TAIWANESE JOURNAL OF MATHEMATICS
Vol. 21, No. 6, pp. 1283–1324, December 2017
DOI: 10.11650/tjm/8009

Coreflexive Modules and Semidualizing Modules with Finite Projective Dimension

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Abstract. Let R and S be rings and ${}_S\omega_R$ a semidualizing bimodule. For a subclass $\mathcal{T}$ of the class of ω-coreflexive modules and $n \geq 1$, we introduce and study modules of ω-$\mathcal{T}$-class n. By using the properties of such modules, we get some equivalent characterizations for ω_S having finite projective dimension. In particular, we prove that the projective dimension of ω_S is at most n if and only if any module of ω-$\mathcal{T}$-class n is ω-coreflexive. Moreover, we get some equivalent characterizations for ω_S having finite projective dimension at most two or one in terms of the properties of (adjoint) ω-coreflexive and ω-cotorsionless modules. Finally, we give some partial answers to the Wakamatsu tilting conjecture.

1. Introduction

It is well known that the (Auslander) transpose is one of the most powerful tools in representation theory of artin algebras and Gorenstein homological algebra, see [2, 3, 8], and references therein. However, this notion does not have its dual version as many notions in classical homological algebra do. So, a natural question is: How to dualize the (Auslander) transpose of modules appropriately? To this aim, we introduced in [18, 20] the notions of the cotranspose and adjoint cotranspose of modules with respect to a semidualizing bimodule ω. Then we showed in [18–20] that many interesting notions and results related to the (Auslander) transpose have counterparts related to the (adjoint) cotranspose. For example, the counterparts of torsionless, reflexive and n-torsionfree modules are ω-cotorsionless, ω-coreflexive and n-ω-cotorsionfree modules, respectively. As a continue of these three papers, this paper is devoted to developing a further general theory introduced in them.

Wakamatsu in [21] introduced and studied the so-called generalized tilting modules, which are usually called Wakamatsu tilting modules, see [5, 16]. The Wakamatsu tilting

Received November 1, 2016; Accepted February 21, 2017.
Communicated by Kunio Yamagata.
2010 *Mathematics Subject Classification.* 18G25, 16E10, 16E30.
Key words and phrases. HT-projections, modules of ω-$\mathcal{T}$-class n, (adjoint) ω-coreflexive modules, (adjoint) ω-cotorsionless modules, (adjoint) n-ω-cospherical modules, projective dimension, Wakamatsu tilting conjecture.
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conjecture is an important homological conjecture in representation theory of artin algebras, which states that for a Wakamatsu tilting module ${}_R\omega$ over an artin algebra R, the projective (or injective) dimensions of ${}_R\omega$ and $\omega_{\mathrm{End}({}_R\omega)}$ are identical [5, 16]. This conjecture situates between the famous finitistic dimension conjecture and the Gorenstein symmetry conjecture; in particular, the latter one is a special case of the Wakamatsu tilting conjecture. All these conjectures remain still open. By [21, Theorem], the Wakamatsu tilting conjecture is equivalent to that for a Wakamatsu tilting module ${}_R\omega$ over an artin algebra R, the projective (or injective) dimension of ${}_R\omega$ is finite if and only if so is the projective (or injective) dimension of $\omega_{\mathrm{End}({}_R\omega)}$. Huang in [10] generalized this equivalent version to left and right noetherian rings.

Observe that the Wakamatsu tilting conjecture makes sense for arbitrary rings. Let R and S be arbitrary rings. By [22, Corollary 3.2], we have that a bimodule ${}_R\omega_S$ is semidualizing if and only if ${}_R\omega$ is Wakamatsu tilting with $S = \mathrm{End}({}_R\omega)$, and if and only if ω_S is Wakamatsu tilting with $R = \mathrm{End}(\omega_S)$. It was proved in [21, Theorem (1)] that for a semidualizing bimodule ${}_R\omega_S$, the projective dimensions of ${}_R\omega$ and ω_S are identical provided that both of them are finite. So, over arbitrary rings R and S, the Wakamatsu tilting conjecture is equivalent to that for a semidualizing bimodule ${}_R\omega_S$, the projective dimension of ${}_R\omega$ is finite if and only if so is the projective dimension of ω_S. In this paper, we will study when the projective dimension of ω_S is at most n by using the properties of modules of ω-$\mathcal{T}$-class n, (adjoint) ω-cotorsionless and ω-coreflexive modules.

This paper is organized as follows.

In Section 2, we give some terminology and some preliminary results.

Let R and S be rings and ${}_S\omega_R$ a semidualizing bimodule. In Section 3, we introduce and study Hom-Tensor projections and Tensor-Hom injections as duals of double dual embeddings in [13]. Let M be a left R-module and F a left S-module. An epimorphism $\omega \otimes_S F \overset{1_\omega \otimes \phi}{\twoheadrightarrow} \omega \otimes_S \mathrm{Hom}_R(\omega, M)$ of left R-modules is called a *Hom-Tensor projection* if it is obtained by applying the functor $\omega \otimes_S -$ to an epimorphism $F \overset{\phi}{\twoheadrightarrow} \mathrm{Hom}_R(\omega, M)$ of left S-modules. We prove that the kernel of a Hom-Tensor projection with F adjoint ω-coreflexive and $\omega \otimes_S F$ 1-ω-cospherical is the ω-counit submodule of a 1-ω-cospherical left R-module; conversely, the ω-counit submodule of a 1-ω-cospherical left R-module is the kernel of a special Hom-Tensor projection. We also get an adjoint version of this result about Tensor-Hom injections.

Jans introduced in [13] the notion of modules of D-class n in terms of the properties of double dual embeddings, and proved that for a left and right noetherian ring R and $n \geq 1$, the right self-injective dimension of R is at most n if and only if any finitely generated left R-module of D-class n is reflexive; and the global dimension of R is at most $n+1$ if and only if $\mathrm{Hom}_R(M, R)$ is projective for any finitely generated left R-module

M of D-class n. Motivated by Jans's philosophy, in Section 4 we introduce and study modules of ω-$\mathcal{T}$-class n in terms of the properties of Hom-Tensor projections, where $\mathcal{T}$ is a subclass of the class of adjoint ω-coreflexive left S-modules and $n \geq 1$. We prove that if U_n is a left R-module of ω-$\mathcal{T}$-class n, then there exists a collection of exact sequences $0 \to \mathrm{Hom}_R(\omega, U_i) \to F_{i-1} \to \mathrm{Hom}_R(\omega, U_{i-1}) \to 0$ $(2 \leq i \leq n)$ of left S-modules with all $F_i \in \mathcal{T}$ and U_i left R-modules; conversely, if there exists a collection of exact sequences as above, then U_n can be selected of ω-$\mathcal{T}$-class n. Let $\mathcal{T}$ be a subclass of the weak Auslander class with respect to ω containing all projective left S-modules. We prove that the projective dimension of ω_S is at most n if and only if any left R-module of ω-$\mathcal{T}$-class n is ω-coreflexive, and if and only if $\mathrm{Tor}_n^S(\omega, V) = 0$ for any adjoint ω-cotorsionless left S-module V. As a supplement to this result, we get that the projective dimension of ω_S is at most $n+1$ if and only if $\mathrm{Tor}_1^S(\omega, \mathrm{Hom}_R(\omega, U_n)) = 0$ for any left R-module U_n of ω-$\mathcal{T}$-class n.

In Section 5, we first obtain some useful exact sequences to describe the kernel and cokernel of the canonical valuation homomorphism $\omega \otimes_S \mathrm{Hom}_R(\omega, M) \to M$ with M a left R-module; and then prove that any n-ω-cospherical left R-module is ω-coreflexive provided that either the projective dimension of ω_S is at most n or ω_S admits a projective resolution ultimately closed at n.

In Section 6, we characterize when ω_S has small projective dimension in terms of the properties of (adjoint) ω-coreflexive modules and ω-cotorsionless modules. We prove that if the projective dimension of ${}_R\omega$ is at most two, then the projective dimension of ω_S is at most two if and only if any 2-ω-cospherical left R-module is ω-coreflexive, if and only if any adjoint ω-coreflexive left S-module is adjoint 2-ω-cospherical, if and only if any left R-module of ω-$\mathcal{T}$-class 2 is ω-coreflexive, if and only if $\mathrm{Tor}_2^S(\omega, V) = 0$ for any adjoint ω-cotorsionless left S-module V, and if and only if $\mathrm{Tor}_1^S(\omega, \mathrm{Hom}_R(\omega, U)) = 0$ for any ω-cotorsionless left R-module U. Moreover, we get that the projective dimension of ω_S is at most one if and only if any 1-ω-cospherical left R-module is ω-cotorsionless (or ω-coreflexive), if and only if any ω-cotorsionless left R-module is ω-coreflexive, and if and only if $\mathrm{Tor}_1^S(\omega, V) = 0$ for any adjoint ω-cotorsionless left S-module module V.

In Section 7, we study the Wakamatsu tilting conjecture in some special cases. Let S be a left artinian ring, $R = S$ and $m, n \geq 1$. We prove that if the projective dimension of ${}_S\omega$ is at most n and the Ext-grade of $\mathrm{Tor}_m^S(\omega, N)$ with respect to ω is at most $n-1$ for any finitely presented left S-module N, then the projective dimensions of ${}_S\omega$ and ω_S are identical. Then we apply this result to get that if the projective dimension of ${}_S\omega$ is at most n and the projective dimension of $\mathrm{Hom}_S(P_i(\omega), \omega)$ is finite for any $0 \leq i \leq n-2$, where $P_i(\omega)$ is the $(i+1)$-st term in a minimal projective resolution of ${}_S\omega$, then the projective dimensions of ${}_S\omega$ and ω_S are identical. As a consequence, we get that if the projective dimension of ${}_S\omega$ is at most one, then the projective dimensions of ${}_S\omega$ and ω_S are identical. Finally, we get that for an artin algebra S, if the right self-injective dimension of S is at most n and the projective dimensions of the first $n-1$ terms in a minimal injective resolution of S_S are finite, then the left and right self-injective dimensions of S are identical.

2. Preliminaries

Throughout this paper, all rings are associative rings with unites. For a ring R, we use $\operatorname{Mod} R$ (resp. $\operatorname{Mod} R^{\mathrm{op}}$) to denote the class of left (resp. right) R-modules. Araya, Takahashi and Yoshino in [1] initialed the study of semidualizing bimodules over noetherian rings. Then Holm and White in [9] extended this notion to associative rings.

Definition 2.1. [1,9] Let R and S be rings. An (R-S)-bimodule ${}_R\omega_S$ is called *semidualizing* if

(1) An (R-S)-bimodule ${}_R\omega_S$ is called *semidualizing* if the following conditions are satisfied.

  (a1) ${}_R\omega$ admits a degreewise finite R-projective resolution.

  (a2) ω_S admits a degreewise finite S-projective resolution.

  (b1) The homothety map ${}_RR_R \xrightarrow{{}_R\gamma} \operatorname{Hom}_{S^{\mathrm{op}}}(\omega,\omega)$ is an isomorphism.

  (b2) The homothety map ${}_SS_S \xrightarrow{\gamma_S} \operatorname{Hom}_{R}(\omega,\omega)$ is an isomorphism.

  (c1) $\operatorname{Ext}_R^{\geq 1}(\omega,\omega)=0$.

  (c2) $\operatorname{Ext}_{S^{\mathrm{op}}}^{\geq 1}(\omega,\omega)=0$.

(2) A semidualizing bimodule ${}_R\omega_S$ is called *faithful* if the following conditions are satisfied.

  (e1) If $M \in \operatorname{Mod} R$ and $\operatorname{Hom}_R(\omega, M)=0$, then $M=0$.

  (e2) If $N \in \operatorname{Mod} S^{\mathrm{op}}$ and $\operatorname{Hom}_{S^{\mathrm{op}}}(\omega, N)=0$, then $N=0$.

Let R be a ring. Recall from [21,22] that a module ω in $\operatorname{Mod} R$ is called *generalized tilting* (it is usually called *Wakamatsu tilting*, see [5,16]) if it satisfies the conditions (a1) and (c1) in Definition 2.1, and there exists an exact sequence

$$0 \to {}_RR \to W^0 \to W^1 \to \cdots \to W^i \to \cdots$$

in $\operatorname{Mod} R$ with all W^i isomorphic to direct summands of finite sums of copies of ${}_R\omega$, such that it remains still exact after applying the functor $\operatorname{Hom}_R(-, {}_R\omega)$. The notion of semidualizing are equivalent to that of Wakamatsu tilting (see the introduction).

By [9, Proposition 3.1], we have that any semidualizing bimodule over a commutative ring is faithful. The following example illustrates that there exist sufficiently many (faithful) semidualizing bimodules.

Example 2.2. (1) For any ring R, ${}_RR_R$ is semidualizing.

(2) Let R be an artin algebra, and let $\{T_1, \ldots, T_n\}$ be a complete set of non-isomorphic simple left R-module. Then $\omega := \bigoplus_{i=1}^{n} I^0(T_i)$ is Wakamatsu tilting, where $I^0(T_i)$ is the injective envelope of T_i for any $1 \leq i \leq n$. By [22, Corollary 3.2], we have that ${}_R\omega_S$ is semidualizing, where $S = \mathrm{End}({}_R\omega)$.

(3) Let k be a field. Then both $A = k[x,y]/(x,y)^2$ and $S = A[u,v]/(u,v)^2$ are commutative artinian non-Gorenstein local rings; and $\mathrm{Hom}_A(S, A)$ and $S \otimes_A \mathrm{Hom}_k(A, k)$ are mutually non-isomorphic semidualizing (S,S)-bimodules with infinite projective and injective dimensions (see [17, Example 2.3.2]).

(4) Let R be a flat S-algebra over a commutative ring S. If ${}_SE_S$ is a semidualizing bimodule, then $E \otimes_S R$ is a faithfully semidualzing (R,R)-bimodule (see [9, Proposition 3.2]).

From now on, R and S are arbitrary associative rings with unit and ${}_R\omega_S$ is a semidualizing bimodule. We write $(-)_* := \mathrm{Hom}(\omega, -)$.

Let $M \in \mathrm{Mod}\, R$. Then we have a canonical valuation homomorphism

$$\theta_M \colon \omega \otimes_S M_* \to M$$

defined by $\theta_M(x \otimes f) = f(x)$ for any $x \in \omega$ and $f \in M_*$. M is called *ω-cotorsionless* if θ_M is an epimorphism; and M is called *ω-coreflexive* if θ_M is an isomorphism (see [18]). We use $\mathrm{Cot}_\omega(R)$ and $\mathrm{Cor}_\omega(R)$ to denote the subclasses of $\mathrm{Mod}\, R$ consisting of ω-cotorsionless modules and ω-coreflexive modules, respectively.

Let $N \in \mathrm{Mod}\, S$. Then we have a canonical valuation homomorphism

$$\mu_N \colon N \to (\omega \otimes_S N)_*$$

defined by $\mu_N(y)(x) = x \otimes y$ for any $y \in N$ and $x \in \omega$. N is called *adjoint ω-cotorsionless* if μ_N is a monomorphism; and N is called *adjoint ω-coreflexive* if μ_N is an isomorphism. We use $\mathrm{Acot}_\omega(S)$ and $\mathrm{Acor}_\omega(S)$ to denote the subclasses of $\mathrm{Mod}\, S$ consisting of adjoint ω-cotorsionless modules and adjoint ω-coreflexive modules, respectively.

Definition 2.3. (1) The *weak Auslander class $w\mathcal{A}_\omega(S)$* with respect to ω consists of all left S-modules N satisfying

(A1) $\mathrm{Tor}^S_{i\geq 1}(\omega, N) = 0$, and

(A2) $N \in \mathrm{Acor}_\omega(S)$.

(2) (see [9]) The *Auslander class* $\mathcal{A}_\omega(S)$ with respect to ω consists of all left S-modules N satisfying (A1), (A2) and

(A3) $\mathrm{Ext}_R^{\geq 1}(\omega, \omega \otimes_S N) = 0$.

We will heavily use the following two lemmas in the sequel.

Lemma 2.4. *Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be two functors between categories $\mathcal{C}$ and $\mathcal{D}$ such that F is a left adjoint of G, $\mu\colon 1_{\mathcal{C}} \to GF$ and $\theta\colon FG \to 1_{\mathcal{D}}$ are the unit and the counit of adjunction arrows, respectively. Then we have*

(1) $G\theta \cdot \mu G = 1_G$.

(2) $\theta F \cdot F\mu = 1_F$.

(3) *There exists an equivalence of categories*

$$\mathrm{Acor}_\omega(S) \underset{G:=(-)_*}{\overset{F:=\omega\otimes_S -}{\rightleftarrows}} \mathrm{Cor}_\omega(R).$$

Proof. See [15, p. 82, Theorem 1(ii)] for the assertions (1) and (2). The assertion (3) is a direct consequence of (1) and (2). □

Following [9], set

$$\begin{aligned}
\mathcal{F}_\omega(R) &:= \{\omega \otimes_S F \mid F \text{ is flat in } \mathrm{Mod}\, S\}, \\
\mathcal{P}_\omega(R) &:= \{\omega \otimes_S P \mid P \text{ is projective in } \mathrm{Mod}\, S\}, \\
\mathcal{I}_\omega(S) &:= \{I_* \mid I \text{ is injective in } \mathrm{Mod}\, R\}, \\
{}_R\omega^{\perp} * &:= \left\{M \in \mathrm{Mod}\, R \mid \mathrm{Ext}_R^{\geq 1}(\omega, M) = 0\right\}.
\end{aligned}$$

The modules in $\mathcal{F}_C(R)$, $\mathcal{P}_\omega(R)$ and $\mathcal{I}_\omega(S)$ are called ω-*flat*, ω-*projective* and ω-*injective* respectively. We use $\mathcal{I}(R)$ to denote the subclass of $\mathrm{Mod}\, R$ consisting of injective modules, and use $\mathcal{P}(S)$ and $\mathcal{F}(S)$ to denote the subclasses of $\mathrm{Mod}\, S$ consisting of projective modules and flat modules, respectively. For a module $M \in \mathrm{Mod}\, R$, we use $\mathrm{Add}_R\, M$ to denote the subclass of $\mathrm{Mod}\, R$ consisting of all direct summands of direct sums of copies of M.

