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Limit Theorems for Multiplicative Cascades in a Random Environment

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Abstract. Let $\zeta = (\zeta_0, \zeta_1, \ldots)$ be a sequence of independent and identically distributed random variables. For $r \geq 2$, let μ_r be Mandelbrot's (limit) measure of multiplicative cascades defined with positive weights indexed by nodes of a regular r-ary tree, and let $Z^{(r)}$ be the mass of μ_r. We study asymptotic properties of $Z^{(r)}$ and the sequence of random measures $(\mu_r)_r$ as $r \to \infty$. We obtain some laws of large numbers and a central limit theorem. The results extend ones established by Liu and Rouault (2000) and by Liu, Rio and Rouault (2003).

1. Introduction and main results

As usual, we write $\mathbb{N}^* = \{1, 2, \ldots\}$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R} = (-\infty, \infty)$ and

$$\mathbb{U} = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n$$

for the union of all finite sequences, where $(\mathbb{N}^*)^0 = \{\emptyset\}$ contains the null sequence $\emptyset$. We describe the model of *Mandelbrot's multiplicative cascades in a random environment* as follows. Let $\zeta = (\zeta_0, \zeta_1, \ldots) = (\zeta_n)_{n \geq 0}$ be a sequence of independent and identically distributed (i.i.d.) random variables taking values in some space Θ, so that each realization of ζ_n corresponds to a probability distribution $F_n(\zeta) = F(\zeta_n)$ on $\mathbb{R}_+$. Suppose that when the environment ζ is given, $\{W_u, u \in \mathbb{U}\}$ is a family of totally independent random variables with values in $\mathbb{R}_+$; all the random variables are defined on some probability space $(\Gamma, \mathbb{P}_\zeta)$; for $u \in \mathbb{U}$, each W_{ui} $(1 \leq i \leq r)$ has distribution $F_n(\zeta) = F(\zeta_n)$ if $|u| = n$, where $|u|$ denotes the length of u. For simplicity, we write W_i for $W_{\emptyset i}$, $1 \leq i \leq r$. The total probability space can be formulated as the product space $(\Gamma \times \Theta, \mathbb{P})$, where $\mathbb{P} = \mathbb{P}_\zeta \otimes \tau$ in the sense that for all measurable and positive functions g, we have

$$\int g \, \mathrm{d}\mathbb{P} = \iint g(\zeta, y) \, \mathrm{d}\mathbb{P}_\zeta(y) \, \mathrm{d}\tau(\zeta),$$

where τ is the law of the environment ζ. The expectation with respect to $\mathbb{P}_\zeta$ (resp. $\mathbb{P}$) will be denoted by $\mathbb{E}_\zeta$ (resp. $\mathbb{E}$).

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Suppose that $\mathbb{E}_\zeta W_1 = 1$ almost surely (a.s.) and $\mathbb{P}(W_1 = 1) < 1$.

Let $\mathcal{F}_0$ be the trivial σ-algebra, and for $n > 1$, let $\mathcal{F}_{n-1}$ be the σ-algebra generated by $\{W_{u_1}, \dots, W_{u_1 \cdots u_{n-1}} : 1 \le u_1, \dots, u_{n-1} \le r\}$. For $r = 2, 3, \dots$, let $Z^{(r)}$ be the Mandelbrot's variable in the random environment ζ associated with W_u ($u \in U/\emptyset$) and parameter r:

$$Z^{(r)} := \lim_{n\to\infty} Y_n^{(r)},$$

where

$$Y_n^{(r)} = \sum_{1\le u_1,\dots,u_n\le r} \frac{W_{u_1} \cdots W_{u_1\cdots u_n}}{r^n}.$$

Let $\mathbb{P}_{\theta\zeta}$ be the probability for the shifted environment $\theta\zeta$. It is easily seen that $Z = Z^{(r)}$ satisfies the following distributional equation:

$$Z^{(r)} = \frac{1}{r}\sum_{i=1}^{r} W_i Z_i^{(r)}, \tag{E}$$

where $Z_i^{(r)}$ are non-negative random variables, which can be chosen independent of each other and independent of $\{W_i, 1 \le i \le r\}$ under $\mathbb{P}_\zeta$. Z is a non-negative random variable independent of $Z_i^{(r)}$ and independent of $\{W_i, 1 \le i \le r\}$ under $\mathbb{P}_\zeta$, $\mathbb{P}_\zeta\{Z_i^{(r)} \in \cdot\} = \mathbb{P}_{\theta\zeta}\{Z^{(r)} \in \cdot\}$. In terms of Laplace transforms $\phi_\zeta^{(r)}(t) = \mathbb{E}_\zeta \exp\{tZ^{(r)}\}$, the equation reads

$$\phi_\zeta^{(r)}(t) = \left[\mathbb{E}_\zeta \phi_{\theta\zeta}^{(r)}(tW_1/r)\right]^r \quad \text{a.s. } t \le 0.$$

In the deterministic environment case, the model was first introduced by Mandelbrot (1974, [19]) and is referred to as "microcanonique". For one choice of W_1, $Y_n^{(r)}$ represents a stochastic model for turbulence of Yaglom (1974, [20]), and if $0 < \mathbb{P}(W_1 = 1) = 1 - \mathbb{P}(W_1 = 0)$, $r^n Y_n^{(r)}$ is the n-th generation size of a simple birth-death process. For fixed r, the properties of $Z^{(r)}$ and related subjects have been studied by many authors; see, for example, Kahane and Peyrière (1976, [10]), Durrett and Liggett (1983, [7]), Guivarc'h (1990, [8]), Holley and Waymire (1992, [9]). See also Collet and Koukiou (1992, [6]), Liu (1997, [13]; 1998, [14]; 2000, [15]), Menshikov et al. (2005, [21]), Barral et al. (2010, [2,3]) for more general results and for related topics.

Let λ be the Lebesgue measure on $[0, 1]$. Fix $r \ge 2$. For every $n \ge 1$, let μ_r^n be the random measure on $[0, 1]$, having on each r-adic interval $A^r_{u_1\cdots u_n} = [\sum_{k=1}^n (u_k - 1)r^{-k}, \sum_{k=1}^n (u_k - 1)r^{-k} + r^{-n}]$ the density $W_{u_1} \cdots W_{u_1\cdots u_n}$ with respect to the Lebesgue measure. In other words,

$$\mu_r^n(f) = \int f \,\mathrm{d}\mu_r^n = \sum_{1\le u_1,\dots,u_n\le r} W_{u_1} \cdots W_{u_1\cdots u_n} \int_{A^r_{u_1\cdots u_n}} f \,\mathrm{d}\lambda \tag{1.1}$$

for each $f \in \mathscr{L}^1([0, 1], \lambda)$. The mass of μ_r^n is $Y_n^{(r)} = \mu_r^n(1)$.

