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# Painlevé–Kuratowski Stability of Approximate Solution Sets for Perturbed Set Optimization Problems Under General Ordering Sets by Recession Cone

Zai-Yun Peng*, Chong-Yang Shao, Yue Zeng and Yi-Bin Xiao

Abstract. The aim of this paper is to study the stability of perturbed set optimization problems via general ordering sets. Firstly, for a set optimization problem (SOP) via general ordering sets, four kinds of concepts about the minimal solutions of (SOP) are given. Then, some properties of the four kinds of solution sets and the level set of objective mappings are investigated. Finally, by employing the recession cone technique, sufficient conditions of upper Painlevé–Kuratowski convergence of minimal approximate solution sets, Painlevé–Kuratowski convergence of weak minimal approximate solution sets of (SOP) are obtained, where the feasible set is perturbed. Some examples are given to illustrate the mainly results in the paper.

## 1. Introduction

For the past few years, the set-valued optimization problem, as an extension of vector optimization problems, has a growing interest for their wide applications in various fields, such as welfare economics, robust optimization, game theory, mathematical finance, etc. More details on set-valued optimization problems and its applications can be found in [4, 20, 21, 23], for instance.

Generally speaking, there exist mainly two solution criteria for set-valued optimization problems: the vector optimization criterion and set optimization criterion. The former considers the efficient elements (vectors) in the union of all values of the objective set-valued mapping. This criterion has been widely studied, see, e.g., [4, 7, 8, 13, 27]. However the vector optimization criterion is not always suitable for all types of set-valued optimization problems. For example, we may not compare the overall level of two football teams by virtue of the vector optimization criterion, since the fact tell us that a team has one of the best player does not always mean it is the best team. In order to overcome this flaw, the latter criterion introduced by Kuroiwa in [22], is based on a comparison among the values (sets) of objective set-valued mappings by set order relations. A set-valued optimization

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*Corresponding author.

problem considered by the set optimization criterion is called a set optimization problem (SOP, for short). Up to now, SOP has been intensively investigated by many researches, see [5, 9, 26, 36, 40, 41, 43–45] and the references therein.

It is well known that the set order relation is an important tool to study set optimization problems. Until now, there are many kinds of set order relations used in study of set optimization problems, mainly including but not limited to lower set less order relation $\preceq_l$ in [5, 25, 45], upper set less order relation $\preceq_u$ in [5, 25, 36, 40], set less or KNY order relation $\preceq_s$ (see [5]), certainly less order relation $\preceq_c$ (see [5]), possibly less order relation $\preceq_p$ in [41]. Note that the aforementioned set order relations are generally based on a cone (named ordering cone). With the deepening of research, one can find that some practical problems, especially some economic problems (see [1, 10]), are not suitable to be characterized by cone-defined order relations. Hence, ordering cone has been gradually extended to general ordering set, such as improvement sets (see [6, 26] for instance), coradiant sets (see [43, 44]) and so on. Furthermore, in 2019, Khushboo and Lalitha [21] introduced the set order relation $\preceq_S^l$, which relies on an arbitrary nonempty proper subset $S$ in image space, and established an existence result of minimal solutions for a class of SOP. Obviously, the set order relation $\preceq_S^l$ is a generalization of some of set order relations, for example $\preceq_l$, $\preceq_E$. Therefore, it is meaningful to study other aspects (e.g., stability) of SOP under the set order relation $\preceq_S$, which defined by an arbitrary nonempty proper subset $S$ in image space.

On the other hand, the stability analysis is an important topic in the study of optimization and related problems, and has been studied extensively by researchers, see, e.g., [8, 14–18, 25, 26, 29–33, 43]. Up to now, the research on the stability of set optimization problems has achieved abundant results in the literatures, such as well-posedness [15, 16, 25, 43, 45], continuity of solution mappings [16, 24, 36, 40–42], connectedness of solution sets [14, 17] and so on. However, to the best of our knowledge, for an important aspect of the stability analysis of set optimization problems, the Painlevé–Kuratowski convergence results are relatively few (see [15, 18, 28]). Han et al. [15] obtained the Painlevé–Kuratowski convergence of the approximate solution sets for set optimization problems with the continuity and convexity of objective mapping by the set order relation $\preceq_l$. Han [18] investigated the Painlevé–Kuratowski convergence of the solution sets for perturbed set optimization problems by the set order relation $\preceq_l$. Recently, Peng et al. [28] studied the Painlevé–Kuratowski convergence of (weak)-minimal solutions for set optimization problems via improvement sets.

Motivated by the research of [15, 18, 21], by set order relation $\preceq_S$, we propose to investigate Painlevé–Kuratowski convergence of a class of perturbed SOP. We present four kinds of concepts of minimal solutions with respect to $\preceq_S$, and discuss the relationships among these solution sets. By virtue of the level set of objective mappings, we establish Painlevé–Kuratowski convergence of approximate solution sets for this perturbed SOP by recession cone technique.

This paper in four sections is organized as follows. Section 2 presents some basic notions and preliminaries results required in the sequel, introduces four kinds of concepts of minimal solutions and the level set of objective set-valued mappings for a class of SOP. Section 3 investigates some related properties of these solution sets and the level set. Under some mild assumptions, Section 4 discusses some sufficient conditions for the Painlevé–Kuratowski convergence of solution sets for perturbed SOP via general ordering sets. By employing the recession cone technique, we also give some illustrative examples along the paper.

## 2. Preliminaries

Throughout this paper, unless specified otherwise, let $X$ and $Y$ be two real normed vector spaces. The topological boundary, topological closure and topological interior of a set $A \subseteq Y$ are defined by $\partial S$, $\mathrm{cl}\, A$ and $\mathrm{int}\, A$, respectively. Let $S$ be a nonempty proper, closed subset of $Y$ with $\mathrm{int}\, S^{\infty} \neq \emptyset$, where $S^{\infty} = \{y \in Y : s + ty \in S, \forall\, s \in S, t \geq 0\}$ is the recession cone of $S$. Let $\mathcal{P}_0(Y)$ be the set of all nonempty subsets of $Y$. The lower set order relation and weak lower set order relation associated with $S$ on $\mathcal{P}_0(Y)$ are defined, respectively, by

$$
\begin{aligned}
A \leq_S B \quad &\Longleftrightarrow \quad B \subseteq A + S, \\
A <_S B \quad &\Longleftrightarrow \quad B \subseteq A + \mathrm{int}\, S.
\end{aligned}
$$

Let $u \in \mathrm{int}\, S^{\infty}$ be fixed. For any nonnegative real number $\varepsilon$, the $\varepsilon$-lower set order relation and weak $\varepsilon$-lower set order relation associated with $S$ on $\mathcal{P}_0(Y)$ are defined, respectively, by

$$
\begin{aligned}
A \leq_S^{\varepsilon} B \quad &\Longleftrightarrow \quad B \subseteq A + S + \varepsilon u, \\
A <_S^{\varepsilon} B \quad &\Longleftrightarrow \quad B \subseteq A + \mathrm{int}\, S + \varepsilon u.
\end{aligned}
$$

For the recession cone of nonempty subset $S$ of $Y$, we collect some basic results as follows (see, [11], [34, Section 8] and [37, p. 306]).

**Lemma 2.1.** *For any nonempty subset $S$ of $Y$, one has*

(i) $S^{\infty} \neq \emptyset$ *and* $S + S^{\infty} \subseteq S$;

(ii) $S^{\infty}$ *is a convex cone;*

(iii) *if $S$ is closed, then $S^\infty$ is closed;*

(iv) *if $S$ is convex, then $S^\infty = \{y \in Y : s + ty \in S, \exists\, s \in S, \forall\, t \geq 0\}$;*

(v) *if $S$ is a pointed, closed and convex cone, then $S^\infty = S$;*

(vi) *if $S$ is bounded (i.e., there exists some $t > 0$ such that $S \subseteq t\mathbb{B}_Y$, where $\mathbb{B}_Y$ is the open unit ball with center $0 \in Y$), then $S^\infty = \{0\}$.*

**Lemma 2.2.** [38, Proposition 2.2] *Let $A$ and $B$ be two nonempty sets in a topological vector space. If $\operatorname{int} A \neq \emptyset$, then $\operatorname{int} A + B \subset \operatorname{int}(A + B)$.*

*Remark* 2.3. According to Lemmas 2.1(i) and 2.2, it is easy to obtained that $S + \operatorname{int} S^\infty \subseteq \operatorname{int} S \neq \emptyset$ and $\operatorname{int} S + S^\infty \subseteq \operatorname{int} S$.

*Remark* 2.4. For the relation $<_S^\varepsilon$, the following statements are true.