Lemma 2.5. (cf. [14, Proposition 2.4(1)] and [9, Lemma 4.1 and Corollary 6.1]).

(1) $\mathrm{Add}_R\, \omega = \mathcal{P}_\omega(R) \subseteq \mathcal{F}_\omega(R) \cup \mathcal{I}(R) \subseteq \mathrm{Cor}_\omega(R) \cap {}_R\omega^{\perp}$.

(2) $\mathcal{P}(S) \subseteq \mathcal{F}(S) \cup \mathcal{I}_\omega(S) \subseteq \mathcal{A}_\omega(S) \subseteq w\mathcal{A}_\omega(S) \subseteq \mathrm{Acor}_\omega(S)$.

Motivated by the notion of n-spherical modules given in [2], we introduce the following

Definition 2.6. Let $n \geq 1$.

(1) (see [18]) A module $M \in \operatorname{Mod} R$ is called *n-ω-cospherical* if $\operatorname{Ext}_R^{1 \leq i \leq n}(\omega, M) = 0$.

(2) A module $N \in \operatorname{Mod} S$ is called *adjoint n-ω-cospherical* if $\operatorname{Tor}^S_{1 \leq i \leq n}(\omega, N) = 0$.

We shall say that any module in $\operatorname{Mod} R$ is *0-ω-cospherical*, and any module in $\operatorname{Mod} S$ is *adjoint 0-ω-cospherical*.

Let $M \in \operatorname{Mod} R$. We use

$$0 \longrightarrow M \xrightarrow{f^{-1}(M)} I^0(M) \xrightarrow{f^0(M)} I^1(M) \xrightarrow{f^1(M)} \cdots \xrightarrow{f^{i-1}(M)} I^i(M) \xrightarrow{f^i(M)} \cdots$$

to denote a minimal injective resolution of M in $\operatorname{Mod} R$.

Definition 2.7. [18] Let $M \in \operatorname{Mod} R$ and $n \geq 1$.

(1) $\operatorname{cTr}_\omega M := \operatorname{Coker} f^0(M)_*$ is called the *cotranspose* of M with respect to ${}_R\omega_S$.

(2) M is called *n-ω-cotorsionfree* if $\operatorname{cTr}_\omega M$ is adjoint n-ω-cospherical.

By [18, Proposition 3.2] (see Corollary 5.2(1) below), we have that for a module $M \in \operatorname{Mod} R$, M is 1-ω-cotorsionfree if and only if it is ω-cotorsionless; and M is 2-ω-cotorsionfree if and only if it is ω-coreflexive. Note that the notion of ω-coreflexive modules has appeared in [4].

Let $N \in \operatorname{Mod} S$ and we use

$$\cdots \xrightarrow{f_i(N)} F_i(N) \xrightarrow{f_{i-1}(N)} \cdots \xrightarrow{f_1(N)} F_1(N) \xrightarrow{f_0(N)} F_0(N) \xrightarrow{f_{-1}(N)} N \longrightarrow 0 \tag{2.1}$$

to denote a minimal flat resolution of N in $\operatorname{Mod} S$, where each $F_i(N) \twoheadrightarrow \operatorname{Coker} f_i(N)$ is a flat cover of $\operatorname{Coker} f_i(N)$. The existence of such a resolution is guaranteed by the fact that any module has a flat cover (see [6]). Based on the fact that $(\omega \otimes_S -, \operatorname{Hom}_R(\omega, -))$ is an adjoint pair, the counterpart of Definition 2.7 was given in [20] as follows.

Definition 2.8. [20] Let $N \in \operatorname{Mod} S$ and $n \geq 1$.

(1) $\operatorname{acTr}_\omega N := \operatorname{Ker}(1_\omega \otimes f_0(N))$ is called the *adjoint cotranspose* of N with respect to ${}_R\omega_S$.

(2) N is called *adjoint n-ω-cotorsionfree* if $\operatorname{acTr}_\omega N$ is n-ω-cospherical.

By Corollary 5.2(2) below, we have that for a module $N \in \operatorname{Mod} S$, N is adjoint 1-ω-cotorsionfree if and only if it is adjoint ω-cotorsionless; and N is adjoint 2-ω-cotorsionfree if and only if it is adjoint ω-coreflexive.

The following result about the properties of (adjoint) ω-cotorsionless and ω-coreflexive is useful.

Proposition 2.9. (1) *Let*

$$0 \longrightarrow K \xrightarrow{\lambda} F \xrightarrow{\phi} N \longrightarrow 0$$

be an exact sequence in $\operatorname{Mod} S$ *with* $F \in \operatorname{Acor}_{\omega}(S)$ *and* $N \in \operatorname{Acot}_{\omega}(S)$. *Then* $N \cong \operatorname{Im}(1_{\omega} \otimes \phi)_*$ *and* $K \cong H_*$, *where* $H = \operatorname{Ker}(1_{\omega} \otimes \phi)$.

(2) *Let*

$$0 \longrightarrow M \xrightarrow{\psi} I \xrightarrow{\alpha} H \longrightarrow 0$$

be an exact sequence in $\operatorname{Mod} R$ *with* $I \in \operatorname{Cor}_{\omega}(R)$ *and* $M \in \operatorname{Cot}_{\omega}(R)$. *Then* $M \cong \operatorname{Im}(1_{\omega} \otimes \psi_*)$ *and* $H \cong \omega \otimes_S K$, *where* $K = \operatorname{Coker} \psi_*$.

Proof. (1) By assumption, we have the following exact sequence

$$0 \longrightarrow H \xrightarrow{\delta} \omega \otimes_S F \xrightarrow{1_{\omega} \otimes \phi} \omega \otimes_S N \longrightarrow 0$$

in $\operatorname{Mod} R$ with $H = \operatorname{Ker}(1_{\omega} \otimes \phi)$. Consider the following exact commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\lambda"] \arrow[d, dashed, "h"] & F \arrow[r, "\phi"] \arrow[d, "\mu_F"] & N \arrow[r] \arrow[d, "\mu_N"] & 0 \\
0 \arrow[r] & H_* \arrow[r, "\delta_*"] & (\omega \otimes_S F)_* \arrow[r, "(1_{\omega} \otimes \phi)_*"] & (\omega \otimes_S N)_*, &
\end{tikzcd}$$

where h is an induced homomorphism. Because μ_F is an isomorphism and μ_N is a monomorphism by assumption, we have that $N \cong \operatorname{Im} \mu_N \cong \operatorname{Im}(1_{\omega} \otimes \phi)_*$ and h is an isomorphism by the snake lemma.

(2) By assumption, we have the following exact sequence

$$0 \longrightarrow M_* \xrightarrow{\psi_*} I_* \xrightarrow{\pi} K \longrightarrow 0$$

in $\operatorname{Mod} S$ with $K = \operatorname{Coker} \psi_*$. Consider the following commutative diagram with exact rows

$$\begin{tikzcd}
 & \omega \otimes_S M_* \arrow[r, "1_{\omega} \otimes \psi_*"] \arrow[d, "\theta_M"] & \omega \otimes_S I_* \arrow[r, "1_{\omega} \otimes \pi"] \arrow[d, "\theta_I"] & \omega \otimes_S K \arrow[r] \arrow[d, dashed, "\gamma"] & 0 \\
0 \arrow[r] & M \arrow[r, "\psi"] & I \arrow[r, "\alpha"] & H \arrow[r] & 0,
\end{tikzcd}$$

where γ is an induced homomorphism. Because θ_I is an isomorphism and θ_M is an epimorphism by assumption, we have that $M = \operatorname{Im} \theta_M \cong \operatorname{Im}(1_{\omega} \otimes \psi_*)$ and γ is an isomorphism by the snake lemma. $\square$

3. Hom-Tensor projections and Tensor-Hom injections

We begin with the following definition which will be convenient for our exposition.

Definition 3.1. Let $M \in \operatorname{Mod} R$ and $F \in \operatorname{Mod} S$. An epimorphism

$$1_\omega \otimes \phi\colon \omega \otimes_S F \twoheadrightarrow \omega \otimes_S M_*$$

in $\operatorname{Mod} R$ is called a *Hom-Tensor projection* (*HT-projection* for short) if it is obtained by applying the functor $\omega \otimes_S -$ to an epimorphism $\phi\colon F \twoheadrightarrow M_*$ in $\operatorname{Mod} S$.

To study the properties of HT-projections, we need the following

Lemma 3.2. *Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be two functors between abelian categories $\mathcal{C}$ and $\mathcal{D}$ such that F is a left adjoint of G, $\mu\colon 1_{\mathcal{C}} \to GF$ and $\theta\colon FG \to 1_{\mathcal{D}}$ are the unit and the counit of adjunction arrows, respectively. Then for $A, B \in \mathcal{D}$, the following statements are equivalent.*

(1) $A \cong \operatorname{Im} \theta_B$.

(2) *θ_A is an epimorphism and there exists a monomorphism $f\colon A \rightarrowtail B$ in $\mathcal{D}$ such that $G(f)$ is an isomorphism.*

Proof. (1) $\Rightarrow$ (2): Let $A \cong \operatorname{Im} \theta_B$ and $g\colon A \to \operatorname{Im} \theta_B$ be an isomorphism in $\mathcal{D}$. Since $\theta_{FG(B)}$ is epic by Lemma 2.4(2) and A is a quotient object of $FG(B)$, we have θ_A is epic. Let $\theta_B = i \cdot p$ be the natural epic-monic decomposition of θ_B with $p\colon FG(B) \twoheadrightarrow \operatorname{Im} \theta_B$ and $i\colon \operatorname{Im} \theta_B \hookrightarrow B$. Then $f := i \cdot g$ is monic. Note that $G(\theta_B) = G(i) \cdot G(p)$ and $G(\theta_B)$ is a retraction by Lemma 2.4(1). It yields that $G(i)$ is an epimorphism and hence an isomorphism. Thus $G(f) = G(i) \cdot G(g)$ is an isomorphism.

(2) $\Rightarrow$ (1): Let θ_A be epic and $f\colon A \rightarrowtail B$ be a monomorphism in $\mathcal{D}$ such that $G(f)$ is an isomorphism. Consider the following commutative diagram with the bottom row exact

$$\begin{tikzcd}
 & FG(A) \arrow[r, "FG(f)"] \arrow[d, "\theta_A"] & FG(B) \arrow[d, "\theta_B"] \\
0 \arrow[r] & A \arrow[r, "f"] & B.
\end{tikzcd}$$

Since $G(f)$ is an isomorphism, $FG(f)$ is also an isomorphism. So we have

$$\operatorname{Im} \theta_B = \operatorname{Im}(\theta_B \cdot (FG(f))) = \operatorname{Im}(f \cdot \theta_A) = \operatorname{Im} f \cong A. \qquad \square$$

For a module $M \in \operatorname{Mod} R$, we call $\operatorname{Im} \theta_M$ the *ω-counit submodule* of M. The following addresses the relation between HT-projections and the ω-counit submodules of 1-ω-cospherical modules.

Theorem 3.3. *Let $M \in \operatorname{Mod} R$ and $F \in \operatorname{Mod} S$. If*

$$1_\omega \otimes \phi\colon \omega \otimes_S F \twoheadrightarrow \omega \otimes_S M_*$$

is a HT-projection with $F \in \operatorname{Acor}_\omega(S)$ *and* $\omega \otimes_S F$ 1-ω*-cospherical in* $\operatorname{Mod} R$*, then* $H := \operatorname{Ker}(1_\omega \otimes \phi)$ *is isomorphic to the* ω*-counit submodule of a* 1-ω*-cospherical module in* $\operatorname{Mod} R$.

Conversely, if H *is isomorphic to the* ω*-counit submodule of a* 1-ω*-cospherical module in* $\operatorname{Mod} R$*, then there exists an exact sequence*

$$0 \longrightarrow H \longrightarrow E \xrightarrow{\alpha} Y \longrightarrow 0$$

in $\operatorname{Mod} R$ *with* E *injective and* $\alpha\colon E \twoheadrightarrow Y$ *a HT-projection.*

Proof. Let

$$0 \longrightarrow H \longrightarrow \omega \otimes_S F \xrightarrow{1_\omega \otimes \phi} \omega \otimes_S M_* \longrightarrow 0$$

be an exact sequence in $\operatorname{Mod} R$ with $1_\omega \otimes \phi$ a HT-projection, $F \in \operatorname{Acor}_\omega(S)$, $\omega \otimes_S F$ 1-ω-cospherical in $\operatorname{Mod} R$ and $H = \operatorname{Ker}(1_\omega \otimes \phi)$. Then we have the following exact sequence

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\phi} M_* \longrightarrow 0 \tag{3.1}$$

in $\operatorname{Mod} S$, where $K = \operatorname{Ker} \phi$. Because $F \in \operatorname{Acor}_\omega(S)$ and $M_* \in \operatorname{Acot}_\omega(S)$ by assumption and Lemma 2.4(1) respectively, we have $K \cong H_*$ by Proposition 2.9(1). Applying the functor $\omega \otimes_S -$ to (3.1) yields that H is isomorphic to a quotient module of $\omega \otimes_S K$. Using Lemma 2.4(2) and [18, Corollary 3.8], we get $H \in \operatorname{Cot}_\omega(R)$. Let $L = \operatorname{Im} \theta_M$ and let $\theta_M = i \cdot p$ be the natural epic-monic decomposition of θ_M with $p\colon \omega \otimes_S M_* \twoheadrightarrow L$ and $i\colon L \hookrightarrow M$. Then

$$i_* \cdot p_* \cdot \mu_{M_*} = (\theta_M)_* \cdot \mu_{M_*} = 1_{M_*}$$

by Lemma 2.4(1). It implies that i_* is an epimorphism, and hence an isomorphism. So $p_* \cdot \mu_{M_*}$ is also an isomorphism. Set $H' = \operatorname{Ker}(p \cdot (1_\omega \otimes \phi))$. Consider the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r] \arrow[d, dashed, "\lambda"] & \omega \otimes_S F \arrow[r, "1_\omega \otimes \phi"] \arrow[d, equal] & \omega \otimes_S M_* \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & H' \arrow[r] & \omega \otimes_S F \arrow[r, "p \cdot (1_\omega \otimes \phi)"] & L \arrow[r] & 0,
\end{tikzcd} \tag{3.2}$$

where λ is an induced homomorphism which is monic. Because $(1_\omega \otimes \phi)_* \cdot \mu_F = \mu_{M_*} \cdot \phi$ and $\omega \otimes_S F$ is 1-ω-cospherical in $\operatorname{Mod} R$, applying the functor $\operatorname{Hom}_R(\omega, -)$ to (3.2) gives the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H_* \arrow[r] \arrow[d, "\lambda_*"] & (\omega \otimes_S F)_* \arrow[r, "\phi \cdot \mu_F^{-1}"] \arrow[d, equal] & M_* \arrow[rr] \arrow[d, "p_* \cdot \mu_{M_*}"] & & 0 \\
0 \arrow[r] & H'_* \arrow[r] & (\omega \otimes_S F)_* \arrow[r, "p_* \cdot (1_\omega \otimes \phi)_*"] & L_* \arrow[r] & \operatorname{Ext}_R^1(\omega, H') \arrow[r] & 0.
\end{tikzcd}$$

Because $p_* \cdot \mu_{M_*}$ is an isomorphism, we have that $\mathrm{Ext}^1_R(\omega, H') = 0$ and λ_* is also an isomorphism. Then it follows from Lemma 3.2 that H is isomorphic to the ω-counit submodule of a 1-ω-cospherical module H'.

Conversely, assume that H is isomorphic to the ω-counit submodule of a 1-ω-cospherical module H' in $\mathrm{Mod}\, R$. By Lemma 3.2, there exists a monomorphism $f\colon H \rightarrowtail H'$ such that f_* is an isomorphism. Consider the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r, "\psi"] \arrow[d, "f"] & E \arrow[r, "\alpha"] \arrow[d, equal] & Y \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & H' \arrow[r, "e"] & E \arrow[r, "\beta"] & Y' \arrow[r] & 0,
\end{tikzcd}$$

where E is injective, e is an embedding, $\psi = e \cdot f$, $Y = \mathrm{Coker}\, \psi$ and $Y' = \mathrm{Coker}\, e$.

We claim that $\alpha\colon E \twoheadrightarrow Y$ is a HT-projection. Since H' is 1-ω-cospherical, we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H_* \arrow[r, "\psi_*"] \arrow[d, "f_*"] & E_* \arrow[r, "\pi"] \arrow[d, equal] & Z \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & H'_* \arrow[r, "e_*"] & E_* \arrow[r, "\beta_*"] & Y'_* \arrow[r] & 0,
\end{tikzcd}$$

where $Z = \mathrm{Coker}\, \psi_*$. Since f_* is an isomorphism, we have $Z \cong Y'_*$. By Proposition 2.9(2) and its proof, we have that $Y \cong \omega \otimes_S Z$ and $\alpha\colon E \twoheadrightarrow Y$, up to isomorphism, is formed by tensoring $\pi\colon E_* \twoheadrightarrow Z$ $(\cong Y'_*)$ with $\omega \otimes_S -$. The claim is proved. $\square$

As a consequence of Theorem 3.3, we have the following

Corollary 3.4. *Let $M \in \mathrm{Mod}\, R$ and $F \in \mathrm{Mod}\, S$, and let*

$$1_\omega \otimes \phi\colon \omega \otimes_S F \twoheadrightarrow \omega \otimes_S M_*$$

be a HT-projection with $F \in \mathrm{Acor}_\omega(S)$ and $\omega \otimes_S F$ 1-ω-cospherical in $\mathrm{Mod}\, R$. Then $H := \mathrm{Ker}(1_\omega \otimes \phi)$ is a ω-cotorsionless and 1-ω-cospherical module in $\mathrm{Mod}\, R$ provided that one of the following conditions is satisfied.

