

Volume Inequalities for Asymmetric Orlicz Zonotopes

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Abstract. In this paper, we deal with the asymmetric Orlicz zonotopes by using the method of shadow system. We establish the volume product inequality and volume ratio inequality for asymmetric Orlicz zonotopes, along with their equality cases.

1. Introduction

A classical problem in convex geometry is to find the maximizer or minimizer of the volume product among convex bodies. The celebrated Blaschke-Santaló inequality characterizes ellipsoids are the maximizers of this function on convex bodies. However, finding the minimizer of this function is an open problem in convex geometry. Only in two dimensional case, this problem is solved by Mahler (see, e.g., [38, 39]). Moreover, it is conjectured by him that simplices are the solutions of this function for all dimensional n , which is called the Mahler's conjecture. Although it is extremely difficult to attack, but it attracts lots of author's interests, many substantial inroads have been made. One can refer to e.g., [1, 5, 9, 10, 16, 19, 25–27, 40–42] for more about this conjecture.

One aspect of the researches for the Mahler's conjecture is to induce the study of the volume product of zonotopes or zonoids, that is the Minkowski sums of origin-symmetric line segments in \mathbb{R}^n , and their limits with respect to the Hausdorff distance (see, e.g., [19, 41, 44]). Although the restriction to zonotopes and zonoids is a regrettable drawback, but there seems no approach for general convex bodies for this problem. On the other hand, inequalities for zonoids can be applied to stochastic geometry (see [45]).

In the last century, the volume product inequalities in Euclidean space, \mathbb{R}^n , have widely been generalized with the development of the L_p -Minkowski theory. See, for example, [7–9, 15, 29–34, 48, 49, 52] for more details about the volume product inequalities in the L_p -Minkowski theory. The L_p -volume product inequalities for zonotopes, together with its

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dual volume ratio inequality, were established by Campi and Cronchi [10]. These results extend the results of Reisner [41]. However, all of these results are restricted to the origin-symmetric setting. The asymmetric extension of the L_p -volume product inequality and L_p -volume ratio inequality, along with the characterization of its extremals are established by Weberndorfer in [51], and the Campi and Cronchi's results as a special case. The seminal work in studying the asymmetric geometric inequalities are very important in convex geometry. For example, in the paper of Ludwig [28], she has obtained a characterization of the asymmetric L_p -centroid body and asymmetric L_p -projection body, which establishes the classification of the $SL(n)$ invariant Minkowski valuation on convex set. After that, the asymmetric geometric inequalities involving the volume and other geometric invariant have emerged. For instance, the asymmetric L_p -centroid body operator turned out to be an extension of L_p version of the Blaschke-Santaló inequality for all convex bodies, whereas established by Lutwak and Zhang [37] for origin-symmetric setting. One can refer to [22–24, 46] for more details.

Beginning with the articles [21, 35, 36] of Haberl, Lutwak, Yang and Zhang, a wider extension of the L_p -Brunn-Minkowski theory emerged, called the Orlicz Brunn-Minkowski theory. In these papers, the Orlicz Busemann projection inequality and Orlicz Busemann centroid inequality were established. Recently, in a paper of Gardner, Hug and Weil [18], a systematic study is made on the Orlicz Minkowski addition, the Orlicz Brunn-Minkowski inequality and Orlicz Minkowski inequality are obtained. See, e.g., [2–4, 12–14, 18, 21, 53, 54] about the Orlicz Brunn-Minkowski theory.

In view of the importance of the volume product inequality in convex geometry, we tried to consider the naturally posed problem in the wide interest of the Orlicz Brunn-Minkowski theory. What is like the volume product inequality or volume ratio inequality for asymmetric Orlicz zonotopes? In this context, the main goal of this paper is to establish the volume product inequality and volume ratio inequality for asymmetric Orlicz zonotopes.

Throughout this paper, let \mathcal{C} be the class of convex, strictly increasing functions $\varphi: [0, \infty) \rightarrow [0, \infty)$ satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$.

Suppose that Λ is a finite set of vectors from $\mathbb{R}^n \setminus \{o\}$, the asymmetric Orlicz zonotope $Z_\varphi^+ \Lambda$ is the unique compact convex set with support function

$$h_{Z_\varphi^+ \Lambda}(u) = \inf \left\{ \lambda > 0 : \sum_{w \in \Lambda} \varphi \left(\frac{\langle w, u \rangle_+}{\lambda} \right) \leq 1 \right\},$$

where $u \in \mathbb{R}^n$ and $\langle w, u \rangle_+ = \max\{0, \langle w, u \rangle\}$ denotes the positive part of the Euclidean scalar product.

In particular, if we take $\varphi(t) = t^p$, $p \geq 1$, then $Z_\varphi^+ \Lambda$ is precisely the L_p -asymmetric zonotope $Z_p^+ \Lambda$ defined in [51].

In this paper, our main results are the volume product inequality and volume ratio inequality for the asymmetric Orlicz zonotopes.

Let $\Lambda_{\perp} = \{e_1, \dots, e_n\}$ denote the canonical basis of \mathbb{R}^n , $Z_{\varphi}^{+,*}\Lambda$ denotes the polar body of $Z_{\varphi}^{+}\Lambda$ with respect to the Santaló point. For the asymmetric Orlicz zonotopes, we establish the following volume product inequality.

Theorem 1.1. *Suppose $\varphi \in \mathcal{C}$ and Λ is a finite and spanning multiset. Then*

$$(1.1) \quad V(Z_{\varphi}^{+,*}\Lambda)V(Z_1^{+}\Lambda) \geq V(Z_{\varphi}^{+,*}\Lambda_{\perp})V(Z_1^{+}\Lambda_{\perp}).$$

Equality holds with $\varphi \neq \text{Id}$ if and only if Λ is a $\text{GL}(n)$ image of the canonical basis Λ_{\perp} . If $\varphi = \text{Id}$, the identity function, the equality holds if and only if $Z_1^{+}\Lambda$ is a parallelepiped.

We follow the notations of paper [51]. A set Λ of vectors from \mathbb{R}^n is called *obtuse* if every pair of distinct vectors u, v from Λ satisfies

$$\langle u, v \rangle_+ = 0.$$

Another result concerning to the volume ratio for asymmetric Orlicz zonotopes associated with the obtuse sets Λ says that it attains its maximum if Λ is a canonical basis of \mathbb{R}^n .

Theorem 1.2. *Suppose $\varphi \in \mathcal{C}$ and Λ is a finite and spanning set. Then*

$$(1.2) \quad \frac{V(Z_{\varphi}^{+}\Lambda)}{V(Z_1^{+}\Lambda)} \leq \frac{V(Z_{\varphi}^{+}\Lambda_{\perp})}{V(Z_1^{+}\Lambda_{\perp})}.$$

Equality holds if and only if Λ is a $\text{GL}(n)$ image of an obtuse set.

The paper is organized as follows. In Section 2, we introduce the asymmetric Orlicz zonotopes and show some of their properties. The shadow system and some results of them are given in Section 3. Section 4 deals with the equality case of the volume product inequality and volume ratio inequality for Orlicz zonotopes. The final proofs of the main theorems are presented in Section 5.

2. Preliminaries

For quick reference we recall some basic definition and notations in convex geometry that is required for our results. Good references are Gardner [17], Gruber [20], Schneider [43].

Let \mathcal{K}^n denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean n -space, \mathbb{R}^n . If K is a convex body, denote by $V(K)$ its n -dimensional volume, and by $h_K(\cdot) : S^{n-1} \rightarrow \mathbb{R}$ the support function of K ; i.e., for $u \in S^{n-1}$,

$$h_K(u) = \max\{\langle u, x \rangle : x \in K\},$$

where $\langle u, x \rangle$ denotes the standard inner product in \mathbb{R}^n . It is shown that the sublinear support function characterizes a convex body and, conversely, every sublinear function on \mathbb{R}^n is the support function of a nonempty compact convex set.

Two convex bodies K, L satisfy $K \subseteq L$ if and only if $h_K(\cdot) \leq h_L(\cdot)$. By the definition of the support function, it follows immediately that the support function of the image $gK := \{gy : y \in K\}$ is given by

$$h_{gK}(x) = h_K(g^T x)$$

for $g \in \text{GL}(n)$. Here g^T denotes the transpose of g .

Let K be a convex body, for every interior point s of K ,

$$K^s = \{y \in \mathbb{R}^n : \langle y, x - s \rangle \leq 1 \text{ for all } x \in K\}$$

defines a convex body that is called the polar body of K with respect to s . A well-known result of Santaló states that, in every convex body $K \in \mathbb{R}^n$, there exists a unique point $s(K) \in K$, the Santaló point, such that

$$V(K^{s(K)}) = \min_{s \in K} V(K^s).$$

To shorten the notation, we shall denote $K^{s(K)}$ by K^* . It is well known that the polarization with respect to the Santaló point is translation invariant and $\text{GL}(n)$ contravariant, that is,

$$(K + y)^* = K^* \quad \text{and} \quad (gK)^* = g^{-T} K^*,$$

for $y \in \mathbb{R}^n$ and $g \in \text{GL}(n)$.