For fixed $r \geq 2$, almost surely the sequence of random measures $\{\mu_r^n, n \geq 1\}$ is weakly convergent, as $n \to \infty$. Let μ_r^∞ be the Borel extension of this weak limit. The random Borel measure μ_r^∞ on $[0,1]$ is called the *Mandelbrot measure for multiplicative cascades in a random environment*. The mass of μ_r^∞ is $Z^{(r)} = \mu_r^\infty(1)$.

In the deterministic environment case, this measure and its extensions have been studied by many authors, see, for example, Kahane and Peyrière (1976, [10]), Waymire and Williams (1996, [22]), Barral (1999, [1]), Liu (2000, [15]), Liu, Rio and Rouault (2003, [17]).

Fix $1 \leq k \leq r$. If the weights $W_{u_1} \cdots W_{u_1 \cdots u_n}$ in (1.1) are replaced by $W_{ku_1} \cdots W_{ku_1 \cdots u_n}$, the corresponding measures will be denoted by $\mu_r^n \circ T_k$ $(1 \leq n < \infty)$, i.e.,

$$(\mu_r^n \circ T_k)(f) = \int f \,\mathrm{d}(\mu_r^n \circ T_k) = \sum_{1 \leq u_1, \ldots, u_n \leq r} W_{ku_1} \cdots W_{ku_1 \cdots u_n} \int_{A^r_{u_1 \cdots u_n}} f \,\mathrm{d}\lambda,$$

and its weak limit (as $n \to \infty$) by $\mu_r^\infty \circ T_k$. Notice that the measures μ_r^n and μ_r^∞ depend on the marked r-ary tree with marks $W_{u_1 \cdots u_n}$ associated with each node $u_1 \cdots u_n$, while $\mu_r^n \circ T_k$ and $\mu_r^\infty \circ T_k$ depend on its shift at k. T_k may be considered the shift operator to the node k in the space of marked trees. For fixed r and f, the random variables $(\mu_r^\infty \circ T_k)(f)$, $1 \leq k \leq r$, are independent of each other and independent of $\{W_i, 1 \leq i \leq r\}$ under $\mathbb{P}_\zeta$, and $\mathbb{P}_\zeta \{(\mu_r^\infty \circ T_k)(f) \in \cdot\} = \mathbb{P}_{\theta\zeta} \{\mu_r^\infty(f) \in \cdot\}$. For $k = 1, 2, \ldots, r$, let τ_k^r be the operator acting on functions from $[0,1]$ to $\mathbb{R}$, defined by

$$\tau_k^r f(x) = f\left(\frac{k-1+x}{r}\right), \quad x \in [0,1].$$

Since $t \in A^r_{u_1 \cdots u_n}$ if and only if $r(t - \frac{u_1 - 1}{r}) \in A^r_{u_2 \cdots u_n}$, we have, for f in $\mathscr{L}^1([0,1], \lambda)$,

$$\mu_r^n(f) = \sum_{k=1}^r W_k \sum_{1 \leq u_2, \ldots, u_n \leq r} W_{ku_2} \cdots W_{ku_2 \cdots u_n} \int_{A^r_{u_2 \cdots u_n}} \frac{1}{r} f\left(\frac{s+k-1}{r}\right) \mathrm{d}s,$$

so that for each $1 \leq n < \infty$,

$$\mu_r^n(f) = \frac{1}{r} \sum_{k=1}^r W_k (\mu_r^{n-1} \circ T_k)(\tau_k^r f), \tag{1.2}$$

with the convention $\mu_r^0 \circ T_k = \lambda$. Taking the limit as $n \to \infty$ in (1.2), we see that a.s. for every $f \in \mathscr{C}([0,1])$,

$$\mu_r^\infty(f) = \frac{1}{r} \sum_{k=1}^r W_k (\mu_r^\infty \circ T_k)(\tau_k^r f). \tag{1.3}$$

In the deterministic environment case, this equation and its version for masses $Z^{(r)}$,

$$Z^{(r)} = \frac{1}{r} \sum_{k=1}^r W_k (Z^{(r)} \circ T_k), \tag{1.4}$$

have been studied by many authors (cf. 1976, [10]; 1983, [7]; 1990, [8]; 1998, [14]; 2001, [16]). Asymptotic properties of the masses $Z^{(r)}$ as $r \to \infty$, have been studied by some authors, see, for example, Liu and Rouault (2000, [18]), Liu, Rio and Rouault (2003, [17]).

The purpose of this paper is to give limit theorems for the process $\left\{Z^{(r)} : r \geq 2\right\}$ and the sequence of random measures $(\mu_r^n)_r$ as $r \to \infty$.

Theorem 1.1 (A central limit theorem)**.** *If $\mathbb{E}W_1^2 < \infty$, then as $r \to \infty$,*

$$\frac{\sqrt{r}}{\sqrt{\mathbb{E}_\zeta W_1^2 - 1}}(Z^{(r)} - 1) \text{ converges in law to the normal law } \mathcal{N}(0,1) \text{ under } \mathbb{P}_\zeta.$$

In the deterministic environment case, Theorem 1.1 reduces to Theorem 1.2 of Liu and Rouault (2000, [18]).

2. Convergence in L^2

The following result will be used in the next section.

Theorem 2.1. *If $\mathbb{E}W_1^2 < r < \infty$, then*

$$\mathbb{E}(Z^{(r)} - 1)^2 = \mathbb{E}(Z^{(r)})^2 - 1 = \frac{\mathbb{E}W_1^2 - 1}{r - \mathbb{E}W_1^2}.$$

In particular,

$$\lim_{r \to \infty} Z^{(r)} = 1 \quad \text{in } L^2.$$

In the deterministic environment case, Theorem 2.1 reduces to Theorem 3.1 of Liu and Rouault (2000, [18]).

The proof of Theorem 2.1 will be based on the following lemmas.

Lemma 2.2. *Let $r \geq 2$ be fixed. Assume that $\mathbb{E}W_1 \log W_1 \in [-\infty, \infty)$. Then the following assertions are equivalent:*

(a) $\mathbb{E}W_1 \log W_1 < \log r$;

(b) $\mathbb{E}_\zeta Z^{(r)} = 1$ *a.s.;*

(b′) $\mathbb{E}Z^{(r)} = 1$;

(c) $\mathbb{P}_\zeta(Z^{(r)} = 0) < 1$ *a.s.;*

(c′) $\mathbb{P}(Z^{(r)} = 0) < 1$.

This is a special case of Theorem 7.1 of Biggins and Kyprianou (2004, [4]) or Theorem 2.5 of Kuhlbusch (2004, [11]).