(1) *$M \in \mathrm{Cor}_\omega(R)$.*

(2) *$\omega \otimes_S M_* \in \mathrm{Cor}_\omega(R)$ and ${}_R\omega_S$ is faithful.*

Conversely, if H is a ω-cotorsionless and 1-ω-cospherical module in $\mathrm{Mod}\, R$ and

$$0 \to H \to E \to Y \to 0$$

is an exact sequence in $\mathrm{Mod}\, R$ with E injective, then $E \twoheadrightarrow Y$ is a HT-projection.

Proof. By Theorem 3.3, we have that $H \in \mathrm{Cot}_\omega(R)$. From the exact sequence

$$0 \longrightarrow H \longrightarrow \omega \otimes_S F \xrightarrow{1_\omega \otimes \phi} \omega \otimes_S M_* \longrightarrow 0$$

in $\mathrm{Mod}\, R$, we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
F \arrow[r, "\phi"] \arrow[d, "\mu_F"] & M_* \arrow[r] \arrow[d, "\mu_{M_*}"] & 0 & \\
(\omega \otimes_S F)_* \arrow[r, "(1_\omega \otimes \phi)_*"] & (\omega \otimes_S M_*)_* \arrow[r] & \mathrm{Ext}^1_R(\omega, H) \arrow[r] & 0,
\end{tikzcd}$$

where μ_F is an isomorphism.

Case 1. Let $M \in \mathrm{Cor}_\omega(R)$. Then by Lemma 2.4(3), we have that $M_* \in \mathrm{Acor}_\omega(S)$ and μ_{M_*} is an isomorphism.

Case 2. Let $\omega \otimes_S M_* \in \mathrm{Cor}_\omega(R)$ and ${}_R\omega_S$ be faithful. Then $\theta_{\omega \otimes_S M_*}$ is an isomorphism. Since $\theta_{\omega \otimes_S M_*} \cdot (1_\omega \otimes \mu_{M_*}) = 1_{\omega \otimes_S M_*}$ by Lemma 2.4(2), we have that $1_\omega \otimes \mu_{M_*}$ is an isomorphism. Since ω is faithful, we have that μ_{M_*} is an epimorphism by [9, Lemma 3.1], and hence an isomorphism by Lemma 2.4(1).

Consequently, in either case, $(1_\omega \otimes \phi)_*$ is epic and $\mathrm{Ext}^1_R(\omega, H) = 0$, that is, H is 1-ω-cospherical.

The converse part of the corollary stems from the proof of the corresponding part of Theorem 3.3 using the fact that H is its own ω-counit submodule. □

In the rest of this section, we state, but do not prove, adjoint counterparts of the above notions and results about HT-projections.

Definition 3.5. Let $N \in \mathrm{Mod}\, S$ and $I \in \mathrm{Mod}\, R$. A monomorphism

$$\psi_* \colon (\omega \otimes_S N)_* \rightarrowtail I_*$$

in $\mathrm{Mod}\, S$ is called a *Tensor-Hom-injection* (*TH-injection* for short) if it is obtained by applying the functor $\mathrm{Hom}_R(\omega, -)$ to the monomorphism $\psi \colon \omega \otimes_S N \rightarrowtail I$ in $\mathrm{Mod}\, R$.

To study the properties of TH-injections, we need the following

Lemma 3.6. *Under the same assumptions as that in Lemma* 3.2*, for $M, N \in \mathcal{C}$, the following statements are equivalent.*

(1) $N \cong \mathrm{Im}\, \mu_M$.

(2) *μ_N is a monomorphism and there exists an epimorphism $g \colon M \twoheadrightarrow N$ in $\mathcal{C}$ such that $F(g)$ is an isomorphism.*

For a module $N \in \operatorname{Mod} S$, we call $\operatorname{Im} \mu_N$ the *ω-unit quotient module* of N. The following addresses the relation between TH-injections and the ω-unit quotient modules of adjoint 1-ω-cospherical modules.

Theorem 3.7. *Let $N \in \operatorname{Mod} S$ and $I \in \operatorname{Mod} R$. If*

$$\psi_* \colon (\omega \otimes_S N)_* \rightarrowtail I_*$$

is a TH-injection with $I \in \operatorname{Cor}_\omega(R)$ and I_* adjoint 1-ω-cospherical in $\operatorname{Mod} S$, then $K := \operatorname{Coker} \psi_*$ is isomorphic to the ω-unit quotient module of an adjoint 1-ω-cospherical module in $\operatorname{Mod} S$.

Conversely, if K is isomorphic to the ω-unit quotient module of an adjoint 1-ω-cospherical module in $\operatorname{Mod} S$, then there exists an exact sequence

$$0 \longrightarrow X \stackrel{\lambda}{\longrightarrow} P \longrightarrow K \longrightarrow 0$$

in $\operatorname{Mod} S$ with P projective and $\lambda \colon X \rightarrowtail P$ is a TH-injection.

As a consequence of Theorem 3.7, we have the following

Corollary 3.8. *Let $N \in \operatorname{Mod} S$ and $I \in \operatorname{Mod} R$, and let*

$$\psi_* \colon (\omega \otimes_S M)_* \rightarrowtail I_*$$

be a TH-injection with $I \in \operatorname{Cor}_\omega(R)$ and I_* adjoint 1-ω-cospherical in $\operatorname{Mod} S$. Then $K := \operatorname{Coker} \psi_*$ is an adjoint ω-cotorsionless and adjoint 1-ω-cospherical module in $\operatorname{Mod} S$ provided that one of the following conditions is satisfied.

(1) *$M \in \operatorname{Acor}_\omega(S)$.*

(2) *$(\omega \otimes_S M)_* \in \operatorname{Acor}_\omega(S)$ and ${}_R\omega_S$ is faithful.*

Conversely, if K is an adjoint ω-cotorsionless and adjoint 1-ω-cospherical module in $\operatorname{Mod} S$ and

$$0 \to X \to F \to K \to 0$$

is an exact sequence in $\operatorname{Mod} S$ with P projective, then $X \rightarrowtail F$ is a TH-injection.

4. Modules of ω-$\mathcal{T}$-class n and finite projective dimension

Motivated by the notion of modules of D-class n introduced in [13], in this section, we first introduce the notion of modules of ω-$\mathcal{T}$-class n as follows. Then we give some equivalent characterizations for ω_S having finite projective dimension in terms of the properties of modules of ω-$\mathcal{T}$-class n.

Definition 4.1. Let $\mathcal{T}$ be a subclass of $\mathrm{Acor}_\omega(S)$. An ω-cotorsionless module U_n in $\mathrm{Mod}\, R$ is said to be of *C-$\mathcal{T}$-class n* if there exist $F_1, \ldots, F_{n-1} \in \mathcal{T}$ and $U_2, \ldots, U_{n-1} \in \mathrm{Cot}_\omega(R)$ such that

$$0 \to U_n \to \omega \otimes_S F_{n-1} \to \omega \otimes_S U_{n-1*} \to 0,$$
$$0 \to U_{n-1} \to \omega \otimes_S F_{n-2} \to \omega \otimes_S U_{n-2*} \to 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots ,$$
$$0 \to U_2 \to \omega \otimes_S F_1 \to \omega \otimes_S U_{1*} \to 0$$

are exact with all the above epimorphisms HT-projections. We shall say that any ω-cotorsionless module is of *ω-$\mathcal{T}$-class* 1.

It seems that it is not easy to grasp the definition of modules of ω-$\mathcal{T}$-class n. The following result is helpful to comprehend it, which will be used frequently in the sequel.

Theorem 4.2. *Let $\mathcal{T}$ be a subclass of $\mathrm{Acor}_\omega(S)$. If a module $U_n \in \mathrm{Mod}\, R$ is of ω-$\mathcal{T}$-class n, then there exists a collection of exact sequences*

$$0 \to U_{i*} \to F_{i-1} \to U_{i-1*} \to 0 \qquad (2 \le i \le n) \tag{4.1}$$

in $\mathrm{Mod}\, S$ with all $F_i \in \mathcal{T}$ and $U_i \in \mathrm{Mod}\, R$.

Conversely, if there exists a collection of exact sequences as in (4.1)*, then U_n can be selected of ω-$\mathcal{T}$-class n.*

Proof. Let $U_n \in \mathrm{Mod}\, R$ be of ω-$\mathcal{T}$-class n. Consider the exact sequences in Definition 4.1. For any $2 \le i \le n$, since $\omega \otimes_S F_{i-1} \twoheadrightarrow \omega \otimes_S U_{i-1*}$ is a HT-projection, we have the following commutative diagram with exact rows

$$\begin{tikzcd}
 & & F_{i-1} \arrow[r] \arrow[d, "\mu_{F_{i-1}}"] & U_{i-1*} \arrow[r] \arrow[d, "\mu_{U_{i-1*}}"] & 0 \\
0 \arrow[r] & U_{i*} \arrow[r] & (\omega \otimes_S F_{i-1})_* \arrow[r] & (\omega \otimes_S U_{i-1*})_*. &
\end{tikzcd}$$

Note that $\mu_{F_{i-1}}$ is an isomorphism by assumption and that $\mu_{U_{i-1*}}$ is a monomorphism by Lemma 2.4(1). Then we get an exact sequence

$$0 \to U_{i*} \to F_{i-1} \to U_{i-1*} \to 0 \qquad (2 \le i \le n).$$

Conversely, assume that there exists a collection of exact sequences as in (4.1). First, consider the following exact sequence

$$0 \longrightarrow H_1 \longrightarrow F_1 \xrightarrow{\phi_1} U_{1*} \longrightarrow 0$$

in $\operatorname{Mod} S$ with $H_1 = \operatorname{Ker} \phi_1$. Set $U_2 = \operatorname{Ker}(1_\omega \otimes \phi_1)$. Then we have an exact sequence

$$0 \longrightarrow U_2 \longrightarrow \omega \otimes_S F_1 \stackrel{1_\omega \otimes \phi_1}{\longrightarrow} \omega \otimes_S U_{1*} \longrightarrow 0$$

in $\operatorname{Mod} S$. Then $1_\omega \otimes \phi_1$ is a HT-projection and U_2 is of ω-$\mathcal{T}$-class 2. Notice that $\omega \otimes_S H_1 \in \operatorname{Cot}_\omega(R)$ by Lemma 2.4(2), so $U_2 \in \operatorname{Cot}_\omega(R)$ since it is isomorphic to a quotient module of $\omega \otimes_S H_1$. Because $F_1 \in \operatorname{Acor}_\omega(S)$ and $U_{1*} \in \operatorname{Acot}_\omega(S)$ by assumption and Lemma 2.4(1) respectively, it follows from Proposition 2.9(1) and its proof that $H_1 \cong U_{2*}$ and $U_{1*} \cong \operatorname{Im}(1_\omega \otimes \phi_1)_*$. So we get an exact sequence

$$0 \longrightarrow U_{2*} \longrightarrow F_1 \stackrel{(1_\omega \otimes \phi_1)_* \cdot \mu_{F_1}}{\longrightarrow} U_{1*} \longrightarrow 0$$

in $\operatorname{Mod} S$.

Next, consider the following exact sequence

$$0 \longrightarrow H_2 \longrightarrow F_2 \stackrel{\phi_2}{\longrightarrow} U_{2*} \longrightarrow 0$$

in $\operatorname{Mod} S$ with $H_2 = \operatorname{Ker} \phi_2$. Set $U_3 = \operatorname{Ker}(1_\omega \otimes \phi_2)$. By using an argument similar to above, we get an exact sequence

$$0 \longrightarrow U_{3*} \longrightarrow F_2 \stackrel{(1_\omega \otimes \phi_2)_* \cdot \mu_{F_2}}{\longrightarrow} U_{2*} \longrightarrow 0$$

in $\operatorname{Mod} S$ with U_3 of ω-$\mathcal{T}$-class 3. Continuing this process, we get the desired assertion. $\square$

The following two lemmas are useful for proving the next theorem.

Lemma 4.3. *Let $N \in \operatorname{Acot}_\omega(S)$ and $L \in \operatorname{Cot}_\omega(R)$. If either N or L is given, then the other exists such that these two modules are connected by the following exact sequences*

$$0 \longrightarrow N \stackrel{\mu_N}{\longrightarrow} (\omega \otimes_S N)_* \longrightarrow \operatorname{Ext}^1_R(\omega, L) \longrightarrow 0,$$
$$0 \longrightarrow \operatorname{Tor}^S_1(\omega, N) \longrightarrow \omega \otimes_S L_* \stackrel{\theta_L}{\longrightarrow} L \longrightarrow 0.$$

Proof. Given $N \in \operatorname{Acot}_\omega(S)$, consider the following exact sequence

$$0 \to N_1 \to P \to N \to 0$$

in $\operatorname{Mod} S$ with P projective. Then we get the following exact sequence

$$0 \to L \to \omega \otimes_S P \to \omega \otimes_S N \to 0$$

in $\operatorname{Mod} R$ with $L = \operatorname{Ker}(\omega \otimes_S P \to \omega \otimes_S N)$. Notice that $\omega \otimes_S N_1 \in \operatorname{Cot}_\omega(R)$ by Lemma 2.4(2) and that L is isomorphic to a quotient module of $\omega \otimes_S N_1$, so $L \in \operatorname{Cot}_\omega(R)$. Now consider the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & N_1 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "\mu_P"] & N \arrow[r] \arrow[d, "\mu_N"] & 0 & \\
0 \arrow[r] & L_* \arrow[r] & (\omega \otimes_S P)_* \arrow[r] & (\omega \otimes_S N)_* \arrow[r] & \operatorname{Ext}^1_R(\omega, L) \arrow[r] & 0.
\end{tikzcd}$$

Since μ_P is an isomorphism and μ_N is a monomorphism by Lemma 2.5(2) and assumption respectively, we have the following two exact sequences

$$0 \longrightarrow N \xrightarrow{\mu_N} (\omega \otimes_S N)_* \longrightarrow \mathrm{Ext}_R^1(\omega, L) \longrightarrow 0,$$
$$0 \to L_* \to (\omega \otimes_S P)_* \ (\cong P) \to N \to 0.$$

Then we get the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1^S(\omega, N) \arrow[r] & \omega \otimes_S L_* \arrow[r] \arrow[d, "\theta_L"] & \omega \otimes_S (\omega \otimes_S P)_* \arrow[r] \arrow[d, "\theta_{\omega \otimes_S P}"] & \omega \otimes_S N \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & L \arrow[r] & \omega \otimes_S P \arrow[r] & \omega \otimes_S N \arrow[r] & 0,
\end{tikzcd}$$

where $\theta_{\omega \otimes_S P}$ is an isomorphism by Lemma 2.5(1). It yields the following exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^S(\omega, N) \longrightarrow \omega \otimes_S L_* \xrightarrow{\theta_L} L \longrightarrow 0.$$

If L is given, then we get the assertion dually. □

Lemma 4.4. *Let $\phi\colon F \twoheadrightarrow N$ be an epimorphism in $\mathrm{Mod}\, S$ with $F \in \mathrm{Acor}_\omega(S)$ and $N \in \mathrm{Acot}_\omega(S)$. Then we have the following exact sequence*

$$\mathrm{Tor}_1^S(\omega, F) \longrightarrow \mathrm{Tor}_1^S(\omega, N) \longrightarrow \omega \otimes_S H_* \xrightarrow{\theta_H} H$$

in $\mathrm{Mod}\, R$, where $H = \mathrm{Ker}(1_\omega \otimes \phi)$.

Proof. By assumption, we have the following exact sequence

$$0 \longrightarrow H \xrightarrow{\alpha} \omega \otimes_S F \xrightarrow{1_\omega \otimes \phi} \omega \otimes_S N \longrightarrow 0$$

in $\mathrm{Mod}\, R$. Then we get the following commutative diagram with exact rows

$$\begin{tikzcd}
& & F \arrow[r, "\phi"] \arrow[d, "\mu_F"] & N \arrow[r] \arrow[d, "\mu_N"] & 0 \\
0 \arrow[r] & H_* \arrow[r, "\alpha_*"] & (\omega \otimes_S F)_* \arrow[r, "(1_\omega \otimes \phi)_*"] & (\omega \otimes_S N)_*. &
\end{tikzcd}$$

Because $F \in \mathrm{Acor}_\omega(S)$ and $N \in \mathrm{Acot}_\omega(S)$ by assumption, μ_F is an isomorphism and μ_N is a monomorphism. So we get the following exact sequence

$$0 \longrightarrow H_* \xrightarrow{\alpha_*} (\omega \otimes_S F)_* \ (\cong F) \xrightarrow{\phi \cdot \mu_F{}^{-1}} N \longrightarrow 0$$

in $\mathrm{Mod}\, S$ and the following commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Tor}_1^S(\omega, F) \arrow[r] & \mathrm{Tor}_1^S(\omega, N) \arrow[r] & \omega \otimes_S H_* \arrow[r, "1_\omega \otimes \alpha_*"] \arrow[d, "\theta_H"] & \omega \otimes_S (\omega \otimes_S F)_* \arrow[d, "\theta_{\omega \otimes_S F}"] \\
& 0 \arrow[r] & H \arrow[r, "\alpha"] & \omega \otimes_S F.
\end{tikzcd}$$

Also because $F \in \mathrm{Acor}_\omega(S)$, we have $\omega \otimes_S F \in \mathrm{Cor}_\omega(R)$ by Lemma 2.4(3). So $\theta_{\omega\otimes_S F}$ is an isomorphism and we get the desired exact sequence. $\square$

From now on, we fix $\mathcal{T}$ a subclass of $w\mathcal{A}_\omega(S)$ containing all projective left S-modules, that is, $\mathcal{P}(S) \subseteq \mathcal{T} \subseteq w\mathcal{A}_\omega(S)$. We use $\mathrm{pd}_{S^{\mathrm{op}}}\,\omega$ and $\mathrm{fd}_{S^{\mathrm{op}}}\,\omega$ to denote the projective and flat dimensions of ω_S, respectively. The following result establishes a relationship between the finiteness of $\mathrm{pd}_{S^{\mathrm{op}}}\,\omega$ and the properties of modules of ω-$\mathcal{T}$-class n, ω-coreflexive modules and adjoint ω-cotorsionless modules.