Suppose Λ is a set in \mathbb{R}^n , it is called multiset if its members are allowed to appear more than once. More precisely, a multiset Λ is identified with its multiplicity function $1_\Lambda : \mathbb{R}^n \rightarrow \mathbb{N} \cup \{0\}$, that generalizes the characteristic function of sets. We say that a vector is an element of a multiset if the corresponding multiplicity function evaluated at the vector is greater than zero, and call a multiset finite if it contains only a finite number of vectors. If these vectors span \mathbb{R}^n , then we say that the multiset is spanning.

The operation between multisets can be defined using the multiset function. For instance, the union $\Lambda_1 \uplus \Lambda_2$ of Λ_1 and Λ_2 is defined as

$$1_{\Lambda_1 \uplus \Lambda_2}(x) = 1_{\Lambda_1}(x) + 1_{\Lambda_2}(x)$$

and $\Lambda_1 - \Lambda_2$ is defined as

$$1_{\Lambda_1 - \Lambda_2}(x) = \max\{0, 1_{\Lambda_1}(x) - 1_{\Lambda_2}(x)\}.$$

We write multisets in usual set notation, that is, $\Lambda = \{v_1, \dots, v_m\}$.

The asymmetric L_p -zonotopes associated with finite and spanning multisets $\Lambda = \{v_1, \dots, v_m\}$ are defined by Weberndorfer [51]. Here we extend the notations to asymmetric Orlicz zonotopes.

Let \mathcal{C} be the class of convex, strictly increasing functions $\varphi: [0, \infty) \rightarrow [0, \infty)$ satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$. Here the normalization is a matter of convenience and other choices are possible. It is not hard to conclude that $\varphi \in \mathcal{C}$ is continuous on $[0, \infty)$.

Asymmetric Orlicz zonotopes associated with finite and spanning multisets $\Lambda = \{v_1, \dots, v_m\}$ are defined by

$$(2.1) \quad h_{Z_\varphi^+ \Lambda}(u) = \inf \left\{ \lambda > 0 : \sum_{i=1}^m \varphi \left(\frac{\langle v_i, u \rangle_+}{\lambda} \right) \leq 1 \right\}$$

for all $u \in S^{n-1}$. Moreover, if $\langle v_i, u \rangle_+ = 0$ for all $i = 1, 2, \dots, m$, we define $h_{Z_\varphi^+ \Lambda}(u) = 0$.

In fact, by the convexity of φ and the sub-additivity of $\langle v, \cdot \rangle_+$, we have

$$\varphi \left(\frac{\langle v, u_1 + u_2 \rangle_+}{\lambda_1 + \lambda_2} \right) \leq \frac{\lambda_1}{\lambda_1 + \lambda_2} \varphi \left(\frac{\langle v, u_1 \rangle_+}{\lambda_1} \right) + \frac{\lambda_2}{\lambda_1 + \lambda_2} \varphi \left(\frac{\langle v, u_2 \rangle_+}{\lambda_2} \right),$$

it follows that the support function defined in (2.1) is sublinear, which grants the existence of convex body $Z_\varphi^+ \Lambda$.

In particular, if $p \geq 1$ and take $\varphi(t) = t^p$, then it turns out that $Z_\varphi^+ \Lambda = Z_p^+ \Lambda$.

Note that $\varphi \in \mathcal{C}$ is strictly convex and increasing on $[0, \infty)$, it follows that the function

$$\lambda \mapsto \sum_{i=1}^m \varphi \left(\frac{\langle v_i, u \rangle_+}{\lambda} \right)$$

is strictly decreasing on $[0, \infty)$. The next lemma easily follows.

Lemma 2.1. *Suppose $\varphi \in \mathcal{C}$ and $\Lambda = \{v_1, \dots, v_m\}$ spans \mathbb{R}^n . For $u_0 \in S^{n-1}$, then*

- (i) $\sum_{i=1}^m \varphi(\langle v_i, u_0 \rangle_+ / \lambda_0) = 1$ if and only if $\lambda_0 = h_{Z_\varphi^+ \Lambda}(u_0)$;
- (ii) $\sum_{i=1}^m \varphi(\langle v_i, u_0 \rangle_+ / \lambda_0) > 1$ if and only if $\lambda_0 < h_{Z_\varphi^+ \Lambda}(u_0)$;
- (iii) $\sum_{i=1}^m \varphi(\langle v_i, u_0 \rangle_+ / \lambda_0) < 1$ if and only if $\lambda_0 > h_{Z_\varphi^+ \Lambda}(u_0)$.

A simple observe of definition (2.1) is that the operator Z_φ^+ on finite and spanning multisets is $GL(n)$ equivariant, that is, $Z_\varphi^+ g\Lambda = gZ_\varphi^+ \Lambda$ holds for all $g \in GL(n)$. In fact,

$$\begin{aligned} h_{Z_\varphi^+ g\Lambda}(u) &= \inf \left\{ \lambda > 0 : \sum_{i=1}^m \varphi \left(\frac{\langle gv_i, u \rangle_+}{\lambda} \right) \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0 : \sum_{i=1}^m \varphi \left(\frac{\langle v_i, g^T u \rangle_+}{\lambda} \right) \leq 1 \right\} \\ &= h_{Z_\varphi^+ \Lambda}(g^T u) = h_{gZ_\varphi^+ \Lambda}(u) \end{aligned}$$

holds for all $u \in S^{n-1}$. Moreover, the asymmetric Orlicz zonotopes defined by (2.1) are closely related to the origin symmetric Orlicz zonotopes $Z_\varphi\Lambda$ defined in [50]. Specially,

$$\varphi\left(\frac{|\langle v_i, u \rangle|}{\lambda}\right) = \varphi\left(\frac{\langle v_i, u \rangle_+ + \langle -v_i, u \rangle_+}{\lambda}\right) = \varphi\left(\frac{\langle v_i, u \rangle_+}{\lambda}\right) + \varphi\left(\frac{\langle -v_i, u \rangle_+}{\lambda}\right),$$

which implies that $Z_\varphi\Lambda = Z_\varphi^+(\Lambda \uplus -\Lambda)$.

In the following, the volume product and volume ratio for Orlicz zonotopes associated with the multisets Λ always refer to $V(Z_\varphi^{+,*}\Lambda)V(Z_1^+\Lambda)$ and $V(Z_\varphi^+\Lambda)/V(Z_1^+\Lambda)$, respectively.

3. Shadow system of multiset

The notion of shadow system, introduced by Rogers and Shephard (see [42, 47]), play an important role in proving geometric inequalities in convex geometry. For example, this method was used by Campi, Gronchi, Meyer and Reisner (see, e.g., [6–10, 40]). A shadow system X_t of points from \mathbb{R}^n is a family of sets which can be defined as follows:

$$X_t = \{x_i + t\beta_i v : x_i \in \mathbb{R}^n\}$$

where $t \in [t_1, t_2]$, $\beta_i \in \mathbb{R}$ and $v \in S^{n-1}$. Here t can be seen as a time-like parameter and β_i as the speed of the point x_i along the direction v .

Let $\Lambda = \{v_1, \dots, v_m\}$ be a finite multiset such that $\Lambda \setminus \{v_1\}$ is spanning. Following the ideas of Campi and Gronchi [10], define $\Lambda_t^a = \{w_1(t), \dots, w_m(t)\}$, where

$$(3.1) \quad w_i(t) = \begin{cases} (1 + ta)v_1 & i = 1, \\ v_i - t \frac{\langle v_1, v_i \rangle}{\|v_1\|^2} v_1 & \text{otherwise.} \end{cases}$$

Here t varies in $[-a^{-1}, 1]$, and

$$(3.2) \quad a = \frac{\sum_{2 \leq i_1 < \dots < i_n \leq m} |[v_{i_1}, \dots, v_{i_n}]|}{\sum_{2 \leq i_2 < \dots < i_n \leq m} |[v_1, v_{i_2}, \dots, v_{i_n}]|},$$

here $[v_{i_1}, \dots, v_{i_n}]$ denotes the determinant of the matrix whose rows are v_{i_1}, \dots, v_{i_n} . From the definition we have $\Lambda_0^a = \Lambda$, and $w_1(1)$ is orthogonal to the remaining vectors in Λ_1^a , while $w_1(-a^{-1}) = o$. Moreover, by the construction (3.1), Λ_t^a , $t \in [-a^{-1}, 1]$, is a shadow system of multisets along the direction $v = v_1/\|v_1\| \in S^{n-1}$.

