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Existence of Solutions to Fully Nonlinear Elliptic Equations with Gradient Nonlinearity

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Abstract. In this article, we study the existence and multiplicity of nontrivial solutions to the problem

$$\begin{cases} -\epsilon^2 F(x, D^2u) = f(x,u) + \psi(Du) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n > 2$. We show that the problem possesses nontrivial solutions for small value of ϵ provided f and ψ are continuous and f has a positive zero. We employ degree theory arguments and Liouville type theorem for the multiplicity of the solutions.

1. Introduction

In this paper, we are interested to study the existence and multiplicity of nontrivial solutions to the following singularly perturbed problem:

$$\begin{cases} -\epsilon^2 F(x, D^2u) = f(x,u) + \psi(Du) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n > 2$, F is convex in M and satisfies the structural conditions given next.

In recent years, there has been a growing interest on the existence, uniqueness and qualitative questions to fully nonlinear elliptic equations. The existence of viscosity solutions to fully nonlinear elliptic equations has been extensively studied in last three decades. Let us recall the celebrated papers which introduced the notion of the viscosity solutions, see [9, 10] to fully nonlinear elliptic equations.

There has been a good amount of interests on the singularly perturbed problems for Laplace equation, see the works of W. M. Ni and I. Takagi, see [20–22] and the references

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therein, where the variational structure of the problem is exploited. Recently, S. Alarcón, L. Iturriaga and A. Quaas made the first effort to study singular perturbed fully nonlinear elliptic equations, see [1]. More precisely, they considered the following singular perturbed problem

$$\begin{cases} -\epsilon^2 \mathcal{M}^+_{\lambda,\Lambda}(D^2 u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\mathcal{M}^+_{\lambda,\Lambda}$ is Pucci's extremal operator defined in (1.4), and showed the existence of two solutions and its asymptotic behaviour. For the recent developments on this area, we refer to a survey paper [26]. The semilinear/quasilinear equations in divergence form are effectively considered by L. Boccardo, F. Murat and J.-P. Puel in a series of papers [4–6]. In the context of fully nonlinear elliptic equations such problems appear in [14]. Further, using the results of [14], S. Koike and A. Swiech [17] allowed the quadratic nonlinearity in the gradient term. There are various difficulties in considering such kind of nonlinearities in gradient because of the lack of maximum principle. The authors put the restriction to prove the maximum principle and prove the existence of solution for small source term. Recently, B. Sirakov in [24] considered the operator with quadratic growth in the gradient term and studied the existence and uniqueness of the solution to Dirichlet problem. He showed the existence for all source term $f \in L^n(\Omega)$ and all boundary values $\psi \in C(\partial\Omega)$ in case of a proper operator. While in the case of nonproper operator with quadratic growth in the gradient, he proved the existence results for those $f \in L^n(\Omega)$ and $\psi \in C(\partial\Omega)$ only which are sufficiently small in the respective norms.

Since in this paper, we deal with the nonlinearity which has zeros, so we recall the earlier works in this direction. P. L. Lions [18] considered the Laplace equations with a nonnegative nonlinearity having a zero at a positive value and proved the existence of two solutions through topological degree arguments in the subcritical case. Iturriaga et al. [16] proved the existence of two positive solutions for p-Laplace operator, where the nonlinearity depends on x but only in the subcritical case. They obtained the solutions to the asymptotical problem at the origin for ϵ small and also established that both solutions converge at least 1 as $\epsilon \to 0$. Recently, using the similar ideas as above, Alarcón et al. [1] proved the existence of two positive viscosity solutions to (1.2) and also established that both solutions converge at least 1 as $\epsilon \to 0$.

So in this context, it is natural to ask whether we can establish the existence of a solution to singularly perturbed fully nonlinear problem which has gradient term? The aim of this paper is to answer this question. We remark that there are various difficulties arise in extending results from semilinear to fully nonlinear equations. For instance, fully nonlinear equations lack the variational structure, which is highly exploited to semilinear equations in the available literature. We point out that nonlinear growth in the gradient term has already been considered by many authors for semilinear and quasilinear elliptic equations but not much work is known for fully nonlinear elliptic equations.

Motivated from the above research works and suggested by the papers [1, 8], we also use truncation procedure to prove the existence of two solutions to (1.1). We establish one solution by the method of subsolution and supersolution and the second solution is obtained by using degree theory arguments. In order to apply degree theory, we get an a-priori bound by using the blow up method introduced by Gidas and Spruck, and then a Liouville type theorem is applied to get the conclusion.

Our results extend and complement the previous results in two ways. We consider the more general uniformly elliptic operator F than the Pucci's extremal operator and the equation involves the gradient nonlinearity. Let us assume that F in (1.1) satisfies (2.1) as well as the following structural conditions:

$$\begin{cases} \mathcal{M}^-_{\lambda,\Lambda}(M-N) \le F(x,M) - F(x,N) \le \mathcal{M}^+_{\lambda,\Lambda}(M-N), \\ F(x,tM) = tF(x,M) \quad \text{for } t \ge 0, \end{cases} \tag{1.3}$$

where M, N are $n \times n$ real symmetric matrices and $\mathcal{M}^\pm_{\lambda,\Lambda}$ are Pucci extremal operators. For a given $0 < \lambda \le \Lambda$, Pucci's extremal operators are defined as follows:

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i \quad \text{and} \quad \mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i. \tag{1.4}$$

The organization of this paper is as follows. Section 2 deals with the preliminaries and available results which have been used in this paper. In Section 3, we prove the auxiliary results. The main theorem is proved in Section 4. We make the following assumptions on f and ψ.

(A1) $f\colon \overline{\Omega} \times [0,+\infty) \to [0,+\infty)$ is a continuous function and $f(x,\cdot)$ is locally Lipschitz in $(0,\infty)$ for all $x \in \overline{\Omega}$ and $f(x,0) = f(x,1) = 0$ and $f(x,t) > 0$ for $t \notin \{0,1\}$.

(A2) $\liminf_{t\to 0+} f(x,t)/t = 1$ uniformly for $x \in \overline{\Omega}$.

(A3) There exists a continuous function $a\colon \overline{\Omega} \to (0,\infty)$ and $\sigma \in (1, \widetilde{n}/(\widetilde{n}-2))$ such that

$$\lim_{t\to 1} \frac{f(x,t)}{|t-1|^\sigma} = a(x).$$

(A4) There exist $\widetilde{k} > 0$ and $T > 0$ such that the map $t \to f(x,t) + \widetilde{k}t$ is increasing for $t \in [0,T]$ and $x \in \Omega$.