Lemma 2.3. *Let $r \geq 2$ be fixed. For $\alpha > 1$, the following assertions are equivalent:*

(a) $\mathbb{E}\left(\sum_{i=1}^{r} W_i\right)^\alpha < \infty$ *and* $\mathbb{E}W_1^\alpha < r^{\alpha-1}$;

(b) $\mathbb{E}\left(\sup_{n\geq 1} Y_n^{(r)}\right)^\alpha < \infty$;

(c) $0 < \mathbb{E}(Z^{(r)})^\alpha < \infty$.

This is given by Theorem 2.2.2 of Liang (2010, [12]).

Proof of Theorem 2.1. Since the function $f(s) = \log \mathbb{E}W_1^s$ is convex, we have $f(2)-f(1) \geq f'(1)$, which gives $\mathbb{E}W_1 \log W_1 \leq \log \mathbb{E}W_1^2$. Therefore, the condition $\mathbb{E}W_1^2 < r < \infty$ implies $\mathbb{E}W_1 \log W_1 < \log r$, so that by Lemmas 2.2 and 2.3, $\mathbb{E}Z^{(r)} = 1$ and $\mathbb{E}(Z^{(r)})^2 < \infty$. By equation (E), we have,

$$(Z^{(r)})^2 = \frac{1}{r^2}\left(\sum_{i=1}^{r} W_i Z_i^{(r)}\right)^2 = \frac{1}{r^2}\left[\sum_{i=1}^{r} W_i^2 (Z_i^{(r)})^2 + \sum_{\substack{1\leq i,j\leq r\\ i\neq j}} W_i W_j Z_i^{(r)} Z_j^{(r)}\right],$$

$$\begin{aligned}\mathbb{E}(Z^{(r)})^2 &= \frac{1}{r^2}\left[r\mathbb{E}\mathbb{E}_\zeta W_1^2 \mathbb{E}_{\theta\zeta}(Z^{(r)})^2 + r(r-1)\mathbb{E}(\mathbb{E}_\zeta W_1)^2(\mathbb{E}_{\theta\zeta} Z^{(r)})^2\right]\\ &= \frac{1}{r}\mathbb{E}\mathbb{E}_\zeta W_1^2 \mathbb{E}_{\theta\zeta}(Z^{(r)})^2 + \frac{r-1}{r}\\ &= \frac{1}{r}\mathbb{E}W_1^2 \mathbb{E}(Z^{(r)})^2 + \frac{r-1}{r}.\end{aligned}$$

So $\mathbb{E}(Z^{(r)})^2 = (r-1)/(r-\mathbb{E}W_1^2)$. Since $\mathbb{E}(Z^{(r)}-1)^2 = \mathbb{E}(Z^{(r)})^2 - 1$, the desired conclusion holds. □

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Proof of Theorem 1.1. Let $r_0 = \mathbb{E}W_1^2$. By the proof of Theorem 2.1, for $r \in (r_0, \infty)$, we have $\mathbb{E}W_1 \log W_1 < \log r$, so that by Lemmas 2.2 and 2.3, for $r \in [r_0, \infty)$, we see that

$$\mathbb{E}_\zeta Z^{(r)} = 1 \quad \text{a.s.},$$
$$\mathbb{E}(Z^{(r)})^2 < \infty$$

and

$$\mathbb{E}_\zeta (Z^{(r)})^2 = \frac{1}{r}\mathbb{E}_\zeta W_1^2 \mathbb{E}_{\theta\zeta}(Z^{(r)})^2 + \frac{r-1}{r} \quad \text{a.s.}$$

By equation (E),

$$rZ^{(r)} - r = \sum_{i=1}^{r}(W_i Z_i^{(r)} - 1).$$

Let $S_r = \sum_{i=1}^{r}(W_i Z_i^{(r)} - 1)$ $(r \geq r_0)$ and let $s_r \geq 0$ be defined by

$$s_r^2 = \sum_{i=1}^{r} \mathbb{E}_\zeta (W_i Z_i^{(r)} - 1)^2.$$

We notice that $W_i Z_i^{(r)} - 1$ are totally independent and identically distributed random variables under $\mathbb{P}_\zeta$ with

$$\mathbb{E}_\zeta \left[W_i Z_i^{(r)} - 1 \right] = \mathbb{E}_\zeta (W_i Z_i^{(r)}) - 1 = \mathbb{E}_\zeta W_1 \mathbb{E}_{\theta\zeta} Z^{(r)} - 1 = 0 \quad \text{a.s. for } r \in [r_0, \infty),$$

and that

$$s_r^2 = r \mathbb{E}_\zeta (W_1 Z_1^{(r)} - 1)^2 = r \left[\mathbb{E}_\zeta W_1^2 \mathbb{E}_\zeta (Z_1^{(r)})^2 - 1 \right] = r \left[\mathbb{E}_\zeta W_1^2 \mathbb{E}_{\theta\zeta} (Z^{(r)})^2 - 1 \right] \quad \text{a.s.}$$

for $r \in [r_0, \infty)$. We shall verify Lindeberg's condition for the sequence $\{S_r : r \geq r_0\}$. For all $\varepsilon > 0$ and $r \in [r_0, \infty)$, we have

$$\begin{aligned}
&\sum_{k=1}^{r} \frac{1}{s_r^2} \int_{\left\{ \left| W_k Z_k^{(r)} - 1 \right| \geq \varepsilon s_r \right\}} \left[W_k Z_k^{(r)} - 1 \right]^2 \mathrm{d}\mathbb{P}_\zeta \\
&= \frac{r}{s_r^2} \int_{\left\{ \left| W_1 Z_1^{(r)} - 1 \right| \geq \varepsilon s_r \right\}} \left[W_1 Z_1^{(r)} - 1 \right]^2 \mathrm{d}\mathbb{P}_\zeta \\
&= \frac{1}{\mathbb{E}_\zeta W_1^2 \mathbb{E}_{\theta\zeta} (Z^{(r)})^2 - 1} \int_{A_r} \left[W_1 Z_1^{(r)} - 1 \right]^2 \mathrm{d}\mathbb{P}_\zeta \\
&= \frac{\mathbb{E}_\zeta \left[W_1 Z_1^{(r)} - 1 \right]^2 \mathbf{1}_{\{A_r\}}}{\mathbb{E}_\zeta W_1^2 \mathbb{E}_{\theta\zeta} (Z^{(r)})^2 - 1},
\end{aligned} \tag{3.1}$$

where $A_r = \left\{ \left| W_1 Z_1^{(r)} - 1 \right| \geq \varepsilon \sqrt{r \left[\mathbb{E}_\zeta W_1^2 \mathbb{E}_{\theta\zeta} (Z^{(r)})^2 - 1 \right]} \right\}$. Notice that for $r \in [r_0, \infty)$,

