\int_{t_1}^{t_2}|(t_2-s)^{\alpha-1}|^2(2M_k+2Tr(Q)m(s)M_g(q^*))ds$$
$$+6\epsilon^2\int_0^{t_1}|(t_1-s)^{\alpha-1}|^2(2M_k+2Tr(Q)m(s)M_g(q^*))ds.$$

In view of Lemma 2.4 and the fact that $T_\alpha(\cdot)$ is compact, strongly continuous operator, the operator $T_\alpha(s)$ is continuous in the uniform operator topology on $(0,b]$. So, as $t_2-t_1\to 0$, with ϵ sufficiently small, the right hand side of the above inequality is independent of $z\in B_q$ and tends to zero. The equicontinuous for the cases $t_1<t_2\le 0$ or $t_1\le 0\le t_2\le b$ are very simple. Thus, the set $\{D_2z,\quad z\in B_q\}$ is equicontinuous.

Step 4. The set $\{(D_2z)(t),\quad z\in B_q\}$ is relatively compact in B_q.
Let $0<t\le b$ be fixed and ϵ be a real number satisfying $0<\epsilon<t$. For $\delta>0$, for $z\in B_q$, we define

$$\begin{aligned}&(D_2^{\epsilon,\delta}z)(t)\\
&=\alpha\int_0^{t-\epsilon}\int_\delta^\infty\theta(t-s)^{\alpha-1}\eta_\alpha(\theta)T((t-s)^\alpha\theta)f(s,z_s+\widehat{\rho}_s)ds+\alpha\int_0^{t-\epsilon}\int_\delta^\infty\theta(t-s)^{\alpha-1}\\
&\quad\times\eta_\alpha(\theta)T((t-s)^\alpha\theta)[\int_{-\infty}^s g(s,\tau,z_\tau+\widehat{\rho}_\tau)dW(\tau)]ds,\\
&=T(\epsilon^\alpha\delta)\alpha\int_0^{t-\epsilon}\int_\delta^\infty\theta(t-s)^{\alpha-1}\eta_\alpha(\theta)T((t-s)^\alpha\theta-\epsilon^\alpha\delta)f(s,z_s+\widehat{\rho}_s)ds+T(\epsilon^\alpha\delta)\alpha\\
&\quad\times\int_0^{t-\epsilon}\int_\delta^\infty\theta(t-s)^{\alpha-1}\eta_\alpha(\theta)T((t-s)^\alpha\theta-\epsilon^\alpha\delta)[\int_{-\infty}^s g(s,\tau,z_\tau+\widehat{\rho}_\tau)dW(\tau)]ds.\end{aligned}$$

We know that $\|D_2z(t)\|_H^2\le q^{**}$ and consequently, for $z\in B_q$, we find that

$$\begin{aligned}&D_2z(t)\\
&=T(\epsilon^\alpha\delta)\alpha\int_0^{t-\epsilon}\int_\delta^\infty\theta(t-s)^{\alpha-1}\eta_\alpha(\theta)T((t-s)^\alpha\theta-\epsilon^\alpha\delta)f(s,z_s+\widehat{\rho}_s)ds+T(\epsilon^\alpha\delta)\alpha\\
&\quad\times\int_0^{t-\epsilon}\int_\delta^\infty\theta(t-s)^{\alpha-1}\eta_\alpha(\theta)T((t-s)^\alpha\theta-\epsilon^\alpha\delta)[\int_{-\infty}^s g(s,\tau,z_\tau+\widehat{\rho}_\tau)dW(\tau)]ds\\
&\quad+\alpha\int_{t-\epsilon}^t\int_0^\infty\theta(t-s)^{\alpha-1}\eta_\alpha(\theta)T((t-s)^\alpha\theta)f(s,z_s+\widehat{\rho}_s)ds\\
&\quad+\alpha\int_{t-\epsilon}^t\int_0^\infty\theta(t-s)^{\alpha-1}\eta_\alpha(\theta)T((t-s)^\alpha\theta)[\int_{-\infty}^s g(s,\tau,z_\tau+\widehat{\rho}_\tau)dW(\tau)]ds,\\
&\quad\in T(\epsilon^\alpha\delta)B_{q^{**}}(0,H)+C_\epsilon,\end{aligned}$$

where diam $(C_\epsilon)\le\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{\epsilon^{2\alpha}}{\alpha^2}(M_f(1+q^*)+2M_k)+2Tr(Q)\Big\{\frac{M\alpha}{\Gamma(1+\alpha)}\Big\}^2\frac{\epsilon^\alpha}{\alpha}\int_{t-\epsilon}^t(t-s)^{\alpha-1}M_g(q^*)$