(A5) Let $\psi\colon \mathbb{R}^n \to \mathbb{R}_0$ is a Lipschitz continuous function with

$$\lim_{|p|\to 0} \psi(p) = 0 \quad \text{and} \quad \lim_{|p|\to\infty} \psi(p) = 0.$$

Here in (A3), $\widetilde{n}$ is defined as follows:

$$\widetilde{n} = \frac{\lambda}{\Lambda}(n-1) - 1.$$

A simple example of function verifying the above assumptions is

$$f(x,t) = (|x|\, t^p + \log(1+t))\, e^t\, |t-1|^\sigma \quad \text{with } p > 1 \text{ and } \psi(p) = |p|\, e^{-|p|}.$$

The main theorem, which we will prove in the last section is the following:

Theorem 1.1. *Assume that Ω is a smooth bounded domain in $\mathbb{R}^n$ and F satisfies* (1.3) *and* (2.1)*. Then, under* (A1)–(A5)*, there exists $\epsilon^* > 0$ such that the problem* (1.1) *has at least two positive viscosity solutions $u_{1,\epsilon}$ and $u_{2,\epsilon}$ for $0 < \epsilon < \epsilon^*$. Moreover, these solutions satisfy $\|u_{1,\epsilon}\|_{L^\infty} \to 1^-$ and $\|u_{2,\epsilon}\|_{L^\infty} \to 1^+$ as $\epsilon \to 0$.*

2. Preliminaries

We begin this section with the definitions and auxiliary results which have been used in proving the main results of this paper.

Definition 2.1. [12, 13, 19] Let Ω be an open subset of $\mathbb{R}^n$. We say that a continuous function $u \in C(\Omega)$ is a viscosity subsolution (resp. viscosity supersolution) of (1.1) in Ω, when the following condition holds: if $x_0 \in \Omega$, $\phi \in C^2(\Omega)$ and $u - \phi$ has a local maximum at x_0 (resp. local minimum at x_0) then

$$-\epsilon^2 F(x_0, D^2\phi(x_0)) \le f(x_0, u(x_0)) + \psi(D\phi(x_0)),$$
$$\big(\text{resp. } -\epsilon^2 F(x_0, D^2\phi(x_0)) \ge f(x_0, u(x_0)) + \psi(D\phi(x_0))\big).$$

Next, we state the comparison principle and strong maximum principle.

Theorem 2.2. [25, Proposition 3.1] *Let us assume that $G\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ and F, G are continuously differentiable in x and satisfy the following conditions*

$$\frac{\partial}{\partial x}(F(x,M) + G(x,u,p)) \le C(1 + |p|^2 + \|M\|),$$

$$\begin{aligned} &\mathcal{M}^-_{\lambda,\Lambda}(M-N) - \mu(|p|+|q|)\,|p-q| - \gamma_1\,|p-q| - \gamma_2\,|u-v| \\ &\le H(x,u,p,M) - H(x,v,q,N), \\ &\mathcal{M}^+_{\lambda,\Lambda}(M-N) + \mu(|p|+|q|)\,|p-q| + \gamma_1\,|p-q| + \gamma_2\,|u-v| \\ &\ge H(x,u,p,M) - H(x,v,q,N) \end{aligned} \tag{2.1}$$

and $H(x,u,p,M) = F(x,M) + G(x,u,p)$. If u is a subsolution and v is a supersolution for

$$F(x, D^2u) + G(x,u,Du) - ku = f \quad \text{in } \Omega,$$

where $f \in C(\partial\Omega)$, $k > 0$ and $u \le v$ on $\partial\Omega$, then $u \le v$ in $\overline{\Omega}$.

Theorem 2.3. [24, Corollary 3.1] *Suppose that $f \in C(\overline{\Omega})$, then the following problem*

$$\begin{cases} F(x, D^2u) + G(x, u, Du) - ku = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

has a unique viscosity solution.

Remark 2.4. The above Theorems 2.2 and 2.3 still remain true if G is nonincreasing in u.

Theorem 2.5. [25, Theorem 4] *Suppose that F, G satisfy the condition* (2.1) *with $\mu = 0$ then any viscosity solution of* (2.2) *satisfies the following estimate*

$$\|u\|_{C^{1,\alpha}(\Omega)} \le C(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}).$$

Theorem 2.6. [3, Theorem A1] *Let Ω be a regular domain and let $u \in C(\overline{\Omega})$ be a nonnegative solution to*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - L\,|Du| - \delta u \le 0 \quad \text{in } \Omega,$$

where $L, \delta \ge 0$. Then either u vanishes identically in Ω or $u(x) > 0$ for all $x \in \Omega$. Moreover, in the latter case for any $x_0 \in \partial\Omega$ such that $u(x_0) = 0$ we have

$$\liminf_{t\to 0} \frac{u(x_0 - t\nu) - u(x_0)}{t} < 0,$$

where ν is the outer normal to $\partial\Omega$.

Liouville type theorems: Let us define

$$\widetilde{n} = \frac{\lambda}{\Lambda}(n-1) + 1, \quad p^+ = \frac{\widetilde{n}}{\widetilde{n} - 2} \quad \text{and} \quad \widetilde{p}^+ = \frac{\lambda(n-2) + \Lambda}{\lambda(n-2) - \Lambda}.$$

Theorem 2.7. *Suppose that $n \ge 3$ and $1 < p \le p^+$ (or $1 < p < \infty$ if $\widetilde{n} \le 2$), then the only viscosity supersolution of*

$$\begin{cases} F(x, D^2u) + u^p = 0 & \text{in } \mathbb{R}^n, \\ u \ge 0 & \text{in } \mathbb{R}^n \end{cases}$$

is $u \equiv 0$.

For the proof of Theorem 2.7, we refer to Theorem 4.1 in [11]. The simple modification shows that Theorem 3.2 in [23] can be proved for uniformly elliptic F in the next theorem.

Theorem 2.8. *Let u be a nontrivial classical bounded solution of*

$$\begin{cases} F(D^2u) + f(u) = 0 & \text{in } \mathbb{R}^n_+, \\ u \ge 0 & \text{in } \mathbb{R}^n_+, \\ u = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

where f is a locally Lipschitz continuous function with $f(0) \geq 0$. If the problem

$$F(D^2u) + f(u) = 0$$

has a nontrivial nonnegative bounded solution in $\mathbb{R}^n_+$ such that $u = 0$ on $\partial\mathbb{R}^n_+$, then this problem has a positive solution in $\mathbb{R}^{n-1}$.