The important result of Campi and Gronchi is that for $t \in [-a^{-1}, 1]$, the asymmetric L_1 -zonotopes associated to Λ_t^a preserve the volume, and then was extended to asymmetric L_p -zonotopes by Weberndorfer. From now on, we use Λ_t to denote Λ_t^a , the orthogonalization of Λ with respect to v_1 if a is determined by (3.2).

In the following, we will show that the asymmetric Orlicz zonotopes associated to the shadow system of a multiset along direction v is independent of t .

Lemma 3.1. *Suppose that Λ_t , $t \in [-a^{-1}, 1]$ is a shadow system of multisets along the direction $v \in S^{n-1}$. Then the orthogonal projection of $Z_\varphi^+ \Lambda_t$ onto v^\perp is independent of t .*

Proof. By the definition of Λ_t , for $x \in v^\perp$ we have

$$\begin{aligned} h_{Z_\varphi^+ \Lambda_t}(x) &= \inf \left\{ \lambda > 0 : \sum_{i=1}^m \varphi \left(\frac{\langle v_i + t\beta_i v, x \rangle_+}{\lambda} \right) \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0 : \sum_{i=1}^m \varphi \left(\frac{\langle v_i, x \rangle_+}{\lambda} \right) \leq 1 \right\} \\ &= h_{Z_\varphi^+ \Lambda}(x), \end{aligned}$$

which shows the result. □

Before characterizing the shadow system of convex bodies along with direction v distinguish with others, we introduce the uppergraph function $\bar{g}_v(K, \cdot)$ and the lowergraph function $\underline{g}_v(K, \cdot)$ of a convex body K by

$$\begin{aligned} \bar{g}_v(K, x) &:= \sup \{ \lambda \in \mathbb{R} : x + \lambda v \in K \}, \\ \underline{g}_v(K, x) &:= \inf \{ \lambda \in \mathbb{R} : x + \lambda v \in K \}. \end{aligned}$$

An alternative representation of the above formulas are obtained by Weberndorfer [51]. Let $w \in v^\perp$, then

$$\begin{aligned} \bar{g}_v(K, x) &= \inf_{w \in v^\perp} \{ h_K(v + w) - \langle x, w \rangle \}, \\ \underline{g}_v(K, x) &= - \inf_{w \in v^\perp} \{ h_K(-v - w) + \langle x, w \rangle \} \end{aligned} \tag{3.3}$$

for all $x \in v^\perp$.

Now we present the characterization of a shadow system obtained by Campi and Gronchi.

Proposition 3.2. [7] *Let K_t , $t \in [-a^{-1}, 1]$, be one parameter family of convex bodies such that $K_t|_{v^\perp}$ is independent of t . Then K_t , $t \in [-a^{-1}, 1]$, is a shadow system of convex bodies along the direction v if and only if for every $x \in K_0|_{v^\perp}$, the functions $t \rightarrow \bar{g}_v(K_t, x)$ and $t \rightarrow -\underline{g}_v(K_t, x)$ are convex and*

$$\underline{g}_v(K_{\lambda s + \mu t}, x) \leq \lambda \bar{g}_v(K_s, x) + \mu \underline{g}_v(K_t, x) \leq \bar{g}_v(K_{\lambda s + \mu t}, x) \tag{3.4}$$

for every $s, t \in [-a^{-1}, 1]$ and $\lambda, \mu \in (0, 1)$ such that $\lambda + \mu = 1$.

A remarkable result about the volume of a shadow system is due to Shephard.

Lemma 3.3. [47] *Every mixed volume involving n shadow systems along the same direction is a convex function of the parameter. In particular, the volume $V(K_t)$ and all quermassintegrals $W_i(K_t)$, $i = 1, 2, \dots, n$, of a shadow system are convex functions of t .*

This result was largely used by Campi and Gronchi [7–10] and Chen [11, 14]. In the following, we show that the support function of asymmetric Orlicz zonotopes associated with a shadow system of multisets Λ_t , $t \in [-a^{-1}, 1]$, is a Lipschitz function of t .

Let $f = (f_1, \dots, f_n): \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a real-valued function, $\bar{v} = (v_1, \dots, v_n)$, $\beta = (\beta_1, \dots, \beta_n)$ and $v(t) = (v_1(t), \dots, v_n(t))$ are vectors in \mathbb{R}^n , where $v_i(t) = v_i + t\beta_i v$ for $i = 1, 2, \dots, n$, as defined before. For notational convenience we define

$$(3.5) \quad \|f(t)\|_\varphi := \inf \left\{ \lambda > 0 : \sum_{i=1}^m \varphi \left(\frac{|f_i(t)|}{\lambda} \right) \leq 1 \right\}$$

for real-valued functions f on \mathbb{R}^n , and $[\cdot]_+ := \max\{\cdot, 0\}$. By Lemma 2.1, if $c > 0$, we have $\|cf\|_\varphi = |c|\|f\|_\varphi$. Moreover, if $f \leq g$ for all $t \in R$, we have

$$\|f\|_\varphi \leq \|g\|_\varphi.$$

Lemma 3.4. *Suppose $\varphi \in \mathcal{C}$ and Λ_t , $t \in [-a^{-1}, 1]$, is a shadow system of multiset $\Lambda = \{v_1, \dots, v_m\}$ along the direction v and speed function β . If $t_1, t_2 \in [-a^{-1}, 1]$ and $x \in \mathbb{R}^n$, then*

$$\left| h_{Z_\varphi^+ \Lambda_{t_1}}(x) - h_{Z_\varphi^+ \Lambda_{t_2}}(x) \right| \leq \|\beta \langle v, x \rangle\|_\varphi |t_1 - t_2|.$$

Proof. From definition (3.5) and Lemma 2.1, we have

$$\begin{aligned} \|f(t)\|_\varphi = \lambda_1 &\iff \sum_{i=1}^m \varphi \left(\frac{|f_i(t)|}{\lambda_1} \right) = 1, \\ \|g(t)\|_\varphi = \lambda_2 &\iff \sum_{i=1}^m \varphi \left(\frac{|g_i(t)|}{\lambda_2} \right) = 1. \end{aligned}$$

The convexity of φ shows

$$(3.6) \quad \varphi \left(\frac{|f_i(t) + g_i(t)|}{\lambda_1 + \lambda_2} \right) \leq \frac{\lambda_1}{\lambda_1 + \lambda_2} \varphi \left(\frac{|f_i(t)|}{\lambda_1} \right) + \frac{\lambda_2}{\lambda_1 + \lambda_2} \varphi \left(\frac{|g_i(t)|}{\lambda_2} \right).$$

Summing both sides of (3.6) with respect to $i = 1, 2, \dots, m$, gives

$$\sum_{i=1}^m \varphi \left(\frac{|f_i(t) + g_i(t)|}{\lambda_1 + \lambda_2} \right) \leq 1.$$

By Lemma 2.1 again we have

$$(3.7) \quad \|f(t) + g(t)\|_\varphi \leq \|f(t)\|_\varphi + \|g(t)\|_\varphi.$$

If we take $f = f - g + g$, we have

$$\|f(t)\|_\varphi - \|g(t)\|_\varphi \leq \|f(t) - g(t)\|_\varphi,$$

which means

$$(3.8) \quad \left| \|f(t)\|_\varphi - \|g(t)\|_\varphi \right| \leq \|f(t) - g(t)\|_\varphi.$$

Moreover, together with the definition of the support function $h_{Z_\varphi^+ \Lambda_t}(x)$ and (3.5), we have

$$h_{Z_\varphi^+ \Lambda_t}(x) = \|\langle v(t), x \rangle_+\|_\varphi.$$

Then, by (3.8) we have

$$\begin{aligned} \left| h_{Z_\varphi^+ \Lambda_{t_1}}(x) - h_{Z_\varphi^+ \Lambda_{t_2}}(x) \right| &= \left| \|\langle v(t_1), x \rangle_+\|_\varphi - \|\langle v(t_2), x \rangle_+\|_\varphi \right| \\ &\leq \|\langle v(t_1), x \rangle_+ - \langle v(t_2), x \rangle_+\|_\varphi \\ &\leq \|\beta \langle x, v \rangle_+(t_1 - t_2)\|_\varphi = \|\beta \langle x, v \rangle_+\|_\varphi |t_1 - t_2|. \end{aligned}$$

We complete the proof. □

Theorem 3.5. *Suppose $\varphi \in \mathcal{C}$, Λ_t , $t \in [-a^{-1}, 1]$, is a shadow system of multisets along the direction $v \in S^{n-1}$. Then $Z_\varphi^+ \Lambda_t$, $t \in [-a^{-1}, 1]$, is a shadow system of convex bodies along the direction v .*

Proof. Let x be a point in $Z_\varphi^+ \Lambda_0|_{v^\perp}$, and $\nu, \mu \in (0, 1)$ satisfy $\nu + \mu = 1$. By Lemma 3.1, it remains to show that the hypotheses of Proposition 3.2 on properties of the graph functions are satisfied.