$\sup_{s\in J} m(s)ds$, which proves that $D_2 z$ is relatively compact in B_q. From the hypotheses (H_1) and (H_4), we know that D_2 is continuous. Therefore, from the Arzela-Ascoli theorem, the operator D_2 is completely continuous. From Theorem 2.8, D has a fixed point and which is a mild solution of (1). ■

Thus, by Theorem 3.1, for any $\lambda > 0$ the operator D has a fixed point x^λ in B_q, which is clearly a mild solution of the following equation

$$
\begin{aligned}
& x^\lambda(t) \\
= \; & \widehat{T}_\alpha(t)[\phi(0) - \mu(x) - h(0,\phi)] + h(t, x_t) \\
& + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) B \Big\{ B^*(b-s)^{\alpha-1} T_\alpha^*(b-s) \\
& \times \Big[(\lambda I + \psi_0^b)^{-1} \Big(E\overline{x}_b - \widehat{T}_\alpha(b)(\phi(0) - \mu(x) - h(0,\phi)) - h(b, x_b) \Big) \Big] \\
& - B^*(b-s)^{\alpha-1} T_\alpha^*(b-s) \int_0^t (\lambda I + \psi_s^b)^{-1} (b-s)^{\alpha-1} T_\alpha(b-s) f(s, x_s) ds \\
& - B^*(b-s)^{\alpha-1} T_\alpha^*(b-s) \int_0^t (\lambda I + \psi_s^b)^{-1} \\
& \times (b-s)^{\alpha-1} T_\alpha(b-s) [\int_{-\infty}^s g(s, \tau, x_\tau) dW(\tau)] ds \\
& + B^*(b-s)^{\alpha-1} T_\alpha(b-s) \int_0^t (\lambda I + \psi_s^b)^{-1} \\
& \times \gamma(s) dW(s) \Big\} ds + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x_s) ds \\
& + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \Big[\int_{-\infty}^s g(s, \tau, x_\tau) dW(\tau) \Big] ds, \\
= \; & \widehat{T}_\alpha(t)[\phi(0) - \mu(x) - h(0,\phi)] + h(t, x_t) + \psi_0^t T_\alpha^*(b-t)(\lambda I + \psi_0^b)^{-1} \Big(E\overline{x}_b - \widehat{T}_\alpha(b) \\
& \times (\phi(0) - \mu(x) - h(0,\phi)) - h(b, x_b) \Big) \\
& + \int_0^t [I - \psi_s^t T_\alpha^*(b-t)(\lambda I + \psi_s^b)^{-1} T_\alpha(b-t)] \\
& \times (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x_s) ds \\
& + \int_0^t [I - \psi_s^t T_\alpha^*(b-t)(\lambda I + \psi_s^b)^{-1} T_\alpha(b-t)](t-s)^{\alpha-1} \\
& \times T_\alpha(t-s) [\int_{-\infty}^s g(s, \tau, x_\tau) dW(\tau)] ds + \int_0^t \psi_s^t T_\alpha^*(b-t)(\lambda I + \psi_s^b)^{-1} \gamma(s) dW(s).
\end{aligned} \tag{5}
$$

Theorem 3.2. *Under the hypotheses $(H_1) - (H_5)$ and Theorem 3.1 hold, the functions f and g are uniformly bounded in H and $L_Q(K, H)$ then the system (1) is approximately controllable on J.*

Proof. Let $\overline{x}^{\lambda}$ be a solution of (1), then writing equation (5) at $t=b$ yields

$$\begin{aligned}\overline{x}^{\lambda}(b) &= \overline{x}_b - \lambda(\lambda I+\psi_0^b)^{-1}\Big(E\overline{x}_b - \widehat{T}_{\alpha}(b)(\phi(0)-\mu(\overline{x}^{\lambda})-h(0,\phi)) - h(b,\overline{x}_b^{\lambda})\Big)\\ &\quad -\int_0^b \lambda(\lambda I+\psi_s^b)^{-1}(b-s)^{\alpha-1}T_{\alpha}(b-s)f(s,\overline{x}_s^{\lambda})ds\\ &\quad -\int_0^b \lambda(\lambda I+\psi_s^b)^{-1}(b-s)^{\alpha-1}T_{\alpha}(b-s)\\ &\quad \times\Big[\int_{-\infty}^s g(s,\tau,\overline{x}_{\tau}^{\lambda})dW(\tau)\Big]ds - \int_0^b \lambda(\lambda I+\psi_s^b)^{-1}\gamma(s)dW(s)\end{aligned}$$