Using Theorems 2.7 and 2.8, we get the following Liouville type theorem in the half space.

Theorem 2.9. *Suppose $n \geq 3$, then the problem*

$$\begin{cases} F(D^2u) + u^p = 0 & \text{in } \mathbb{R}^n_+, \\ u \geq 0 & \text{in } \partial\mathbb{R}^n_+ \end{cases}$$

does not have nontrivial nonnegative bounded solution provided $1 < p \leq \widetilde{p}^+$ (or $1 < p < \infty$ if $\lambda(n-2) \leq \Lambda$).

The proofs of Theorems 2.9 and 2.8 follow on the same line as Theorem 3.2 and Theorem 1.5 in [23]. The following Liouville type theorems have been borrowed from [1].

Proposition 2.10. [1, Theorem 2.1] *Let u be a viscosity solution of the inequality*

$$-\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \geq f(u) \quad \text{in } \mathbb{R}^n,$$

where f is continuous nonnegative function. Then either $\inf_{\mathbb{R}^n} u = -\infty$ or $\inf_{\mathbb{R}^n} u$ is a zero of f.

The next lemma can be proved using the similar arguments as Theorem 1.2 in [1], since it is short and interesting, so we repeat it here.

Lemma 2.11. *Let $f\colon [0,\infty) \to [0,\infty)$ be a continuous function satisfying the following three assumptions:*

(i) *$f(t) = 0$ if $t = 0$ or $t = 1$, and $f(t) > 0$ if $t \neq 1$, $t > 0$.*

(ii) *There exist constants $\gamma > 0$ and $\sigma \in (1, \widetilde{n})$ such that $f(t) \geq \gamma(t-1)^\sigma$ for $t > 1$.*

(iii) *There exists a constant $\kappa > 0$ such that $\liminf_{t\to 0+} f(t)/t \geq \kappa$.*

Any bounded solution of the problem

$$\begin{cases} -F(x, D^2u) \geq f(u) & \text{in } \mathbb{R}^n, \\ u \geq 0, \end{cases} \tag{2.3}$$

where F satisfies (1.3), is either the constant function $u \equiv 0$ or else $u \equiv 1$.

Proof. Let u be a bounded solution of (2.3). Observe that u is also a viscosity supersolution of

$$\begin{cases} -\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \geq f(u) & \text{in } \mathbb{R}^n, \\ u \geq 0. \end{cases}$$

By Proposition 2.10, we have that the minimum of u must satisfy $f(\min u) = 0$. So there are two possibilities: either $\min u = 0$ or 1. Let us first consider the case $\min u \equiv 0$ and consider $\rho\colon [0, +\infty) \to \mathbb{R}$, such that $0 \leq \rho(r) \leq 1$, $\rho \in C^\infty$, ρ nonincreasing $\rho(r) = 1$ if $0 \leq r \leq 1/2$ and $\rho(r) = 0$ if $r \geq 1$. As ρ is a radial function so that the eigenvalues of $D^2\rho(|x|)$ are $\rho'(|x|)/|x|$ and $\rho''(|x|)$ of multiplicities $n-1$ and one, respectively. Also $\rho'(|x|) \leq 0$ as ρ is nonincreasing.

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2\rho(|x|)) = \begin{cases} \Lambda(n-1)\frac{\rho'(|x|)}{|x|} + \lambda\rho''(|x|) & \text{if } \rho''(|x|) \geq 0, \\ \Lambda(r)(n-1)\frac{\rho'(|x|)}{|x|} + \Lambda\rho''(|x|) & \text{if } \rho''(|x|) \leq 0. \end{cases}$$

So it is obvious that there exists $C > 0$ such that

$$-\mathcal{M}^-_{\lambda,\Lambda}(D^2\rho(|x|)) \leq C.$$

Now, we define

$$\beta(x) = m\left(\frac{R}{2}\right)\rho\left(\frac{|x|}{R}\right),$$

where $m(r) := \min_{|x|\leq r} u(x)$. Then by the scaling property of $\mathcal{M}^-$ we have

$$-\mathcal{M}^-_{\lambda,\Lambda}(D^2\beta(x)) \leq \frac{Cm(R/2)}{R^2}.$$

In addition, $\beta(x) \leq 0 \leq u(x)$ if $|x| > R$ and $\beta(x) = m(R/2) \leq u(x)$ if $|x| \leq R/2$. Thus, there exists a global minimum of $u(x) - \beta(x)$ achieved at a point x_R with $|x_R| < R$. Note that $u(x_R) - \beta(x_R) \leq 0$ and so $u(x_R) \leq \beta(x_R) \leq m(R/2)$. If we define $\phi(x) = \beta(x) - \beta(x_R) + u(x_R)$, we obtain that ϕ is a test function for u at x_R and thus

$$f(u(x_R)) \leq -F(x, D^2\phi(x_R)) = -F(x, D^2\beta(x_R)) \leq -\mathcal{M}^-_{\lambda,\Lambda}(D^2\beta(x_R)) \leq \frac{Cm(R/2)}{R^2}.$$

So, since $0 < u(x_R) \leq m(R/2)$, by (i) and (iii) there exists $M_0 > 0$, large enough so that

$$\frac{\kappa}{2}u(x_R) < f(u(x_R)) \leq \frac{Cm(R/2)}{R^2}$$

for any $R > M_0$. This implies that

$$\frac{\kappa}{2}u(x_R) \leq \frac{Cm(R/2)}{R^2} < \frac{cm(R)}{R^2},$$

which is impossible if $m(R) \neq 0$ for all $R > M_0$ because $\kappa > 0$. Thus there is some $R > 0$ sufficiently large so that $m(R) = 0$. Further, since u also satisfies

$$-\mathcal{M}^-_{\lambda,\Lambda}(D^2 u) \geq 0 \quad \text{in } B_L(0)$$

in viscosity sense for any $L \geq R$. So by the Strong Maximum Principle, we have $u \equiv 0$ in $B_L(0)$ for any $L > R$, i.e., $u \equiv 0$ in $\mathbb{R}^n$.