(i) For any $\varepsilon > 0$, the relation $\leq_S^\varepsilon$ is not reflexive if $0 \notin \operatorname{int} S$, as for any bounded set $B$ in $\mathcal{P}_0(Y)$, $B \leq_S^\varepsilon B$ does not hold.

(ii) When $0 \notin \operatorname{int} S$, the relation $<_S^\varepsilon$ is not reflexive for any $\varepsilon > 0$.

(iii) The relation $<_S^\varepsilon$ is reflexive for some $\varepsilon > 0$ (not all) if $0 \in \operatorname{int} S$.

Let $K$ be a nonempty subset of $X$ and $F$ be a nonempty set-valued mapping from $K$ to $Y$ with compact values (i.e., $F(x)$ is a compact set for each $x \in K$). Let us consider the set optimization problem via the general order set $S$ as follows:

$$\text{(SOP)} \qquad \min_S \ F(x) \ \ s.t. \ \ x \in K.$$

**Definition 2.5.** For any $\varepsilon \geq 0$, a point $x_0 \in K$ is said to be

(i) a minimal solution of (SOP) if for any $x \in K$,

$$F(x) \leq_S F(x_0) \quad \Longrightarrow \quad F(x_0) \leq_S F(x);$$

(ii) a minimal approximate solution of (SOP) if for any $x \in K$,

$$F(x) \leq_S^\varepsilon F(x_0) \quad \Longrightarrow \quad F(x_0) \leq_S^\varepsilon F(x);$$

(iii) a weak minimal solution of (SOP) if for any $x \in K$,

$$F(x) <_S F(x_0) \quad \Longrightarrow \quad F(x_0) <_S F(x);$$

(iv) a weak minimal approximate solution of (SOP) if for any $x \in K$,

$$F(x) <_S^\varepsilon F(x_0) \quad \Longrightarrow \quad F(x_0) <_S^\varepsilon F(x).$$

The minimal solution set, minimal approximate solution set, weak minimal solution set and weak minimal approximate solution set of (SOP) are denoted, respectively, by $E(K)$, $E(\varepsilon, K)$, $W(K)$ and $W(\varepsilon, K)$. Clearly, $E(K) = E(0, K)$ and $W(K) = W(0, K)$.

First, for $\varepsilon \geq 0$, we define the $\varepsilon$-level set of $F$ at $x \in K$ by

$$L_F(\varepsilon, x, K) = \{y \in K : F(y) \leq_S^{\varepsilon} F(x)\} \cup \{x\}.$$

Notice that for any $\varepsilon \geq 0$ and $x \in K$, the set $\{y \in K : F(y) \leq_S^{\varepsilon} F(x)\}$ may be empty, but the $\varepsilon$-level set $L_F(\varepsilon, x, K)$ of $F$ is always nonempty due to $x \in L_F(\varepsilon, x, K)$.

Next, we recall some basic definitions and facts which will be used in the sequel.

**Definition 2.6.** [12, Definitions 2.5.1 and 2.5.12] Let $X$ and $Y$ be two topological vector spaces. Suppose that $G$ is a set-valued mapping from $A \subseteq X$ to $Y$ and $x_0 \in A$.

(i) $G$ is said to be upper semicontinuous (u.s.c, for short) at $x_0$ if for any open set $V$ with $G(x_0) \subseteq V$, there exists a neighbourhood $U$ of $x_0$ in $X$ such that $G(x) \subseteq V$ for all $x \in U$;

(ii) $G$ is said to be lower semicontinuous (l.s.c, for short) at $x_0$ if for any open set $V$ with $G(x_0) \cap V \neq \emptyset$, there exists a neighbourhood $U$ of $x_0$ in $X$ such that $G(x) \cap V \neq \emptyset$ for all $x \in U$;

(iii) $G$ is said to be Hausdorff upper semicontinuous (H-u.s.c, for short) at $x_0$ if

$$\forall\, V \in \mathcal{V}_Y, \exists\, U \in \mathcal{V}(x_0), \forall\, x \in U : G(x) \subseteq G(x_0) + V,$$

where $\mathcal{V}_Y$ denotes the class of balanced neighbourhoods of $0 \in Y$;

(iv) $G$ is said to be Hausdorff lower semicontinuous (H-l.s.c, for short) at $x_0$ if

$$\forall\, V \in \mathcal{V}_Y, \exists\, U \in \mathcal{V}(x_0), \forall\, x \in U : G(x_0) \subseteq G(x) + V.$$

We say $G$ is l.s.c (resp. u.s.c, H-l.s.c, H-u.s.c) on $A$ if it is l.s.c (resp. u.s.c, H-l.s.c, H-u.s.c) at each $x \in A$. $G$ is said to be continuous (resp. H-continuous) on $A$ if it is both l.s.c and u.s.c (resp. H-u.s.c and H-l.s.c) on $A$.

Aubin and Ekeland given some equivalent characterizations of l.s.c and u.s.c in [2, pp. 108–109], also see [12, Propositions 2.5.6 and 2.5.9].

**Lemma 2.7.** *Let $X$ and $Y$ be two metric spaces, $G\colon X \rightrightarrows Y$ be a set-valued mapping and $x_0 \in A$.*

(i) *$G$ is l.s.c at $x_0$ if and only if for any sequence $\{x_n\} \subseteq X$ with $x_n \to x_0$ and any $y_0 \in G(x_0)$, there exists $y_n \in G(x_n)$ such that $y_n \to y_0$.*

(ii) *$G(x_0)$ is compact, $G$ is u.s.c at $x_0$ if and only if for any sequence $\{x_n\} \subseteq X$ with $x_n \to x_0$ and any $y_n \in G(x_n)$, there exist $y_0 \in G(x_0)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0$.*

**Definition 2.8.** [12, Definition 2.5.16] Suppose that $X$ and $Y$ are two Hausdorff topological vector spaces. Let $C$ be a convex cone in $Y$ and $G\colon X \rightrightarrows Y$ be a set-valued mapping. $G$ is to be

(i) Hausdorff $C$-upper semicontinuous (H-$C$-u.s.c, for short) at $x_0 \in X$ if

$$\forall V \in \mathcal{V}_Y, \exists U \in \mathcal{V}(x_0), \forall x \in U : G(x) \subseteq G(x_0) + V + C;$$

(ii) Hausdorff $C$-lower semicontinuous (H-$C$-l.s.c, for short) at $x_0 \in X$ if

$$\forall V \in \mathcal{V}_Y, \exists U \in \mathcal{V}(x_0), \forall x \in U : G(x_0) \subseteq G(x) + V + C;$$

(iii) H-$C$-continuous on $X$ if it is both H-$C$-u.s.c and H-$C$-l.s.c at every $x \in X$.

*Remark* 2.9. From Proposition 2.2 and Remark 2.2 of [39], we know that a set-valued mapping $G$ is continuous, then it is H-$C$-continuous, and the converse may be not hold. In other words, the concept of H-$C$-continuity for set-valued mappings is strictly larger than the concept of continuity for set-valued mappings. Next we give Example 2.10 in infinite dimensional space to show the case.

**Example 2.10.** (i) Let $X = Y = l^\infty = \big\{(x_1, x_2, \dots) : x_i \in \mathbb{R}, \sup\{|x_i| : i = 1, 2, \dots\} < +\infty\big\}$ be the set of all bounded sequence of real numbers and $C = \{(x_1, x_2, \dots) \in l^\infty : x_i \geq 0, i = 1, 2, \dots\}$. The set-valued mapping $G\colon X \rightrightarrows Y$ is defined as follows:

$$G(x) = \begin{cases} (0, \beta_x p + p] & \text{if } x \in C, \\ [-p, 0] & \text{if } x \notin C, \end{cases}$$

where $\beta_x = \sup_i |x_i|$ with $x = (x_1, x_2, \dots) \in X$, $(0, x] = \{\alpha 0 + (1-\alpha)x : \alpha \in (0, 1]\}$ and $[-p, 0] = \{\alpha(-p) + (1-\alpha)0 : \alpha \in [0, 1]\}$ with $p = (1, 1, \dots) \in Y$.

It is not difficult to find that $G$ is H-$C$-l.s.c at 0. However, for any neighbourhood $U$ of 0, there exists some $z \notin C$ with $z \in U$. It follows that $G(z) = [-p, 0]$. Take the open set $U(p, 1/2) = \{(x_1, x_2, \dots) \in l^\infty : |x_i - 1| < 1/2, i = 1, 2, \dots\}$, then

$$G(z) \cap U(p, 1/2) = \emptyset,$$

which implies $G$ is not l.s.c at 0.