By assumption, the shadow system Λ_t is equal to, say, $\{v_1(t), \dots, v_m(t)\}$ where $v_i(t) = v_i + t\beta_i v$. With these definitions, the support function of $Z_\varphi^+ \Lambda_t$ can be written as

$$h_{Z_\varphi^+ \Lambda_t}(u) = \inf \left\{ \lambda > 0 : \sum_{i=1}^m \varphi \left(\frac{\langle v_i(t), u \rangle_+}{\lambda} \right) \leq 1 \right\} = \|\langle v(t), u \rangle_+\|_\varphi.$$

To establish the convexity of the uppergraph and lowergraph function as functions of t , we first prove the uppergraph function is a convex function of t . Notice that Λ_t is also a shadow system in direction $-v$. Then the vector v can be replaced by $-v$ and by application of the identity $\bar{g}_{-v}(\cdot, x) = -\underline{g}_v(\cdot, x)$, we can obtain that the lowergraph function is a convex function of t .

By the definition of $\bar{g}_v(Z_\varphi^+ \Lambda_t, x)$, we have

$$\begin{aligned} \bar{g}_v(Z_\varphi^+ \Lambda_{\nu s + \mu t}, x) &= \inf_{w_1, w_2 \in v^\perp} \left\{ h_{Z_\varphi^+ \Lambda_{\nu s + \mu t}}(v + \nu w_1 + \mu w_2) - \langle x, \nu w_1 + \mu w_2 \rangle \right\} \\ &= \inf_{w_1, w_2 \in v^\perp} \left\{ \|\langle v(\nu s + \mu t), v + \nu w_1 + \mu w_2 \rangle_+\|_\varphi - \langle x, \nu w_1 + \mu w_2 \rangle \right\}. \end{aligned}$$

By the inequality $\max\{u + v, 0\} \leq \max\{u, 0\} + \max\{v, 0\}$ and definition (3.5), we have

$$\begin{aligned} & \|\langle v(\nu s + \mu t), v + \nu w_1 + \mu w_2 \rangle_+ \|_\varphi \\ &= \inf \left\{ \lambda > 0 : \sum_{i=1}^m \varphi \left(\frac{\langle v_i + (\nu s + \mu t)\beta_i v, v + \nu w_1 + \mu w_2 \rangle_+}{\lambda} \right) \leq 1 \right\}. \end{aligned}$$

The convexity of φ and (3.7) imply that

$$(3.9) \quad \|\langle v(\nu s + \mu t), v + \nu w_1 + \mu w_2 \rangle_+ \|_\varphi \leq \nu \|\langle v(s), v + w_1 \rangle_+ \|_\varphi + \mu \|\langle v(t), v + w_2 \rangle_+ \|_\varphi.$$

Thus, together with (3.9) and the expression of $\bar{g}_v(K, x)$ we have

$$\begin{aligned} & \inf_{w_1, w_2 \in v^\perp} \{ \|\langle v(\nu s + \mu t), v + \nu w_1 + \mu w_2 \rangle_+ \|_\varphi - \langle x, \nu w_1 + \mu w_2 \rangle \} \\ & \leq \inf_{w_1 \in v^\perp} \{ \nu \|\langle v(s), v + w_1 \rangle_+ \|_\varphi - \nu \langle x, w_1 \rangle \} \\ & \quad + \inf_{w_2 \in v^\perp} \{ \mu \|\langle v(t), v + w_2 \rangle_+ \|_\varphi - \mu \langle x, w_2 \rangle \}, \end{aligned}$$

which means

$$\bar{g}_v(Z_\varphi^+ \Lambda_{\nu s + \mu t}, x) \leq \nu \bar{g}_v(Z_\varphi^+ \Lambda_s, x) + \mu \bar{g}_v(Z_\varphi^+ \Lambda_t, x).$$

Hence $t \rightarrow \bar{g}_v(Z_\varphi^+ \Lambda_t, x)$ is convex.

Next we verify the inequalities (3.4) of Proposition 3.2. First we show

$$(3.10) \quad \nu \bar{g}_v(Z_\varphi^+ \Lambda_s, x) + \mu \underline{g}_v(Z_\varphi^+ \Lambda_t, x) \leq \bar{g}_v(Z_\varphi^+ \Lambda_{\nu s + \mu t}, x).$$

To see this, let $w \in v^\perp$,

$$(3.11) \quad \nu \bar{g}_v(Z_\varphi^+ \Lambda_s, x) = \inf_{w \in v^\perp} \{ \|\nu \langle v(s), v + w \rangle_+ \|_\varphi - \nu \langle x, w \rangle \}.$$

Let $w = \nu^{-1}(w_1 - \mu w_2)$, $w_1, w_2 \in v^\perp$, in (3.11), we have

$$\nu \bar{g}_v(Z_\varphi^+ \Lambda_s, x) = \inf_{w_1, w_2 \in v^\perp} \{ \|\langle v(s), \nu v + w_1 - \mu w_2 \rangle_+ \|_\varphi - \langle x, w_1 - \mu w_2 \rangle \},$$

where

$$\begin{aligned} \langle v(s), \nu v + w_1 - \mu w_2 \rangle_+ &= \langle v_i + s\beta_i v, (1 - \mu)v + w_1 - \mu w_2 \rangle_+ \\ &= \mu \langle v_i + t\beta_i v, -v - w_2 \rangle_+ + \langle v_i + (\nu s + \mu t)\beta_i v, v + w_1 \rangle_+. \end{aligned}$$

By (3.7) we have,

$$\begin{aligned} & \nu \bar{g}_v(Z_\varphi^+ \Lambda_s, x) \\ &= \inf_{w_1, w_2 \in v^\perp} \{ \|\langle v(s), \nu v + w_1 - \mu w_2 \rangle_+ \|_\varphi - \langle x, w_1 - \mu w_2 \rangle \} \\ &= \inf_{w_1, w_2 \in v^\perp} \{ \|\mu \langle v_i + t\beta_i v, -v - w_2 \rangle_+ + \langle v_i + (\nu s + \mu t)\beta_i v, v + w_1 \rangle_+ \|_\varphi - \langle x, w_1 - \mu w_2 \rangle \} \\ &\leq \mu \inf_{w \in v^\perp} \{ \|\langle v(t), -v - w \rangle_+ \|_\varphi - \langle x, -w \rangle \} + \inf_{w \in v^\perp} \{ \|\langle v(\nu s + \mu t), v + w \rangle_+ \|_\varphi - \langle x, w \rangle \} \\ &= -\mu \underline{g}_v(Z_\varphi^+ \Lambda_t, x) + \bar{g}_v(Z_\varphi^+ \Lambda_{\nu s + \mu t}, x), \end{aligned}$$

which implies the inequality (3.10).

The left-hand of inequality (3.4) can be derived from inequality (3.10) by replacing v by $-v$, and using the following facts

$$\underline{g}_{-v}(\cdot, x) = -\bar{g}_v(\cdot, x) \quad \text{and} \quad \bar{g}_{-v}(\cdot, x) = -\underline{g}_v(\cdot, x).$$

Now we complete the proof. □

Now Theorem 3.5 together with Lemma 3.1 imply the following theorem.

Theorem 3.6. *Suppose $\varphi \in \mathcal{C}$ and Λ is a finite and spanning multiset. If $\Lambda_t, t \in [-a^{-1}, 1]$, is an orthogonalization of Λ defined by (3.1), then*

- (i) *The volume $V(Z_\varphi^{+,*}\Lambda_t)^{-1}$ is a convex function of t . In particular, the inverse volume product for asymmetric Orlicz zonotopes associated with Λ_t is a convex function of t .*
- (ii) *The volume $V(Z_\varphi^+\Lambda_t)^{-1}$ is a convex function of t . In particular, the volume ratio for asymmetric Orlicz zonotopes associated with Λ_t is a convex function of t .*

Theorem 3.6 shows that the inverse volume product and the volume ratio for asymmetric Orlicz zonotopes are nondecreasing if Λ is replaced by either $\Lambda_{-a^{-1}}$ or Λ_1 , because convex functions attain global maxima at the boundary of compact intervals.

4. The equality condition

The following lemma is crucial for the proof of our main results.

Lemma 4.1. *Suppose $\varphi \in \mathcal{C}$, and Λ is a finite and spanning multiset. Replace all vectors in Λ that point in the same direction by their sum, and denote this new multiset by $\bar{\Lambda}$. Then the following inequalities*

$$(4.1) \quad \frac{V(Z_\varphi^+\Lambda)}{V(Z_1^+\Lambda)} \leq \frac{V(Z_\varphi^+\bar{\Lambda})}{V(Z_1^+\bar{\Lambda})},$$

$$V(Z_\varphi^{+,*}\Lambda)V(Z_1^+\Lambda) \geq (V_\varphi^{+,*}\bar{\Lambda})(V(Z_1^+\bar{\Lambda}))$$

hold. Equality holds, when $\varphi \neq \text{Id}$, if and only if $\Lambda = \bar{\Lambda}$.