Theorem 4.5. *For any $n \geq 1$, the following statements are equivalent.*

(1) $\mathrm{pd}_{S^{\mathrm{op}}}\,\omega \leq n$.

(2) *Any module of ω-$\mathcal{P}(S)$-class n in $\mathrm{Mod}\,R$ is ω-coreflexive.*

(3) *Any module of ω-$\mathcal{T}$-class n in $\mathrm{Mod}\,R$ is ω-coreflexive.*

(4) $\mathrm{Tor}_n^S(\omega, V) = 0$ *for any $V \in \mathrm{Acot}_\omega(S)$.*

(5) $\mathrm{Tor}_{n+1}^S(\omega, N) = 0$ *for any $N \in \mathrm{Mod}\,S$.*

Proof. (1) $\Leftrightarrow$ (5): It is trivial since $\mathrm{pd}_{S^{\mathrm{op}}}\,\omega = \mathrm{fd}_{S^{\mathrm{op}}}\,\omega$. The implication (3) $\Rightarrow$ (2) is also trivial.

(2) $\Rightarrow$ (4): If $n = 1$, then the assertion follows from Lemma 4.3. Now let $V \in \mathrm{Acot}_\omega(S)$ and $n \geq 2$. By the proof of Lemma 4.3, there exists an exact sequence

$$0 \to U_{1*} \to P \to V \to 0$$

in $\mathrm{Mod}\,S$ with P projective. By Theorem 4.2 and its proof, we have the following two exact sequences

$$0 \longrightarrow U_{n*} \longrightarrow P_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow P_1 \longrightarrow U_{1*} \longrightarrow 0,$$
$$0 \longrightarrow U_n \longrightarrow \omega \otimes_S P_{n-1} \xrightarrow{1_\omega \otimes f_{n-1}} \omega \otimes_S U_{n-1*} \longrightarrow 0$$

with all $P_i \in \mathrm{Mod}\,S$ projective, $U_n \in \mathrm{Mod}\,R$ of ω-$\mathcal{P}(S)$-class n and $U_{n-1*} = \mathrm{Im}\,f_{n-1}$, such that $1_\omega \otimes f_{n-1}$ is a HT-projection. Then by Lemma 4.4, we have the following exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^S(\omega, U_{n-1*}) \longrightarrow \omega \otimes_S U_{n*} \xrightarrow{\theta_{U_n}} U_n \longrightarrow 0.$$

By (2), $U_n \in \mathrm{Cor}_\omega(R)$ and θ_{U_n} is an isomorphism. So $\mathrm{Tor}_1^S(\omega, U_{n-1*}) = 0$, and hence

$$\mathrm{Tor}_n^S(\omega, V) \cong \mathrm{Tor}_{n-1}^S(\omega, U_{1*}) \cong \mathrm{Tor}_1^S(\omega, U_{n-1*}) = 0.$$

(4) $\Rightarrow$ (3): Let $U_n \in \operatorname{Mod} R$ be of ω-$\mathcal{T}$-class n. Then by Theorem 4.2, there exists an exact sequence

$$0 \longrightarrow U_{n*} \longrightarrow T_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow T_1 \longrightarrow U_{1*} \longrightarrow 0$$

in $\operatorname{Mod} S$ with all $T_i \in \mathcal{T}$ such that $U_n \cong \operatorname{Ker}(1_\omega \otimes f_{n-1})$. By Lemma 4.4, we have the following exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^S(\omega, U_{n-1*}) \longrightarrow \omega \otimes_S U_{n*} \xrightarrow{\theta_{U_n}} U_n \longrightarrow 0, \tag{4.2}$$

where $U_{n-1*} = \operatorname{Im} f_{n-1}$. In addition, we have the following exact sequence

$$0 \longrightarrow U_{1*} \longrightarrow I^0(U_1)_* \xrightarrow{f^0(U_1)_*} I^1(U_1)_* \longrightarrow \operatorname{cTr}_\omega U_1 \longrightarrow 0$$

in $\operatorname{Mod} S$. By Lemma 2.5(2), we have

$$\operatorname{Tor}_{\geq 1}^S(\omega, I^0(U_1)_*) = 0 = \operatorname{Tor}_{\geq 1}^S(\omega, I^1(U_1)_*).$$

Put $V = \operatorname{Im} f^0(U_1)_*$. Then $V \in \operatorname{Acot}_\omega(S)$. So by (4), we have

$$\operatorname{Tor}_1^S(\omega, U_{n-1*}) \cong \operatorname{Tor}_{n-1}^S(\omega, U_{1*}) \cong \operatorname{Tor}_n^S(\omega, V) = 0.$$

It follows from (4.2) that θ_{U_n} is an isomorphism and $U_n \in \operatorname{Cor}_\omega(R)$.

(4) $\Leftrightarrow$ (5): Let $N \in \operatorname{Mod} S$ and

$$0 \to V \to P \to N \to 0$$

be an exact sequence in $\operatorname{Mod} S$ with P projective. Then $V \in \operatorname{Acot}_\omega(S)$. Conversely, let $V \in \operatorname{Acot}_\omega(S)$. Then by [20, Lemma 3.7(1)], there exists an exact sequence

$$0 \to V \to E \to N \to 0$$

in $\operatorname{Mod} S$ with E ω-injective. Note that $\operatorname{Tor}_{\geq 1}^S(\omega, E) = 0$ by Lemma 2.5(2). Now the assertion follows easily from the dimension shifting. $\square$

As a consequence of Theorem 4.5, we have the following

Corollary 4.6. *For any $n \geq 1$, the following statements are equivalent.*

(1) *$U_{n*} \in \operatorname{Acor}_\omega(S)$ for any U_n of ω-$\mathcal{P}(S)$-class n in $\operatorname{Mod} R$.*

(2) *$U_{n*} \in \operatorname{Acor}_\omega(S)$ for any U_n of ω-$\mathcal{T}$-class n in $\operatorname{Mod} R$.*

(3) *$[\operatorname{Tor}_n^S(\omega, V)]_* = 0$ for any $V \in \operatorname{Acot}_\omega(S)$.*

If $\operatorname{pd}_{S^{op}} \omega \leq n$, then these equivalent conditions are satisfied.

Proof. (1) $\Rightarrow$ (3): Let $V \in \mathrm{Acot}_\omega(S)$. From the proof of the implications (2) $\Rightarrow$ (4) in Theorem 4.5, we know that there exists $U_n \in \mathrm{Mod}\, R$ be of ω-$\mathcal{P}(S)$-class n such that $\mathrm{Ker}\, \theta_{U_n} \cong \mathrm{Tor}_n^S(\omega, V)$. It implies

$$\mathrm{Ker}(\theta_{U_n})_* \cong (\mathrm{Ker}\, \theta_{U_n})_* \cong [\mathrm{Tor}_n^S(\omega, V)]_*.$$

By (1), we have $U_{n*} \in \mathrm{Acor}_\omega(S)$. So $\mu_{U_{n*}}$ is an isomorphism, and hence $(\theta_{U_n})_*$ is also an isomorphism by Lemma 2.4(1). It follows that $[\mathrm{Tor}_n^S(\omega, V)]_* = 0$.

(2) $\Rightarrow$ (1): It is trivial because $\mathcal{P}(S) \subseteq \mathcal{T}$.

(3) $\Rightarrow$ (2): Let $U_n \in \mathrm{Mod}\, R$ be of ω-$\mathcal{T}$-class n. Then by Theorem 4.2, there exists an exact sequence

$$0 \longrightarrow U_{n*} \longrightarrow T_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow T_1 \longrightarrow U_{1*} \longrightarrow 0$$

in $\mathrm{Mod}\, S$ with all $T_i \in \mathcal{T}$ such that $U_n \cong \mathrm{Ker}(1_\omega \otimes f_{n-1})$. Because $\mathcal{T} \subseteq w\mathcal{A}_\omega(S)$, by Lemma 4.4 we have the following exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^S(\omega, U_{n-1*}) \longrightarrow \omega \otimes_S U_{n*} \xrightarrow{\theta_{U_n}} U_n \longrightarrow 0, \tag{4.3}$$

where $U_{n-1*} = \mathrm{Im}\, f_{n-1}$. In addition, we have the following exact sequence

$$0 \longrightarrow U_{1*} \longrightarrow I^0(U_1)_* \xrightarrow{f^0(U_1)_*} I^1(U_1)_* \longrightarrow \mathrm{cTr}_\omega\, U_1 \longrightarrow 0$$

in $\mathrm{Mod}\, S$. Put $V = \mathrm{Im}\, f^0(U_1)_*$. Then $V \in \mathrm{Acot}_\omega(S)$. So by (4.3) and the assumption of (3), we have

$$\begin{aligned}\mathrm{Ker}(\theta_{U_n})_* &\cong (\mathrm{Ker}\, \theta_{U_n})_* \cong [\mathrm{Tor}_1^S(\omega, U_{n-1*})]_* \\ &\cong [\mathrm{Tor}_{n-1}^S(\omega, U_{1*})]_* \cong [\mathrm{Tor}_n^S(\omega, V)]_* = 0.\end{aligned}$$

It follows from Lemma 2.4(1) that $\mu_{U_{n*}}$ is an isomorphism and $U_{n*} \in \mathrm{Acor}_\omega(S)$.

The last assertion follows immediately from Theorem 4.5. □

The following result is a supplement to Theorem 4.5.

Theorem 4.7. *For any $n \geq 1$, the following statements are equivalent.*

(1) $\mathrm{pd}_{S^{op}}\, \omega \leq n+1$.

(2) $\mathrm{Tor}_1^S(\omega, U_{n*}) = 0$ *for any module U_n of ω-$\mathcal{P}(S)$-class n in* $\mathrm{Mod}\, R$.

(3) $\mathrm{Tor}_1^S(\omega, U_{n*}) = 0$ *for any module U_n of ω-$\mathcal{T}$-class n in* $\mathrm{Mod}\, R$.

Proof. (1) $\Rightarrow$ (3): Let $U_n \in \mathrm{Mod}\, R$ be of ω-$\mathcal{T}$-class n. Then by Theorem 4.2, there exists an exact sequence

$$0 \to U_{n*} \to T_{n-1} \to \cdots \to T_1 \to U_{1*} \to 0$$

in $\mathrm{Mod}\, S$ with all $T_i \in \mathcal{T}$. Then $\mathrm{Tor}^S_{\geq 1}(\omega, T_i) = 0$ for any $1 \leq i \leq n-1$. On the other hand, we have the following exact sequence

$$0 \longrightarrow U_{1*} \longrightarrow I^0(U_1)_* \stackrel{f^0(U_1)_*}{\longrightarrow} I^1(U_1)_* \longrightarrow \mathrm{cTr}_\omega\, U_1 \longrightarrow 0$$

in $\mathrm{Mod}\, S$. Note that $\mathrm{Tor}^S_{\geq 1}(\omega, I^0(U_1)_*) = 0 = \mathrm{Tor}^S_{\geq 1}(\omega, I^1(U_1)_*)$ by Lemma 2.5(2). So by (1), we have

$$\mathrm{Tor}^S_1(\omega, U_{n*}) \cong \mathrm{Tor}^S_{n+2}(\omega, \mathrm{cTr}_\omega\, U_1) = 0.$$

(3) $\Rightarrow$ (2): It is trivial.

(2) $\Rightarrow$ (1): Let $N \in \mathrm{Mod}\, S$. Then we have the following commutative diagram with exact rows

$$\begin{tikzcd}
 & & F_1(N) \arrow[r, "f_0(N)"] \arrow[d, "\mu_{F_1(N)}"] & F_0(N) \arrow[r] \arrow[d, "\mu_{F_0(N)}"] & N \arrow[r] & 0 \\
0 \arrow[r] & (\mathrm{acTr}_\omega\, N)_* \arrow[r, "\alpha"] & (\omega \otimes_S F_1(N))_* \arrow[r, "(1_\omega \otimes f_0(N))_*"] & (\omega \otimes_S F_0(N))_*, & &
\end{tikzcd}$$

where $\mu_{F_0(N)}$ and $\mu_{F_1(N)}$ are isomorphisms by Lemma 2.5(2). So we get the following exact sequence

$$0 \longrightarrow (\mathrm{acTr}_\omega\, N)_* \stackrel{{\mu_{F_1(N)}}^{-1}\cdot\alpha}{\longrightarrow} F_1(N) \stackrel{f_0(N)}{\longrightarrow} F_0(N) \longrightarrow N \longrightarrow 0$$

in $\mathrm{Mod}\, S$. By Theorem 4.2, we have the following exact sequence

$$0 \to U_{n*} \to P_{n-1} \to \cdots \to P_1 \to (\mathrm{acTr}_\omega\, N)_* \to 0$$

in $\mathrm{Mod}\, S$ with all P_i projective such that U_n is of ω-$\mathcal{P}(S)$-class n. Then by (2), we have

$$\mathrm{Tor}^S_{n+2}(\omega, N) \cong \mathrm{Tor}^S_1(\omega, U_{n*}) = 0.$$

It implies that $\mathrm{pd}_{S^{\mathrm{op}}}\, \omega = \mathrm{fd}_{S^{\mathrm{op}}}\, \omega \leq n+1$. $\square$

For a module $N \in \mathrm{Mod}\, S$, the $\mathcal{A}_\omega(S)$-*projective dimension* $\mathcal{A}_\omega(S)$-$\mathrm{pd}_R\, N$ of N is defined as

$$\inf \big\{n \mid \text{there exists an exact sequence } 0 \to A_n \to \cdots \to A_1 \to A_0 \to N \to 0 \text{ in } \mathrm{Mod}\, S \text{ with all } A_i \in \mathcal{A}_\omega(S)\big\}.$$

If no such n exists, then set $\mathcal{A}_\omega(S)$-$\mathrm{pd}_R\, N = \infty$. As a byproduct of Theorem 4.2, we get the following

Proposition 4.8. *For any $n \geq 1$, the following statements are equivalent.*

(1) $\mathcal{A}_\omega(S)$-$\mathrm{pd}_S N \leq n+1$ *for any* $N \in \mathrm{Mod}\, S$.

(2) $U_{n*} \in \mathcal{A}_\omega(S)$ *for any* U_n *of* ω-$\mathcal{A}_\omega(S)$-*class* n *in* $\mathrm{Mod}\, R$.

(3) $U_{n*} \in \mathcal{A}_\omega(S)$ *for any* U_n *of* ω-$\mathcal{P}(S)$-*class* n *in* $\mathrm{Mod}\, R$.

Proof. (1) $\Rightarrow$ (2): Let $U_n \in \mathrm{Mod}\, R$ be of ω-$\mathcal{A}_\omega(S)$-class n in $\mathrm{Mod}\, R$. Then by Theorem 4.2, there exists an exact sequence

$$0 \to U_{n*} \to A_{n-1} \to \cdots \to A_1 \to U_{1*} \to 0$$

in $\mathrm{Mod}\, S$ with all $A_i \in \mathcal{A}_\omega(S)$ and $U_1 \in \mathrm{Mod}\, R$. On the other hand, we have the following exact sequence

$$0 \longrightarrow U_{1*} \longrightarrow I^0(U_1)_* \xrightarrow{f^0(U_1)_*} I^1(U_1)_* \longrightarrow \mathrm{cTr}_\omega\, U_1 \longrightarrow 0$$

in $\mathrm{Mod}\, S$. So we get the following exact sequence

$$0 \longrightarrow U_{n*} \longrightarrow A_{n-1} \longrightarrow \cdots \longrightarrow A_1 \longrightarrow I^0(U_1)_* \xrightarrow{f^0(U_1)_*} I^1(U_1)_* \longrightarrow \mathrm{cTr}_\omega\, U_1 \longrightarrow 0$$

in $\mathrm{Mod}\, S$, where $I^0(U_1)_*, I^1(U_1)_* \in \mathcal{A}_\omega(S)$ by Lemma 2.5(2). Because $\mathcal{A}_\omega(S)$ is projectively resolving and closed under direct summands by [9, Theorem 6.2 and Proposition 4.2], we have $U_{n*} \in \mathcal{A}_\omega(S)$ by [2, Lemma 3.12].

(2) $\Rightarrow$ (3): It is trivial.