Finally, in the case that $\inf_{\mathbb{R}^n} u = 1$, by setting $u_0 = u - 1$, we see that u_0 is a viscosity solution of

$$\begin{cases} -\mathcal{M}^-_{\lambda,\Lambda}(D^2 u_0) \geq f(u_0 + 1) & \text{in } \mathbb{R}^n, \\ u_0 \geq 0, \end{cases}$$

which satisfies $m_{u_0}(R) := \min_{|x| \leq R} u_0(x) \to 0$ as $R \to \infty$. Arguing as in the previous case and using (ii), we obtain that

$$\gamma u_0(x_R)^\sigma \leq f(u_0(x_R) + 1) \leq \frac{C m_{u_0}(R)}{R^2},$$

which implies

$$m_{u_0}(R) R^{2/(\sigma-1)} \leq c,$$

i.e.,

$$m_{u_0}(R) R^{\widetilde{p}-2} \leq c R^{\widetilde{p}-2-2/(\sigma-1)} \quad \text{for all } R \geq 0, \tag{2.4}$$

where $\widetilde{p} = \frac{\Lambda}{\lambda}(n-1) + 1$. By Corollary 3.1 in [11], $m_{u_0}(R) R^{\widetilde{k}-2}$ is increasing in R, which is a contradiction to (2.4) because $\widetilde{p} - 2 - 2/(\sigma - 1) < 0$ for $\sigma \in (1, \widetilde{n})$. □

Note that by Assumption (A3) in the main theorem, there exist $R > 1$ and $\gamma_1 > 0$ such that $f(x,t) \geq \gamma_1 |t-1|^\sigma$ for $t \in [1, R]$. Without loss of generality, we may assume that $R \leq T$ from Assumption (A4). Then we truncate f as follows:

$$f_R(x,t) = \begin{cases} f(x,t^+) & \text{if } t \leq R, \\ \frac{f(x,R)}{R^\sigma} t^\sigma & \text{if } t \geq R, \end{cases}$$

where $t^+ = \max\{0, t\}$. Also, without any loss of generality, we may assume that

$$\liminf_{t \to 0+} \frac{f_R(x,t)}{t} \geq 1 \quad \text{uniformly for } x \in \overline{\Omega}.$$

With this definition, f_R has a power growth at infinity with exponent less than $\widetilde{n}/(\widetilde{n} - 2)$ and also satisfies the following properties:

(1) $f_R(x,t) \geq \gamma_2 |t-1|^\sigma$ for $t \geq 1$, where $\gamma_2 = \min\{\gamma_1, \inf_{x \in \Omega} f(x,R)/R^\sigma\} > 0$,

(2) the map $t \mapsto f_R(x,t) + \widetilde{k} t$ is increasing for $t \in [0, +\infty]$, where $\widetilde{k}$ is as in Assumption (A4).

3. Auxiliary results

Now, we consider the following auxiliary problem

$$(T_{\epsilon,\tau}) \qquad \begin{cases} -\epsilon^2 F(x, D^2 u) = f_R(x,u) + \epsilon^2 \tau u^+ + \psi(Du) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where τ is a nonnegative parameter.

Remark 3.1. In order to find the solution of the main problem (1.1) it suffices to show that $(T_{\epsilon,0})$ has a solution u satisfying $\|u\|_{L^\infty(\Omega)} \leq R$.

In the proof of next lemma, we have borrowed arguments of Lemma 3.1 in [1] and the fact that $\psi(p) \to 0$ as $|p| \to 0$.

Lemma 3.2. *Under Assumptions* (A1), (A2) *and* (A5), *for a given* $\widetilde{\epsilon} > 0$, *there exists a constant* $D_{\widetilde{\epsilon}}$ *such that if* u *is a viscosity solution of the problem* $(T_{\epsilon,\tau})$ *with* $0 < \epsilon < \widetilde{\epsilon}$ *and* $\tau \geq 0$, *then*

$$\|u\|_{L^\infty} \leq D_{\widetilde{\epsilon}},$$

and therefore (using Theorem 2.5*) there is a positive constant* C_ϵ *such that*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_\epsilon.$$

Proof. Suppose for sake of contradiction that there exists a sequence $\{(u_n, \epsilon_n, \tau_n)\}_{n\in N}$ with u_n being a positive viscosity solution of

$$(3.1) \qquad \begin{cases} -\epsilon_n^2 F(x_n, D^2 u_n) = f_R(x_n, u_n) + \epsilon_n^2 \tau_n u_n^+ + \psi(Du_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

such that $S_n := \max_{\overline{\Omega}} u_n(x_n) \to \infty$ as $n \to \infty$, where $\{x_n\} \subset \Omega$ is a sequence of points where the maximum is attained. We remark that since we are supposing $\tau > 0$, at this point this sequence may not be bounded away from the boundary. Now let $\delta_n = \operatorname{dist}(x_n, \partial\Omega)$ and define $w_n(y) = S_n^{-1} u_n(A_n y + x_n)$, where A_n will be fixed later. Hence w_n satisfies

$$(3.2) \qquad \begin{aligned} -F(x_n, D^2 w_n) = {} & \frac{A_n^2}{\epsilon_n^2 S_n} \{ f_R(A_n y + x_n, S_n w_n(y)) + \psi((S_n/A_n) D w_n) \} \\ & + \tau_n A_n^2 w_n(y) \quad \text{in } \Omega_n \end{aligned}$$

where $\Omega_n = A_n^{-1}(\Omega - x_n)$. We choose $A_n^2 = \epsilon_n^2 S_n^{1-\sigma} f(x_n, R)^{-1} R^\sigma$. Since $S_n \to \infty$, $0 < \epsilon_n < \widetilde{\epsilon}$ and $\tau_n \leq \mu_1^+$ (because no positive solution of $(T_{\epsilon,\tau})$ exists for $\tau > \mu_1^+$), we conclude that $A_n \to 0$ and $\tau_n A_n^2 \to 0$. Using the fact that

$$\frac{A_n^2}{\epsilon_n^2 S_n} = \frac{R^\sigma}{S_n^\sigma f(x_n, R)} \to 0,$$

and ψ is bounded near ∞ and so the right-hand side of (3.2) becomes

$$\frac{R^{\sigma} f_R(A_n y + x_n, S_n w_n(y))}{f(x_n, R) S_n^{\sigma}} + o(1),$$

and by continuity property of f and definition of f_R, it is bounded and $\|w_n\|_{L^\infty} \leq 1$ so by regularity results (Theorem 2.5), there exists a constant K such that $\|w_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq K$ for all n and for some α. So by compactness of embedding $C^{1,\alpha} \subset C^1$ for all α we have that up to subsequence $w_n \to w$ in compact subsets of $\mathbb{R}^n$ or $\mathbb{R}^n_+$ according to whether the limit of δ_n/A_n is infinity or not, that is, Ω_n tends to $\mathbb{R}^n$ or to a half space. Finally, taking limit in (3.1) and by stability result, we have that w satisfies in the viscosity sense either

$$F(x_0, D^2 w) + w^{\sigma} = 0 \quad \text{in } \mathbb{R}^n, \tag{3.3}$$

or

$$\begin{cases} F(x_0, D^2 w) + w^{\sigma} = 0 & \text{in } \mathbb{R}^n_+, \\ w = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{3.4}$$