It follows from the assumption on f and g are uniformly bounded on J. Then, there is a subsequence, still denoted by $f(s,\overline{x}_s^{\lambda})$ and $\int_0^s g(s,\tau,\overline{x}_{\tau}^{\lambda})dW(\tau)$ which converges weakly to say, $f(s,w)$ in H, $g(s,\tau,w)$ in $L(K,H)$. The compactness of $T_{\alpha}(t), t>0$ which implies that $T_{\alpha}(b-s)f(s,\overline{x}_s^{\lambda}) \to T_{\alpha}(b-s)f(s,w)$, $T_{\alpha}(b-s)g(s,\tau,\overline{x}_{\tau}^{\lambda}) \to T_{\alpha}(b-s)g(s,\tau,w)$. On the other hand, by hypothesis (H_5), for all $0\le t\le b$, $\lambda(\lambda I+\psi_t^b)^{-1} \to 0$ strongly as $\lambda\to 0^+$ and $\|\lambda(\lambda I+\psi_t^b)^{-1}\|\le 1$. Therefore, by the Lebesque dominated convergence theorem it follows that

$$\begin{aligned}&E\|\overline{x}^{\lambda}(b)-\overline{x}_b\|^2\\ &\le E\|\lambda(\lambda I+\psi_0^b)^{-1}\Big(E\overline{x}_b - \widehat{T}_{\alpha}(b)(\phi(0)-\mu(\overline{x}^{\lambda})-h(0,\phi)) - h(b,\overline{x}_b^{\lambda})\Big)\\ &\quad -\int_0^b \lambda(\lambda I+\psi_s^b)^{-1}(b-s)^{\alpha-1}T_{\alpha}(b-s)f(s,\overline{x}_s^{\lambda})ds\\ &\quad -\int_0^b \lambda(\lambda I+\psi_s^b)^{-1}(b-s)^{\alpha-1}T_{\alpha}(b-s)\\ &\quad \times[\int_{-\infty}^s g(s,\tau,\overline{x}_{\tau}^{\lambda})dW(\tau)]ds - \int_0^b \lambda(\lambda I+\psi_s^b)^{-1}\gamma(s)dW(s)\|^2 \to 0 \quad \text{as} \quad \lambda\to 0^+.\end{aligned}$$

So $\overline{x}^{\lambda}(b)\to\overline{x}_b$ holds, which shows that the system (1) is approximately controllable and hence the theorem is proved. ■

Remark 3.3. In real world problems, impulsive effects also exist in addition to stochastic effects. The theory of impulsive differential equations is much richer than the theory of classical differential equations without impulse effects. The impulsive differential equations serve as basic models to study the dynamics of evolution processes that are subject to sudden changes in their states [33]. The applications of the impulsive differential equations arise in epidemiology, pharmacokinetics, fed-batch culture in fermentative production and population dynamics, etc. (see [21] and the references therein). Among the previous research, little is concerned with differential equations with fractional order and impulses [36, 39]. Moreover, impulsive control, which is based on the theory of impulsive differential equations has gained renewed interests recently for its promising applications toward controlling systems exhibiting chaotic behavior. The results in Theorem 3.2 can be extended to study the approximate controllability of fractional neutral stochastic integro-differential control systems with impulsive effects by employing the idea and technique as in Theorem 3.2.