Proof. Let multiset $\Lambda = \{v_1, \dots, v_m\}$ and $\bar{\Lambda} = \{w_1, \dots, w_k\}$. By the construction of $\bar{\Lambda}$ we have

$$w_j = \sum_{i \in I_j} v_i,$$

where $I_j, j = 1, 2, \dots, k$, is a partition of $\{1, 2, \dots, m\}$ and the vectors in every $\{v_i : i \in I_j\}$ point in the same direction. If $\varphi = \text{Id}$, this means $\varphi(t) = t$, here we write $Z_{\text{Id}}^+\Lambda = Z_1^+\Lambda$. It is easy to know that

$$(4.2) \quad Z_1^+\Lambda = Z_1^+\bar{\Lambda}.$$

Now assume that $\varphi \neq \text{Id}$. Let $u \in S^{n-1}$, and set

$$h_{Z_\varphi^+\Lambda}(u) = \lambda \quad \text{and} \quad h_{Z_\varphi^+\bar{\Lambda}}(u) = \bar{\lambda}.$$

By the definition of $Z_\varphi^+\bar{\Lambda}$, and note that all v_i point in the same direction for $i \in I_j$, then we have

$$\left\langle \sum_{i \in I_j} v_i, u \right\rangle_+ = \sum_{i \in I_j} \langle v_i, u \rangle_+.$$

It follows that

$$\sum_{j=1}^k \varphi \left(\frac{\left\langle \sum_{i \in I_j} v_i, u \right\rangle_+}{\bar{\lambda}} \right) = \sum_{j=1}^k \varphi \left(\frac{\sum_{i \in I_j} \langle v_i, u \rangle_+}{\bar{\lambda}} \right) = 1.$$

Since the fact that φ is convex and increasing, we have that if $x_1, \dots, x_l \in [0, \infty)$, then

$$\varphi(x_1 + \dots + x_l) \geq \varphi(x_1) + \dots + \varphi(x_l).$$

Equality holds if and only if φ is a linear function. So we obtain

$$(4.3) \quad 1 = \sum_{j=1}^k \varphi \left(\frac{\sum_{i \in I_j} \langle v_i, u \rangle_+}{\bar{\lambda}} \right) \geq \sum_{i=1}^m \varphi \left(\frac{\langle v_i, u \rangle_+}{\bar{\lambda}} \right).$$

Since $h_{Z_\varphi^+\Lambda}(u) = \lambda$, by Lemma 2.1, we obtain $\lambda \leq \bar{\lambda}$. Since $\varphi \neq \text{Id}$, equality holds only if all sum over $i \in I_j$ contain at most one positive summand, that means $\Lambda = \bar{\Lambda}$. In fact, if $\Lambda \neq \bar{\Lambda}$, say v_1 and v_2 point in the same direction, by the convexity of φ , then $h_{Z_\varphi^+\bar{\Lambda}}(v_1) > h_{Z_\varphi^+\Lambda}(v_1)$. Hence we obtain $Z_\varphi^+\Lambda \subset Z_\varphi^+\bar{\Lambda}$.

On the other hand, if $\Lambda \neq \bar{\Lambda}$, equality holds in (4.3) if and only if

$$\sum_{j=1}^k \varphi(x_j) = \varphi \left(\sum_{i=1}^k x_i \right)$$

holds for arbitrary k and $x_i \in \mathbb{R}$. Combining with the convexity and the normalization of φ , and solving this functional equation we know that $\varphi(t) = t$. Then

$$Z_\varphi^+\Lambda = Z_1^+\Lambda = Z_1^+\bar{\Lambda}.$$

The first inequality of (4.1) now follows immediately. To the second inequality of (4.1), if $\varphi \neq \text{Id}$, note that

$$Z_\varphi^{+,*}\Lambda = (Z_\varphi^+\Lambda - s(Z_\varphi^+\Lambda))^o \supseteq (Z_\varphi^+\bar{\Lambda} - s(Z_\varphi^+\Lambda))^o,$$

equality holds if and only if $\Lambda = \bar{\Lambda}$. Thus

$$V(Z_\varphi^{+,*}\Lambda) \geq V((Z_\varphi^+\bar{\Lambda} - s(Z_\varphi^+\Lambda))^o) \geq V(Z_\varphi^{+,*}\bar{\Lambda}).$$

Together with (4.2) show the second inequality of (4.1). □

In the following, we observe that a set that can be written as a disjoint union $\Lambda_\perp \cup \{v_1, \dots, v_l\}$ is obtuse if and only if there are disjoint nonempty subset I_1, \dots, I_l of $\{1, 2, \dots, n\}$ and positive numbers μ_i such that, for every $j \in \{1, 2, \dots, l\}$,

$$v_j = \sum_{i \in I_j} -\mu_i e_i.$$

The following lemma shows that every spanning obtuse set has a linear image of above type.

Lemma 4.2. [51] *Suppose Λ is a spanning obtuse set, then the following three statements hold:*

- (i) *If $B \subset \Lambda$ is a basis, then the vectors in $\Lambda \setminus B$ are pairwise orthogonal and have nonpositive components with respect to the basis B .*
- (ii) *Every $\text{GL}(n)$ image of Λ that contains the canonical basis Λ_\perp is obtuse.*
- (iii) *Suppose in addition that Λ contains the canonical basis. For every $y \in Z_\varphi^+\Lambda$ there is a $g \in \text{GL}(n)$ such that gy has nonnegative coordinates with respect to the canonical basis and $\Lambda_\perp \subset g\Lambda$.*

This lemma is established by Weberndorfer in [51].

One of the immediate implications of the above lemma is that a spanning obtuse set contains at least n and not more than $2n$ vectors. Now we give the equality condition of our main results.

Lemma 4.3. *Suppose $\varphi \in \mathcal{C}$ and Λ is a spanning obtuse set. Then*

$$\frac{V(Z_\varphi^+\Lambda)}{V(Z_1^+\Lambda)} = \frac{V(Z_\varphi^+\Lambda_\perp)}{V(Z_1^+\Lambda_\perp)}.$$

Proof. Let Λ be a spanning obtuse set. By $\text{GL}(n)$ invariance of the volume ratio for asymmetric Orlicz zonotopes and Lemma 4.2, we may assume that $\Lambda = \{w_1, \dots, w_m\}$, where $n \leq m \leq 2n$, contains the canonical basis $\Lambda_\perp = \{e_1, \dots, e_n\}$. In the following, if we can establish the dissection formula

$$(4.4) \quad Z_\varphi^+ \Lambda = \bigcup_{1 \leq i_1 < \dots < i_n \leq m} Z_\varphi^+ \{w_{i_1}, \dots, w_{i_n}\}.$$

Then, we have

$$(4.5) \quad V(Z_\varphi^+ \Lambda) = \sum_{1 \leq i_1 < \dots < i_n \leq m} V(Z_\varphi^+ \{v_{i_1}, \dots, v_{i_n}\}).$$

The $\text{GL}(n)$ equivariance of Z_φ^+ together with (4.5) for $\varphi(t) = t$, we have

$$\frac{V(Z_\varphi^+ \Lambda)}{V(Z_1^+ \Lambda)} = \frac{\sum_{1 \leq i_1 < \dots < i_n \leq m} V(Z_\varphi^+ \{v_{i_1}, \dots, v_{i_n}\})}{\sum_{1 \leq i_1 < \dots < i_n \leq m} V(Z_1^+ \{v_{i_1}, \dots, v_{i_n}\})} = \frac{V(Z_\varphi^+ \Lambda_\perp)}{V(Z_1^+ \Lambda_\perp)}.$$

Here we used the $\text{GL}(n)$ equivariance of Z_φ^+ and the fact $V(Z_\varphi^+ \{v_{i_1}, \dots, v_{i_n}\}) = 0$, if $\{v_{i_1}, \dots, v_{i_n}\}$ is not a $\text{GL}(n)$ image of canonical basis Λ_\perp . Hence we have

$$\frac{V(Z_\varphi^+ \Lambda)}{V(Z_1^+ \Lambda)} = \frac{V(Z_\varphi^+ \Lambda_\perp)}{V(Z_1^+ \Lambda_\perp)}.$$

In the following, we will show that the dissection formula (4.4) holds. Let $y \in \bigcup_{1 \leq i_1 < \dots < i_n \leq m} Z_\varphi^+ \{w_{i_1}, \dots, w_{i_n}\}$, then it must belong to, say, $Z_\varphi^+ \{w_1, \dots, w_n\}$. In order to prove $y \in Z_\varphi^+ \Lambda$. Let

$$h_{Z_\varphi^+ \{w_1, \dots, w_n\}}(u) = \lambda_0 \quad \text{and} \quad h_{Z_\varphi^+ \Lambda}(u) = \lambda_1.$$