But Equation (3.3) contradicts the Liouville type Theorem 2.7 while (3.4) contradicts the Liouville type Theorem 2.9 in the half space because

$$1 < \sigma < \frac{\widetilde{n}}{\widetilde{n} - 2} < \frac{\widetilde{n} - \Lambda/\lambda}{\widetilde{n} - 2 - \Lambda/\lambda}.$$

These contradictions prove that $\|u\|_\infty \leq C$ for any solution of the problem $(T_{\tau,\epsilon})$ with $\epsilon < \widetilde{\epsilon}$ and $\tau \geq 0$. Finally, using the $C^{1,\alpha}$ estimate we obtain a constant C_ϵ such that

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_\epsilon$$

for some $\alpha \in (0, 1)$. □

Now, we look for a family of supersolutions of $(T_{\epsilon,\tau})$. For this purpose we consider a function χ the solution of

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2\chi) = 1 & \text{in } \Omega, \\ \chi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

and $A = \|\chi\|_{C^1(\Omega)}$. For the existence of such function see, for example, Theorem 17.18 in [15].

Lemma 3.3. *Under Assumptions* (A3) *and* (A5)*, for any $\epsilon > 0$ there exist $\tau^*_\epsilon, \delta_\epsilon > 0$ such that $v_\xi = 1 + \xi + \frac{\delta_\epsilon}{2A}\chi$ is a supersolution of $(T_{\epsilon,\tau})$ for any $\xi \in [-\delta_\epsilon, \delta_\epsilon/3]$ and $\tau \in [0, \tau^*_\epsilon)$. Moreover, we may choose δ_ϵ as a nonincreasing function of ϵ^{-1}.*

Proof. For fixed $\epsilon > 0$, by Assumption (A3), we have

$$\lim_{t \to 1} \frac{f_R(x,t)}{\epsilon^2 |t-1|} = 0,$$

and so for each m, there exists $\delta_{1,m}$ such that

$$\epsilon^{-2} f_R(x,t) < \frac{|t-1|}{8Ae^m} \quad \text{for } |t-1| \leq \delta_{1,m},$$

that is,

$$\epsilon^{-2} f_R(x,t) < \frac{\delta_{1,m}}{8Ae^m} \quad \text{for } |t-1| \leq \delta_{1,m}.$$

Now using (A5), for $\delta_{1,m}/(8Ae^m)$ as above, we find some constant $\delta_{2,m}$ such that

$$\epsilon^{-2} \psi(p) < \frac{\delta_{1,m}}{8Ae^m} \quad \text{for } |p| < \delta_{2,m}.$$

Note that $\delta_{1,m}$ nondecreasing while $\delta_{1,m}/e^m$ decreases in m. Since $\delta_{1,m}/e^m$ decreases so $\delta_{2,m}$ also nondecreasing in m. Further, it can also be seen that both $\delta_{1,m}$ and $\delta_{2,m}$ nonincreasing in ϵ^{-1}.

Let us define

$$\delta_m = \min \{\delta_{1,m}, \delta_{2,m}\},$$

so δ_m is nondecreasing in m and nonincreasing in ϵ^{-1}. We also have

$$\epsilon^{-2} f_R(x,t) < \frac{\delta_{1,m}}{8Ae^m} \quad \text{for } |t-1| < \delta_m,$$
$$\epsilon^{-2} \psi(p) < \frac{\delta_{1,m}}{8Ae^m}, \quad |p| < \delta_m.$$

This implies that

$$\epsilon^{-2} f_R(x,t) + \epsilon^{-2} \psi(p) < \frac{\delta_{1,m}}{4Ae^m} \quad \text{for } |t-1| < \delta_m,\ |p| < \delta_m, \tag{3.6}$$

holds for each m. Choose m sufficiently large, say, m_0 such that

$$\delta_{m_0} > \frac{\delta_{1,m_0}}{e^{m_0}},$$

which is always possible because δ_m is nondecreasing and positive for each m while $\delta_{1,m}/e^m$ decreases exponentially as m increases. Let us denote

$$\delta = \delta_{m_0} \quad \text{and} \quad \delta_1 = \frac{\delta_{1,m_0}}{e^{m_0}}$$

and note that δ is nonincreasing in ϵ^{-1}. With this notation, (3.6) takes the form

$$\epsilon^{-2} f_R(x,t) + \epsilon^{-2} \psi(p) < \frac{\delta_1}{4A} \quad \text{for } |t-1| < \delta,\ |p| < \delta, \tag{3.7}$$

and $\delta_1 < \delta$. Let us choose $\tau_\epsilon^* > 0$ such that

$$\tau u < \frac{\delta_1}{4A} \quad \text{for } u \in [1-\delta, 1+\delta]. \tag{3.8}$$

Let us define $v_\xi = 1 + \xi + \chi\frac{\delta}{2A}$ and observe that

$$v_\xi - 1 = \xi + \chi\frac{\delta}{2A} \leq \xi + \frac{\delta}{2A} < \delta \text{ for } \xi \in [-\delta, \delta/3], \text{ and}$$
$$\text{by noting that } A = \|\chi\|_{C^1(\Omega)} \text{ we get, } |Dv_\xi| = \left|\frac{\delta}{2A}D\chi\right| < \delta.$$

Let us take $t = v_\xi$ and $p = Dv_\xi$ in Equation (3.7) and adding (3.7) and (3.8), we get

$$\epsilon^{-2} f_R(x, v_\xi) + \epsilon^{-2}\psi(Dv_\xi) + \tau v_\xi < \frac{\delta}{2A}.$$

Using (3.5), we also have

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2 v_\xi) = \frac{\delta}{2A}.$$

Thus

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2 v_\xi) = \frac{\delta}{2A} > \epsilon^{-2} f_R(x, v_\xi) + \epsilon^{-2}\psi(Dv_\xi) + \tau v_\xi.$$

By (1.3), we know that $-F(x, M) \geq -\mathcal{M}^+_{\lambda,\Lambda}(M)$ for each symmetric matrix M and $x \in \Omega$. Thus v_ξ defines a family of supersolution of $(T_{\epsilon,\tau})$ for each $\tau \in (0, \tau_\epsilon^*]$ and $\xi \in [-\delta, \delta/3]$, where $\delta = \delta_\epsilon$. □

Next, we study existence of the first solution for $(T_{\tau,\epsilon})$ via sub- and supersolutions method. For this, we need Assumption (A4) and the following theorem proved in [3].