(3) $\Rightarrow$ (1): Let $N \in \mathrm{Mod}\, S$ and

$$0 \longrightarrow K_n \longrightarrow P_n \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_1 \xrightarrow{f_0} P_0 \longrightarrow N \longrightarrow 0$$

be an exact sequence in $\mathrm{Mod}\, S$ with all P_i projective. Then for any $1 \leq i \leq n$, we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K_i \arrow[r] \arrow[d, dashed] & P_i \arrow[r, "f_{i-1}"] \arrow[d, "\mu_{P_i}"] & P_{i-1} \arrow[d, "\mu_{P_{i-1}}"] \\
0 \arrow[r] & U_{i*} \arrow[r] & (\omega \otimes_S P_i)_* \arrow[r, "(1_\omega \otimes f_{i-1})_*"] & (\omega \otimes_S P_{i-1})_*,
\end{tikzcd}$$

$K_i = \mathrm{Ker}\, f_{i-1}$ and $U_i = \mathrm{Ker}(1_\omega \otimes f_{i-1})$. By Lemma 2.5(2), we have that all μ_{P_i} are isomorphisms. So $K_i \cong U_{i*}$ for any $1 \leq i \leq n$. Then by Theorem 4.2, U_n can be selected of ω-$\mathcal{P}(S)$-class n. So K_n $(\cong U_{n*}) \in \mathcal{A}_\omega(S)$ by (3), and hence $\mathcal{A}_\omega(S)$-$\mathrm{pd}_S N \leq n+1$. $\square$

5. Some useful exact sequences

In this section, we give some exact sequences, which will be used frequently in the sequel. The following result is fundamental.

Proposition 5.1. *Let*

$$0 \longrightarrow M \longrightarrow U^0 \xrightarrow{f} U^1 \tag{5.1}$$

be an exact sequence in $\mathrm{Mod}\, R$ *satisfying the following conditions:*

(1) *Both* U^0 *and* U^1 *are in* $\mathrm{Cor}_\omega(R)$.

(2) ${U^0}_*$ *is adjoint* 1-ω-*cospherical and* ${U^1}_*$ *is adjoint* 2-ω-*cospherical.*

Then there exists an exact sequence

$$0 \longrightarrow \mathrm{Tor}_2^S(\omega, H) \longrightarrow \omega \otimes_S M_* \xrightarrow{\theta_M} M \longrightarrow \mathrm{Tor}_1^S(\omega, H) \longrightarrow 0$$

in $\mathrm{Mod}\, R$*, where* $H = \mathrm{Coker}\, f_*$.

Proof. By applying the functor $(-)_*$ to (5.1), We get an exact sequence

$$0 \longrightarrow M_* \longrightarrow {U^0}_* \xrightarrow{f_*} {U^1}_* \longrightarrow H \longrightarrow 0$$

in $\mathrm{Mod}\, S$. Let

$$f = i \cdot p$$

with $p\colon U^0 \twoheadrightarrow \mathrm{Im}\, f$ and $i\colon\ \mathrm{Im}\, f \hookrightarrow U^1$ and

$$f_* = i' \cdot p'$$

with $p'\colon U^0_* \twoheadrightarrow \mathrm{Im}\, f_*$ and $i'\colon\ \mathrm{Im}\, f_* \hookrightarrow {U^1}_*$ be the natural epic-monic decompositions of f and f_*, respectively. Since $\mathrm{Tor}_1^S(\omega, {U^0}_*) = 0$ and θ_{U^0} is an isomorphism by assumption, we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1^S(\omega, \mathrm{Im}\, f_*) \arrow[r] & \omega \otimes_R M_* \arrow[r] \arrow[d, "\theta_M"] & \omega \otimes_S {U^0}_* \arrow[r, "1_\omega \otimes p'"] \arrow[d, "\theta_{U^0}"] & \omega \otimes_S \mathrm{Im}\, f_* \arrow[r] \arrow[d, dashed, "h"] & 0 \\
& 0 \arrow[r] & M \arrow[r] & U^0 \arrow[r, "p"] & \mathrm{Im}\, f \arrow[r] & 0,
\end{tikzcd}$$

where h is an induced homomorphism. Then

$$p \cdot \theta_{U^0} = h \cdot (1_\omega \otimes p').$$

In addition, by the snake lemma, we have

$$\mathrm{Ker}\, \theta_M \cong \mathrm{Tor}_1^S(\omega, \mathrm{Im}\, f_*) \quad \text{and} \quad \mathrm{Coker}\, \theta_M \cong \mathrm{Ker}\, h.$$

On the other hand, since $\mathrm{Tor}_1^S(\omega, {U^1}_*) = 0 = \mathrm{Tor}_2^S(\omega, {U^1}_*)$ by assumption, by applying the functor $\omega \otimes_S -$ to the exact sequence

$$0 \longrightarrow \mathrm{Im}\, f_* \xrightarrow{i'} {U^1}_* \longrightarrow H \longrightarrow 0,$$

we get the following exact sequence:

$$0 \longrightarrow \mathrm{Tor}_1^S(\omega, H) \longrightarrow \omega \otimes_S \mathrm{Im}\, f_* \xrightarrow{1_\omega \otimes i'} \omega \otimes_S {U^1}_* \longrightarrow \omega \otimes_S H \longrightarrow 0$$

and the isomorphism

$$\mathrm{Tor}_1^S(\omega, \mathrm{Im}\, f_*) \cong \mathrm{Tor}_2^S(\omega, H).$$

Because

$$\begin{CD} \omega \otimes_S {U^0}_* @>{1_\omega \otimes f_*}>> \omega \otimes_S {U^1}_* \\ @V{\theta_{U^0}}VV @VV{\theta_{U^1}}V \\ U^0 @>{f}>> U^1 \end{CD}$$

is a commutative diagram, we have

$$f \cdot \theta_{U^0} = \theta_{U^1} \cdot (1_\omega \otimes f_*).$$

Because $f_* = i' \cdot p'$, we get

$$1_\omega \otimes f_* = 1_\omega \otimes (i' \cdot p') = (1_\omega \otimes i') \cdot (1_\omega \otimes p').$$

Thus we have

$$i \cdot h \cdot (1_\omega \otimes p') = i \cdot p \cdot \theta_{U^0} = f \cdot \theta_{U^0} = \theta_{U^1} \cdot (1_\omega \otimes f_*) = \theta_{U^1} \cdot (1_\omega \otimes i') \cdot (1_\omega \otimes p').$$

Because $1_\omega \otimes p'$ is epic, we get $i \cdot h = \theta_{U^1} \cdot (1_\omega \otimes i')$. Notice that i is monic and θ_{U^1} is an isomorphism, so

$$\mathrm{Coker}\, \theta_M \cong \mathrm{Ker}\, h \cong \mathrm{Ker}(1_\omega \otimes i') \cong \mathrm{Tor}_1^S(\omega, H).$$

Consequently we obtain the desired exact sequence. □

In the following, we give some applications of Proposition 5.1.

Corollary 5.2. (1) (see [18, Proposition 3.2]) *Let $M \in \mathrm{Mod}\, R$. Then there exists an exact sequence*

$$0 \longrightarrow \mathrm{Tor}_2^S(\omega, \mathrm{cTr}_\omega M) \longrightarrow \omega \otimes_S M_* \xrightarrow{\theta_M} M \longrightarrow \mathrm{Tor}_1^S(\omega, \mathrm{cTr}_\omega M) \longrightarrow 0$$

in $\mathrm{Mod}\, R$.

(2) *Let $N \in \mathrm{Mod}\, S$. Then there exists an exact sequence*

$$0 \longrightarrow \mathrm{Ext}_R^1(\omega, \mathrm{acTr}_\omega N) \longrightarrow N \xrightarrow{\mu_N} (\omega \otimes_S N)_* \longrightarrow \mathrm{Ext}_R^2(\omega, \mathrm{acTr}_\omega N) \longrightarrow 0$$

in $\mathrm{Mod}\, S$.

Proof. The assertion (1) follows from Lemma 2.5 and Proposition 5.1, and the assertion (2) follows from Lemma 2.5 and [19, Proposition 6.7]. □

Corollary 5.3. (1) *Let $N \in \operatorname{Mod} S$. Then there exists an exact sequence*

$$0 \longrightarrow \operatorname{Tor}_2^S(\omega, N) \longrightarrow \omega \otimes_S (\operatorname{acTr}_\omega N)_* \stackrel{\theta_{\operatorname{acTr}_\omega N}}{\longrightarrow} \operatorname{acTr}_\omega N \longrightarrow \operatorname{Tor}_1^S(\omega, N) \longrightarrow 0$$

in $\operatorname{Mod} R$.

(2) *Let $M \in \operatorname{Mod} R$. Then there exists an exact sequence*

$$0 \longrightarrow \operatorname{Ext}_R^1(\omega, M) \longrightarrow \operatorname{cTr}_\omega M \stackrel{\mu_{\operatorname{cTr}_\omega M}}{\longrightarrow} (\omega \otimes_S \operatorname{cTr}_\omega M)_* \longrightarrow \operatorname{Ext}_R^2(\omega, M) \longrightarrow 0$$

in $\operatorname{Mod} S$.

Proof. (1) Let $N \in \operatorname{Mod} S$. Then we have the following exact sequence

$$0 \longrightarrow \operatorname{acTr}_\omega N \longrightarrow \omega \otimes_S F_1(N) \stackrel{1_\omega \otimes f_0(N)}{\longrightarrow} \omega \otimes_S F_0(N) \longrightarrow \omega \otimes_S N \longrightarrow 0$$

in $\operatorname{Mod} R$ with both $\omega \otimes_S F_1(N)$ and $\omega \otimes_S F_0(N)$ in $\mathcal{F}_\omega(R)$. By Lemma 2.5(1), we have that both $\omega \otimes_S F_1(N)$ and $\omega \otimes_S F_0(N)$ are in $\operatorname{Cor}_\omega(R)$. On the other hand, by Lemma 2.5(2), we have that $(\omega \otimes_S F)_* \cong F$ for any flat module F in $\operatorname{Mod} S$. So we have

$$\operatorname{Tor}_{\geq 1}^S(\omega, (\omega \otimes_S F_0(N))_*) = 0 = \operatorname{Tor}_{\geq 1}^S(\omega, (\omega \otimes_S F_1(N))_*).$$

Now the assertion follows from Proposition 5.1.

(2) See [19, Corollary 6.8]. □

For the case $n = 0$, the first assertion in the following result is exactly Corollary 5.2.

Proposition 5.4. *Let $M \in \operatorname{Mod} R$ be n-ω-cospherical with $n \geq 0$. Then we have*

(1) *There exists an exact sequence*

$$\begin{aligned} 0 \longrightarrow \operatorname{Tor}_{n+2}^S(\omega, \operatorname{Coker} f^n(M)_*) \longrightarrow \omega \otimes_S M_* \stackrel{\theta_M}{\longrightarrow} M \\ \longrightarrow \operatorname{Tor}_{n+1}^S(\omega, \operatorname{Coker} f^n(M)_*) \longrightarrow 0 \end{aligned}$$

in $\operatorname{Mod} R$.

(2) $\operatorname{Coker} f^n(M)_*$ *is adjoint n-ω-cospherical.*

Proof. Let $M \in \operatorname{Mod} R$ be n-ω-cospherical. Then $\operatorname{Ext}_R^{1 \leq i \leq n}(\omega, M) = 0$ and we get the following exact sequence

$$\begin{aligned} 0 \longrightarrow M_* \longrightarrow I^0(M)_* \stackrel{f^0(M)_*}{\longrightarrow} I^1(M)_* \stackrel{f^1(M)_*}{\longrightarrow} \cdots \\ \stackrel{f^{n-1}(M)_*}{\longrightarrow} I^n(M)_* \stackrel{f^n(M)_*}{\longrightarrow} I^{n+1}(M)_* \longrightarrow \operatorname{Coker} f^n(M)_* \longrightarrow 0 \end{aligned} \tag{5.2}$$

in $\operatorname{Mod} S$ with $\operatorname{cTr}_\omega M = \operatorname{Coker} f^0(M)_*$.

(1) Because $\operatorname{Tor}^S_{\geq 1}(\omega, I_*) = 0$ for any injective module in $\operatorname{Mod} R$ by Lemma 2.5(2), we have $\operatorname{Tor}^S_i(\omega, \operatorname{cTr}_\omega M) \cong \operatorname{Tor}^S_{n+i}(\omega, \operatorname{Coker} f^n(M)_*)$ for any $i \geq 1$. Now the assertion follows from Corollary 5.2.

(2) Applying the functor $\omega \otimes_S -$ to (5.2) we get the following commutative diagram

$$\begin{tikzcd}
\omega \otimes_S I^0(M)_* \arrow[r, "1_\omega \otimes f^0(M)_*"] \arrow[d, "\theta_{I^0(M)}"] & \cdots \arrow[r] & \omega \otimes_S I^n(M)_* \arrow[r, "1_\omega \otimes f^n(M)_*"] \arrow[d, "\theta_{I^n(M)}"] & \omega \otimes_S I^{n+1}(M)_* \arrow[r] \arrow[d, "\theta_{I^{n+1}(M)}"] & \omega \otimes_S \operatorname{Coker} f^n(M)_* \arrow[r] & 0 \\
I^0(M) \arrow[r, "f^0(M)"] & \cdots \arrow[r] & I^n(M) \arrow[r, "f^n(M)"] & I^{n+1}(M).
\end{tikzcd}$$

All columns in this diagram are isomorphisms by Lemma 2.5(1). So the upper row is exact, which implies $\operatorname{Tor}^S_{1 \leq i \leq n}(\omega, \operatorname{Coker} f^n(M)_*) = 0$ and $\operatorname{Coker} f^n(M)_*$ is adjoint n-ω-cospherical. $\square$

Let $N \in \operatorname{Mod} S^{\mathrm{op}}$ and let

$$\cdots \xrightarrow{g_{n+1}} P_n \xrightarrow{g_n} \cdots \xrightarrow{g_2} P_1 \xrightarrow{g_1} P_0 \xrightarrow{g_0} N \longrightarrow 0$$

be a projective resolution of N in $\operatorname{Mod} S^{\mathrm{op}}$. If there exists $n \geq 1$ such that $\operatorname{Im} g_n \cong \bigoplus_{j=1}^m U_j$ with each U_j isomorphic to a direct summand of some $\operatorname{Im} g_{i_j}$ with $i_j < n$, then we say N *has a projective resolution ultimately closed* at n (see [12]).

We now are in a position to prove the following

Theorem 5.5. *Let $n \geq 1$. Then any n-ω-cospherical module in* $\operatorname{Mod} R$ *is ω-coreflexive provided that one of the following conditions is satisfied.*

(1) $\operatorname{pd}_{S^{\mathrm{op}}} \omega \leq n$.

(2) *ω_S admits a projective resolution ultimately closed at n.*

Proof. (1) It follows directly from Proposition 5.4(1).

(2) Let

$$\cdots \xrightarrow{g_{n+1}} P_n \xrightarrow{g_n} \cdots \xrightarrow{g_2} P_1 \xrightarrow{g_1} P_0 \xrightarrow{g_0} \omega \longrightarrow 0$$

be a projective resolution of ω in $\operatorname{Mod} S^{\mathrm{op}}$ ultimately closed at n. Then $\operatorname{Im} g_n \cong \bigoplus_{j=1}^m U_j$ with each U_j isomorphic to a direct summand of some $\operatorname{Im} g_{i_j}$ with $i_j < n$. Now let $M \in \operatorname{Mod} R$ be n-ω-cospherical. Then $\operatorname{Ext}_R^{1 \leq i \leq n}(\omega, M) = 0$ and we have

$$\begin{aligned}
\operatorname{Tor}^S_{n+1}(\omega, \operatorname{Coker} f^n(M)_*) &\cong \operatorname{Tor}^S_1(\operatorname{Im} g_n, \operatorname{Coker} f^n(M)_*) \\
&\cong \operatorname{Tor}^S_1\left(\bigoplus_{j=1}^m U_j, \operatorname{Coker} f^n(M)_*\right) \\
&\cong \bigoplus_{j=1}^m \operatorname{Tor}^S_1(U_j, \operatorname{Coker} f^n(M)_*).
\end{aligned}$$

By Proposition 5.4(2), we have

$$\operatorname{Tor}_1^S(\operatorname{Im} g_{i_j}, \operatorname{Coker} f^n(M)_*) \cong \operatorname{Tor}_{i_j+1}^S(\omega, \operatorname{Coker} f^n(M)_*) = 0.$$

Note that U_j is isomorphic to a direct summand of some $\operatorname{Im} g_{i_j}$. Then we have $\operatorname{Tor}_1^S(U_j, \operatorname{Coker} f^n(M)_*) = 0$ for any $1 \leq j \leq m$, and so $\operatorname{Tor}_{n+1}^S(\omega, \operatorname{Coker} f^n(M)_*) = 0$. By Proposition 5.4(2), we conclude that $\operatorname{Tor}_{1\leq i\leq n+1}^S(\omega, \operatorname{Coker} f^n(M)_*) = 0$. Similar to the above argument we get $\operatorname{Tor}_{n+2}^S(\omega, \operatorname{Coker} f^n(M)_*) = 0$. Consequently, by Proposition 5.4(1), we have that θ_M is an isomorphism and M is ω-coreflexive. $\square$

Corollary 5.6. *For any $n \geq 1$, a module $M \in \operatorname{Mod} R$ satisfying $\operatorname{Ext}_R^{0\leq i\leq n}(\omega, M) = 0$ implies $M = 0$ provided that one of the following conditions is satisfied.*

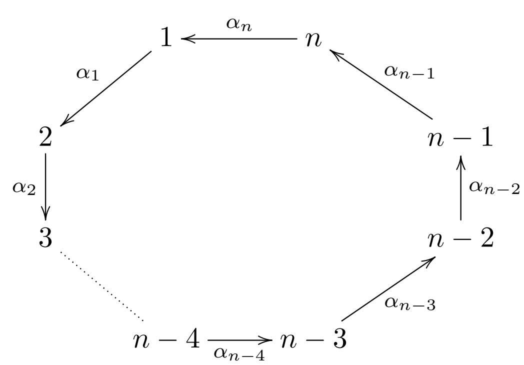
(1) $\operatorname{pd}_{S^{\mathrm{op}}} \omega \leq n$.