(ii) Let $X = Y = l^1 = \{(\xi_1, \xi_2, \dots) : \xi_i \in \mathbb{R}, \sum_{i=1}^{\infty} |\xi_i| < +\infty\}$ and $C = \{(\xi_1, \xi_2, \dots) \in l^1 : \xi_i \geq 0, i = 1, 2, \dots\}$. We consider the set-valued mapping $G\colon X \rightrightarrows Y$ as follows:

$$G(x) = \begin{cases} [-q, 0] & \text{if } x = (\xi_1, \xi_2, \dots) \in C, \\ [q, \alpha_x q] & \text{if } x = (\xi_1, \xi_2, \dots) \notin C, \end{cases}$$

where $q = (1, 1, 0, 0, \dots) \in l^1$, $\alpha_x = \sup_i |\xi_i| + 1$, and the line segment $[y, z]$ is defined by

$$[y, z] = \{\lambda y + (1 - \lambda) z : \lambda \in [0, 1]\}$$

with $y = (y_1, y_2, \dots) \in X$ and $z = (z_1, z_2, \dots) \in X$.

After computation, $G$ is H-$C$-u.s.c at 0. Consider the open set $V = [-q, 0] + B(0, 1/4)$. Obviously, $G(0) \subseteq V$. On the other hand, for any neighbourhood $U$ of 0, there's always some $x' = (\xi_1', \xi_2', \dots) \in U$ such that $\xi_i < 0$ holds for some $i \in \mathbb{N}$. It follows that $x' \notin C$. Thence $G(x') = [q, \alpha_{x'} q] \nsubseteq V$. This together with the arbitrariness of $U$ that $G$ is not u.s.c at 0.

Similarly, from definitions of H-continuity and H-$C$-continuity, we have the following proposition.

**Proposition 2.11.** *Each H-continuous set-valued mapping is always H-$C$-continuous.*

*Remark* 2.12. The converse of Proposition 2.11 does not hold in general. We give the following Example 2.13 to show the case.

**Example 2.13.** (i) Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$. And the set-valued mapping $G\colon X \rightrightarrows Y$ is defined as follows:

$$G(x) = \begin{cases} [0, 1] & \text{if } x \in A, \\ [1, 2] & \text{if } x \notin A, \end{cases}$$

where $A$ is the set of all rational numbers in $\mathbb{R}$.

For any rational number $x_0$, it can be check easily that $G$ is H-$C$-u.s.c at $x_0$, but $G$ is not H-u.s.c at $x_0$.

(ii) Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$. And the set-valued mapping $G\colon X \rightrightarrows Y$ is defined as follows:

$$G(x) = \begin{cases} [0, 2] & \text{if } x \geq 0, \\ (0, 1] & \text{if } x < 0. \end{cases}$$

It can be check easily that $G$ is H-$C$-l.s.c at 0, but $G$ is not H-l.s.c at 0.

According to the corresponding results (see [3, Lemma 2.2.3], [15, Lemma 2.6], [39, Proposition 2.2 and Remark 2.2], Proposition 2.11, Remark 2.12), the relations between several types of upper/lower semicontinuity of a set-valued mapping $G$ at $x$ under the condition P (i.e., $G(x)$ is compact) are shown as Figure 2.1.

$$
\begin{tikzcd}
\text{u.s.c} \arrow[rr, shift left] \arrow[dr] & & \text{H-u.s.c} \arrow[ll, shift left, "\text{P}"] \arrow[dl] & \text{H-l.s.c} \arrow[rr, shift left] \arrow[dr] & & \text{l.s.c} \arrow[ll, shift left, "\text{P}"] \arrow[dl] \\
& \text{H-C-u.s.c} & & & \text{H-C-l.s.c} &
\end{tikzcd}
$$

Figure 2.1: The relations between several types of upper/lower semicontinuity.

In the study of stability of (SOP), we need also the following necessary background on convergence and convexity.

**Definition 2.14.** [35, p. 111, Section B, Chapter 4] Let $\{A_n\}$ be a sequence of nonempty subsets of $X$ and $A \subseteq X$. We say that the sequence $\{A_n\}$ converges in the sense of Painlevé–Kuratowski to A ($A_n \xrightarrow{PK} A$) if

$$\mathrm{Ls}\, A_n \subseteq A \subseteq \mathrm{Li}\, A_n,$$

where

$$\mathrm{Ls}\, A_n = \left\{ a \in X : a = \lim_{k\to\infty} a_{n_k}, a_{n_k} \in A_{n_k} \right\}, \quad \mathrm{Li}\, A_n = \left\{ a \in X : a = \lim_{n\to\infty} a_n, a_n \in A_n \right\}.$$

The inclusion relation $\mathrm{Ls}\, A_n \subseteq A$ is referred as the upper part of Painlevé–Kuratowski convergence, denoted by $A_n \overset{PK}{\rightharpoonup} A$, and $A \subseteq \mathrm{Li}\, A_n$ is referred as the lower part of Painlevé–Kuratowski convergence, written as $A_n \overset{PK}{\rightharpoondown} A$. Clearly, $\mathrm{Li}\, A_n \subseteq \mathrm{Ls}\, A_n$.

The other convergent concept for sequence of sets is Hausdorff convergence. For $x \in X$ and two nonempty subsets $A$ and $B$ of $X$, set

$$h(x, A) = \inf_{a\in A} d(x, a) \quad \text{and} \quad e(A, B) = \sup_{a\in A} h(a, B).$$

A sequence of nonempty subsets $\{A_n\}$ of $X$ converges to a set $A \subseteq X$ in the sense of Hausdorff ($A_n \xrightarrow{H} A$) if and only if $e(A_n, A) \to 0$ and $e(A, A_n) \to 0$. The condition $e(A_n, A) \to 0$ is the upper part of Hausdorff convergence, and denoted by $A_n \overset{H}{\rightharpoonup} A$, while the conclude relation $e(A, A_n) \to 0$ is the lower part of Hausdorff convergence, and denoted by $A_n \overset{H}{\rightharpoondown} A$.

**Lemma 2.15.** [18, Lemma 2.3] *Let $\{A_n\}$ be a sequence of nonempty subsets of $\mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$. Then $A \subseteq \mathrm{Li}\, A_n$ if and only if for any open set $W$ with $W \cap A \neq \emptyset$, there exists $N \in \mathbb{N}$, such that $A_n \cap W \neq \emptyset$ for any $n > N$.*

**Lemma 2.16.** [19, Lemma 2.1 and Corollary 2.1] *Let $A$ be a nonempty subset of $X$ and $\{A_n\}$ be a sequence of nonempty subsets of $X$. Then the following assertions hold:*

(i) *if $A_n \overset{H}{\rightharpoonup} A$ and $A$ is closed, then $\mathrm{Ls}\, A_n \subseteq A$;*

(ii) $A_n \xrightarrow{H} A$ *if and only if for any* $\varepsilon > 0$, *there exists* $N_\varepsilon \in \mathbb{N}$ *such that*

$$A_n \subseteq A + \varepsilon \mathbb{B}_X, \quad \forall n > N_\varepsilon,$$

*where* $\mathbb{B}_X$ *is the open unit ball with center* $0 \in X$.

**Definition 2.17.** [15, Definition 2.3] Let $A$ be a nonempty convex subset of a real normed vector space $X$, and $C$ be a pointed, closed and convex cone with nonempty interior $\operatorname{int} C$ in a real normed vector space $Y$. A set-valued mapping $G\colon A \rightrightarrows Y$ is said to be

(i) natural quasi $C$-convex on $A$ if for any $x, y \in A$ and for any $t \in [0,1]$, there exists some $\lambda \in [0,1]$ such that

$$\lambda G(x) + (1-\lambda)G(y) \subseteq G(tx + (1-t)y) + C;$$

(ii) strictly natural quasi $C$-convex on $A$ if for any $x, y \in A$ with $x \neq y$ and for any $t \in (0,1)$, there exists some $\lambda \in [0,1]$ such that

$$\lambda G(x) + (1-\lambda)G(y) \subseteq G(tx + (1-t)y) + \operatorname{int} C.$$

**Lemma 2.18.** [15, Lemma 2.5] *Let* $A$ *and* $B$ *be two nonempty subsets of real normed vector space* $Y$. *If* $0 < \alpha < \beta$, $B$ *is convex and* $A + \beta\mathbb{B}_Y \subseteq B + \alpha\mathbb{B}_Y$, *then* $A \subseteq \operatorname{int} B$.

## 3. Some characterizations on solution sets and the level set for (SOP)

In this section, we devote to discuss some properties of the level set and solution sets for (SOP). We start with the following two lemmas.

**Lemma 3.1.** *Assume that* $S+S \subseteq S$, $0 \in S$, $K$ *is nonempty compact and for each* $x \in K$, *the set* $\{y \in K : F(y) \leq_S F(x)\}$ *is closed. Then* $E(K) \neq \emptyset$.