Theorem 3.4. [3, Theorem 2.2] *There exist a function $\phi_1^+ \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$, and $\mu_1^+ > 0$ satisfying*

$$\begin{cases} F(x, D^2\phi_1^+) = -\mu_1^+\phi_1^+ & \text{in } \Omega, \\ \phi_1^+ = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

Moreover,

(1) $\phi_1^+ > 0$ *in* Ω *and all positive solutions to* (3.9) *are of the form* $(\mu_1^+, \alpha\phi_1^+)$, *with* $\alpha > 0$,

(2) $\mu_1^+ = \sup\{\mu \mid \exists\phi > 0 \text{ in } \Omega \text{ satisfying } F(x, D^2\phi) + \mu\phi \leq 0\}$.

Next onwards, we will denote this eigenvalue and eigenfunction of $-F$ with Dirichlet boundary condition in Ω by μ_1^+ and ϕ_1^+, respectively. Without any loss of generality, we may assume that $\|\phi_1^+\|_{L^\infty(\Omega)} = 1$.

Proposition 3.5. *Let* (A1)–(A5) *hold. Then the problem* $(T_{\epsilon,\tau})$ *has a positive solution* $u_{1,\epsilon,\tau} < 1$ *for* $0 < \epsilon < \bar{\epsilon} < (1/\mu_1^+)^{1/2}$ *and* $0 \le \tau < \tau_\epsilon^*$*. Moreover, the following property holds: given* $0 < \bar{\epsilon} < (1/\mu_1^+)^{1/2}$ *there exists* $\rho > 0$ *such that* $\rho\phi_1^+ \le u_{1,\epsilon,\tau} < 1$ *for any* $0 < \epsilon < \bar{\epsilon}$ *and* $\tau \in [0, \tau_\epsilon^*)$*.*

Proof. Using (A2), we may find $\rho > 0$ such that $f_R(x,t) > \epsilon^2\mu_1^+ t$ for any $t \in (0,\rho)$ and $0 < \epsilon < \bar{\epsilon} < (1/\mu_1^+)^{1/2}$; then $\rho\phi_1^+$ is a subsolution to the problem $(T_{\epsilon,\tau})$ for any $\tau \ge 0$ and $0 < \epsilon < \bar{\epsilon}$. From Lemma 3.3, for $\tau \in [0, \tau_\epsilon^*)$, we have the supersolution $v_{-\delta} < 1$. Since δ_ϵ is nonincreasing in ϵ^{-1}, we may choose ρ such such $\rho\phi_1^+ < v_{-\delta_\epsilon/2}$ for any $0 < \epsilon < \bar{\epsilon}$. Let us denote by X the Banach space of $C^{1,\alpha}$ functions on $\overline{\Omega}$ which are 0 on $\partial\Omega$, endowed with the norm

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} = \sup_{x\in\overline{\Omega}} |u(x)| + \sup_{x\in\overline{\Omega}} |Du(x)| + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|Du(x) - Du(y)|}{|x-y|^\alpha}.$$

Also, we will write $u \ll v$ to say that $u < v$ in Ω and $\frac{\partial u}{\partial \nu} > \frac{\partial v}{\partial \nu}$ on $\partial\Omega$, where ν denotes the unitary outward normal vector to $\partial\Omega$. Let us define a map $U_\tau \colon X \to X$ defined as follows: $U_\tau(v) = u$ where u is the unique solution to the Dirichlet problem

$$\begin{cases} -\epsilon^2 F(x, D^2u) - \psi(Du) + \gamma u = f_R(x,v) + (\gamma + \epsilon^2\tau)v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

For the existence, see Theorem 2.3. This is compact because if v_n is a uniformly bounded sequence so $\sup_n \left(\left\| f_R(x,v_n) + (\gamma + \epsilon^2\tau)v_n \right\|_{L^\infty(\Omega)} \right) < \infty$. By choosing appropriate value of k in Lemma 3.1 [24], we can find a common subsolution and supersolution for all the value of n. Hence the sequence of solutions will lie between the common sub- and supersolution for all n. Hence it is uniformly bounded. Now we can apply Theorem 2.5 to get a uniform bound for the sequence of solutions in $C^{1,\alpha}$. This fact and compact embedding yields the required result. Let us define $D := \left\{ u \in X : \rho\phi_1^+ \le u \le v_{-\delta_\epsilon/2} \right\}$ and by using the comparison principle, we get $U_\tau \colon D \to D$ is increasing map. Thus, using the monotone iteration method, see Theorem 2.2.2 in [2], we obtain a solution $0 \le u_{1,\epsilon,\tau} \le v_{-\delta_\epsilon/2} < 1$. Thus we obtained a solution which satisfies the claimed properties. □

Proposition 3.6. *Assume that* (A1)–(A5) *hold. If* $0 < \epsilon < (1/\mu_1^+)^{1/2}$ *and* $\tau_0 \in (0, \tau_\epsilon^*)$ *then* (T_{ϵ,τ_0}) *has a second positive solution* u_{2,ϵ,τ_0}*. Moreover* $\|u_{2,\epsilon,\tau_0}\|_\infty > 1$*.*

Proof. Let us fix $0 < \epsilon < (1/\mu_1^+)^{1/2}$ and consider the bounded open set

$$\mathcal{O} = \left\{ u \in X \;\middle|\; \|u\|_X < C_\epsilon + B_\epsilon + 1,\, u > \rho\phi_1^+ \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} < \frac{\partial \rho\phi_1^+}{\partial \nu} \text{ on } \partial\Omega \right\},$$

where $C_\epsilon, B_\epsilon > 0$ will be chosen below and ρ is as in the proof of Proposition 3.5, so $\rho\phi_1^+ < 1$ and it is a strict subsolution for all problems $(T_{\epsilon,\tau})$ with $\tau \ge 0$ (in particular by using (A2) $\epsilon^2\mu_1^+\rho\phi_1^+ < f_R(x,\rho\phi_1^+))$. We want to use degree theory for the operator $(I-U_\tau)$. We need that $0 \neq (I-U_\tau)(\partial\mathcal{O})$, (i.e., no solution of $(T_{\epsilon,\tau})$ lies on $\partial\mathcal{O}$) so that the $\deg(I-U_\tau,\mathcal{O},0)$ will be well defined and by homotopy invariance degree is independent on τ. In order to show this, we will use a prior estimate obtained in Lemma 3.2. Let us take C_ϵ such that