(2) *ω_S admits a projective resolution ultimately closed at n.*

Proof. If $M \in \operatorname{Mod} R$ satisfies $\operatorname{Ext}_R^{0\leq i\leq n}(\omega, M) = 0$, then $M \in \operatorname{Cor}_\omega(R)$ by Theorem 5.5. So $M \cong \omega \otimes_S M_* = 0$. $\square$

Obviously, for a module $N \in \operatorname{Mod} S^{\mathrm{op}}$, if $\operatorname{pd}_{S^{\mathrm{op}}} N \leq n$, then N admits a projective resolution ultimately closed at $n+1$. However, the converse does not hold in general as illustrated by the following example.

Example 5.7. Let R be a finite-dimensional algebra over an algebraically closed field given by the quiver:



modulo the ideal generated by $\{\alpha_{i+1}\alpha_i, \alpha_1\alpha_n \mid 1 \leq i \leq n-1\}$. For any $1 \leq i \leq n$, we use $S(i)$ and $P(i)$ to denote the simple R-module and the indecomposable projective R-module corresponding to the vertex i, respectively. Then R is a self-injective algebra with infinite global dimension. For any $1 \leq i \leq n$, the following exact sequence

(5.3)

$$\cdots \to P(i) \to P(i-1) \to \cdots \to P(1) \to P(n) \to P(n-1) \to \cdots \to P(i) \to S(i) \to 0$$

is a minimal projective resolution of $S(i)$ with $\mathrm{Im}(P(i) \to P(i-1)) \cong S(i)$. So $\mathrm{pd}_R S(i) = \infty$ and (5.3) is ultimately closed at m for any $m \geq n$.

From (5.3), we know that

$$\cdots \to \bigoplus_{i=1}^{n} P(i) \to \cdots \to \bigoplus_{i=1}^{n} P(i) \to \bigoplus_{i=1}^{n} P(i) \to \bigoplus_{i=1}^{n} S(i) \to 0 \tag{5.4}$$

is a minimal projective resolution of $\bigoplus_{i=1}^{n} S(i)$ with $\mathrm{Im}(\bigoplus_{i=1}^{n} P(i) \to \bigoplus_{i=1}^{n} P(i)) \cong \bigoplus_{i=1}^{n} S(i)$. So $\mathrm{pd}_R \bigoplus_{i=1}^{n} S(i) = \infty$ and (5.4) is ultimately closed at m for any $m \geq 1$.

6. ω-coreflexive modules and small projective dimension

In this section, by investigating the relationship between ω-coreflexive modules and adjoint ω-coreflexive modules, we give some equivalent characterizations for ω_S having projective dimension at most two. We begin with the following

Proposition 6.1. *The following statements are equivalent.*

(1) *Any* 2-ω-*cospherical module in* $\mathrm{Mod}\, R$ *is* ω-*coreflexive.*

(2) *Any adjoint* ω-*coreflexive module in* $\mathrm{Mod}\, S$ *is adjoint* 2-ω-*cospherical.*

Proof. (1) $\Rightarrow$ (2): Let $N \in \mathrm{Acor}_\omega(S)$. Then $\mathrm{acTr}_\omega N \in \mathrm{Mod}\, R$ is 2-ω-cospherical. So by (1), we have that $\mathrm{acTr}_\omega N \in \mathrm{Cor}_\omega(R)$. By Corollary 5.3, there exists an exact sequence

$$0 \longrightarrow \mathrm{Tor}_2^S(\omega, N) \longrightarrow \omega \otimes_S \mathrm{acTr}_\omega N_* \overset{\theta_{\mathrm{acTr}_\omega N}}{\longrightarrow} \mathrm{acTr}_\omega N \longrightarrow \mathrm{Tor}_1^S(\omega, N) \longrightarrow 0.$$

It induces that

$$\mathrm{Tor}_1^S(\omega, N) = 0 = \mathrm{Tor}_2^S(\omega, N)$$

and N is adjoint 2-ω-cospherical.

(2) $\Rightarrow$ (1): Let $M \in \mathrm{Mod}\, R$ be 2-ω-cospherical. Then

$$\mathrm{Ext}_R^1(\omega, M) = 0 = \mathrm{Ext}_R^2(\omega, M).$$

By Corollary 5.3(2), there exists an exact sequence

$$0 \longrightarrow \mathrm{Ext}_R^1(\omega, M) \longrightarrow \mathrm{cTr}_\omega M \overset{\mu_{\mathrm{cTr}_\omega M}}{\longrightarrow} (\omega \otimes_S \mathrm{cTr}_\omega M)_* \longrightarrow \mathrm{Ext}_R^2(\omega, M) \longrightarrow 0.$$

So $\mu_{\mathrm{cTr}_\omega M}$ is an isomorphism and $\mathrm{cTr}_\omega M \in \mathrm{Acor}_\omega(S)$. Hence by (2), we have

$$\mathrm{Tor}_1^S(C, \mathrm{cTr}_\omega M) = 0 = \mathrm{Tor}_2^S(\omega, \mathrm{cTr}_\omega M).$$

It follows from Corollary 5.2 that θ_M is an isomorphism and $M \in \mathrm{Cor}_\omega(R)$. $\square$

Dually, we have the following

Proposition 6.2. *The following statements are equivalent.*

(1) *Any adjoint* 2-ω*-cospherical module in* $\operatorname{Mod} S$ *is adjoint* ω*-coreflexive.*

(2) *Any* ω*-coreflexive module in* $\operatorname{Mod} R$ *is* 2-ω*-cospherical.*

By Propositions 6.1 and 6.2, we have the following

Corollary 6.3. *The following statements are equivalent.*

(1) *A module in* $\operatorname{Mod} R$ *is* 2-ω*-cospherical if and only if it is* ω*-coreflexive.*

(2) *A module in* $\operatorname{Mod} S$ *is adjoint* ω*-coreflexive if and only if it is adjoint* 2-ω*-cospherical.*

In the following, we establish a direct connection between ω-coreflexive modules and adjoint ω-coreflexive modules.

Proposition 6.4. *For any* $N \in \operatorname{Mod} S$*, the following statements are equivalent.*

(1) $\omega \otimes_S N \in \operatorname{Cor}_\omega(R)$.

(2) $(\omega \otimes_S N)_* \in \operatorname{Acor}_\omega(S)$.

Proof. (1) $\Rightarrow$ (2): By Lemma 2.4(3).

(2) $\Rightarrow$ (1): By Lemma 2.4(2), we have

$$\theta_{\omega\otimes_S N} \cdot (1_\omega \otimes \mu_N) = 1_{\omega\otimes_S N}.$$

So $\theta_{\omega\otimes_S N}$ is an epimorphism and

$$\operatorname{Ker} \theta_{\omega\otimes_S N} \cong \operatorname{Coker}(1_\omega \otimes \mu_N) \cong \omega \otimes_S \operatorname{Coker} \mu_N.$$

On the other hand, since $(\theta_{\omega\otimes_S N})_* \cdot \mu_{(\omega\otimes_S N)_*} = 1_{(\omega\otimes_S N)_*}$ by Lemma 2.4(1), we have

$$(\operatorname{Ker} \theta_{\omega\otimes_S N})_* \cong \operatorname{Ker}(\theta_{\omega\otimes_S N})_* \cong \operatorname{Coker} \mu_{(\omega\otimes_S N)_*}.$$

So $(\omega \otimes_S \operatorname{Coker} \mu_N)_* \cong \operatorname{Coker} \mu_{(C\otimes_S N)_*} = 0$ by (2). Thus $\omega \otimes_S \operatorname{Coker} \mu_N = 0$ by [19, Corollary 6.6(2)], and therefore $\theta_{\omega\otimes_S N}$ is a monomorphism. Consequently, we conclude that $\theta_{\omega\otimes_S N}$ is an isomorphism and $\omega \otimes_S N \in \operatorname{Cor}_\omega(R)$. □

Dually, we have the following

Proposition 6.5. *For any* $M \in \operatorname{Mod} R$*, the following statements are equivalent.*

(1) $M_* \in \operatorname{Acor}_\omega(S)$.

(2) $\omega \otimes_S M_* \in \mathrm{Cor}_\omega(R)$.

As a consequence of Propositions 6.4 and 6.5, we have the following

Corollary 6.6. *The following statements are equivalent.*

(1) $\omega \otimes_S N \in \mathrm{Cor}_\omega(R)$ *for any* $N \in \mathrm{Mod}\, S$.

(2) $M_* \in \mathrm{Acor}_\omega(S)$ *for any* $M \in \mathrm{Mod}\, R$.

Proof. (1) $\Rightarrow$ (2): Let $M \in \mathrm{Mod}\, R$. Then $\omega \otimes_S M_* \in \mathrm{Cor}_\omega(R)$ by (1). Thus $M_* \in \mathrm{Acor}_\omega(S)$ by Proposition 6.5.

(2) $\Rightarrow$ (1): Let $N \in \mathrm{Mod}\, S$. Then $(\omega \otimes_S N)_* \in \mathrm{Acor}_\omega(S)$ by (2). Thus $\omega \otimes_S N \in \mathrm{Cor}_\omega(R)$ by Proposition 6.4. $\square$

Lemma 6.7. *If* $\mathrm{pd}_R\, \omega \leq 2$, *then* $\mathrm{Ext}_R^{\geq 1}(\omega, \omega \otimes_S N) = 0$ *for any* $N \in \mathrm{Mod}\, S$.

Proof. Let $N \in \mathrm{Mod}\, S$. Then we have the following exact sequence

$$0 \longrightarrow \mathrm{acTr}_\omega\, N \longrightarrow \omega \otimes_S F_1(N) \stackrel{1_\omega \otimes f_0(N)}{\longrightarrow} \omega \otimes_S F_0(N) \longrightarrow \omega \otimes_S N \longrightarrow 0$$

in $\mathrm{Mod}\, R$. By Lemma 2.5(2), we have

$$\mathrm{Ext}_R^{\geq 1}(\omega, \omega \otimes_S F_0(N)) = 0 = \mathrm{Ext}_R^{\geq 1}(\omega, \omega \otimes_S F_1(N)).$$

Because $\mathrm{pd}_R\, C \leq 2$ by assumption, we have

$$\mathrm{Ext}_R^i(\omega, \omega \otimes_S N) \cong \mathrm{Ext}_R^{i+2}(\omega, \mathrm{acTr}_\omega\, N) = 0$$

for any $i \geq 1$. $\square$

The following is the main result in this section.

Theorem 6.8. *If* $\mathrm{pd}_R\, \omega \leq 2$, *then the following statements are equivalent.*

(1) $\mathrm{pd}_{S^{\mathrm{op}}}\, \omega \leq 2$.

(2) *Any* 2-ω-*cospherical module in* $\mathrm{Mod}\, R$ *is* ω-*coreflexive.*

(3) *A module in* $\mathrm{Mod}\, R$ *is* 2-ω-*cospherical module if and only if it is* ω-*coreflexive.*

(4) *Any adjoint* ω-*coreflexive module in* $\mathrm{Mod}\, S$ *is adjoint* 2-ω-*cospherical.*

(5) *A module in* $\mathrm{Mod}\, S$ *is adjoint* ω-*coreflexive if and only if it is adjoint* 2-ω-*cospherical.*

(6) *Any module of* ω-$\mathcal{P}(S)$-*class* 2 *in* $\mathrm{Mod}\, R$ *is* ω-*coreflexive.*

(7) *Any module of ω-$\mathcal{T}$-class 2 in* $\operatorname{Mod} R$ *is ω-coreflexive.*

(8) $\operatorname{Tor}_2^S(\omega, V) = 0$ *for any* $V \in \operatorname{Acot}_\omega(S)$.

(9) $\operatorname{Tor}_3^S(\omega, N) = 0$ *for any* N *in* $\operatorname{Mod} S$.

(10) $\operatorname{Tor}_1^S(\omega, U_*) = 0$ *for any* $U \in \operatorname{Cot}_\omega(R)$.

Proof. By Theorems 4.5 and 4.7, we have (1) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) $\Leftrightarrow$ (10). The assertions (1) $\Rightarrow$ (2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5) follow from Theorem 5.5, Proposition 6.1 and Corollary 6.3, respectively. The implications (3) $\Rightarrow$ (2) and (5) $\Rightarrow$ (4) are trivial.

(2) + (4) $\Rightarrow$ (1): Let $N \in \operatorname{Mod} S$. Then $\operatorname{Ext}_R^{\geq 1}(\omega, \omega \otimes_S N) = 0$ by Lemma 6.7. So $\omega \otimes_S N \in \operatorname{Cor}_\omega(R)$ by (2). Then it follows from Corollary 6.6 that $(\operatorname{acTr}_\omega N)_* \in \operatorname{Acor}_\omega(S)$. So $\operatorname{Tor}_1^S(\omega, (\operatorname{acTr}_\omega N)_*) = 0$ by (4). Since $(\omega \otimes_S F_1(N))_* \cong F_1(N)$ and $(\omega \otimes_S F_0(N))_* \cong F_0(N)$ by Lemma 2.5(2), it induces that $\operatorname{Ker} f_0(N) \cong (\operatorname{acTr}_\omega N)_*$. So we have that

$$\operatorname{Tor}_3^S(\omega, N) \cong \operatorname{Tor}_1^S(\omega, (\operatorname{acTr}_C N)_*) = 0$$

and $\operatorname{pd}_{S^{\mathrm{op}}} \omega \leq 2$.

(2) $\Rightarrow$ (3): Let $M \in \operatorname{Cor}_\omega(R)$. Then $M \cong \omega \otimes_S M_*$. By Lemma 6.7, we have $\operatorname{Ext}_R^i(\omega, M) \cong \operatorname{Ext}_R^i(\omega, \omega \otimes_S M_*) = 0$ for any $i \geq 1$. $\square$

As a consequence of Theorem 6.8, we have the following

Corollary 6.9. $\operatorname{pd}_R \omega = \operatorname{pd}_{S^{\mathrm{op}}} \omega \leq 2$ *if and only if for* $M \in \operatorname{Mod} R$, *the following statements are equivalent.*

(1) $M \in \operatorname{Cor}_\omega(R)$.

(2) *There exists an exact sequence*

$$U_1 \to U_0 \to M \to 0$$

in $\operatorname{Mod} R$ *with all* $U_i \in \operatorname{Add}_R \omega \cup \operatorname{Inj} R$.

(3) M *is 2-ω-cospherical.*

Proof. Let $\operatorname{pd}_R \omega = \operatorname{pd}_{S^{\mathrm{op}}} \omega \leq 2$. Then (1) $\Leftrightarrow$ (3) by Theorem 6.8, and (1) $\Rightarrow$ (2) by [18, Lemma 3.6]. Now let

$$U_1 \to U_0 \to M \to 0$$

be an exact sequence in $\operatorname{Mod} R$ with all $U_i \in \operatorname{Add}_R \omega \cup \operatorname{Inj} R$, and let $K = \operatorname{Ker}(U_1 \to U_0)$. Then by Lemma 2.5(1), we have $\operatorname{Ext}_R^i(\omega, M) \cong \operatorname{Ext}_R^{i+2}(\omega, K) = 0$ for any $i \geq 1$. So we have (2) $\Rightarrow$ (3).

Conversely, for any $K \in \operatorname{Mod} R$, consider the following exact sequence

$$0 \longrightarrow K \longrightarrow I^0(K) \xrightarrow{f^0} I^1(K) \longrightarrow M \longrightarrow 0,$$

where $M = \operatorname{Coker} f^0$. Then by the equivalence between (2) and (3), we have $\operatorname{Ext}^3_R(\omega, K) \cong \operatorname{Ext}^1_R(\omega, M) = 0$. It implies $\operatorname{pd}_R \omega \leq 2$. So by Theorem 6.8 and assumption, we have $\operatorname{pd}_{S^{\mathrm{op}}} \omega \leq 2$. It follows from [21, Theorem (1)] that $\operatorname{pd}_R \omega = \operatorname{pd}_{S^{\mathrm{op}}} \omega$. □

In the following result, we give some equivalent characterizations for ω_S or ${}_R\omega$ being projective.

Proposition 6.10. (1) *The following statements are equivalent.*

(1a) ω_S *is projective.*

(1b) *Any module in* $\operatorname{Mod} R$ *is* ω*-coreflexive.*

(1c) *Any module in* $\operatorname{Mod} R$ *is* ω*-cotorsionless.*

(2) *The following statements are equivalent.*

(2a) ${}_R\omega$ *is projective.*

(2b) *Any module in* $\operatorname{Mod} S$ *is adjoint* ω*-coreflexive.*

(2c) *Any module in* $\operatorname{Mod} S$ *is adjoint* ω*-cotorsionless.*

Proof. (1) The implication (1a) $\Rightarrow$ (1b) follows from Corollary 5.2(1), and the implication (1b) $\Rightarrow$ (1c) is trivial.

(1c) $\Rightarrow$ (1a): Let $N \in \operatorname{Mod} S$. By (1c), $\operatorname{acTr}_\omega N \in \operatorname{Cot}_\omega(R)$ and $\theta_{\operatorname{acTr}_\omega N}$ is an epimorphism. So by Corollary 5.3(1), we have that $\operatorname{Tor}^S_1(\omega, N) = 0$ and ω_S is flat, and hence projective.

(2) The implication (2a) $\Rightarrow$ (2b) follows from Corollary 5.2(2), and the implication (2b) $\Rightarrow$ (2c) is trivial.