$$\left[W_1 Z_1^{(r)} - 1 \right]^2 = W_1^2 \left[(Z_1^{(r)})^2 - 1 \right] - 2W_1 \left[Z_1^{(r)} - 1 \right] + (W_1 - 1)^2, \tag{3.2}$$

$$\mathbb{E} \left(W_1^2 \left| (Z_1^{(r)})^2 - 1 \right| \right) = \mathbb{E} \left(\mathbb{E}_\zeta W_1^2 \mathbb{E}_{\theta\zeta} \left| (Z^{(r)})^2 - 1 \right| \right) = \mathbb{E} W_1^2 \mathbb{E} \left| (Z^{(r)})^2 - 1 \right| \to 0, \tag{3.3}$$

$$\begin{aligned}
\mathbb{E} \left| -2W_1 \left[Z_1^{(r)} - 1 \right] \right| &= 2\mathbb{E} \left(\mathbb{E}_\zeta W_1 \mathbb{E}_{\theta\zeta} \left| Z^{(r)} - 1 \right| \right) = 2\mathbb{E} W_1 \mathbb{E} \left| Z^{(r)} - 1 \right| \\
&= 2\mathbb{E} \left| Z^{(r)} - 1 \right| \to 0.
\end{aligned}$$

Let $\{r'\}$ be any subsequence of $\{r\}$. Notice that from (3.3), we can choose a subsequence $\{r''\}$ of $\{r'\}$ with $r'' \to \infty$ for which

$$\mathbb{E}_\zeta \left(W_1^2 \left| (Z_1^{(r'')})^2 - 1 \right| \right) \to 0 \quad \text{a.s.} \tag{3.4}$$

Similarly, we also have that

$$\mathbb{E}_\zeta \left| -2W_1 \left[Z_1^{(r'')} - 1 \right] \right| \to 0 \quad \text{a.s.} \tag{3.5}$$

By Markov's inequality, we have

$$\mathbb{E}_\zeta \mathbf{1}_{\{A_r\}} = \mathbb{P}_\zeta \{A_r\} \le \frac{\mathbb{E}_\zeta \left[W_1 Z_1^{(r)} - 1\right]^2}{\varepsilon^2 r \left(\mathbb{E}_\zeta W_1^2 \mathbb{E}_{\theta\zeta}(Z^{(r)})^2 - 1\right)} = \frac{1}{\varepsilon^2 r} \to 0 \quad \text{a.s.}$$

Thus

$$\mathbf{1}_{\{A_r\}} \to 0 \quad \text{in probability under } \mathbb{P}_\zeta.$$

Therefore by the dominated convergence theorem, we see that

$$\mathbb{E}_\zeta (W_1 - 1)^2 \mathbf{1}_{\{A_r\}} \to 0 \quad \text{a.s.} \tag{3.6}$$

By (3.1), (3.2), (3.4), (3.5) and (3.6), we have

$$\begin{aligned}
&\lim_{r''\to\infty} \sum_{k=1}^{r''} \frac{1}{s_{r''}^2} \int_{\left\{\left|W_k Z_k^{(r'')} - 1\right| \ge \varepsilon s_{r''}\right\}} \left[W_k Z_k^{(r'')} - 1\right]^2 \mathrm{d}\mathbb{P}_\zeta \\
&= \lim_{r''\to\infty} \frac{\mathbb{E}_\zeta \left[W_1 Z_1^{(r'')} - 1\right]^2 \mathbf{1}_{\{A_{r''}\}}}{\mathbb{E}_\zeta W_1^2 \mathbb{E}_{\theta\zeta}(Z^{(r'')})^2 - 1} \\
&= 0.
\end{aligned}$$

So by Lindeberg's theorem, $S_{r''}/s_{r''}$ converges in law to the normal law $\mathcal{N}(0,1)$ under $\mathbb{P}_\zeta$. Since

$$\frac{s_{r''}^2}{r''(\mathbb{E}_\zeta W_1^2 - 1)} = \frac{\mathbb{E}_\zeta W_1^2 \mathbb{E}_{\theta\zeta}(Z^{(r'')})^2 - 1}{\mathbb{E}_\zeta W_1^2 - 1} \to 1 \quad \text{a.s. as } r'' \to \infty$$

by Theorem 2.1, this implies that, as $r'' \to \infty$,

$$\frac{\sqrt{r''}}{\sqrt{\mathbb{E}_\zeta W_1^2 - 1}} (Z^{(r'')} - 1) = \frac{S_{r''}}{s_{r''}} \cdot \frac{s_{r''}}{\sqrt{r''(\mathbb{E}_\zeta W_1^2 - 1)}}$$

converges in law to $\mathcal{N}(0,1)$ under $\mathbb{P}_\zeta$. Since the limit is independent of the subsequence taken, as $r \to \infty$,

$$\frac{\sqrt{r}}{\sqrt{\mathbb{E}_\zeta W_1^2 - 1}} (Z^{(r)} - 1) = \frac{S_r}{s_r} \cdot \frac{s_r}{\sqrt{r(\mathbb{E}_\zeta W_1^2 - 1)}}$$

converges in law to $\mathcal{N}(0,1)$ under $\mathbb{P}_\zeta$. □

4. The Mandelbrot measures for multiplicative cascades in a random environment

In this section $r \ge 2$ is fixed unless the contrary is mentioned.

Let $f \in \mathscr{L}^1([0,1],\lambda)$ be fixed. The sequence $\{(\mu_r^n(f), \mathcal{F}_n), n \ge 1\}$ is a martingale. By the martingale convergence theorem, considering the positive and negative parts of f, we see that the limit

$$\mu_r(f) = \lim_{n\to\infty} \mu_r^n(f) \tag{4.1}$$

exists $\mathbb{P}_\zeta$-a.s. Let D be a countable dense subset of $\mathscr{C}([0,1])$ equipped with the supremum norm $\|\cdot\|_\infty$. Then $\mathbb{P}_\zeta$-a.s. (4.1) holds for all $f \in \mathscr{C}([0,1])$ since $|\mu_r^n(f)| \le \|f\|_\infty \mu_r^n(1)$ and $|\mu_r(f)| \le \|f\|_\infty \mu_r(1)$. Hence $\mathbb{P}_\zeta$-a.s.

$$\mu_r^\infty(f) = \mu_r(f) \quad \text{for all } f \in \mathscr{C}([0,1])$$

(for any Borel measure μ and any integrable function f, we always write $\mu(f) = \int f \, \mathrm{d}\mu$).

In the deterministic environment case, Kahane and Peyrière [10] proved that the positive martingale $\{\mu_r^n(1)\}_n$ is uniformly integrable if and only if $\mathbb{E}W_1 \log W_1 < \log r$. In that case $\mu_r^n(1) \to \mu_r^\infty(1)$ a.s. and in L^1.