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le C_\epsilon$$

for all possible solutions of $(T_{\epsilon,\tau})$, which is possible by Theorem 2.5. We claim that any solution u of $(T_{\epsilon,\tau})$ such that $u \ge \rho\phi_1^+$ in Ω satisfies $u > \rho\phi_1^+$ in Ω and $\frac{\partial u}{\partial\nu} < \frac{\partial\rho\phi_1^+}{\partial\nu}$ on $\partial\Omega$ and so it does not lie on $\partial\mathcal{O}$. By computation and noting that $\psi \ge 0$ and in view of Theorem 3.4, we get

$$-\epsilon^2F(x,D^2(\rho\phi_1^+)) + k\rho\phi_1^+ \le \epsilon^2\mu_1^+\rho\phi_1^+ + (k+\epsilon^2\tau)\rho\phi_1^+ + \psi(\rho D\phi_1^+) \quad \text{in } \Omega.$$

Further, by using $\epsilon^2\mu_1^+\rho\phi_1^+ < f_R(x,\rho\phi_1^+)$, we get

$$-\epsilon^2F(x,D^2(\rho\phi_1^+)) + k\rho\phi_1^+ \le f_R(x,\rho\phi_1^+) + (k+\epsilon^2\tau)\rho\phi_1^+ + \psi(\rho D\phi_1^+) \quad \text{in } \Omega.$$

As $u \ge \rho\phi_1^+$ and $v \to f_R(x,v) + (\gamma+\epsilon^2\tau)v$ is increasing we have

$$f_R(x,u) + (\gamma+\epsilon^2\tau)u \ge f_R(x,\rho\phi_1^+) + (\gamma+\epsilon^2\tau)\rho\phi_1^+.$$

This implies that

$$-\epsilon^2F(x,D^2u) - \psi(Du) + \gamma u \ge -\epsilon^2F(x,D^2\rho\phi_1^+) - \psi(D\rho\phi_1^+) + \gamma\rho\phi_1^+$$

and then by using Höpf type Lemma 2.6, we get required result for the proof of claim. Using the fact $f_R(x,u) + \psi(Du) \ge 0$ and definition of μ_1^+ in Theorem 3.4 it is clear that $(T_{\epsilon,\tau})$ has no nonnegative solution for $\tau > \mu_1^+$, so by homotopy invariance we have

$$\deg(I-U_\tau,\mathcal{O},0) = 0 \quad \text{for } \tau \ge 0.$$

Now we fix $\tau = \tau_0$ and consider a supersolution $\phi := v_{\xi=0} > 1$ from Lemma 3.3 and we assume that no solution of $(T_{\epsilon,\tau})$ touches it, for if u is a solution such that $u(x_0) = \phi(x_0)$ for some $x_0 \in \Omega$, then from claim $u \equiv \phi$ and in this case u is required solution so we have done.

Now otherwise using the $C^{1,\alpha}$-estimate we obtain that we may choose the constant B_ϵ such that

$$\|U_\tau v\|_X \le B_\epsilon, \quad \forall\, v \in X : 0 \le v \le \phi; \tag{3.10}$$

and consider the open and convex subset of $\mathcal{O}$

$$\mathcal{O}' = \{u \in \mathcal{O} \mid u < \phi \text{ in } \Omega\}$$

and we claim that $\deg(I - U_{\tau_0}, \mathcal{O}', 0) = 1$. Observe that U_{τ_0} maps $\overline{\mathcal{O}'}$ into $\overline{\mathcal{O}'}$. Indeed, if $v \in \overline{\mathcal{O}'}$ then $\|U_{\tau_0} v\|_X \le B_\epsilon$ by (3.10), this implies by definition that $U_{\tau_0} v \in \mathcal{O}$. If we consider $u = U_{\tau_0} v$ we have

$$\begin{aligned} -\epsilon^2 F(x, D^2\phi) + k\phi &\ge f_R(x,\phi) + (k+\epsilon^2\tau)\phi + \psi(D\phi), \\ -\epsilon^2 F(x, D^2 u) + ku &= f_R(x,v) + (k+\epsilon^2\tau)v + \psi(Dv), \\ -\epsilon^2 F(x, D^2(\rho\phi_1^+)) + ku &\le f_R(x,\rho\phi_1^+) + (k+\epsilon^2\tau)\rho\phi_1^+ + \psi(\rho D\phi_1^+), \end{aligned}$$

then, since $\rho\phi_1^+ \le v \le \phi$, the comparison principle and the increasing property of the right-hand sides of the equations and inequations implies that $\rho\phi_1^+ \le U_{\tau_0} v \le \phi$. Now let $u_0 \in \mathcal{O}'$ and consider the constant mapping

$$K\colon \overline{\mathcal{O}'} \to \overline{\mathcal{O}'}$$

defined by $K(u) = u_0$. Let us consider a homotopy

$$H(\mu, v) = \mu U_{\tau_0}(v) + (1-\mu)Kv, \quad \mu \in [0,1]$$

between $I - U_{\tau_0}$ and $I - K$ in $\overline{\mathcal{O}'}$. Now we claim that $0 \notin (I - H(\mu,\cdot))(\partial\mathcal{O}')$ for all $\mu \in [0,1]$. For, if $v \in \partial\mathcal{O}'$ such that

$$(I - H(\mu, v)) = 0$$

and

$$\mu U_{\tau_0}(v) - (1-\mu)u_0 = v.$$

For $\mu = 1$, then $\phi = v = U_{\tau_0}$, which is not possible because no solution touches ϕ. Now if $\mu \ne 1$ then $\mu U_{\tau_0}(v) - (1-\mu)u_0 \in \mathcal{O}'$ from the convexity of $\mathcal{O}'$ as $U_{\tau_0}(v)$, $u_0 \in \mathcal{O}'$, which is again not possible because $v \in \partial\mathcal{O}'$. Using the homotopy invariance we get

$$\deg(I - U_{\tau_0}) = \deg(I - K) = 1.$$

As $0 \notin (I - K_{\tau_0})(\partial\mathcal{O})$ and $0 \notin (I - U_{\tau_0})(\partial\mathcal{O}')$ so by excision property we have

$$\deg(I - U_{\tau_0}, \mathcal{O}, 0) = \deg(I - U_{\tau_0}, \mathcal{O} - \overline{\mathcal{O}'}, 0) + \deg(I - U_{\tau_0}, \mathcal{O}', 0).$$

It follows that $\deg(I - U_{\tau_0}, \mathcal{O} - \overline{\mathcal{O}'}, 0) = -1$ so (T_{ϵ,τ_0}) has a solution $u_2 \in \mathcal{O} - \overline{\mathcal{O}'}$. In particular, $u_2(x_0) > \phi(x_0) > 1$ at some point $x_0 \in \Omega$, otherwise it would be on $\partial\mathcal{O}'$. □

Lemma 3.7. *Assume that assumptions as in Propositions* 3.5, 3.6 *hold. Then, for given* $0 < \epsilon < (1/\mu_1^+)^{1/2}$, *there exists a solution* $u_{2,\epsilon,0}$ *for the problem* $(T_{\epsilon,0})$ *with* $\|u_{2,\epsilon,0}\|_\infty \ge 1$.