*Proof.* It follows from $S + S \subseteq S$, $0 \in S$ and Theorem 5.1 in [9] that $E(K) \neq \emptyset$. □

**Lemma 3.2.** *Assume that* $S + S \subseteq S$, $0 \notin \operatorname{int} S$, $x_0 \in K$.

(i) *For any* $\varepsilon > 0$, $x_0 \in E(\varepsilon, K)$ *if and only if there does not exist any* $y \in K$ *such that* $F(y) \leq_S^\varepsilon F(x_0)$;

(ii) *For any* $\varepsilon > 0$, $x_0 \in W(\varepsilon, K)$ *if and only if there does not exist any* $y \in K$ *such that* $F(y) <_S^\varepsilon F(x_0)$.

*Proof.* (i) The sufficiency is straightforward. We only need to prove the necessity. Let $u \in \operatorname{int} S^{\infty}$. Suppose to the contrary that there exists $y \in K$ such that $F(y) \leq_S^{\varepsilon} F(x_0)$, that is,

$$F(x_0) \subseteq F(y) + S + \varepsilon u. \tag{3.1}$$

Since $x_0 \in E(\varepsilon, K)$, we have $F(x_0) \leq_S^{\varepsilon} F(y)$, i.e.,

$$F(y) \subseteq F(x_0) + S + \varepsilon u. \tag{3.2}$$

From Remark 2.3,

$$S + \operatorname{int} S^{\infty} \subseteq \operatorname{int} S. \tag{3.3}$$

It follows from (3.1), (3.2) and (3.3) that

$$\begin{aligned} F(x_0) &\subseteq F(y) + S + \varepsilon u \subseteq F(x_0) + S + S + 2\varepsilon u \\ &\subseteq F(x_0) + S + S + \operatorname{int} S^{\infty} + \varepsilon u \subseteq F(x_0) + S + \operatorname{int} S + \varepsilon u \\ &\subseteq F(x_0) + \operatorname{int} S + \varepsilon u, \end{aligned}$$

which is a contradiction as $0 \notin \operatorname{int} S$ and $F(x_0)$ is bounded.

Exploiting the lines of the proof of (i), the statement (ii) holds and we omit it here. □

The following lemmas are necessary for us to establish our main results.

**Lemma 3.3.** *Assume that $S + S \subseteq S$, $0 \notin \operatorname{int} S$. Then*

(i) *for any $\varepsilon \geq 0$, $E(K) \subseteq E(\varepsilon, K)$ and $W(K) \subseteq W(\varepsilon, K)$;*

(ii) *for any $\varepsilon \geq 0$, $E(\varepsilon, K) \subseteq W(\varepsilon, K)$;*

(iii) *for any $\varepsilon > 0$, $W(K) \subseteq E(\varepsilon, K)$.*

*Proof.* Let $u \in \operatorname{int} S^{\infty}$.

(i) The conclusion is obvious when $\varepsilon = 0$. Let $\varepsilon > 0$ and $x \in E(K)$. For any $y \in K$, we consider the following two cases:

*Case* 1*:* $F(y) \leq_S F(x)$. We conclude that $F(y) \not\leq_S^{\varepsilon} F(x)$. If not, $F(x) \subseteq F(y) + S + \varepsilon u$, it follows from $x \in E(K)$ that $F(x) \subseteq F(x) + S + S + \varepsilon u \subseteq F(x) + \operatorname{int} S + \frac{\varepsilon}{2} u$. It is impossible.

*Case* 2*:* $F(y) \not\leq_S F(x)$. That is, $F(x) \not\subseteq F(y) + S$. By the relation $S + \varepsilon u \subseteq S$, we deduce

$$F(x) \not\subseteq F(y) + S + \varepsilon u,$$

which means that $F(y) \not\leq_S^{\varepsilon} F(x)$.

From Cases 1 and 2, one arrives at $F(y) \not\leq_S^{\varepsilon} F(x)$ for any $y \in K$. By virtue of Lemma 3.2(i), $x \in E(\varepsilon, K)$. Therefore $E(K) \subseteq E(\varepsilon, K)$ for each $\varepsilon \geq 0$. Similarly, one can prove that $W(K) \subseteq W(\varepsilon, K)$.

(ii) Let's take any $x \in E(\varepsilon, K)$ and any $y \in K$.

Case 1: $\varepsilon = 0$, we have $E(0, K) = E(K)$, and $W(0, K) = W(K)$. If $F(y) \not<_S F(x)$, $x \in W(K)$ trivially holds. If $F(y) <_S F(x)$, then $F(x) \subseteq F(y) + \operatorname{int} S \subseteq F(y) + S$, and so $F(y) \leq_S F(x)$. From $x \in E(K)$, we can get $F(x) \leq_S F(y)$. This implies that

$$F(y) \subseteq F(x) + S \subseteq F(y) + S + \operatorname{int} S \subseteq F(x) + S + S + \operatorname{int} S \subseteq F(x) + \operatorname{int} S,$$

i.e., $F(x) <_S F(y)$. Therefore $x \in W(K)$.

Case 2: $\varepsilon > 0$, by Lemma 3.2(i), $F(y) \not\leq_S^{\varepsilon} F(x)$, that is, $F(x) \not\subseteq F(y) + S + \varepsilon u$. This together with $\operatorname{int} S \subseteq S$ implies that $F(x) \not\subseteq F(y) + \operatorname{int} S + \varepsilon u$. From Lemma 3.2(ii) and the arbitrariness of $y$, we obtain $x \in W(\varepsilon, K)$. Hence, the relation $E(\varepsilon, K) \subseteq W(\varepsilon, K)$ holds for any $\varepsilon \geq 0$.

(iii) Let $\varepsilon > 0$, $y \in K$ be fixed and $x \in W(K)$. We need to show that $x \in E(\varepsilon, K)$. From $x \in W(K)$, we have $F(y) \not<_S F(x)$ or $F(y) <_S F(x)$ implies that $F(x) <_S F(y)$.

When $F(y) \not<_S F(x)$, we have $F(x) \not\subset F(y) + \operatorname{int} S$ by the definition of the relation $\not<_S$. It follows from $S + \varepsilon u \subset \operatorname{int} S$ that $F(x) \not\subset F(y) + S + \varepsilon u$, that is, $F(y) \not\leq_S^{\varepsilon} F(x)$. Therefore, $x \in E(\varepsilon, K)$ because $y \in K$ is arbitrary.

For the other case, we have $F(y) \subset F(x) + \operatorname{int} S$ and $F(x) \subset F(y) + \operatorname{int} S$. We next need to consider two cases. One is $F(x) \not\subset F(y) + S + \varepsilon u$, i.e., $F(y) \not\leq_S^{\varepsilon} F(x)$, which implies $x \in E(\varepsilon, K)$. And the second is $F(x) \subset F(y) + S + \varepsilon u$, i.e., $F(y) \leq_S^{\varepsilon} F(x)$. Then

$$\begin{aligned} F(y) &\subset F(x) + \operatorname{int} S \subset F(y) + S + \operatorname{int} S + \varepsilon u \\ &\subset F(x) + S + \operatorname{int} S + \operatorname{int} S + \varepsilon u \subset F(x) + S + \varepsilon u, \end{aligned}$$

i.e., $F(x) \leq_S^{\varepsilon} F(y)$. It follows from Definition 2.5(ii) and $y$ is arbitrary that $x \in E(\varepsilon, K)$. □

*Remark* 3.4. Lemma 3.3 extends the corresponding results (i.e., Remarks 2.3–2.5) of [18] from the cone case to the general ordering set case.

**Lemma 3.5.** *Assume that* $0 \notin \operatorname{int} S$, $S + S \subset S$. *Then, for any* $\varepsilon \geq 0$, $x_0 \in K$,

$$E(\varepsilon, L_F(\varepsilon, x_0, K)) \subseteq E(\varepsilon, K).$$

*Proof.* If $\varepsilon = 0$, the conclusion is straightforward.

If $\varepsilon > 0$, by contradiction, let

$$v \in E(\varepsilon, L_F(\varepsilon, x_0, K)) \tag{3.4}$$

and $v \notin E(\varepsilon, K)$. Then, there exists $z \in K$ such that $F(z) \leq_S^\varepsilon F(v)$, i.e.,

$$F(v) \subseteq F(z) + S + \varepsilon u. \tag{3.5}$$

We claim that $z \in L_F(\varepsilon, x_0, K)$. As $v \in E(\varepsilon, L_F(\varepsilon, x_0, K))$, we need to consider the following two cases:

*Case* 1*:* $v = x_0$. Then $F(z) \leq_S^\varepsilon F(v) = F(x_0)$.