By the definition of the support function, we have

$$\sum_{i=1}^n \varphi\left(\frac{\langle w_i, u \rangle_+}{\lambda_0}\right) = 1 \quad \text{and} \quad \sum_{j=1}^m \varphi\left(\frac{\langle w_j, u \rangle_+}{\lambda_1}\right) = 1.$$

Since φ is increasing,

$$\sum_{i=1}^n \varphi\left(\frac{\langle w_i, u \rangle_+}{\lambda_1}\right) \leq \sum_{j=1}^m \varphi\left(\frac{\langle w_j, u \rangle_+}{\lambda_1}\right).$$

By Lemma 2.1 we have $\lambda_1 \geq \lambda_0$. This means

$$Z_\varphi^+ \{w_1, \dots, w_n\} \subseteq Z_\varphi^+ \Lambda.$$

We prove $Z_\varphi^+ \Lambda$ contains the right-hand side of (4.4). Now it remains to prove that $Z_\varphi^+ \Lambda$ is a subset of the right-hand side of (4.4). Let $y \in Z_\varphi^+ \Lambda$, it is sufficient to show that

there is a $g \in \text{GL}(n)$ such that $y \in Z_\varphi^+ g^{-1} \Lambda_\perp$ and $g^{-1} \Lambda_\perp \subseteq \Lambda$. By Lemma 4.2, there is a $g \in \text{GL}(n)$ such that gy has nonnegative coordinates with respect to the canonical basis and $\Lambda_\perp \subseteq g\Lambda$. Moreover, $g\Lambda$ is obtuse, then we can write

$$g\Lambda = \Lambda_\perp \cup \{w_1, \dots, w_{m-n}\},$$

and there are disjoint subsets I_1, \dots, I_{m-n} of $\{1, 2, \dots, n\}$, and positive number μ'_i such that, for $1 \leq j \leq m-n$,

$$w_j = \sum_{i \in I_j} -\mu'_i e_i.$$

Let $h_{Z_\varphi^+ g\Lambda} = \lambda_0$. Then we have

$$\sum_{i=1}^n \varphi\left(\frac{\langle e_i, u \rangle_+}{\lambda_0}\right) + \sum_{i=j}^{m-n} \varphi\left(\frac{\langle w_j, u \rangle_+}{\lambda_0}\right) = 1.$$

Note that the convexity and strict monotonicity of φ imply that, there exists a constant $\nu > 0$ such that

$$\sum_{j=1}^n \varphi\left(\frac{\nu \langle -\mu'_j e_j, u \rangle_+}{\lambda_0}\right) \geq \sum_{j=1}^{m-n} \varphi\left(\frac{\langle w_j, u \rangle_+}{\lambda_0}\right).$$

We write $\mu_j = \nu \mu'_j$, $j = 1, 2, \dots, n$, and define the set $\tilde{\Lambda} = \{e_1, \dots, e_n, -\mu_1 e_1, \dots, -\mu_n e_n\}$. Obviously, $\tilde{\Lambda}$ is an obtuse set. Moreover, we have

$$\sum_{i=1}^n \varphi\left(\frac{\langle e_i, u \rangle_+}{\lambda_0}\right) + \sum_{j=1}^n \varphi\left(\frac{\langle -\mu_j e_j, u \rangle_+}{\lambda_0}\right) \geq 1.$$

Then by Lemma 2.1 we have $\lambda_0 \leq h_{Z_\varphi^+ \tilde{\Lambda}}(u)$. Then

$$(4.6) \quad Z_\varphi^+ g\Lambda \subseteq Z_\varphi^+ \tilde{\Lambda} = Z_\varphi^+ \{e_1, \dots, e_n, -\mu_1 e_1, \dots, -\mu_n e_n\}.$$

It remains to show that $Z_\varphi^+ g\Lambda \subseteq Z_\varphi^+ \Lambda_\perp$. First note that Λ is obtuse, w_i has negative coordinates with respect to the canonical basis Λ_\perp . In order to simplify the computation, we assume that $\Lambda'(\mu) = \{e_1, \dots, e_n, -\mu e_1\}$, where $\mu \geq 0$. For $x \in e_1^\perp \cap e_2^\perp$, by (3.3) we have

$$(4.7) \quad \bar{g}_{e_2}(Z_\varphi^+ \Lambda'(\mu), x) = \inf_{w \in e_2^\perp} \left\{ h_{Z_\varphi^+ \Lambda'(\mu)}(e_2 + w) - \langle x, w \rangle \right\}.$$

Here

$$(4.8) \quad \begin{aligned} & h_{Z_\varphi^+ \Lambda'(\mu)}(e_2 + w) \\ &= \inf \left\{ \lambda > 0 : \varphi\left(\frac{\langle e_1, w \rangle_+}{\lambda}\right) + \varphi\left(\frac{\langle -\mu e_1, w \rangle_+}{\lambda}\right) + \sum_{i=2}^n \varphi\left(\frac{\langle e_i, e_2 + w \rangle_+}{\lambda}\right) \leq 1 \right\}. \end{aligned}$$

Note that, for all $w \in e_2^\perp$, the scalar product $\langle x, w \rangle$ does not depend on the first component of w . The monotonicity of φ together with the expression of (4.8) show that it suffices to compute the infimum of (4.7) over all $w \in e_1^\perp \cap e_2^\perp$. It is now obvious that the uppergraph function $\bar{g}_{e_2}(Z_\varphi^+ \Lambda'(\mu), x)$ is independent of μ for every $x \in e^\perp \cap e_2^\perp$. The same argument applied to the lowergraph function leads to the same conclusion, so we infer that

$$Z_\varphi^+ \Lambda'(\mu) \cap e_1^\perp$$

is independent of μ . Moreover, the support function of $Z_\varphi^+ \Lambda'(\mu)$ evaluated at vectors $w \in e_1^\perp$,

$$h_{Z_\varphi^+ \Lambda'(\mu)}(w) = \inf \left\{ \lambda > 0 : \sum_{i=2}^n \varphi \left(\frac{\langle e_2, w \rangle_+}{\lambda} \right) \leq 1 \right\}$$

is a constant function of μ . Equivalently,

$$Z_\varphi^+ \Lambda'(\mu)|_{e_1^\perp}$$

is independent of μ .

If $\mu = 1$, the convex body $Z_\varphi^+ \Lambda'(1)$ is symmetric with respect to reflections in the hyperplane e_1^\perp . For $y \in Z_\varphi^+ \Lambda'(\mu)$, since $Z_\varphi^+ \Lambda'(\mu) \cap e_1^\perp$ and $Z_\varphi^+ \Lambda'(\mu)|_{e_1^\perp}$ are independent of μ , then we have

$$y|_{e_1^\perp} \in Z_\varphi^+ \Lambda'(\mu)|_{e_1^\perp} = Z_\varphi^+ \Lambda'(1)|_{e_1^\perp} = Z_\varphi^+ \Lambda'(1) \cap e_1^\perp = Z_\varphi^+ \Lambda'(\mu) \cap e_1^\perp$$

for all μ . In particular, $\underline{g}_{e_1}(Z_\varphi^+ \Lambda'(\mu), y|_{e_1^\perp})$ is negative for all μ . Moreover, the uppergraph function $\bar{g}_{e_1}(Z_\varphi^+ \Lambda'(\mu), y|_{e_1^\perp})$ is independent of μ . As $h_{Z_\varphi^+ \Lambda'(\mu)}(e_1 + w)$ is independent of μ , for $w \in e_1^\perp$, we have

$$\begin{aligned} y &\in \left\{ y|_{e_1^\perp} + re_1 : 0 \leq r \leq \bar{g}_{e_1}(Z_\varphi^+ \Lambda'(\mu), y|_{e_1^\perp}) \right\} \\ &= \left\{ y|_{e_1^\perp} + re_1 : 0 \leq r \leq \bar{g}_{e_1}(Z_\varphi^+ \Lambda'(0), y|_{e_1^\perp}) \right\} \subset Z_\varphi^+ \Lambda'(0). \end{aligned}$$

Then we have $Z_\varphi^+ \Lambda'(\mu) \subseteq Z_\varphi^+ \Lambda_\perp$. This together with (4.6) we have $Z_\varphi^+ \Lambda'(\mu) = Z_\varphi^+ \Lambda_\perp$. Repeating this argument for μ_2, \dots, μ_n , if they are not zero, we obtain that gy is contained in $Z_\varphi^+ \Lambda_\perp$, which shows the equality of (4.4).