Proof. Given $0 < \epsilon < (1/\mu_1^+)^{1/2}$, consider a sequence $\tau_n \to 0$ and corresponding solutions $u_n = u_{2,\epsilon,\tau_n}$ obtained from Proposition 3.6. We know that $\|u_n\|_\infty > 1$ and by Lemma 3.2, we have a uniform bound for $\|u_n\|_{C^{1,\alpha}}$ for some $\alpha \in (0,1)$. So by compact embedding $C^{1,\alpha}(\Omega) \subset C^1(\Omega)$, we have a subsequence $u_n \to u$ in C^1, so by stability property of viscosity solution, see Theorem 2.9 in [7], u is a nonnegative viscosity solution of $T_{\epsilon,0}$. As $\|u_n\|_\infty > 1$, we obtain $\|u\|_\infty \geq 1$. Thus u is a nontrivial nonnegative viscosity solution and so by the strong maximum principle it is a positive solution. □

In the next lemma, we show the maximum of solutions are bounded above by R, for $\epsilon \neq 0$ small enough.

Lemma 3.8. *The solution $u_{2,\epsilon,0}$ from Lemma 3.7 satisfies $\|u_{2,\epsilon,0}\|_\infty \to 1$ as $\epsilon \to 0$. In particular, there exists ϵ^* such that if $0 < \epsilon < \epsilon^*$ then $\|u_{2,\epsilon,0}\|_\infty \leq R$.*

Proof. We prove this lemma by method of contradiction. Let $\eta > 1$ be given and suppose there exists a sequence $\epsilon_n \to 0^+$ such that corresponding solutions $u_n = u_{2,\epsilon_n,0}$ of $(T_{\epsilon,0})$ satisfy $\|u_n\|_\infty > \eta$. Let $x_n \in \Omega$ be a sequence such that $u_n(x_n) = \|u_n\|_\infty > \eta$ and let $d_n = \text{dist}(x_n, \partial\Omega)$. Let $w_n(x) = u_n(x_n + \epsilon_n x)$ and so it satisfies

$$-F(x, D^2 w_n(x)) = f_R(x_n + \epsilon_n x, w_n) + \psi((1/\epsilon_n) D w_n) \quad \text{in } B(0, d_n \epsilon_n^{-1}),$$

and w_n is bounded as in Lemma 3.2. Now by applying the $C^{1,\alpha}$ estimate we obtain a constant C such that

$$\|u_n\|_{C^{1,\alpha}} \leq C \quad \text{for all } n.$$

Again, by compact embedding of $C^{1,\alpha} \subset C^1$, we find up to a subsequence $w_n \to w$ in C^1 in compact subsets of Ω and $x_n \to x_0$ in $\overline{\Omega}$, where now w is a C^1 function defined in $\mathbb{R}^n$ or in the half space $\mathbb{R}^n_+$. Thus, in view of (A5), w is a C-viscosity solution of the problem

$$\begin{cases} -F(x_0, D^2 w) = f_R(x_0, w), \\ w \geq 0, \end{cases} \tag{3.11}$$

in $\mathbb{R}^n$ or in the half space. If such w solves the Problem (3.11) in $\mathbb{R}^n$, then according to Lemma 2.11, we conclude that either $w \equiv 0$ or $w \equiv 1$, which is contradiction to $w_n(0) = u_n(x_n) \geq \eta > 1$. On the other hand, if such w solves the problem above in the half space. Then it becomes a nonnegative bounded solution of (3.11) on $\mathbb{R}^n_+$. So by Theorem 2.8 (for related results, see Theorem 3.2 in [23]), it is a positive solution of (3.11) in $\mathbb{R}^{n-1}$. Then applying Lemma 2.11 in $\mathbb{R}^{n-1}$, we get either $w \equiv 0$ or $w \equiv 1$, which is again a contradiction and so the lemma is proved. □

4. Proof of Theorem 1.1

The first solution is $u_{1,\epsilon,0}$, which is obtained in Proposition 3.5 for Problem $(T_{\epsilon,0})$. From the proof of Proposition 3.5, it is clear that $\|u_{1,\epsilon,0}\|_\infty < 1$. At the same time by Assumptions (A1) and (A2), if t_ϵ is the largest real number such that

$$f(x,t) > \epsilon^2 \mu_1^+ t \quad \text{for } t \in (0,t_\epsilon) \text{ uniformly for } x \in \overline{\Omega},$$

then

$$t_\epsilon \to 1^- \quad \text{as } \epsilon \to 0^+.$$

Since no positive solution of $T_{\epsilon,0}$ laying below t_ϵ, for if, u is such a solution then

$$F(x, D^2u) + \mu_1^+ u < 0,$$

which is not possible by definition of μ_1^+ in Theorem 3.4. Thus the solution obtained in Proposition 3.5 satisfies $t_\epsilon \leq u_{1,\epsilon,0} < 1$. Thus $u_{1,\epsilon} = u_{1,\epsilon,0}$ is a solution of

$$\begin{cases} -\epsilon^2 F(x, D^2u) = f(x,u) + \psi(Du) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\|u_{1,\epsilon}\|_\infty \to 1^-$ as $\epsilon \to 0^+$. On the other hand, the second solution corresponds to $u_{2,\epsilon,0}$ is a the solution of $T_{\epsilon,0}$, which is given by Lemma 3.7. Lemma 3.7 also shows that $\|u_{2,\epsilon,0}\|_\infty \geq 1$. Besides of this, by Lemma 3.8, $\|u_{2,\epsilon,0}\|_\infty \leq R$ for $\epsilon > 0$ small, where R is the parameter of truncation in the definition of f_R and $\|u_{2,\epsilon,0}\| \to 1^+$ as $\epsilon \to 0^+$. This completes the proof.

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