(2c) $\Rightarrow$ (2a): Let $M \in \operatorname{Mod} R$. By (2c), $\operatorname{cTr}_\omega M \in \operatorname{Acot}_\omega(S)$ and $\mu_{\operatorname{cTr}_\omega M}$ is a monomorphism. So by Corollary 5.3(2), we have that $\operatorname{Ext}^1_R(\omega, M) = 0$ and ${}_R\omega$ is projective. □

Let R be an artin algebra and $\mathbb{D}$ its ordinary duality. Then we have the following facts: (1) ${}_R\mathbb{D}(R)_R$ is a semidualizing bimodule; and (2) R is selfinjective if and only if $\mathbb{D}(R)$ is projective as a left (or right) R-module. The following result is an immediate consequence of Proposition 6.10. Compare it with [11, Corollary 1.2], which states that a left and right noetherian ring R is self-injective if and only if any finitely generated left (or right) R-module A is reflexive, that is, $\operatorname{Hom}_R(\operatorname{Hom}_R(A, R), R)) \cong A$.

Corollary 6.11. *For an artin algebra* R*, the following statements are equivalent.*

(1) *R is selfinjective.*

(2) *Any module in* $\operatorname{Mod} R$ *is* $\mathbb{D}(R)$*-coreflexive.*

(3) *Any module in* $\operatorname{Mod} R$ *is* $\mathbb{D}(R)$*-cotorsionless.*

(4) *Any module in* $\operatorname{Mod} R$ *is adjoint* $\mathbb{D}(R)$*-coreflexive.*

(5) *Any module in* $\operatorname{Mod} R$ *is adjoint* $\mathbb{D}(R)$*-cotorsionless.*

In the following result, we give some equivalent characterizations for ω_S having projective dimension at most one.

Theorem 6.12. *The following statements are equivalent.*

(1) $\operatorname{pd}_{S^{\mathrm{op}}} \omega \leq 1$.

(2) *Any* 1-ω*-cospherical module in* $\operatorname{Mod} R$ *is* ω*-cotorsionless.*

(3) *Any* 1-ω*-cospherical module in* $\operatorname{Mod} R$ *is* ω*-coreflexive.*

(4) *Any* ω*-cotorsionless module in* $\operatorname{Mod} R$ *is* ω*-coreflexive.*

(5) $\operatorname{Tor}_1^S(\omega, V) = 0$ *for any* $V \in \operatorname{Acot}_\omega(S)$.

(6) $\operatorname{Tor}_2^S(\omega, N) = 0$ *for any* $N \in \operatorname{Mod} S$.

Proof. By Theorem 4.5 and Lemma 4.3, we have (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6). The implication (3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (4): Let $M \in \operatorname{Cot}_\omega(R)$. Then θ_M is an epimorphism. By [18, Proposition 3.7] and Lemma 2.5(1), there exists an exact sequence

$$0 \to N \to W \to M \to 0$$

in $\operatorname{Mod} R$ with $W \in \mathcal{P}_\omega(R)$ and N 1-ω-cospherical. Then we get the following commutative diagram with exact rows

$$\begin{tikzcd}
 & \omega \otimes_S N_* \arrow[r] \arrow[d, "\theta_N"] & \omega \otimes_S W_* \arrow[r] \arrow[d, "\theta_W"] & \omega \otimes_S M_* \arrow[r] \arrow[d, "\theta_M"] & 0 \\
0 \arrow[r] & N \arrow[r] & W \arrow[r] & M \arrow[r] & 0,
\end{tikzcd}$$

where θ_W is an isomorphism by Lemma 2.5(1). Because $N \in \operatorname{Cot}_\omega(R)$ and θ_N is an epimorphism by (2), we have that θ_M is a monomorphism, and hence an isomorphism. Thus $M \in \operatorname{Cor}_\omega(R)$.

(4) $\Rightarrow$ (3): Let $M \in \operatorname{Mod} R$ be 1-ω-cospherical. Then the following exact sequence

$$0 \to M \to I^0(M) \to M_1 \to 0$$

in $\operatorname{Mod} R$ yields the following commutative diagram with exact rows

$$\begin{tikzcd}
 & \omega \otimes_S M_* \arrow[r] \arrow[d, "\theta_M"] & \omega \otimes_S I^0(M)_* \arrow[r] \arrow[d, "\theta_{I^0(M)}"] & \omega \otimes_S M_{1*} \arrow[r] \arrow[d, "\theta_{M_1}"] & 0 \\
0 \arrow[r] & M \arrow[r] & I^0(M) \arrow[r] & M_1 \arrow[r] & 0,
\end{tikzcd}$$

where $\theta_{I^0(M)}$ is an isomorphism by Lemma 2.5(1). So θ_{M_1} is an epimorphism and $M_1 \in \operatorname{Cot}_\omega(R)$. By (4), we have that $M_1 \in \operatorname{Cor}_\omega(R)$ and θ_{M_1} is an isomorphism. Thus θ_M is an epimorphism and $M \in \operatorname{Cot}_\omega(R)$. By (4) again, $M \in \operatorname{Cor}_\omega(R)$. □

7. Wakamatsu tilting conjecture over artinian rings

In this section, we aim at studying the Wakamatsu tilting conjecture in some special cases.

Let $N \in \operatorname{Mod} S$. In the minimal flat resolution (2.1) of N in $\operatorname{Mod} S$, for any $i \geq -1$, put $\operatorname{Im} f_i(N) = N_i$, and let $f_i(N) = \alpha_i \cdot \pi_i$ be the natural epic-monic decomposition of $f_i(N)$ with $\pi_i \colon F_{i+1}(N) \twoheadrightarrow N_i$ and $\alpha_i \colon N_i \hookrightarrow F_i(N)$.

Lemma 7.1. *Let $N \in \operatorname{Mod} S$. Then for any $i \geq 0$, we have*

$$(\operatorname{acTr}_\omega N_{i-1})_* \cong N_{i+1} \quad \textit{and} \quad \operatorname{Ext}^1_R(\omega, \operatorname{acTr}_\omega N_i) = 0.$$

Proof. For any $i \geq 0$, we have the following two exact sequences

$$0 \longrightarrow N_{i+1} \xrightarrow{\alpha_{i+1}} F_{i+1}(N) \xrightarrow{f_i(N)} F_i(N) \xrightarrow{\pi_{i-1}} N_{i-1} \longrightarrow 0,$$
$$0 \longrightarrow \operatorname{acTr}_\omega N_{i-1} \xrightarrow{\beta_{i+1}} \omega \otimes_S F_{i+1}(N) \xrightarrow{1_\omega \otimes f_i(N)} \omega \otimes_S F_i(N) \xrightarrow{1_\omega \otimes \pi_{i-1}} \omega \otimes_S N_{i-1} \longrightarrow 0.$$

Then we get the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & N_{i+1} \arrow[r, "\alpha_{i+1}"] \arrow[d, dashed, "h"] & F_{i+1}(N) \arrow[r, "f_i(N)"] \arrow[d, "\mu_{F_{i+1}(N)}"] & F_i(N) \arrow[d, "\mu_{F_i(N)}"] \\
0 \arrow[r] & (\operatorname{acTr}_\omega N_{i-1})_* \arrow[r, "\beta_{i+1*}"] & (\omega \otimes_S F_{i+1}(N))_* \arrow[r, "(1_\omega \otimes f_i(N))_*"] & (\omega \otimes_S F_i(N))_*,
\end{tikzcd} \tag{7.1}$$

where h is an induced homomorphism. Note that $\mu_{F_{i+1}(N)}$ and $\mu_{F_i(N)}$ are isomorphisms by Lemma 2.5(2). So h is an isomorphism and $(\operatorname{acTr}_\omega N_{i-1})_* \cong N_{i+1}$. Because N_i is isomorphic to a submodule of the adjoint ω-coreflexive module $F_i(N)$, N_i is adjoint ω-cotorsionless. It follows from Corollary 5.2(2) that $\operatorname{Ext}^1_R(\omega, \operatorname{acTr}_\omega N_i) = 0$. □

Lemma 7.2. *Let $N \in \operatorname{Mod} S$. Then for any $i \geq 0$, there exists an exact sequence*

$$\eta_i\colon 0 \longrightarrow \operatorname{acTr}_\omega N_i \longrightarrow \omega \otimes_S F_{i+2}(N) \xrightarrow{g_i} \operatorname{acTr}_\omega N_{i-1} \longrightarrow \operatorname{Tor}_{i+1}^S(\omega, N) \longrightarrow 0. \tag{7.2}$$

Proof. Let g_i be the composition

$$\omega \otimes_S F_{i+2}(N) \overset{1_\omega \otimes \pi_{i+1}}{\longrightarrow} \omega \otimes_S N_{i+1} \overset{1_\omega \otimes h}{\longrightarrow} \omega \otimes_S (\operatorname{acTr}_\omega N_{i-1})_* \overset{\theta_{\operatorname{acTr}_\omega N_{i-1}}}{\longrightarrow} \operatorname{acTr}_\omega N_{i-1},$$

where h is as in (7.1). Since $1_\omega \otimes \pi_{i+1}$ is an epimorphism and $1_\omega \otimes h$ is an isomorphism, we have

$$\operatorname{Im} g_i = \operatorname{Im}(\theta_{\operatorname{acTr}_\omega N_{i-1}} \cdot (1_\omega \otimes h) \cdot (1_\omega \otimes \pi_{i+1})) = \operatorname{Im} \theta_{\operatorname{acTr}_\omega N_{i-1}}.$$

So

$$\operatorname{Coker} g_i \cong \operatorname{Tor}_1^S(\omega, N_{i-1}) \cong \operatorname{Tor}_{i+1}^S(\omega, N)$$

by Corollary 5.3(1). For (7.1) we know that

$$\beta_{i+1*} \cdot h = \mu_{F_{i+1}(N)} \cdot \alpha_{i+1},$$

so we have

$$(1_\omega \otimes \beta_{i+1*}) \cdot (1_\omega \otimes h) = (1_\omega \otimes \mu_{F_{i+1}(N)}) \cdot (1_\omega \otimes \alpha_{i+1}).$$

Note that

$$f_{i+1}(N) = \alpha_{i+1} \cdot \pi_{i+1} \quad \text{and} \quad \beta_{i+1} \cdot \theta_{\operatorname{acTr}_\omega N_{i-1}} = \theta_{\omega \otimes_S F_{i+1}(N)} \cdot (1_\omega \otimes \beta_{i+1*}).$$

So by Lemma 2.4(2), we have

$$\begin{aligned} 1_\omega \otimes f_{i+1}(N) &= \theta_{\omega \otimes_S F_{i+1}(N)} \cdot (1_\omega \otimes \mu_{F_{i+1}(N)}) \cdot (1_\omega \otimes f_{i+1}(N)) \\ &= \theta_{\omega \otimes_S F_{i+1}(N)} \cdot (1_\omega \otimes \mu_{F_{i+1}(N)}) \cdot (1_\omega \otimes \alpha_{i+1}) \cdot (1_\omega \otimes \pi_{i+1}) \\ &= \theta_{\omega \otimes_S F_{i+1}(N)} \cdot (1_\omega \otimes \beta_{i+1*}) \cdot (1_\omega \otimes h) \cdot (1_\omega \otimes \pi_{i+1}) \\ &= \beta_{i+1} \cdot \theta_{\operatorname{acTr}_\omega N_{i-1}} \cdot (1_\omega \otimes h) \cdot (1_\omega \otimes \pi_{i+1}) \\ &= \beta_{i+1} \cdot g_i. \end{aligned}$$

Since β_{i+1} is a monomorphism, we have

$$\operatorname{Ker} g_i \cong \operatorname{Ker}(1_\omega \otimes f_{i+1}(N)) = \operatorname{acTr}_\omega N_i.$$

The proof is finished. □

Following [19, Definition 6.2], the Ext-*cograde* of a module M in $\operatorname{Mod} R$ with respect to ω is defined as $\text{E-cograde}_\omega M := \inf\{i \geq 0 \mid \operatorname{Ext}_R^i(\omega, M) \neq 0\}$. If $\operatorname{Ext}_R^{\geq 0}(\omega, M) = 0$, then set $\text{E-cograde}_\omega M = \infty$.

In the following, m and n are positive integers. We use $\operatorname{mod} S$ to denote the class of finitely presented left S-modules.

Lemma 7.3. *Let S be a left coherent ring. If* $\text{E-cograde}_\omega \operatorname{Tor}_m^S(\omega, N) \geq n-1$ *for any* $N \in \operatorname{mod} S$*, then* $\operatorname{Ext}_R^j(\omega, \operatorname{acTr}_\omega N_{i+j-2}) = 0$ *for any $i \geq m$ and $1 \leq j \leq n$.*

Proof. (1) The case for $n = 1$ follows from Lemma 7.1. Now suppose $n \geq 2$. Because S is left coherent and $\text{E-cograde}_\omega \operatorname{Tor}_m^S(\omega, N) \geq n-1$ for any $N \in \operatorname{mod} S$ by assumption, it is immediate that $\text{E-cograde}_\omega \operatorname{Tor}_i^S(\omega, N) \geq n-1$ for any $N \in \operatorname{mod} S$ and $i \geq m$. We divide the exact sequence (7.2) in Lemma 7.2 into the following two exact sequences

$$0 \longrightarrow \operatorname{acTr}_\omega N_i \longrightarrow \omega \otimes_S F_{i+2}(N) \xrightarrow{\nu_i} K_i \longrightarrow 0, \tag{7.3}$$

$$0 \longrightarrow K_i \xrightarrow{\lambda_i}{}_i \operatorname{acTr}_\omega N_{i-1} \longrightarrow \operatorname{Tor}_{i+1}^S(\omega, N) \longrightarrow 0, \tag{7.4}$$

where $K_i = \operatorname{Im} g_i$ and $g_i = \lambda_i \cdot \nu_i$ is the natural epic-monic decomposition of g_i. For $i \geq m$, applying the functor $(-)_*$ to (7.3) yields

$$\operatorname{Ext}_R^j(\omega, K_i) \cong \operatorname{Ext}_R^{j+1}(\omega, \operatorname{acTr}_\omega N_i)$$

for any $j \geq 1$ by Lemma 2.5(1); and then applying the functor $(-)_*$ to (7.4) gives a monomorphism

$$\operatorname{Ext}_R^2(\omega, \operatorname{acTr}_\omega N_i)\ (\cong \operatorname{Ext}_R^1(\omega, K_i)) \rightarrowtail \operatorname{Ext}_R^1(\omega, \operatorname{acTr}_\omega N_{i-1}).$$

Doing similarly for the exact sequences $\eta_{i+1}, \eta_{i+2}, \ldots, \eta_{n+i-2}$, we get a chain of monomorphisms

$$\operatorname{Ext}_R^n(\omega, \operatorname{acTr}_\omega N_{n+i-2}) \rightarrowtail \cdots \rightarrowtail \operatorname{Ext}_R^2(\omega, \operatorname{acTr}_\omega N_i) \rightarrowtail \operatorname{Ext}_R^1(\omega, \operatorname{acTr}_\omega N_{i-1}).$$

Now the assertion follows from Lemma 7.1. $\square$

Lemma 7.4. *Let S be a left coherent ring. If* $\operatorname{pd}_R \omega \leq n$ *and* $\text{E-cograde}_\omega \operatorname{Tor}_m^S(\omega, N) \geq n-1$ *for any $N \in \operatorname{mod} S$, then we have*

(1) $\operatorname{Ext}_R^{\geq 1}(\omega, \operatorname{acTr}_\omega N_i) = 0$ *for any $i \geq m+n-2$.*

(2) *N_i is adjoint ω-coreflexive for any $i \geq m+n-2$.*

(3) $\operatorname{Ext}_R^{\geq 1}(\omega, \omega \otimes_S N_i) = 0$ *for any $i \geq m+n-2$.*

(4) $\text{E-cograde}_\omega \operatorname{Tor}_{i+1}^S(\omega, N) = \infty$ *for any $i \geq m+n-1$.*

Proof. (1) Let $i \geq m+n-2$. It follows from Lemma 7.3 that $\operatorname{Ext}_R^{1 \leq j \leq n}(\omega, \operatorname{acTr}_\omega N_i) = 0$. Since $\operatorname{pd}_R \omega \leq n$, we have $\operatorname{Ext}_R^{\geq n+1}(\omega, \operatorname{acTr}_\omega N_i) = 0$.

(2) It follows from (1) and Corollary 5.2(2).

(3) Since there exists an exact sequence

$$0 \to \operatorname{acTr}_\omega N_i \to \omega \otimes_S F_{i+2}(N) \to \omega \otimes_S F_{i+1}(N) \to \omega \otimes_S N_i \to 0,$$

the assertion follows from (1) and Lemma 2.5(1).

(4) Let g_i be as in the proof of Lemma 7.2 with $i \geq m+n-1$, that is,

$$g_i = \theta_{\mathrm{acTr}_\omega N_{i-1}} \cdot (1_\omega \otimes h) \cdot (1_\omega \otimes \pi_{i+1}).$$

Then we have

$$g_{i*} = (\theta_{\mathrm{acTr}_\omega N_{i-1}})_* \cdot (1_\omega \otimes h)_* \cdot (1_\omega \otimes \pi_{i+1})_*.$$

Because both $\mu_{N_{i+1}}$ and $\mu_{F_{i+2}(N)}$ are isomorphisms by (2) and Lemma 2.5(2), the equality

$$(1_\omega \otimes \pi_{i+1})_* \cdot \mu_{F_{i+2}(N)} = \mu_{N_{i+1}} \cdot \pi_{i+1}$$

implies that $(1_\omega \otimes \pi_{i+1})_*$ is an epimorphism. Because $(\theta_{\mathrm{acTr}_\omega N_{i-1}})_*$ is an epimorphism by Lemma 2.4(1), we have that g_{i*} is also an epimorphism.