Theorem 4.1. *If $\mathbb{E}W_1 \log W_1 < \log r$, then for each fixed $f \in \mathscr{L}^1([0,1],\lambda)$, we have*

$$\lim_{n\to\infty} \mu_r^n(f) = \mu_r^\infty(f) \text{ in } L^1, \quad \text{and} \quad \mu_r^\infty(f) = \mu_r(f)\ \mathbb{P}_\zeta\text{-a.s.}$$

To prove the L^1 convergence, we need the following lemma.

Lemma 4.2. *If $\mathbb{E}W_1 \log W_1 < \log r$, then for each fixed f in $\mathscr{L}^1([0,1],\lambda)$, we have*

$$\mathbb{E}_\zeta \mu_r(f) = \mathbb{E}_\zeta \mu_r^\infty(f) = \lambda(f) \quad a.s.$$

Proof of Lemma 4.2. (a) We first prove that $\mathbb{E}_\zeta \mu_r(f) = \lambda(f)$ a.s. Clearly, for each $1 \le n < \infty$,

$$\mathbb{E}_\zeta \mu_r^n(f) = \lambda(f) \quad \text{a.s.} \tag{4.2}$$

We assume for the moment that $f \in \mathscr{L}^\infty([0,1],\lambda)$. Since $\mathbb{E}W_1 \log W_1 < \log r$, $\mu_r^n(1) \to \mu_r(1)$ in L^1 by Sheffé's theorem, Lemma 2.2 and (4.1) with $f = 1$. Therefore $\{\mu_r^n(1)\}_n$ is uniformly integrable. As $|\mu_r^n(f)| \le \|f\|_\infty \mu_r^n(1)$, this implies that $\{\mu_r^n(f)\}_n$ is also uniformly integrable, so that

$$\mu_r^n(f) \to \mu_r(f) \quad \text{in } L^1 \tag{4.3}$$

by (4.1). Letting $n \to \infty$ in (4.2), we see that $\mathbb{E}_\zeta \mu_r(f) = \lambda(f)$ a.s.

Assume only now $f \in \mathscr{L}^1([0,1],\lambda)$. Fatou's lemma and (4.2) yield $\mathbb{E}_\zeta \mu_r(f) \le \lambda(f)$ a.s. for $f \ge 0$. Therefore the functional $f \mapsto \mathbb{E}_\zeta \mu_r(f)$ is 1-Lipschitz on $\mathscr{L}^1([0,1],\lambda)$. On $\mathscr{L}^\infty([0,1],\lambda)$, it coincides with the continuous functional $f \mapsto \lambda(f)$. By the density of $\mathscr{L}^\infty([0,1],\lambda)$ in $\mathscr{L}^1([0,1],\lambda)$, this implies that $\mathbb{E}_\zeta \mu_r(f) = \lambda(f)$ a.s. for all $f \in \mathscr{L}^1([0,1],\lambda)$.

(b) We then prove that $\mathbb{E}_\zeta \mu_r^\infty(f) = \lambda(f)$ a.s. Set $\overline{\mu}_r^\infty(A) = \mathbb{E}_\zeta \mu_r^\infty(A)$ for $A \in B$ (recall that B is the Borel σ-field on $[0,1]$). The set function $\overline{\mu}_r^\infty$ is well defined by using the proof of Lemma 2.2 of Liu, Rio and Rouault (2003, [17]). The σ-additivity of μ_r^∞ implies that of $\overline{\mu}_r^\infty$. Therefore $\overline{\mu}_r^\infty$ is a Borel measure on $[0,1]$. For $f \in \mathscr{C}([0,1])$, we have

$$\overline{\mu}_r^\infty(f) = \mathbb{E}_\zeta \mu_r^\infty(f) = \mathbb{E}_\zeta \mu_r(f) = \lambda(f) \quad \text{a.s.}$$

Therefore the measure $\overline{\mu}_r^\infty$ and λ coincide, so that $\mathbb{E}_\zeta \mu_r^\infty(f) = \lambda(f)$ a.s. for all $f \in \mathscr{L}^1([0,1],\lambda)$. □

Proof of Theorem 4.1. Fix $f \in \mathscr{L}^1([0,1],\lambda)$. Let $\varepsilon > 0$ be arbitrarily fixed, and take $g \in \mathscr{C}([0,1])$ such that $\lambda(|f-g|) < \varepsilon$. By the triangle inequality and Lemma 4.2,

$$\begin{aligned}\mathbb{E}_\zeta \left|\mu_r^n(f) - \mu_r^\infty(f)\right| &\le \mathbb{E}_\zeta \left|\mu_r^n(f-g)\right| + \mathbb{E}_\zeta \left|\mu_r^n(g) - \mu_r^\infty(g)\right| + \mathbb{E}_\zeta \left|\mu_r^\infty(g-f)\right| \\ &\le 2\lambda(|f-g|) + \mathbb{E}_\zeta \left|\mu_r^n(g) - \mu_r^\infty(g)\right|.\end{aligned} \tag{4.4}$$

Because $g \in \mathscr{C}([0,1])$, we have $\lim_{n\to\infty} \mu_r^n(g) = \mu_r^\infty(g) = \mu_r(g)$ in L^1 (cf. (4.3)). Therefore letting $n \to \infty$ in (4.4), we see that

$$\limsup_{n\to\infty} \mathbb{E}_\zeta \left|\mu_r^n(f) - \mu_r^\infty(f)\right| \le 2\varepsilon,$$

so that $\lim_{n\to\infty} \mu_r^n(f) = \mu_r^\infty(f)$ in L^1. Since $\lim_{n\to\infty} \mu_r^n(f) = \mu_r(f)$ $\mathbb{P}_\zeta$-a.s., it follows that $\mu_r^\infty(f) = \mu_r(f)$ $\mathbb{P}_\zeta$-a.s. □

Lemma 4.3. (Proposition 3.1 in Liu, Rio and Rouault (2003, [17])) *Fix $n \ge 1$ and let $U^1, U^2, \dots, U^n$ be independent and integrable random variables. Let $(U^n_{i_1\cdots i_n})$ be a family of independent random variables indexed by $(n, i_1, \dots, i_n)$, such that for every n, $U^n_{i_1\cdots i_n}$ has the same distribution as U^n.*

(a) *For $r \ge 1$, set*

$$S_r^n = r^{-n} \sum_{1\le i_1,\dots,i_n\le r} U^1_{i_1} \cdots U^n_{i_1\cdots i_n},$$

and let H_r^n be the σ-field generated by $\{S_k^n, k \ge r\}$. Then $\{(S_r^n, H_r^n)\}_{r\ge1}$ is a reverse martingale, and $\lim_{r\to\infty} S_r^n = \mathbb{E}U^1\mathbb{E}U^2\cdots\mathbb{E}U^n$ *a.s. and in L^1.*