4. Example

In this section an example is presented for the approximate controllability results to the following fractional order neutral stochastic partial differential system with nonlocal conditions and infinite delay in Hilbert spaces

$$
(6)\qquad \begin{aligned}
&{}^cD_t^\alpha\Big[z(t,x)-\int_{-\infty}^{t} e^{4(s-t)}z(s,x)ds\Big]\\
&= \frac{\partial^2}{\partial x^2}\Big[z(t,x)-\int_{-\infty}^{t} e^{4(s-t)}z(s,x)ds\Big]+\eta(t,x)\\
&+\int_{-\infty}^{0}\hat{a}(s)\sin z(t+s,x)ds\\
&+\int_{-\infty}^{t}\int_{-\infty}^{t}\xi(t,x,s-t)g(z(s,x))dsd\beta(s,x),\\
&x\in[0,\pi],\quad t\in J=[0,b]\\
&z(t,0)=z(t,\pi)=0\quad t\in J\\
&z(0,x)+\int_0^\pi k_1(x,y)z(t,y)dy=\varphi(t,x),\quad t\in(-\infty,0],
\end{aligned}
$$

where ${}^cD_t^\alpha$ is a Caputo fractional partial derivative of order $0<\alpha<1$, $b>0$, $k_1(x,y)\in L_2([0,\pi]\times[0,\pi])$ and $\int_{-\infty}^{0}|\hat{a}(s)|ds<\infty$. We can set this problem in our formulation by taking $H=L_2([0,\pi])$ defined on a stochastic space $(\Omega,\mathfrak{F},P)$. β is the real standard Wiener process(that is $K=\mathbb{R}$ and $Q=1$). The operator A is defined by $Az=z''$ with domain $D(A)=\{z\in H|z,z'$ are absolutely continuous, $z''\in H, z(0)=z(\pi)=0\}$. It is well known that A generates a strongly continuous semigroup $T(\cdot)$, which is compact, analytic and self adjoint. The spectrum of A consists of the eigen values $-n^2$ for $n\in\mathbb{N}$, with corresponding normalized eigen vectors $z_n(x)=\sqrt{\frac{2}{\pi}}sin(nx)$. In addition, the following properties hold

(a) $\{z_n; n\in\mathbb{N}\}$ is an orthonormal basis of H,

(b) If $\varrho\in D(A)$ then $A\varrho=-\sum_{n=1}^{\infty}n^2<\varrho,z_n>z_n$,

(c) For every $\varrho\in H$, $T(t)\varrho=\sum_{n=1}^{\infty}e^{-n^2t}<\varrho,z_n>z_n$,

(d) The operator $(-A)^{1/2}$ is given by $(-A)^{1/2}\varrho = \sum_{n=1}^{\infty} n < \varrho, z_n > z_n$ on the space $D((-A)^{1/2}) = \{\varrho(\cdot) \in H; \quad \sum_{n=1}^{\infty} n < \varrho, z_n > z_n \in H\}$.

Define the bounded linear operator $B : U \to H$ by $Bu(t)(x) = \eta(t, x)$, $0 \leq x \leq \pi, u \in U$.
Now, we present a special phase space C_v. Let $v(s) = e^{2s}, s < 0$, then $l = \int_{-\infty}^{0} v(s)ds = \frac{1}{2}$. Let $\|\varphi\|_{C_v} = \int_{-\infty}^{0} v(s) \sup_{s\leq\theta\leq 0}(E|\varphi(\theta)|^2)^{1/2}ds$, then it follows from the reference [11] that $(C_v, \|\cdot\|_{C_v})$ is a Banach space. For $(t, \varphi) \in J \times C_v$, where $\varphi(\theta)(x) = \varphi(\theta, x)$, $(\theta, x) \in (-\infty, 0] \times [0, \pi]$, and define the Lipschitz continuous functions $h, f : J \times C_v \to H$, $g : J \times C_v \to L_Q(H)$, for the infinite delay as follows

$$h(t, \varphi)(x) = \int_{-\infty}^{0} e^{-4\theta}\varphi(\theta)(x)d\theta,$$

$$f(t, \varphi)(x) = \int_{-\infty}^{0} \hat{a}(\theta)\sin(\varphi(\theta)(x))d\theta,$$

$$g(t, \varphi)(x) = \int_{-\infty}^{0} \xi(t, x, \theta)g(\varphi(\theta)(x))d\theta,$$

Then, the equation (6) can be rewritten as the abstract form as the system (1). Thus, under the appropriate conditions on the functions h, f, and g are satisfies the hypotheses $(H_1) - (H_4)$. On the other hand, it can be easily seen that the deterministic linear fractional control system corresponding to (6) is approximately controllable on $[0, \pi]$ (see [15, 16]). All conditions of the Theorem 3.2 is satisfied, therefore the system (6) is approximately controllable on $[0, \pi]$.

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