*Case* 2*:* $v \neq x_0$. Then $F(x_0) \subseteq F(v) + S + \varepsilon u$. It follows from (3.5) that

$$F(x_0) \subseteq F(v) + S + \varepsilon u \subseteq F(z) + S + 2\varepsilon u \subseteq F(z) + S + \varepsilon u.$$

Thus, we can get $z \in L_F(\varepsilon, x_0, K)$ from Cases 1 and 2. It follows from $z \in L_F(\varepsilon, x_0, K)$, $F(z) \leq_S^\varepsilon F(v)$ and Lemma 3.2(i) that $v \notin E(\varepsilon, L_F(\varepsilon, x_0, K))$, which contradicts (3.4). □

**Lemma 3.6.** *Assume that* $\varepsilon > 0$*,* $S + S \subseteq S$*,* $0 \notin \operatorname{int} S$*. Then, for any* $x_0 \in K$*,*

$$x_0 \in E(\varepsilon, K) \iff L_F(\varepsilon, x_0, K) = \{x_0\}.$$

*Proof.* By virtue of Lemma 3.2(i), the necessity is true. Now, we prove that the sufficiency. As $\varepsilon > 0$ and $0 \notin \operatorname{int} S$, we have $F(x) \not\leq_S^\varepsilon F(x)$ for each $x \in K$. It follows from $L_F(\varepsilon, x_0, K) = \{x_0\}$ that $\{y \in K : F(y) \leq_S^\varepsilon F(x_0)\}$ is empty. Therefore, there does not exist $y \in K$ such that $F(y) \leq_S^\varepsilon F(x_0)$. In view of Lemma 3.2(i), $x_0 \in E(\varepsilon, K)$. □

**Lemma 3.7.** *Assume that* $K$ *is closed,* $F$ *is H-*$S^\infty$*-u.s.c on* $K$ *with* $S$*-closed values (i.e., for each* $x \in K$*,* $F(x) + S$ *is closed). Then,* $L_F(\varepsilon, x, K)$ *is closed for all* $\varepsilon \geq 0$ *and* $x \in K$*.*

*Proof.* Let $\varepsilon \geq 0$ and $x \in K$. From the definition of $L_F(\varepsilon, x, K)$, we only need to show that the set $M := \{y \in K : F(y) \leq_S^\varepsilon F(x)\}$ is closed. Let sequence $\{x_n\} \subseteq M$ with $x_n \to x_0 \in X$. Clearly, $x_n \in K$. By the closeness of $K$, $x_0 \in K$. On the other hand, for any $n \in \mathbb{N}$, $F(x_n) \leq_S^\varepsilon F(x)$, that is,

$$F(x) \subseteq F(x_n) + S + \varepsilon u, \quad n = 1, 2, \ldots. \tag{3.6}$$

As $F$ is H-$S^\infty$-u.s.c at $x_0$, for any neighbourhood $V$ of origin 0 in $Y$, when $n$ is sufficiently large, one has

$$F(x_n) \subseteq F(x_0) + V + S^\infty. \tag{3.7}$$

By (3.6) and (3.7), for $n$ sufficiently large,

$$F(x) \subseteq F(x_n) + S + \varepsilon u \subseteq F(x_0) + V + S + S^\infty + \varepsilon u.$$

It follows from Lemma 2.1 and the arbitrariness of $V$ that

$$F(x) \subseteq F(x_0) + S + \varepsilon u.$$

That is, $F(x_0) \leq_S^\varepsilon F(x)$, i.e., $x_0 \in M$. This proves that $M$ is closed. Hence, $L_F(\varepsilon, x, K)$ is closed for any $\varepsilon \geq 0$. □

## 4. Painlevé–Kuratowski convergence of approximate solution sets

In this section, under some suitable conditions, we study the upper Painlevé–Kuratowski convergence of minimal approximate solution sets and Painlevé–Kuratowski convergence of weak minimal approximate solution sets for (SOP) under the case that the feasible set is perturbed. In the rest of this paper, we always assume that $X$ is a finite-dimensional space $\mathbb{R}^n$.

**Lemma 4.1.** *Let $\{\varepsilon_n\} \subseteq \mathbb{R}_+$ with $\varepsilon_n \searrow \varepsilon_0$. $\{K_n\}$ is a sequence of nonempty sets in $X$, and $K \subseteq X$ is bounded. Assume that*

(i) *$S + S \subseteq S$, $0 \notin \operatorname{int} S$;*

(ii) *$\operatorname{Ls} K_n \subseteq K$, and there exist $\delta > 0$ and $N \in \mathbb{N}$ such that for any $n > N$, $K_n \subseteq K + \delta\mathbb{B}_X$;*

(iii) *$F$ is H-$S^\infty$-continuous with $S$-closed values on $K$.*

*Then, for any $\alpha > 0$, $x \in K$ and each sequence $\{x_n \in K_n : n \in \mathbb{N}\}$ with $x_n \to x$, there exists $n_0 \in \mathbb{N}$ such that*

$$L_F(\varepsilon_n, x_n, K_n) \subseteq L_F(\varepsilon_0, x, K) + \alpha\mathbb{B}_X, \quad \forall\, n > n_0.$$

*Proof.* Let $u \in \operatorname{int} S^\infty$. Suppose to the contrary, without loss of generality, that there exists $\alpha_0 > 0$ such that

$$L_F(\varepsilon_n, x_n, K_n) \nsubseteq L_F(\varepsilon_0, x, K) + \alpha_0\mathbb{B}_X, \quad \forall\, n \in \mathbb{N}.$$

This implies for each $n \in \mathbb{N}$, there exists

$$y_n \in L_F(\varepsilon_n, x_n, K_n), \tag{4.1}$$

but

$$y_n \notin L_F(\varepsilon_0, x, K) + \alpha_0\mathbb{B}_X. \tag{4.2}$$

Obviously, $y_n \in K_n$. From $x_n \to x$ and $x \in L_F(\varepsilon_0, x, K)$, there exists $N_1 \in \mathbb{N}$ such that

$$x_n \in L_F(\varepsilon_0, x, K) + \alpha_0\mathbb{B}_X, \quad \forall\, n > N_1. \tag{4.3}$$

By (4.2) and (4.3), we obtain that $x_n \neq y_n$ when $n > N_1$. It follows from (4.1) that

$$F(x_n) \subseteq F(y_n) + S + \varepsilon_n u, \quad \forall\, n > N_1. \tag{4.4}$$

From (ii) and $K$ is bounded, $\{y_n\}$ is bounded. Without loss the generality, we can assume that $y_n \to y_0 \in X$. Combining this with $\operatorname{Ls} K_n \subseteq K$, we have $y_0 \in K$.

Next, we show that $F(x) \subseteq F(y_0)+S+\varepsilon_0 u$. Note that $-\theta u+\operatorname{int} S^\infty$ is a neighbourhood $V$ of origin 0 in $Y$ for any $\theta > 0$. Since $F$ is H-$S^\infty$-l.s.c at $x$, there exists a neighbourhood $U_x$ of $x$ such that

$$F(x) \subseteq F(x') - \theta u + \operatorname{int} S^\infty + S^\infty, \quad \forall\, x' \in U_x. \tag{4.5}$$

Due to $F$ is H-$S^\infty$-u.s.c at $y_0$, there exists a neighbourhood $U_{y_0}$ of $y_0$ such that

$$F(y') \subseteq F(y_0) - \theta u + \operatorname{int} S^\infty + S^\infty, \quad \forall\, y' \in U_{y_0}. \tag{4.6}$$

It follows from $x_n \to x$ and $y_n \to y_0$ that there exists $N_2 \in \mathbb{N}$ such that $x_n \in U_x$ and $y_n \in U_{y_0}$ for $n > N_2$. This together with (4.4)–(4.6) that, for $n > \max\{N_1, N_2\}$, we have

$$\begin{aligned} F(x) &\subseteq F(x_n) - \theta u + \operatorname{int} S^\infty + S^\infty \subseteq F(y_n) + S + \varepsilon_n u - \theta u + \operatorname{int} S^\infty + S^\infty \\ &\subseteq F(y_0) - 2\theta u + \varepsilon_n u + 2 \operatorname{int} S^\infty + 2S^\infty + S \subseteq F(y_0) - 2\theta u + \varepsilon_n u + \operatorname{int} S. \end{aligned}$$

Since $\theta > 0$ is arbitrary, $\varepsilon_n \searrow \varepsilon_0$ and $F$ is $S$-closed on $K$, let $\theta \to 0$ and $n \to \infty$, we obtain

$$F(x) \subseteq F(y_0) + S + \varepsilon_0 u,$$

which implies that $y_0 \in L_F(\varepsilon_0, x, K)$. It follows from $y_n \to y_0$ that $y_n \in L_F(\varepsilon_0, x, K) + \alpha_0 \mathbb{B}_X$ for $n$ sufficiently large, which contradicts (4.2). □

*Remark* 4.2. It follows from Lemma 2.16 that Lemma 4.1 remain actually valid if the assumption (ii) is replaced by any of the following assumptions:

(a) $\operatorname{Ls} K_n \subseteq K$ and $\{K_n\}$ is uniform bounded (i.e., there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $K_n \subseteq \delta \mathbb{B}_X$ for each $n > n_0$);

(b) $K$ is closed and $K_n \xrightarrow{H} K$.