(b) *Assume additionally $\mathbb{E}U^n = 0$. If $\boldsymbol{a} = \left\{a^r_{i_1\cdots i_n}, 1 \le i_1, \dots, i_n \le r, r \ge 1\right\}$ is a family of real numbers such that $\|\boldsymbol{a}\|_\infty = \sup_{r\ge1} \max_{1\le i_1,\dots,i_n\le r} \left|a^r_{i_1\cdots i_n}\right| < \infty$, then as $r \to \infty$,*

$$\Gamma_r(\boldsymbol{a}) := r^{-n} \sum_{1\le i_1,\dots,i_n\le r} U^1_{i_1} \cdots U^n_{i_1\cdots i_n} a^r_{i_1\cdots i_n} \to 0 \quad \textit{a.s. and in } L^1.$$

Lemma 4.4. (Lemma 3.2 in Liu, Rio and Rouault (2003, [17])) *Assume that the conditions of Lemma 4.3(b) are satisfied. For $M > 0$, let $\overline{U}^k_{i_1\cdots i_k} := (-M \vee U^k_{i_1\cdots i_k}) \wedge M$. Set $\widetilde{U}^k_{i_1\cdots i_k} := \overline{U}^k_{i_1\cdots i_k} - \mathbb{E}\overline{U}^k_{i_1\cdots i_k}$ and*

$$\Gamma_r^M(\boldsymbol{a}) := r^{-n} \sum_{1\le i_1,\dots,i_n\le r} \overline{U}^1_{i_1} \cdots \overline{U}^{n-1}_{i_1\cdots i_{n-1}} \widetilde{U}^n_{i_1\cdots i_n} a^r_{i_1\cdots i_n}.$$

Then

$$\lim_{M\to\infty} \limsup_{r\ge1} \sup_{\boldsymbol{a}:\|\boldsymbol{a}\|_\infty\le1} \left|\Gamma_r(\boldsymbol{a}) - \Gamma_r^M(\boldsymbol{a})\right| = 0 \quad \textit{a.s.}$$

Lemma 4.5. (Proposition 3.4 in Liu, Rio and Rouault (2003, [17])) *Let* $\{U_{nk}, n \geq 1, 1 \leq k \leq r_n\}$ *be a triangular array of row-wise independent, integrable and centered real random variables such that* $\lim_{n\to\infty} r_n = \infty$. *If the family* $\{U_{nk}, n \geq 1, 1 \leq k \leq r_n\}$ *is uniformly integrable, then as* $n \to \infty$,

$$U_n = \frac{1}{r_n}\sum_{k=1}^{r_n} U_{nk} \to 0 \quad \textit{in } L^1.$$

For $n \leq \infty$ and some subset G of $\mathscr{L}^1([0,1],\lambda)$, we shall study a.s. and L^1 convergence of

$$\|\mu_r^n - \lambda\|_G := \sup_{f\in G} |\mu_r^n(f) - \lambda(f)|$$

as $r \to \infty$. In order to obtain uniform convergence results for finite n, we need finiteness of metric entropy in $\mathscr{L}^1([0,1],\lambda)$.

Definition 4.6. (Definition 3.6 in Liu, Rio and Rouault (2003, [17])) Let (V,d) be an arbitrary semi-metric space and T be a subset of V. The covering number $N(\varepsilon, T, d)$ is the minimal number of balls of radius ε needed to cover T. The entropy number is $H(\varepsilon, T, d) = \log N(\varepsilon, T, d)$. The subset T is said to be totally bounded in (V,d) if $N(\varepsilon, T, d)$ is finite for all $\varepsilon > 0$.

Definition 4.7. (Definition 3.7 in Liu, Rio and Rouault (2003, [17])) For $f, g \in \mathscr{L}^1([0,1], \lambda)$ such that $f \leq g$, the bracket $[f,g]$ is the set of all $h \in \mathscr{L}^1([0,1],\lambda)$ such that $f \leq h \leq g$. It is called an ε-bracket if $\lambda(g-f) \leq \varepsilon$. The class G is said to be totally bounded with brackets in $\mathscr{L}^1([0,1],\lambda)$ if it can be covered by a finite number of ε-brackets, for all $\varepsilon > 0$.

Theorem 4.8. *Let* $1 \leq n < \infty$ *be fixed.*

(a) $\lim_{r\to\infty} Y_n^{(r)} = 1$ $\mathbb{P}_\zeta$*-a.s. and in* L^1.

(b) *For* $f \in \mathscr{L}^1([0,1],\lambda)$,

$$\lim_{r\to\infty} \mu_r^n(f) = \lambda(f) \quad \textit{in } L^1.$$

(c) *If* G *is a class of uniformly bounded functions, totally bounded in* $\mathscr{L}^1([0,1],\lambda)$, *then*

$$\lim_{r\to\infty} \|\mu_r^n - \lambda\|_G = 0 \quad \mathbb{P}_\zeta\textit{-a.s. and in } L^1.$$

In the deterministic environment case, Theorem 4.8 reduces to Theorem 3.8 of Liu, Rio and Rouault (2003, [17]).

Proof of Theorem 4.8. Part (a) is a direct consequence of Lemma 4.3(a).

To prove parts (b) and (c), we first remark that for each $f \in \mathscr{L}^\infty([0,1],\lambda)$ and $1 \le n < \infty$,

$$\lim_{r\to\infty} \big(\mu_r^n(f) - \mu_r^{n-1}(f)\big) = 0 \quad \mathbb{P}_\zeta\text{-a.s. and in } L^1, \tag{4.5}$$

by applying Lemma 4.3(b) to the decomposition

$$\mu_r^n(f) - \mu_r^{n-1}(f) = \sum_{1\le u_1,\dots,u_n\le r} W_{u_1}\cdots W_{u_1\cdots u_{n-1}}(W_{u_1\cdots u_n} - 1)\int_{A^r_{u_1\cdots u_n}} f\,\mathrm{d}\lambda.$$

Since $\mu_r^0 = \lambda$, (4.5) implies that, for each $f \in \mathscr{L}^\infty([0,1],\lambda)$ and $1 \le n < \infty$,

$$\lim_{r\to\infty} (\mu_r^n(f) - \lambda(f)) = 0 \quad \mathbb{P}_\zeta\text{-a.s. and in } L^1. \tag{4.6}$$

By the density of $\mathscr{L}^\infty([0,1],\lambda)$ in $\mathscr{L}^1([0,1],\lambda)$, $\mathbb{E}_\zeta\mu_r^n(f) = \lambda(f)$ a.s. for each $1 \le n < \infty$ and the inequality

$$|\mu_r^n(f) - \lambda(f)| \le \mu_r^n(|f-g|) + |\mu_r^n(g) - \lambda(g)| + \lambda(|g-f|),$$

we see that the L^1 convergence in (4.6) still holds for every f in $\mathscr{L}^1([0,1],\lambda)$, which ends the proof of (b).