Moreover, if we can show that the intersection of any two distinct parts in the dissection (4.4) has volume zero, we can complete the proof. To see this, let $\Lambda^1, \Lambda^2 \in \Lambda$, each contain n vectors and assume that $\Lambda^1 \neq \Lambda^2$. If one of these sets is not spanning, then the intersection $Z_\varphi^+ \Lambda^1 \cap Z_\varphi^+ \Lambda^2$ is a set of volume zero contained in a hyperplane. Otherwise, without loss of generality, $\Lambda^1 = \Lambda_\perp$ and Λ^2 does not contain e_1 . Then by the definition of support function (2.1), we have $h_{Z_\varphi^+ \Lambda^1}(-e_1) = 0$, and $h_{Z_\varphi^+ \Lambda^2}(e_1) = 0$. Then we obtain that $Z_\varphi^+ \Lambda^1 \cap Z_\varphi^+ \Lambda^2$ is a set of volume zero contained in the hyperplane e_1^\perp . So we complete the proof. \square

If take $\varphi(t) = t^p, p \geq 1$, this result reduces to L_p case.

Corollary 4.4. *Suppose $p \geq 1$ and Λ is a spanning obtuse set. Then*

$$\frac{V(Z_p^+ \Lambda)}{V(Z_1^+ \Lambda)} = \frac{V(Z_p^+ \Lambda_\perp)}{V(Z_1^+ \Lambda_\perp)}.$$

In paper [9], Campi and Gronchi proved that if K_t is a shadow system of origin symmetric convex bodies in \mathbb{R}^n , then $V(K_t^*)^{-1}$ is a convex function of t . This result is developed by Meyer and Reisner [40] to more general setting.

Proposition 4.5. [40] *Suppose $K_t, t \in [-a^{-1}, 1]$, is a shadow system of convex bodies along the direction $v = e_1$ and $V(K_t)$ is independent of t . Then the volume of K_t^* is independent of t if and only if there are a real number α and a vector $z \in \mathbb{R}^{1 \times (n-1)}$ such that*

$$K_t = t\alpha e_1 + \begin{pmatrix} 1 & tz \\ 0 & I_{n-1} \end{pmatrix} K_0.$$

Unfortunately, there is no analogue result for volume product of asymmetric Orlicz zonotopes with equality holds.

Lemma 4.6. *Let $\varphi \in \mathcal{C}$ and $\Lambda = \Lambda_\perp \cup \{-\mu e_1\}$. Then*

$$(4.9) \quad V(Z_\varphi^{+,*} \Lambda) V(Z_1^+ \Lambda) \geq V(Z_\varphi^{+,*} \Lambda_\perp) V(Z_1^+ \Lambda_\perp),$$

equality holds if and only if $\varphi = \text{Id}$.

Proof. If $\varphi = \text{Id}$, this means $Z_\varphi^+ \Lambda = Z_1^+ \Lambda = Z_1 \Lambda$. It is an immediate consequence of the fact that all parallelepipeds have the same volume product.

Now assume that $\varphi \neq \text{Id}$, let $\Lambda_t, t \in [-a^{-1}, 1]$, denote the orthogonalization of Λ with respect to e_1 defined by (3.1). Theorem 3.6 shows that the inverse volume product of asymmetric Orlicz zonotopes associate with Λ_t is a convex function of t . As a convex function attains its maxima at the boundary of compact intervals, we obtain

$$\frac{1}{V(Z_\varphi^{+,*} \Lambda) V(Z_1^+ \Lambda)} \leq \max_{t \in \{-a^{-1}, 1\}} \left\{ \frac{1}{V(Z_\varphi^{+,*} \Lambda_t) V(Z_1^+ \Lambda_t)} \right\}.$$

By the $\text{GL}(n)$ invariance of the volume product of asymmetric Orlicz zonotopes and the definition of Λ_t , the right hand side of this inequality is just

$$\frac{1}{V(Z_\varphi^{+,*} \Lambda_\perp) V(Z_1^+ \Lambda_\perp)}.$$

Thus the equality condition of inequality (4.9) means that the $V(Z_\varphi^{+,*} \Lambda_t)$ is a constant function of t . On the other hand, by the definition of (3.1), we have

$$\Lambda_t = \{(1 + ta)e_1, \mu(t - 1)e_1, e_2, \dots, e_n\},$$

here Λ_t , $t \in [-a^{-1}, 1]$, is a spanning obtuse set. Together with Lemma 4.3 and the fact $Z_1^+ \Lambda_t$ is independent of t , we obtain that $V(Z_\varphi^+ \Lambda_t)$ is independent of t . Proposition 4.5 implies that $Z_\varphi^+ \Lambda_t$ are affine images of each other, which means there is a number α and a vector $z \in \mathbb{R}^{1 \times (n-1)}$ such that

$$Z_\varphi^+ \Lambda_t = t\alpha e_1 + \phi_t Z_\varphi^+ \Lambda,$$

where $\phi_t = \begin{pmatrix} 1 & tz \\ 0 & I_{n-1} \end{pmatrix}$. Note that as Z_φ^+ is $GL(n)$ equivariant, we can rewrite it as

$$(4.10) \quad Z_\varphi^+ \Lambda_t = t\alpha e_1 + Z_\varphi^+ \phi_t \Lambda.$$

Equivalent, for all $u \in \mathbb{R}^n$,

$$(4.11) \quad h_{Z_\varphi^+ \Lambda_t}(u) = t\alpha \langle e_1, u \rangle + h_{Z_\varphi^+ \phi_t \Lambda}(u).$$

Now we determine the constant α . Here $t \in [a^{-1}, 1]$, the zonotope $Z_\varphi^+ \Lambda_t$ is symmetric with respect to permutations of all coordinates except the first. Due to (4.10), this implies that z has $n - 1$ equal components, say ξ . Note that the coefficient a has nothing to do with the ξ , without loss of generality, we may assume $\xi \leq 0$, let $u = e_1$ and $t = 1$ in (4.11), after a simple computation we obtain

$$\alpha = \frac{a}{\varphi^{-1}(1)} = a.$$

In order to determine ξ , first, by the normalization of φ and together with Lemma 2.1, we have

$$h_{Z_\varphi^+ \Lambda_1}(e_i) = 1, \quad i = 2, \dots, n.$$

Note that $Z_\varphi^+ \Lambda_1$ is convex, specially, let $|e_2| = 1 = h_{Z_\varphi^+ \Lambda_1}(e_2)$, which means that e_2 is contained in a plane intersect with $Z_\varphi^+ \Lambda_1$, we say,

$$(4.12) \quad \{e_2\} = Z_\varphi^+ \Lambda_1 \cap (e_2 + \text{span}\{e_1\}).$$

On the other hand, by the definition of convex hull conv and the support function of $Z_\varphi^+ \Lambda$, together with Lemma 2.1, we have $Z_\varphi^+ \Lambda$ contains the convex hull of Λ , that is

$$\text{conv}\{\Lambda\} \subseteq Z_\varphi^+ \Lambda.$$

Then we have $Z_\varphi^+ \phi_1 \Lambda$ contains the convex hull of $\phi_1 \Lambda$, $\text{conv}\{\phi_1 \Lambda\}$. In particular, it contains $\phi_1 e_2 = \xi e_1 + e_2$. Combining this observation with (4.10) and (4.12) for $t = 1$, we obtain

$$e_2 = (a + \xi)e_1 + e_2,$$

which means $\xi = -a$.

Now putting $u = e_1 + e_2$ and $t = -a^{-1}$ in equation (4.11). Let $h_{Z_\varphi^+ \Lambda_{-a^{-1}}}(e_1 + e_2) = \lambda$, $h_{Z_\varphi^+ \phi_{-a^{-1}} \Lambda}(e_1 + e_2) = \lambda'$. Note that $\Lambda_{-a^{-1}} = \{\mu(-a^{-1} - 1)e_1, e_2, \dots, e_n\}$ and $\phi_{-a^{-1}} \Lambda = \{e_1, e_1 + e_2, \dots, e_1 + e_n, -\mu e_1\}$, then we have

$$(4.13) \quad \lambda = 1 \quad \text{and} \quad (n - 1)\varphi\left(\frac{1}{\lambda'}\right) + \varphi\left(\frac{2}{\lambda'}\right) = 1.$$

Note that by equation (4.11), λ and λ' should satisfy $\lambda = -1 + \lambda'$. On the other hand, by (4.13), they contradict with equality (4.11). Then $Z_\varphi^+ \Lambda_t$ are not the affine images of each other. So the equality (4.9) does not hold when $\varphi \neq \text{Id}$. We complete the proof. \square

If take $\varphi(t) = t^p$, $p \geq 1$, it is established in [51].

Corollary 4.7. *Let $p \geq 1$ and $\Lambda = \Lambda_\perp \cup \{-\mu e_1\}$. Then*

$$V(Z_p^{+,*} \Lambda)V(Z_1^+ \Lambda) \geq V(Z_p^{+,*} \Lambda_\perp)V(Z_1^+ \Lambda_\perp),$$

equality holds if and only if $p = 1$.