**Lemma 4.3.** *Let $\{\varepsilon_n\} \subseteq \mathbb{R}_+$ with $\varepsilon_n \nearrow \varepsilon_0$, $\{K_n\}$ be a sequence of nonempty sets in $X$ and $K \subseteq X$ with $\operatorname{Ls} K_n \subseteq K$, $S + S \subseteq S$, $0 \notin \operatorname{int} S$. For any $x_0, y_0 \in K$, any sequence $\{x_n \in K_n : n \in \mathbb{N}\}$ with $x_n \to x_0$ and $\{y_n \in K_n : n \in \mathbb{N}\}$ with $y_n \to y_0$, suppose that*

(i) *$F$ is H-$S^\infty$-u.s.c at $x_0$ and H-$S^\infty$-l.s.c at $y_0$;*

(ii) *$F(y_n)$ is $S$-convex (i.e., $F(y_n) + S$ is convex) for each $n \in \mathbb{N}$;*

(iii) *$F(x_0)$ is compact, and $F(x_0) \subseteq F(y_0) + \operatorname{int} S + \varepsilon_0 u$ with $u \in \operatorname{int} S^\infty$.*

*Then, there exists $n_0 \in \mathbb{N}$ such that*

$$F(x_n) \subseteq F(y_n) + \operatorname{int} S + \varepsilon_n u, \quad \forall\, n > n_0.$$

*Proof.* Let $u \in \operatorname{int} S^{\infty}$. From $F(x_0) \subseteq F(y_0)+\operatorname{int} S+\varepsilon_0 u$ and the openness of $z+\operatorname{int} S+\varepsilon_0 u$ for any $z \in F(y_0)$, we have that

$$\{z + \operatorname{int} S + \varepsilon_0 u : z \in F(y_0)\}$$

is an open covering of $F(x_0)$. Since $F(x_0)$ is compact, there exists a finite subset $\{z_1, z_2, \ldots, z_m\}$ of $F(y_0)$ such that

$$F(x_0) \subseteq \bigcup_{i=1}^{m} \{z_i + \operatorname{int} S + \varepsilon_0 u\} = \{z_1, z_2, \ldots, z_m\} + \operatorname{int} S + \varepsilon_0 u.$$

Obviously, $\{z_1, z_2, \ldots, z_m\} + \operatorname{int} S + \varepsilon_0 u$ is open. As a result, for some $\delta > 0$,

$$F(x_0) + 3\delta\mathbb{B}_Y \subseteq \{z_1, z_2, \ldots, z_m\} + \operatorname{int} S + \varepsilon_0 u. \tag{4.7}$$

Note that $F$ is H-$S^{\infty}$-u.s.c at $x_0$ and $F(x_0)$ is compact, it follows from Lemma 2.6 in [15], $F$ is $S^{\infty}$-u.s.c at $x_0$. Therefore, for $\mathbb{B}_Y$, there exists a neighbourhood $U_1$ of $x_0$ such that

$$F(x) \subseteq F(x_0) + \delta\mathbb{B}_Y + S^{\infty}, \quad \forall\, x \in U_1.$$

By $x_n \to x_0$, there exists $N_1 \in \mathbb{N}$ such that $x_n \in U_1$ for any $n > N_1$. That is,

$$F(x_n) \subseteq F(x_0) + \delta\mathbb{B}_Y + S^{\infty}, \quad \forall\, n > N_1. \tag{4.8}$$

From $F$ is H-$S^{\infty}$-l.s.c at $y_0$ and Lemma 2.6 in [15], $F$ is $S^{\infty}$-l.s.c at $y_0$. For $\mathbb{B}_Y$ and any $i \in \{1, 2, \ldots, m\}$, there exists a neighbourhood $U^i(y_0)$ of $y_0$ such that

$$F(y) \cap (z_i + \delta\mathbb{B}_Y - S^{\infty}) \neq \emptyset, \quad \forall\, y \in U^i(y_0).$$

Especially, let $U_2(y_0) = \bigcap_{i=1}^{m} U^i(y_0)$, by $y_n \to y_0$, then there exists $N_2 \in \mathbb{N}$ such that

$$F(y_n) \cap (z_i + \delta\mathbb{B}_Y - S^{\infty}) \neq \emptyset, \quad \forall\, i \in \{1, 2, \ldots, m\},\ \forall\, n > N_2. \tag{4.9}$$

Let $N_3 = \max\{N_1, N_2\}$. We now claim that for any $n > N_3$, we have

$$F(x_n) \subseteq F(y_n) + \operatorname{int} S + \varepsilon_n u.$$

Taking into account (4.7) and (4.8), for $n > N_3$,

$$F(x_n) + 2\delta\mathbb{B}_Y \subseteq F(x_0) + 3\delta\mathbb{B}_Y + S^{\infty} \subseteq \{z_1, z_2, \ldots, z_m\} + \operatorname{int} S + \varepsilon_0 u. \tag{4.10}$$

By (4.10), for each $h_n \in F(x_n) + 2\delta\mathbb{B}_Y$, there exist $k_n \in \{1, 2, \ldots, m\}$ and $s'_n \in \operatorname{int} S$ such that

$$h_n = z_{k_n} + s'_n + \varepsilon_0 u. \tag{4.11}$$

From (4.9), $F(y_n) \cap (z_k + \delta\mathbb{B}_Y - S^\infty) \neq \emptyset$ for any $n > N_3$. Thence, for any fixed $n > N_3$, there exist $v_n \in F(y_n)$, $b_n \in \mathbb{B}_Y$ and $u'_n \in S^\infty$ such that

$$v_n = z_{k_n} + \delta b_n - u'_n. \tag{4.12}$$

It follows from (4.11) and (4.12) that

$$h_n = v_n - \delta b_n + s'_n + u'_n + \varepsilon_0 u \in F(y_n) + \delta\mathbb{B}_Y + \operatorname{int} S + \varepsilon_0 u, \quad \forall\, n > N_3.$$

Applying the arbitrariness of $h_n \in F(x_n) + 2\delta\mathbb{B}_Y$, we derive that

$$F(x_n) + 2\delta\mathbb{B}_Y \subseteq F(y_n) + \delta\mathbb{B}_Y + \operatorname{int} S + \varepsilon_0 u \subseteq F(y_n) + \delta\mathbb{B}_Y + S + \varepsilon_0 u, \quad \forall\, n > N_3.$$

By hypothesis (ii), $F(y_n) + S + \varepsilon_0 u$ is convex. According to Lemma 2.18, one has

$$F(x_n) \subseteq F(y_n) + \operatorname{int} S + \varepsilon_0 u, \quad \forall\, n > N_3.$$

Consequently,

$$F(x_n) \subseteq F(y_n) + \operatorname{int} S + U(\varepsilon_0 u), \quad \forall\, n > N_3,$$

where $U(\varepsilon_0 u)$ is neighbourhood of $\varepsilon_0 u$. By applying $\varepsilon_n \nearrow \varepsilon_0$, there exists $N_4 \in \mathbb{N}$ such that $\varepsilon_n u \in U(\varepsilon_0 u)$ for any $n > N_4$. Let $N_0 = \max\{N_3, N_4\}$, one has

$$F(x_n) \subseteq F(y_n) + \operatorname{int} S + \varepsilon_n u, \quad \forall\, n > N_0.$$

This completes the proof. □

Next, we discuss the upper Painlevé–Kuratowski convergence of weak minimal approximate solution sets.

**Theorem 4.4.** *Let $\{\varepsilon_n\} \subseteq \mathbb{R}_+$ with $\varepsilon_n \nearrow \varepsilon_0 > 0$, $S + S \subseteq S$, $0 \notin \operatorname{int} S$, $\{K_n\}$ be a sequence of nonempty sets in $X$ and $K \subseteq X$ with $K_n \xrightarrow{PK} K$. For any $x_0, y_0 \in K$, any sequence $\{x_n \in K_n : n \in \mathbb{N}\}$ with $x_n \to x_0$ and $\{y_n \in K_n : n \in \mathbb{N}\}$ with $y_n \to y_0$, suppose that*

(i) *$F(x_0)$ is compact;*

(ii) *$F(y_n)$ is $S$-convex for each $n \in \mathbb{N}$;*

(iii) *$F$ is $H$-$S^\infty$-u.s.c at $x_0$ and $H$-$S^\infty$-l.s.c at $y_0$.*

*Then,* $\operatorname{Ls} W(\varepsilon_n, K_n) \subseteq W(\varepsilon_0, K)$.