For part (c), we assume that G is uniformly bounded by 1 for the sake of simplicity. To prove the a.s. convergence, it is enough to show that for every $n < \infty$,

$$\lim_{r\to\infty} \left\|\mu_r^n - \mu_r^{n-1}\right\|_G = 0 \quad \mathbb{P}_\zeta\text{-a.s.} \tag{4.7}$$

From Lemma 4.4, it is sufficient to prove (4.7) when the W_u are bounded by a constant $M \ge 1$. Since G is totally bounded, for every $\varepsilon > 0$ one can find $f_1,\dots,f_N \in \mathscr{L}^1([0,1],\lambda)$ such that for every $f \in G$ there is some f_i such that $\lambda(|f-f_i|) \le \varepsilon$. Actually we can choose the functions f_i in $\mathscr{L}^\infty([0,1],\lambda)$ since it is dense in $\mathscr{L}^1([0,1],\lambda)$.

By definition of μ_r^n, we then have $|\mu_r^n(g)| \le M^n\lambda(|g|)$ for g in $\mathscr{L}^1([0,1],\lambda)$ and $n \ge 0$. Hence, for $f \in G$ and $\lambda(|f-f_i|) \le \varepsilon$,

$$\left|\big(\mu_r^n(f) - \mu_r^{n-1}(f)\big) - \big(\mu_r^n(f_i) - \mu_r^{n-1}(f_i)\big)\right| \le 2M^n\varepsilon. \tag{4.8}$$

Now, from (4.6) $\mathbb{P}_\zeta$-a.s. for every $1 \le i \le N$,

$$\lim_{r\to\infty} \big(\mu_r^n(f_i) - \mu_r^{n-1}(f_i)\big) = 0.$$

Jointly with (4.8) it yields $\mathbb{P}_\zeta$-a.s.

$$\limsup_{r\to\infty} \left\|\mu_r^n - \mu_r^{n-1}\right\|_G \le 2M^n\varepsilon$$

for every ε. This gives the $\mathbb{P}_\zeta$-a.s. convergence of part (c).

To get the L^1 convergence, it is enough to prove, for every fixed $n < \infty$, the uniform integrability of $(\left\|\mu_r^n - \mu_r^{n-1}\right\|_G)_r$. But this is indeed the case because $\left\|\mu_r^n - \mu_r^{n-1}\right\|_G$ is bounded by

$$S_r^n := r^{-n} \sum_{1 \le u_1, \dots, u_n \le r} W_{u_1} \cdots W_{u_1 \cdots u_{n-1}} \left|W_{u_1 \cdots u_n} - 1\right|$$

which by Lemma 4.3 converges in L^1 and is therefore uniformly integrable. □

Theorem 4.9. *Assume* $\mathbb{E} W_1 \log^+ W_1 < \infty$.

(a) $\lim_{r\to\infty} Z^{(r)} = 1$ $\mathbb{P}_\zeta$*-a.s. and in* L^1.

(b) *For* $f \in \mathscr{L}^1([0,1], \lambda)$,

$$\lim_{r\to\infty} \mu_r^\infty(f) = \lambda(f) \quad \text{in } L^1.$$

(c) *If G is a subset of $\mathscr{L}^1([0,1],\lambda)$ such that, for each $\varepsilon > 0$, it can be covered by a finite number of ε-brackets $[f_i, g_i]$, with f_i and g_i measurable, bounded and λ-a.e. continuous, then*

$$\lim_{r\to\infty} \mathbb{E}_\zeta^* \left\|\mu_r^\infty - \lambda\right\|_G = 0 \quad \text{and} \quad \lim_{r\to\infty} \left\|\mu_r^\infty - \lambda\right\|_G = 0 \quad \mathbb{P}_\zeta^*\text{-a.s.},$$

where $\mathbb{P}_\zeta^*$ and $\mathbb{E}_\zeta^*$ denote the corresponding outer conditional probability and outer conditional expectation.

In the deterministic environment case, Theorem 4.9 reduces to Theorem 3.9(a)–(c) of Liu, Rio and Rouault (2003, [17]).

Proof of Theorem 4.9. (a) For $n \le +\infty$, let $H_n^{(r)}$ be the σ-field generated by $Y_n^{(s)}$, $s \ge r$. By Lemma 4.3(a), for each $n < \infty$, $\{(Y_n^{(r)}, H_n^{(r)})\}_{r \ge 1}$ is a reverse martingale. Thus for every integer $p \ge 1$ and every bounded and continuous function $g\colon \mathbb{R}^p \to \mathbb{R}$, we have

$$\mathbb{E}_\zeta \left(Y_n^{(r)} g(Y_n^{(r+1)}, Y_n^{(r+2)}, \dots, Y_n^{(r+p)})\right) = \mathbb{E}_\zeta \left(Y_n^{(r+1)} g(Y_n^{(r+1)}, Y_n^{(r+2)}, \dots, Y_n^{(r+p)})\right). \tag{4.9}$$

Let $r_0 \ge 2$ be such that $\mathbb{E} W_1 \log W_1 < \log r_0$. For each fixed $r \ge r_0$, as $n \to \infty$, $Y_n^{(r)} \to Z^{(r)}$ $\mathbb{P}_\zeta$-a.s. and in L^1. Thus using uniform integrability, we may let $n \to \infty$ in (4.9), showing that $\{(Z^{(r)}, H_\infty^{(r)})\}_{r\ge 1}$ is also a reverse martingale. Therefore $Z^{(r)}$ convergence $\mathbb{P}_\zeta$-a.s. and is uniformly integrable. To identify the limit, we will see in (b) below that $Z^{(r)} \to 1$ in L^1, so that the proof of (a) is finished.

(b) We first prove that for each $f \in \mathscr{L}^\infty([0,1],\lambda)$,

$$\lim_{r\to\infty} \mu_r^\infty(f) = \lambda(f) \quad \text{in } L^1. \tag{4.10}$$

By extension of (1.2) to the associated Borel measures we get the decomposition

$$\mu_r^\infty(f) - \lambda(f) = \frac{1}{r} \sum_{k=1}^{r} \left[W_k (\mu_r^\infty \circ T_k)(\tau_k^r f) - \lambda(\tau_k^r f)\right].$$

Since $|W_k(\mu_r^\infty \circ T_k)(\tau_k^r f) - \lambda(\tau_k^r f)| \leq c_1 Z^{(r)} \circ T_k + c_2$ (c_1, c_2 are constants), the family $\{W_k(\mu_r^\infty \circ T_k)(\tau_k^r f) - \lambda(\tau_k^r f)\}_{k,r}$ is uniformly integrable, so that Lemma 4.5 gives (4.10). By density of $\mathscr{L}^\infty([0,1],\lambda)$ in $\mathscr{L}^1([0,1],\lambda)$, using Lemma 4.2 and

$$|\mu_r^\infty(f) - \lambda(f)| \leq \mu_r^\infty(|f-g|) + |\mu_r^\infty(g) - \lambda(g)| + \lambda(|g-f|)$$

for $g \in \mathscr{L}^\infty([0,1],\lambda)$, we see that (4.10) holds for $f \in \mathscr{L}^1([0,1],\lambda)$.