5. Proofs of the main results

Before giving the main results, let us present the following lemma established by Weberndorfer, which we will use in the proof of our results.

Lemma 5.1. [51] *Suppose Φ is a real-valued $\text{GL}(n)$ invariant function on finite and spanning multisets. Moreover, assume that $\Phi(\Lambda_t)$ is a convex function of t whenever Λ_t , $t \in [-a^{-1}, 1]$, is an orthogonalization of a multiset Λ defined by (3.1). Then for every finite and spanning multiset Λ , there exists a multiset Λ_{e_1} of multiples of e_1 such that*

$$(5.1) \quad \Phi(\Lambda) \leq \Phi(\Lambda_\perp \uplus \Lambda_{e_1}).$$

Moreover,

- (i) *if Λ is not a $\text{GL}(n)$ image of Λ_\perp and equality holds in (5.1), then Λ_{e_1} is not the empty set.*
- (ii) *if Λ is not a $\text{GL}(n)$ image of an obtuse set and equality holds in (5.1), then Λ_{e_1} contains a positive multiple of e_1 .*

Now we are in a position of proving the main results.

Proof of Theorem 1.1. First, if $\varphi = \text{Id}$, it is established in [51], so we only need to show the case $\varphi \neq \text{Id}$.

For $\varphi \neq \text{Id}$, let $\mathcal{P}(\Lambda) = 1/[V(Z_\varphi^{+,*}\Lambda)V(Z_1^+\Lambda)]$ denote the inverse volume product of the asymmetric Orlicz zonotopes, Λ_t , $t \in [-a^{-1}, 1]$, denotes the orthogonalization of Λ . First, by the $\text{GL}(n)$ invariance of $\mathcal{P}(\Lambda)$, there is nothing to show if Λ is a $\text{GL}(n)$ image of the canonical basis Λ_\perp . Otherwise, by Theorem 3.6, we know that $\mathcal{P}(\Lambda_t)$, $t \in [-a^{-1}, 1]$, satisfies the hypotheses of Lemma 5.1. Then there exists a multiset Λ_{e_1} of e_1 such that

$$(5.2) \quad \mathcal{P}(\Lambda) \leq \mathcal{P}(\Lambda_\perp \uplus \Lambda_{e_1}).$$

If Λ_{e_1} is empty then the inequality (1.1) holds. If Λ_{e_1} contains the only positive multiples of e_1 , then $\overline{\Lambda_\perp \uplus \Lambda_{e_1}}$ is a $\text{GL}(n)$ image of Λ_\perp . Then we have

$$\mathcal{P}(\Lambda) \leq \mathcal{P}(\Lambda_\perp \uplus \Lambda_{e_1}) = \mathcal{P}(\Lambda_\perp).$$

It remains to show that if Λ_{e_1} contains negative multiples, we say, $\Lambda_{e_1} = \{-\mu e_1\}$, where $\mu > 0$. By Lemma 4.6 we have

$$\mathcal{P}(\Lambda) \leq \mathcal{P}(\Lambda_\perp \uplus \Lambda_{e_1}) < \mathcal{P}(\Lambda_\perp).$$

Then the inequality of (1.1) holds. Now we deal with the equality condition. Since the equality holds in (5.2) only if Λ_{e_1} is not empty. By Lemma 4.1, we have

$$\mathcal{P}(\Lambda \uplus \Lambda_{e_1}) \leq \mathcal{P}(\overline{\Lambda_\perp \uplus \Lambda_{e_1}}),$$

equality holds if and only if $\Lambda_{e_1} = \{-\mu e_1\}$, where $\mu \geq 0$. Note that Λ_{e_1} is not the empty set, then $\mu > 0$. So we have

$$(5.3) \quad \mathcal{P}(\Lambda) = \mathcal{P}(\Lambda_\perp \uplus \Lambda_{e_1}) = \mathcal{P}(\overline{\Lambda_\perp \cup \{-\mu e_1\}}) = \mathcal{P}(\Lambda_\perp).$$

Note that $\varphi \neq \text{Id}$, by Lemma 4.6, we have the equalities of (5.3) hold if and only if Λ , $\Lambda_\perp \uplus \Lambda_{e_1}$, $\overline{\Lambda_\perp \cup \{-\mu e_1\}}$, and Λ_\perp are $\text{GL}(n)$ images of each other. So we obtain the desired inequality together with its equality conditions. Now we complete the proof. \square

If $\varphi(t) = t^p$, $p \geq 1$, this result reduces to the asymmetric L_p -volume ratio inequality in [51].

Corollary 5.2. *Let $p \geq 1$ and Λ be a finite and spanning multiset. Then*

$$V(Z_p^{+,*}\Lambda)V(Z_1^+\Lambda) \geq V(Z_p^{+,*}\Lambda_\perp)V(Z_1^+\Lambda_\perp).$$

If $p > 1$, equality holds if and only if Λ is a $\text{GL}(n)$ image of the canonical basis Λ_\perp . If $p = 1$, equality holds if and only if $Z_1^+\Lambda$ is a parallelepiped.

Proof of Theorem 1.2. Let $\mathcal{R}(\Lambda) = V(Z_\varphi^+\Lambda)/V(Z_1^+\Lambda)$. Theorem 3.6 implies that $\mathcal{R}(\Lambda)$ satisfies the hypotheses of Lemma 5.1. Then

$$(5.4) \quad \mathcal{R}(\Lambda) \leq \mathcal{R}(\Lambda_\perp \uplus \Lambda_{e_1}),$$

where Λ_{e_1} contains multiples of e_1 . If Λ_{e_1} is empty, then (1.2) holds. If Λ_{e_1} contains the only positive multiples of e_1 , then $\overline{\Lambda_\perp \uplus \Lambda_{e_1}}$ is a $\text{GL}(n)$ image of Λ_\perp . Then we have

$$(5.5) \quad \mathcal{R}(\overline{\Lambda_\perp \uplus \Lambda_{e_1}}) = \mathcal{R}(\Lambda_\perp).$$

Moreover, by Lemma 4.1,

$$(5.6) \quad \mathcal{R}(\Lambda_\perp \uplus \Lambda_{e_1}) \leq \mathcal{R}(\overline{\Lambda_\perp \uplus \Lambda_{e_1}}).$$

Together with (5.4), (5.5) and (5.6) we obtain

$$\mathcal{R}(\Lambda) \leq \mathcal{R}(\Lambda_\perp).$$

Note that if $\Lambda_{e_1} = \{-\mu e_1\}$, where $\mu > 0$, then $\Lambda_\perp \uplus \Lambda_{e_1} = \Lambda_\perp \cup \{-\mu e_1\}$ is an obtuse set. Lemma 4.3 implies that the inequality in (1.2) holds.

Now we deal with the equality case of (1.2). We assume that Λ is not a $\text{GL}(n)$ image of an obtuse set. Note that (5.4) with equality holds only if Λ_{e_1} contains a positive multiple of e_1 . In this case, then $\overline{\Lambda_\perp \uplus \Lambda_{e_1}}$ is a $\text{GL}(n)$ image of Λ_\perp . Then we have

$$\mathcal{R}(\overline{\Lambda_\perp \uplus \Lambda_{e_1}}) = \mathcal{R}(\Lambda_\perp).$$

By Lemma 4.1, we have

$$\mathcal{R}(\Lambda_\perp \uplus \Lambda_{e_1}) \leq \mathcal{R}(\overline{\Lambda_\perp \uplus \Lambda_{e_1}}).$$

Equality holds if and only if $\Lambda_\perp \uplus \Lambda_{e_1} = \overline{\Lambda_\perp \uplus \Lambda_{e_1}}$, which means Λ_{e_1} must be an negative multiples of e_1 , then it contradicts with Λ_{e_1} contains positive multiples of e_1 . We prove that if the equality hold in (1.2), holds then Λ is a $\text{GL}(n)$ image of an obtuse set.

On the other hand, if Λ is a $\text{GL}(n)$ image of an obtuse set, by the $\text{GL}(n)$ invariance of the volume ratio for the Orlicz zonotopes and Lemma 4.3, the equality of (1.2) holds.

Together with the above we have the equality of (1.2) hold if and only if Λ is a $\text{GL}(n)$ image of an obtuse set. We complete the proof. \square

If we take $\varphi(t) = t^p$, $p > 1$, then it reduces to the following.

Corollary 5.3. *Suppose $p > 1$ and Λ is a finite and spanning multiset. Then*

$$\frac{V(Z_p^+\Lambda)}{V(Z_1^+\Lambda)} \leq \frac{V(Z_p^+\Lambda_\perp)}{V(Z_1^+\Lambda_\perp)},$$

equality holds if and only if Λ is a $\text{GL}(n)$ image of an obtuse set.

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