*Proof.* For any $x \in \operatorname{Ls} W(\varepsilon_n, K_n)$, according to the definition of $\operatorname{Ls} W(\varepsilon_n, K_n)$, there exists $x_{n_k} \in W(\varepsilon_{n_k}, K_{n_k})$ such that $x_{n_k} \to x$. Obviously, $x_{n_k} \in K_{n_k}$. Hence $x \in K$ due to $\operatorname{Ls} K_{n_k} \subseteq K$.

Next, we show that $x \in W(\varepsilon_0, K)$. If not, assume that $x \notin W(\varepsilon_0, K)$. Then, by Lemma 3.2, there exists $y \in K$ such that $F(y) <_S^{\varepsilon_0} F(x)$, that is,

$$F(x) \subseteq F(y) + \operatorname{int} S + \varepsilon_0 u.$$

By virtue of $K \subseteq \operatorname{Li} K_n$, there exists $y_n \in K_n$ such that $y_n \to y$. From Lemma 4.3, there exists $k_0 \in \mathbb{N}$ such that

$$F(x_{n_k}) \subseteq F(y_{n_k}) + \operatorname{int} S + \varepsilon_{n_k} u, \quad \forall\, k > k_0.$$

It follows that for any $k > k_0$, $F(y_{n_k}) <_S^{\varepsilon_0} F(x_{n_k})$, which contradicts $x_{n_k} \in W(\varepsilon_{n_k}, K_{n_k})$. Therefore, $\operatorname{Ls} W(\varepsilon_n, K_n) \subseteq W(\varepsilon_0, K)$. □

Next, we verify Theorem 4.4 by the following example.

**Example 4.5.** Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $S = \{(1/3, 1/3)\} + \mathbb{R}^2_+$, $u = (1,1)$, $\varepsilon_0 = 2/3$, $\varepsilon_n = 2/3 - 1/(2n)$, $K = [-1, 3]$, $K_n = [-1-1/n, 3+1/n]$, $A = \{t(0,0) + (1-t)(0,1) : t \in [0,1]\}$. The mapping $F \colon X \rightrightarrows Y$ is defined as

$$F(x) = \{(|x|, x)\} + A.$$

It is easy to see that $\varepsilon_n \nearrow \varepsilon_0$ and all the other assumptions in Theorem 4.4 are satisfied. By calculation, $W(\varepsilon_0, K) = [-1, 1]$ and $W(\varepsilon_n, K_n) = [-1 - 1/n, 1 - 1/(2n)]$. It is not difficult to verify that $\operatorname{Ls} W(\varepsilon_n, K_n) = \operatorname{Ls}[-1-1/n, 1-1/(2n)] \subseteq [-1, 1] = W(\varepsilon_0, K)$. Hence, Theorem 4.4 is applicable.

*Remark* 4.6. Compared to Theorem 3.1 in [18], Theorem 4.4 extended the order cone to the general ordering set. Moreover, the continuity of objective mapping is weakened to the Hausdorff cone continuity.

Next, we establish the lower Painlevé–Kuratowski convergence results for minimal approximate solution sets and weak minimal approximate solution sets.

**Theorem 4.7.** *Let $\{\varepsilon_n\} \subseteq \mathbb{R}_+$ with $\varepsilon_n \searrow \varepsilon_0 > 0$, $\{K_n\}$ be a sequence of nonempty sets in $X$ and be uniform bounded. Suppose that*

(i) *$S + S \subseteq S$ and $0 \in \partial S$;*

(ii) *$K_n$ and $K$ are compact with $K_n \xrightarrow{PK} K$;*

(iii) *$F$ is H-$S^\infty$-continuous with $S$-closed values on $K_n$;*

(iv) *for any* $n \in \mathbb{N}$ *and* $x \in K_n$, $\{y \in K_n : F(y) \leq_S F(x)\}$ *is closed.*

*Then,* $E(\varepsilon_0, K) \subseteq \operatorname{Li} E(\varepsilon_n, K_n)$.

*Proof.* Taking any $x_0 \in E(\varepsilon_0, K)$. Obviously, $x_0 \in K$. By (ii), there exists $x_n \in K_n$ such that $x_n \to x_0$. It follows from Lemma 3.6 that $L_F(\varepsilon_0, x_0, K) = \{x_0\}$. By virtue of Lemma 4.1, for any $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$L_F(\varepsilon_n, x_n, K_n) \subseteq \{x_0\} + \delta \mathbb{B}_X, \quad \forall\, n > n_0. \tag{4.13}$$

As a result, for any $n > n_0$, $L_F(\varepsilon_n, x_n, K_n)$ is bounded. Similar to the proof of Lemma 3.7, one has $\{y_n \in L_F(\varepsilon_n, x_n, K_n) : F(y_n) \leq_S^{\varepsilon_n} F(x_n)\}$ is closed. Combining this with Lemmas 3.1 and 3.3, we get $E(\varepsilon_n, L_F(\varepsilon_n, x_n, K_n))$ is nonempty. Taking any $u_n \in E(\varepsilon_n, L_F(\varepsilon_n, x_n, K_n))$ $(n = 1, 2, \ldots)$, it follows from Lemma 3.5 that $u_n \in E(\varepsilon_n, K_n)$. Applying (4.13) and the arbitrariness of $\delta > 0$, one has $u_n \to x_0$, $\forall\, n > n_0$. Consequently, $x_0 \in \operatorname{Li} E(\varepsilon_n, K_n)$. Therefore, $E(\varepsilon_0, K) \subseteq \operatorname{Li} E(\varepsilon_n, K_n)$. $\square$

To verify Theorem 4.7, we give the following example.

**Example 4.8.** Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 0, x_2 \geq 2\} \cup \mathbb{R}^2_+$. Taking $\varepsilon_0 = 1/2$, $\varepsilon_n = 1/2 + 1/n$, $u = (1, 1)$, $K = [-1, 3]$ and $K_n = [-1, 3 + 1/n]$, $n = 1, 2, \ldots$. Define mapping $F\colon X \rightrightarrows Y$ as follows:

$$F(x) = \{(|x|, x - 1)\} + B,$$

where $B = \{z \in \mathbb{R}^2 : z = t(0, 0) + (1 - t)(0, 1), t \in [0, 1]\}$.

It can be verified that all the hypotheses in Theorem 4.7 are satisfied. By calculation, $S^\infty = \mathbb{R}^2_+$, $E(\varepsilon_0, K) = [-1, 0)$, $E(\varepsilon_n, K_n] = [-1, 1/n)$. Obviously, $E(\varepsilon_0, K) \subseteq \operatorname{Li} E(\varepsilon_n, K_n)$. Thus, Theorem 4.7 is valid.

**Theorem 4.9.** *Let* $\{\varepsilon_n\} \subseteq \mathbb{R}_+$ *with* $\varepsilon_n \searrow \varepsilon_0 > 0$, $\{K_n\}$ *is a sequence of nonempty sets in* $X$, $K \subseteq X$ *is convex. If* $S$ *is a closed, pointed convex cone in* $Y$ *and* $0 \in \partial S$. *Suppose that*

(i) $K_n$, $K$ *are compact and* $K_n \xrightarrow{PK} K$;

(ii) $F$ *is* $H$-$S^\infty$*-continuous and strictly natural quasi* $S^\infty$*-convex on* $K$;

(iii) *for any* $n \in \mathbb{N}$ *and* $x \in K_n$, $\{y \in K_n : F(y) \leq_S F(x)\}$ *is closed. Moreover,* $x \in K$, $\{y \in K : F(y) \leq_S F(x)\}$ *is also closed.*

*Then,* $W(\varepsilon_0, K) \subseteq \operatorname{Li} W(\varepsilon_n, K_n)$.

*Proof.* Let $u \in \operatorname{int} S^{\infty}$. From Lemma 3.1, $E(K)$ is nonempty. Therefore, $E(\varepsilon_0, K) \neq \emptyset$ by Lemma 3.3(i). For any open set $V$ with $W(\varepsilon_0, K) \cap V \neq \emptyset$, we deduce that

$$E(\varepsilon_0, K) \cap V \neq \emptyset. \tag{4.14}$$

Indeed, if not, suppose that $E(\varepsilon_0, K) \cap V = \emptyset$. Then for any $m_0 \in W(\varepsilon_0, K) \cap V$, $m_0 \notin E(\varepsilon_0, K)$. By Lemma 3.2, there exists $p_0 \in K$ such that $F(p_0) \leq_S^{\varepsilon_0} F(m_0)$, that is,

$$F(m_0) \subseteq F(p_0) + S + \varepsilon_0 u \subseteq F(p_0) + \operatorname{int} S. \tag{4.15}$$

We consider the following two cases:

*Case* 1*:* $m_0 \neq p_0$. From the strictly natural quasi $S^{\infty}$-convexity of $F$ on $K$, for any $t \in (0,1)$, there exists $\lambda \in [0,1]$ such that

$$\lambda F(m_0) + (1-\lambda) F(p_0) \subseteq F(tm_0 + (1-t)p_0) + \operatorname{int} S^{\infty}. \tag{4.16}$$

Next, we show that

$$F(m_0) \subseteq F(tm_0 + (1-t)p_0) + \operatorname{int} S^{\infty}, \quad \forall\, t \in (0,1). \tag{4.17}$$

Suppose, by contradiction, that there exist $q_0 \in F(m_0)$ and $t_0 \in (0,1)$ such that

$$q_0 \notin F(t_0 m_0 + (1-t_0)p_0) + \operatorname{int} S^{\infty}. \tag{4.18}$$

We can find $v_0 \in F(p_0)$ and $s_0 \in \operatorname{int} S$ such that $q_0 = v_0 + s_0$ by (4.15). It follows from (4.16) and Lemma 2.1(v) that $q_0 \in F(t_0 m_0 + (1-t_0)p_0) + \operatorname{int} S^{\infty}$, which contradicts (4.18). So (4.17) holds.