(c) Let us first reduce the problem to a simpler one involving only one function. Let $\varepsilon > 0$, and let $\{[f_i, g_i] : 1 \leq i \leq N\}$ be a cover of G by ε-brackets, with f_i and g_i measurable, bounded and λ-a.e. continuous. If $f \in [f_i, g_i]$, then

$$\mu_r^\infty(f) - \lambda(f) \leq \mu_r^\infty(g_i) - \lambda(f_i) = [\mu_r^\infty(g_i) - \lambda(g_i)] + [\lambda(g_i) - \lambda(f_i)]$$

and

$$\mu_r^\infty(f) - \lambda(f) \geq \mu_r^\infty(f_i) - \lambda(g_i) = [\mu_r^\infty(f_i) - \lambda(f_i)] + [\lambda(f_i) - \lambda(g_i)].$$

Therefore

$$\|\mu_r^\infty - \lambda\|_G \leq \max\{|\mu_r^\infty(g_i) - \lambda(g_i)|, |\mu_r^\infty(f_i) - \lambda(f_i)| : 1 \leq i \leq N\} + \varepsilon. \tag{4.11}$$

(c1) To prove the $\mathbb{P}_\zeta^*$-a.s. convergence, it is convenient to introduce the random measures $\widetilde{\mu}_r^n$ defined by

$$\widetilde{\mu}_r^n = \frac{1}{r}\sum_{k=1}^{r} W_k(Y_{n-1}^{(r)} \circ T_k)\delta_{k/r}, 1 \leq n \leq \infty,$$

(recall that by convention $Y_{n-1}^{(r)} \circ T_k = 1$ if $n = 1$, and $= Z^{(r)} \circ T_k$ if $n = \infty$), and to compare it with μ_r^n with the help of (1.2).

Let us first prove that $\mathbb{P}_\zeta$-a.s. for all $t \in [0,1]$,

$$\lim_{r\to\infty} \widetilde{\mu}_r^\infty([0,t]) = t. \tag{4.12}$$

For fixed $t \in (0,1]$ and $1 \leq n < \infty$, set

$${}^tY_n^{(r)} := \frac{r}{[rt]}\widetilde{\mu}_r^n([0,t]) = \frac{1}{[rt]}\sum_{u_1=1}^{[rt]} W_{u_1} \sum_{1\leq u_2,\ldots,u_n\leq r} \frac{W_{u_1u_2}\cdots W_{u_1\cdots u_n}}{r^{n-1}}, \tag{4.13}$$

where $[x]$ is the integer part of x. By Theorem 4.1 and (4.13), if $\mathbb{E}W_1 \log W_1 < \log r$, then as $n \to \infty$, ${}^tY_n^{(r)}$ converges $\mathbb{P}_\zeta$-a.s. and in L^1 to

$${}^tY_\infty^{(r)} := \frac{1}{[rt]}\sum_{k=1}^{[rt]} W_k Z^{(r)} \circ T_k.$$

For $1 \le n \le \infty$, let ${}^tH_n^{(r)}$ be the σ-field generated by $\{{}^tY_n^{(k)}, k \ge r\}$. Let $r \ge t^{-1}$ be such that $\mathbb{E}W_1 \log W_1 < \log r$. Just like $Y_n^{(r)}$, for each fixed $1 \le n \le \infty$, the sequence $\{{}^tY_n^{(r)}\}_{r \ge r_t}$ is a reverse martingale with respect to $\{{}^tH_n^{(r)}\}_{r \ge t^{-1}}$ (the proof is similar with that of (a)), so that it converges $\mathbb{P}_\zeta$-a.s. and in L^1. To identify the limit of ${}^tY_\infty^{(r)}$, we use

$$ {}^tY_\infty^{(r)} - 1 = \frac{1}{[rt]} \sum_{k=1}^{[rt]} (W_k Z^{(r)} \circ T_k - 1) $$

and Lemma 4.5 to conclude that ${}^tY_\infty^{(r)} \to 1$ in L^1. Since

$$ \widetilde{\mu}_r^\infty([0,t]) = \frac{[rt]}{r}\, {}^tY_\infty^{(r)}, $$

it follows that

$$ \lim_{r\to\infty} \widetilde{\mu}_r^\infty([0,t]) = t \quad \mathbb{P}_\zeta\text{-a.s. and in } L^1. $$

By a classical monotonicity argument, this implies (4.12), hence the $\mathbb{P}_\zeta$-a.s. weak convergence of $\widetilde{\mu}_r^\infty$ to λ. To get a similar result for μ_r^∞, observe first that, from (1.3),

$$ \mu_r^\infty(f) - \widetilde{\mu}_r^\infty(f) = \frac{1}{r} \sum_{k=1}^{r} W_k(\mu_r^\infty \circ T_k)(\tau_k^r f - f(k/r)). $$

Since, for $f \in \mathscr{C}([0,1])$,

$$ \sup_{x\in[0,1]} |\tau_k f(x) - f_{k,r}| \le \omega_f(r^{-1}), $$

where $\omega_f(h)$ is the maximal oscillation of f on intervals of size h, $h > 0$, we have

$$ |\mu_r^\infty(f) - \widetilde{\mu}_r^\infty(f)| \le \frac{\omega_f(r^{-1})}{r} \sum_{k=1}^{r} W_k(Z^{(r)} \circ T_k) = \omega_f(r^{-1}) Z^{(r)}, $$

where the last equality holds by (1.4). This yields the $\mathbb{P}_\zeta$-a.s. weak convergence of μ_r^∞ to λ. Therefore (cf. [5, p. 163, Proposition 8.12]) $\mathbb{P}_\zeta^*$-a.s. for all f measurable, bounded and λ-a.s. continuous,

$$ \lim_{r\to\infty} \mu_r^\infty(f) = \lambda(f). $$

Replacing f by f_i, g_i in the above equation and using (4.11), we see that

$$ \mathbb{P}_\zeta^*\text{-a.s. } \limsup_{r\to\infty} \|\mu_r^\infty - \lambda\|_G \le \varepsilon, $$

for every $\varepsilon > 0$, which ends the proof of the $\mathbb{P}_\zeta^*$-a.s. convergence.

(c2) Taking $\mathbb{E}_\zeta^*$ in (4.11) and using (b) gives the L^1-convergence. $\square$

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