Consider the exact sequences (7.2)–(7.4) in Lemmas 7.2 and 7.3. Because $g_{i*} = \lambda_{i*} \cdot \nu_{i*}$, we have that λ_{i*} is an epimorphism, and hence an isomorphism. Applying the functor $(-)_*$ to the exact sequence (7.3) we have

$$\mathrm{Ext}^j_R(\omega, K_i) \cong \mathrm{Ext}^{j+1}_R(\omega, \mathrm{acTr}_\omega N_i) = 0$$

for any $j \geq 1$ by (1) and Lemma 2.5(1). Moreover, applying the functor $(-)_*$ to the exact sequence (7.4) we get a long exact sequence

$$\begin{gathered} 0 \longrightarrow K_{i*} \xrightarrow{\lambda_{i*}} (\mathrm{acTr}_\omega N_{i-1})_* \longrightarrow (\mathrm{Tor}^S_{i+1}(\omega, N))_* \longrightarrow \cdots \\ \cdots \longrightarrow \mathrm{Ext}^j_R(\omega, K_i) \longrightarrow \mathrm{Ext}^j_R(\omega, \mathrm{acTr}_\omega N_{i-1}) \longrightarrow \mathrm{Ext}^j_R(\omega, \mathrm{Tor}^S_{i+1}(\omega, N)) \longrightarrow \cdots . \end{gathered} \tag{7.5}$$

Notice that $i \geq m+n-1$, so also by (1) we have $\mathrm{Ext}^{\geq 1}_R(\omega, \mathrm{acTr}_\omega N_{i-1}) = 0$. Then from the exact sequence (7.5) we get $\mathrm{Ext}^{\geq 1}_R(\omega, \mathrm{Tor}^S_{i+1}(\omega, N)) = 0$. Because λ_{i*} is an isomorphism, we have that $(\mathrm{Tor}^S_{i+1}(\omega, N))_* = 0$ and $\text{E-cograde}_\omega \mathrm{Tor}^S_{i+1}(\omega, N) = \infty$. □

The main result in this section is the following

Theorem 7.5. *Let S be a left artinian ring and $R = S$. If $\mathrm{pd}_S \omega \leq n$ and $\text{E-cograde}_\omega \mathrm{Tor}^S_m(\omega, N) \geq n-1$ for any $N \in \mathrm{mod}\, S$, then $\mathrm{pd}_S \omega = \mathrm{pd}_{S^{\mathrm{op}}} \omega \leq n$.*

Proof. Define a linear map

$$\gamma\colon K_0(\mathrm{mod}\, S) \to K_0(\mathrm{mod}\, S) \quad \text{via} \quad \gamma([M]) = \sum_{i \geq 0} (-1)^i [\mathrm{Ext}^i_S(\omega, M)].$$

Since $\mathrm{pd}_S \omega \leq n$, this map is well defined. By Lemmas 2.5 and 7.4(2)(3), for any $N \in \mathrm{mod}\, S$ and $i \geq m+n-1$ we have

$$[N] = \sum_{j=0}^{i-1} (-1)^j [F_j(N)] + (-1)^i [N_{i-1}]$$

$$
\begin{aligned}
&= \sum_{j=0}^{i-1}(-1)^j[(\omega \otimes_S F_j(N))_*] + (-1)^i[(\omega \otimes_S N_{i-1})_*] \\
&= \sum_{j=0}^{i-1}(-1)^j\gamma([\omega \otimes_S F_j(N)]) + (-1)^i\gamma([\omega \otimes_S N_{i-1}]) \\
&= \gamma\left(\sum_{j=0}^{i-1}(-1)^j[\omega \otimes_S F_j(N)] + (-1)^i[\omega \otimes_S N_{i-1}]\right),
\end{aligned}
$$

which implies that γ is surjective. Because S is left artinian by assumption, it follows from [3, p. 5, Theorem 1.7] that $K_0(\operatorname{mod} S)$ is a finitely generated free abelian group and γ is bijective. On the other hand, for any $Y \in \operatorname{mod} S$, we have that $[Y] = 0$ if and only if $Y = 0$. Since $\operatorname{Ext}_S^{\geq 0}(\omega, \operatorname{Tor}^S_{\geq m+n}(\omega, N)) = 0$ by Lemma 7.4(4), we have $\gamma([\operatorname{Tor}^S_{\geq m+n}(\omega, N)]) = 0$ and $[\operatorname{Tor}^S_{\geq m+n}(\omega, N)] = 0$. So $\operatorname{Tor}^S_{\geq m+n}(\omega, N) = 0$ and $\operatorname{pd}_{S^{\mathrm{op}}} \omega \leq m + n - 1$. Now it follows from [21, Theorem (1)] that $\operatorname{pd}_{S^{\mathrm{op}}} \omega = \operatorname{pd}_S \omega \leq n$. $\square$

In the following, we study when the Ext-cograde condition in Theorem 7.5 is satisfied. We need the following

Lemma 7.6. *Let $Q \in \operatorname{Mod} R$ be finitely generated projective and $t \geq 0$. Then $\operatorname{fd}_{S^{\mathrm{op}}} \operatorname{Hom}_R(Q, \omega) \leq t$ if and only if $\operatorname{Hom}_R(Q, \operatorname{Tor}^S_{t+1}(\omega, N)) = 0$ for any $N \in \operatorname{Mod} S$.*

Proof. Let $N \in \operatorname{Mod} S$ and

$$\boldsymbol{P} =: \cdots \to P_i \to \cdots \to P_1 \to P_0 \to N \to 0$$

be a projective resolution of N in $\operatorname{Mod} S$. Because $Q \in \operatorname{Mod} R$ is finitely generated projective by assumption, the functor $\operatorname{Hom}_R(Q, -)$ is exact. Then we have

$$
\begin{aligned}
\operatorname{Tor}^S_{t+1}(\operatorname{Hom}_R(Q, \omega), N) &\cong H_{t+1}(\operatorname{Hom}_R(Q, \omega) \otimes_S \boldsymbol{P}) \\
&\cong H_{t+1}(\operatorname{Hom}_R(Q, \omega \otimes_S \boldsymbol{P})) \\
&\cong \operatorname{Hom}_R(Q, H_{t+1}(\omega \otimes_S \boldsymbol{P})) \qquad \text{(by [8, p. 33, Excercise 3])} \\
&\cong \operatorname{Hom}_R(Q, \operatorname{Tor}^S_{t+1}(\omega, N)).
\end{aligned}
$$

Now the assertion follows easily. $\square$

Let R be a semiperfect ring. Then any finitely generated left or right R-module has a projective cover. In this case, since ${}_R\omega$ admits a degreewise finite R-projective resolution by Definition 2.1, we may assume that

$$\cdots \xrightarrow{g_i(\omega)} P_i(\omega) \xrightarrow{g_{i-1}(\omega)} \cdots \xrightarrow{g_1(\omega)} P_1(\omega) \xrightarrow{g_0(\omega)} P_0(\omega) \xrightarrow{g_{-1}(\omega)} {}_R\omega \longrightarrow 0$$

is a minimal projective resolution of ${}_R\omega$ in $\operatorname{Mod} R$ with all $P_i(\omega)$ finitely generated. Put $\omega_i := \operatorname{Im} g_i(\omega)$ for any $i \geq -1$ (in particular, $\omega_{-1} = \omega$). Let $n \geq 0$. Recall from [19, Definition 6.2] that the *strong* Ext-*cograde* of a module $M \in \operatorname{Mod} R$ with respect to ω, denoted by $\operatorname{s.E\text{-}cograde}_\omega M$, is said to be at least n if $\operatorname{E\text{-}cograde} X \geq n$ for any quotient module X of M.

Proposition 7.7. *Let R be a semiperfect ring. Then the following statements are equivalent.*

(1) $\operatorname{s.E\text{-}cograde}_\omega \operatorname{Tor}_m^S(\omega, N) \geq n-1$ *for any* $N \in \operatorname{Mod} S$.

(2) $\operatorname{fd}_{S^{\mathrm{op}}} \operatorname{Hom}_R(P_i(\omega), \omega) \leq m-1$ *for any* $0 \leq i \leq n-2$.

Proof. The case for $n = 1$ is trivial. Now suppose $n \geq 2$.

(1) $\Rightarrow$ (2): We proceed by using induction on i.

When $i = 0$, we will prove $\operatorname{fd}_{S^{\mathrm{op}}} \operatorname{Hom}_R(P_0(\omega), \omega) \leq m-1$. Let $N \in \operatorname{Mod} S$. Because $\operatorname{s.E\text{-}cograde}_\omega \operatorname{Tor}_m^S(\omega, N) \geq n-1$ by (1), we have $\operatorname{Hom}_R(\omega, \operatorname{Tor}_m^S(\omega, N)) = 0$. Let $f \in \operatorname{Hom}_R(P_0(\omega), \operatorname{Tor}_m^S(\omega, N))$. Then f induces naturally a homomorphism

$$\overline{f}\colon \omega\ (\cong P_0(\omega)/\omega_0) \to \operatorname{Tor}_m^S(\omega, N)/f(\omega_0)$$

in $\operatorname{Mod} R$. Since $\operatorname{s.E\text{-}cograde}_\omega \operatorname{Tor}_m^S(\omega, N) \geq n-1$ by (1), we have $\overline{f} = 0$. So $P_0(\omega) = \operatorname{Ker} f + \omega_0$. Notice that $P_0(\omega)$ is the projective cover of ω, so ω_0 is superfluous in $P_0(\omega)$. It induces that $\operatorname{Ker} f = P_0(\omega)$ and $f = 0$. Thus we have $\operatorname{Hom}_R(P_0(\omega), \operatorname{Tor}_m^S(\omega, N)) = 0$, and therefore $\operatorname{fd}_{S^{\mathrm{op}}} \operatorname{Hom}_R(P_0(\omega), \omega) \leq m-1$ by Lemma 7.6.

Now suppose that $i \geq 1$ and $N \in \operatorname{Mod} S$. Let X be a quotient module of $\operatorname{Tor}_m^S(\omega, N)$. By (1), we have $\operatorname{Ext}_R^{0 \leq i \leq n-2}(\omega, X) = 0$. Then

$$\operatorname{Ext}_R^1(\omega_{i-2}, X) \cong \operatorname{Ext}_R^i(\omega, X) = 0$$

for any $1 \leq i \leq n-2$. From the exact sequence

$$0 \to \omega_{i-1} \to P_{i-1}(\omega) \to \omega_{i-2} \to 0,$$

we get the following exact sequence

$$\operatorname{Hom}_R(P_{i-1}(\omega), X) \to \operatorname{Hom}_R(\omega_{i-1}, X) \to \operatorname{Ext}_R^1(\omega_{i-2}, X) \to 0. \tag{7.6}$$

By the induction hypothesis, we have $\operatorname{fd}_{S^{\mathrm{op}}} \operatorname{Hom}_R(P_{i-1}(\omega), \omega) \leq m-1$. Then it follows from Lemma 7.6 that $\operatorname{Hom}_R(P_{i-1}(\omega), \operatorname{Tor}_m^S(\omega, N)) = 0$ and $\operatorname{Hom}_R(P_{i-1}(\omega), X) = 0$. So it is derived from (7.6) that $\operatorname{Hom}_R(\omega_{i-1}, X) = 0$. Note that $P_i(\omega)$ is the projective cover of ω_{i-1}. Then by using an argument similar to that in the proof of the case for $i = 0$, we get $\operatorname{Hom}_R(P_i(\omega), \operatorname{Tor}_m^S(\omega, N)) = 0$. Thus $\operatorname{fd}_{S^{\mathrm{op}}} \operatorname{Hom}_R(P_i(\omega), \omega) \leq m-1$ by Lemma 7.6.

(2) $\Rightarrow$ (1): Let X be a quotient module of $\operatorname{Tor}_m^S(\omega, N)$. Then by (2) and Lemma 7.6, we have $\operatorname{Hom}_R(\bigoplus_{i=0}^{n-2} P_i(\omega), \operatorname{Tor}_m^S(\omega, N)) = 0$ and $\operatorname{Hom}_R(\bigoplus_{i=0}^{n-2} P_i(\omega), X) = 0$. Since ω_{i-1} is a quotient module of $P_i(\omega)$ for any $i \geq 0$, we then have $\operatorname{Hom}_R(\bigoplus_{i=0}^{n-2} \omega_{i-1}, X) = 0$. So from (7.6) we get $\operatorname{Ext}_R^1(\bigoplus_{i=1}^{n-2} \omega_{i-2}, X) = 0$. Since $\operatorname{Ext}_R^{i+1}(\omega, X) \cong \operatorname{Ext}_R^1(\omega_{i-1}, X)$ for any $i \geq 0$, we have that $\operatorname{Ext}_R^{0 \leq i \leq n-2}(\omega, X) = 0$ and $\operatorname{s.E\text{-}cograde}_\omega \operatorname{Tor}_m^S(\omega, N) \geq n-1$. $\square$

By applying Theorem 7.5 and Proposition 7.7, we get the following

Theorem 7.8. *Let S be a left artinian ring and $R = S$. If $\operatorname{pd}_S \omega \leq n$ and $\operatorname{pd}_{S^{\mathrm{op}}} \operatorname{Hom}_S(P_i(\omega), \omega) < \infty$ for any $0 \leq i \leq n-2$, then $\operatorname{pd}_{S^{\mathrm{op}}} \omega = \operatorname{pd}_S \omega \leq n$.*

Proof. Without loss of generality, assume $\operatorname{pd}_{S^{\mathrm{op}}} \operatorname{Hom}_S(P_i(\omega), \omega) \leq m$ $(< \infty)$ for any $0 \leq i \leq n-2$. By Proposition 7.7, $\operatorname{s.E\text{-}cograde}_\omega \operatorname{Tor}_{m+1}^S(\omega, N) \geq n-1$ for any $N \in \operatorname{Mod} S$. Then it follows from Theorem 7.5 that $\operatorname{pd}_{S^{\mathrm{op}}} \omega = \operatorname{pd}_S \omega \leq n$. $\square$

Note that in the case for $n = 1$, the condition "$\operatorname{pd}_{S^{\mathrm{op}}} \operatorname{Hom}_S(P_i(\omega), \omega) < \infty$ for any $0 \leq i \leq n-2$" in Theorem 7.8 is automatically satisfied. So we immediately have the following

Corollary 7.9. *Let S be a left artinian ring and $R = S$. If $\operatorname{pd}_S \omega \leq 1$, then $\operatorname{pd}_{S^{\mathrm{op}}} \omega = \operatorname{pd}_S \omega \leq 1$.*

We do not know whether the statements (1a) and (2a) in Proposition 6.10 are equivalent in general. However, by Corollary 7.9, we have the following

Corollary 7.10. *Let S be a left artinian ring and $R = S$. If ${}_S\omega$ is projective, then ω_S is projective.*

Let S be an artin algebra over a commutative artinian ring and $\mathbb{D}$ the usual Matlis duality between $\operatorname{mod} S$ and $\operatorname{mod} S^{\mathrm{op}}$. Then ${}_S\mathbb{D}(S)_S$ is a semidualizing bimodule and $\operatorname{Hom}(-, \mathbb{D}(S))$ maps minimal injective (resp. projective) resolutions of modules in $\operatorname{mod} S$ to minimal projective (resp. injective) resolutions of modules in $\operatorname{mod} S^{\mathrm{op}}$. Let

$$0 \to S_S \to I^0(S_S) \to I^1(S_S) \to \cdots \to I^i(S_S) \to \cdots$$

be a minimal injective resolution of S_S in $\operatorname{Mod} S^{\mathrm{op}}$. Note that ${}_S\mathbb{D}(S)$ and $\mathbb{D}(S)_S$ are injective cogenerators for $\operatorname{Mod} S$ and $\operatorname{Mod} S^{\mathrm{op}}$, respectively. So $\operatorname{pd}_S \mathbb{D}(S) = \operatorname{id}_{S^{\mathrm{op}}} S$ and $\operatorname{pd}_{S^{\mathrm{op}}} \mathbb{D}(S) = \operatorname{id}_S S$ by [8, Theorem 3.2.19]. Now, by putting ${}_S\omega_S = {}_S\mathbb{D}(S)_S$ in Theorem 7.8, we get the following

Corollary 7.11. *Let S be an artin algebra and $\operatorname{id}_{S^{\mathrm{op}}} S \leq n$. If $\operatorname{pd}_{S^{\mathrm{op}}} I^i(S_S) < \infty$ for any $0 \leq i \leq n-2$, then $\operatorname{id}_S S = \operatorname{id}_{S^{\mathrm{op}}} S \leq n$.*

The following corollary is well known, which is a dual version of Corollary 7.9.

Corollary 7.12. (cf. [7, Theorem I]) *Let S be an artin algebra. If* $\mathrm{id}_{S^{\mathrm{op}}} S \leq 1$*, then* $\mathrm{id}_S S = \mathrm{id}_{S^{\mathrm{op}}} S \leq 1$.

Putting $n = 2$ in Corollary 7.11, we have the following

Corollary 7.13. *Let S be an artin algebra and* $\mathrm{id}_{S^{\mathrm{op}}} S \leq 2$*. If* $\mathrm{pd}_{S^{\mathrm{op}}} I^0(S_S) < \infty$*, then* $\mathrm{id}_S S = \mathrm{id}_{S^{\mathrm{op}}} S \leq 2$.

Acknowledgments

This research was partially supported by NSFC (Grant Nos. 11571164, 11501144), a Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions and NSF of Guangxi Province of China (Grant No. 2016GXNSFAA380151). The authors thank the referee for the useful suggestions.

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