Since $m_0 \in V$ and $V$ is open, there exists some $t' \in (0,1)$ such that $t'm_0 + (1-t')p_0 \in V$. As a results, $t'm_0 + (1-t')p_0 \notin E(\varepsilon_0, K)$. By virtue of Lemma 3.2, there is $w_0 \in K$ such that $F(w_0) \leq_S^{\varepsilon_0} F(t'm_0 + (1-t')p_0)$, that is,

$$F(t'm_0 + (1-t')p_0) \subseteq F(w_0) + S + \varepsilon_0 u.$$

From (4.17), one has

$$\begin{aligned} F(m_0) &\subseteq F(t'm_0 + (1-t')p_0) + \operatorname{int} S^{\infty} \subseteq F(w_0) + S + \operatorname{int} S^{\infty} + \varepsilon_0 u \\ &\subseteq F(w_0) + \operatorname{int} S + \varepsilon_0 u, \end{aligned}$$

i.e., $F(w_0) <_S^{\varepsilon_0} F(m_0)$, which contradicts $m_0 \in W(\varepsilon_0, K)$.

*Case* 2*:* $m_0 = p_0$. Similarly, it is easy to prove that there is $w_0 \in K$ such that $F(w_0) <_S^{\varepsilon_0} F(m_0)$. This contradicts $m_0 \in W(\varepsilon_0, K)$.

Therefore, (4.14) is true. From Theorem 4.7 and Lemma 2.15, there exists $N_0 \in \mathbb{N}$ such that

$$V \cap E(\varepsilon_n, K_n) \neq \emptyset, \quad \forall\, n > N_0. \tag{4.19}$$

By Lemma 3.3(ii), $E(\varepsilon_n, K_n) \subseteq W(\varepsilon_n, K_n)$. Therefore, for $n > N_0$, we derive $V \cap W(\varepsilon_n, K_n) \neq \emptyset$ by (4.19). Taking into account Lemma 2.15, we conclude that $W(\varepsilon_0, K) \subseteq \operatorname{Li} W(\varepsilon_n, K_n)$. □

*Remark* 4.10. In 2020, by using natural quasi cone-convexity and continuity for objective mappings, Han et al. obtained the Painlevé–Kuratowski convergence of the approximate solution sets for SOP (see [18]). Theorems 4.7 and 4.9 improve Theorem 3.2 in [18] from the following three aspects:

(i) The natural quasi cone-convexity of object mapping on $K_n$ has been removed;

(ii) The continuity of object mapping is weakened to the Hausdorff cone continuity (see Remark 2.12);

(iii) The order cone be extended to the general ordering set in this paper.

In fact, we can also use the following example to show that Theorem 4.9 is still true whether $S$ is a cone or not.

**Example 4.11.** Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $\varepsilon_0 = 2/3$, $\varepsilon_n = 2/3 + 1/n$, $K = [-1, 3]$ and $K_n = [-1 - 1/n, 3 + 1/n]$, $n = 1, 2, \ldots$. Taking $S = S_1 \cup S_2 \cup \mathbb{R}^2_+$ and $u = (1, 1)$, where

$$S_1 = \left\{(x, y) \in \mathbb{R}^2 : y \geq -\frac{1}{2}x + \frac{1}{2}, x \leq -1\right\}, \quad S_2 = \{(x, y) \in \mathbb{R}^2 : y \geq -x, -1 \leq x \leq 0\}.$$

Define mapping $G\colon X \rightrightarrows Y$ as follows:

$$G(x) = \{(|x|, x - 1)\} + A,$$

where, $A = \{t(0, 0) + (1 - t)(0, 1) : t \in [0, 1]\}$.

Noticed that $S^\infty = \{(x, y) \in \mathbb{R}^2 : y \geq -x, x \leq 0\} \cup \mathbb{R}^2_+$, all assumptions of Theorem 4.9 are fulfilled. After calculation,

$$E(\varepsilon_n, K_n) = [-1 - 1/n, 2/3 + 1/n) \quad \text{and} \quad E(\varepsilon_0, K) = [-1, 2/3).$$

Moreover, $E(\varepsilon_0, K) \subseteq \operatorname{Li} E(\varepsilon_n, K_n) = [-1, 2/3]$. Hence, Theorem 4.9 is valid.

However, $K_n$ is nonconvex for any $n$. Therefore, for Theorem 3.2 of [18], the condition that $G$ is strictly natural quasi $S^\infty$-convex on $K_n$ is not satisfied. As a result, Theorem 3.2 of [18] is unapplicable here.

At last, we provide Example 4.12 to illustrate Remark 4.10.

**Example 4.12.** Let $X = Y = \mathbb{R}$, $S = \mathbb{R}_+$. $K = [-2, 2]$ and $K_n = [-2, 2 + 2/n]$. Taking $\varepsilon_0 = 1$, $\varepsilon_n = 1 + 1/n$ and $u = 1$. Consider the set-valued mapping $G\colon X \rightrightarrows Y$ as follows:

$$G(x) = \begin{cases} [3, 4] & \text{if } x \in [-2, -1], \\ [0, 8] & \text{if } x \in (-1, 1), \\ [2 + x, 3 + x] & \text{else.} \end{cases}$$

Clearly, it is easy to verify that all conditions of Theorems 4.7 and 4.9 are satisfied and $S^\infty = \mathbb{R}_+$. After calculation, one has

$$\begin{aligned} E(\varepsilon_n, K_n) &= (-1, 1 + 1/n), \qquad E(\varepsilon_0, K) = (-1, 1), \\ W(\varepsilon_n, K_n) &= (-1, 1 + 1/n), \qquad W(\varepsilon_0, K) = (-1, 1). \end{aligned}$$

Obviously,

$$E(\varepsilon_0, K) \subseteq \operatorname{Li} E(\varepsilon_n, K_n) = [-1, 1] \quad \text{and} \quad W(\varepsilon_0, K) \subseteq \operatorname{Li} W(\varepsilon_n, K_n) = [-1, 1].$$

Hence, Theorems 4.7 and 4.9 are applicable.

However, it's not hard to get that $G$ is not continuous on $K$. Indeed, for $x_0 = -1$ and $V = (2, 5)$. We have $G(-1) = [3, 4] \subseteq V$. But for any neighbourhood $U$ of $x_0 = -1$, There is always $y \in U \cap (-1, 1) \neq \emptyset$ such that $G(y) = [0, 8] \nsubseteq V$. It follows that $G$ is not u.s.c at $x_0$. Consequently $G$ is not continuous on $K$. Therefore, Theorem 3.2 of [18] is unapplicable here.

## 5. Conclusions

In this paper, by virtue of a general set order relation, the Painlevé–Kuratowski convergence of approximate solution sets to SOP under the feasible set is perturbed is discussed for the first time. With the help of the recession cone technique, we establish the upper Painlevé–Kuratowski convergence of minimum approximate solution sets and the Painlevé–Kuratowski convergence of the weak minimum approximate solution sets under the assumption that $\varepsilon_n$ is monotone. In addition, another interesting work that deserve study is to study other aspects of set optimization problems with the set order relation $\leq_S$, such as connectedness of solution sets and Hausdorff continuity of solution mappings. We will study further in the future.

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Zai-Yun Peng
College of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing 400074, China
*E-mail address*: `pengzaiyun@126.com`

Chong-Yang Shao
College of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan 611731, China
and
College of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing 400074, China
*E-mail address*: `shaocyll@sina.com`

Yue Zeng
College of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing 400074, China

and
College of Mathematics, Sichuan University, Chengdu 610064, China
*E-mail address*: zengyueylmn@163.com

Yi-Bin Xiao
College of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan 611731, China
*E-mail address*: xiaoyb9999@hotmail.com