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Dynamical Properties and Some Classes of Non-porous Subsets of Lebesgue Spaces

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Abstract. In this paper, we introduce several classes of non-σ-porous subsets of a general Lebesgue space. Also, we study some linear dynamics of operators and show that the set of all non-hypercyclic vectors of a sequences of weighted translation operators on L^p-spaces is not σ-porous.

1. Introduction

σ-porous sets, as a collection of very thin subsets of metric spaces, were introduced and studied first time in [8] through a research on boundary behavior of functions, and then were applied in differentiation and Banach spaces theories in [3, 14]. The concepts related to porosity have been active topics in recent decades because they can be adapted for many known notions in several kind of metric spaces; see the monograph [21]. σ-porous subsets of $\mathbb{R}$ are null and of first category, while in every complete metric space without any isolated points these two categories are different [20]. On the other hand, linear dynamics including hypercyclicity in operator theory received attention during the last years; see books [2, 11] and for instance [6, 16, 17]. Recently, F. Bayart in [1] through study of hypercyclic shifts (which was previously studied in [15]; see also [10]) proved that the set of non-hypercyclic vectors of some classes of weighted shift operators on $\ell^2(\mathbb{Z})$ is a non-σ-porous set. This would be a new example of a first category set which is not σ-porous. In this work, by some idea from the proof of [1, Theorem 1] first we introduce a class of non-σ-porous subsets of general Lebesgue spaces, and then we develop the main result of [1] to sequences of weighted translation operators on general Lebesgue spaces in the context of discrete groups and hypergroups. In particular, we prove that if $p \geq 1$, K is a discrete hypergroup, (a_n) is a sequence with distinct terms in K, and $w\colon K \to (0,\infty)$ is a bounded measurable function such that

$$\sum_{n\in\mathbb{N}} \frac{1}{w(a_0)w(a_1)\cdots w(a_n)} \chi_{\{a_{n+1}\}} \in L^p(K),$$

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then the set of all non-hypercyclic vectors of the sequence $(\Lambda_n)_n$ is not σ-porous, where the operators Λ_n are given in Definition 3.8. Also, we study non-σ-porosity of non-hypercyclic vectors of weighted composition operators on $L^p(\mathbb{R}, \tau)$, where τ is the Lebesgue measure on $\mathbb{R}$.

2. Non-σ-porous subsets of Lebesgue spaces

In this section, we will introduce some classes of non-σ-porous subsets of Lebesgue spaces related to a fixed function. First, we recall the definition of the main notion of this paper.

Definition 2.1. Let $0 < \lambda < 1$. A subset E of a metric space X is called λ-*porous* at $x \in E$ if for each $\delta > 0$ there is an element $y \in B(x; \delta) \setminus \{x\}$ such that

$$B(y; \lambda d(x, y)) \cap E = \emptyset.$$

E is called λ-*porous* if it is λ-porous at every element of E. Also, E is called σ-λ-*porous* if it is a countable union of λ-porous subsets of X.

The following lemma plays a key role in the proof of main results of this section. This fact is a special case of [19, Lemma 2]; see also [1, Lemma 2].

Lemma 2.2. *Let $\mathcal{F}$ be a nonempty family of nonempty closed subsets of a complete metric space X such that for each $F \in \mathcal{F}$ and each $x \in X$ and $r > 0$ with $B(x; r) \cap F \neq \emptyset$, there exists an element $J \in \mathcal{F}$ such that*

$$\emptyset \neq J \cap B(x; r) \subseteq F \cap B(x; r)$$

and $F \cap B(x; r)$ is not λ-porous at all elements of $J \cap B(x; r)$. Then, every set in $\mathcal{F}$ is not σ-λ-porous.

Throughout this paper we shall consider an arbitrary number $0 < \lambda \leq 1/2$ and for the simplicity, we shall just write σ-porous instead of σ-λ-porous for a general λ with $0 < \lambda \leq 1/2$. The next theorem is a development of [1, Theorem 1]. The proof of this theorem is motivated by the proof of [1, Theorem 1]. Hence, same as [1], the proof of this theorem is based on Lemma 2.2.

Theorem 2.3. *Let $p \geq 1$, Ω be a locally compact Hausdorff space, μ be a nonnegative Radon measure on Ω, and $A \subseteq \Omega$ be a Borel set such that for every compact subset K of Ω there exists a constant C_K satisfying that*

$$|f|\chi_{A\cap K} \leq C_K \|f\|_p \quad a.e. \quad (f \in L^p(\Omega, \mu)). \tag{2.1}$$

Then, for each measurable function g on Ω with $g\chi_A \in L^p(\Omega,\mu)$, the set

$$\Gamma_g := \big\{f \in L^p(\Omega,\mu) : |f| \geq |g|\chi_A \ a.e.\big\}$$

is not σ-porous in $L^p(\Omega,\mu)$.

Proof. Fix an arbitrary number $0 < \lambda \leq 1/2$, and pick $0 < \beta < \lambda$. Denote

$$\mathcal{F} := \big\{\Gamma_g : g\chi_A \in L^p(\Omega,\mu)\big\}.$$

We will show that the collection $\mathcal{F}$ satisfies the conditions of Lemma 2.2. Let $g \in L^p(\Omega,\mu)$. Without loss of generality, we can assume that g is a nonnegative function. Trivially, $\Gamma_g \neq \emptyset$. Let (f_n) be a sequence in Γ_g and $f_n \to f$ in $L^p(\Omega,\mu)$. Then, by (2.1), $|f| \geq g\chi_A$ a.e., and so $f \in \Gamma_g$. Therefore, every element of the collection $\mathcal{F}$ is a closed subset of $L^p(\Omega,\mu)$. Now, assume that $f \in L^p(\Omega,\mu)$ and $r > 0$ with $B(f;r) \cap \Gamma_g \neq \emptyset$. We find a measurable function h with $0 \leq h\chi_A \in L^p(\Omega,\mu)$ such that

$$\emptyset \neq B(f;r) \cap \Gamma_h \subseteq B(f;r) \cap \Gamma_g,$$

and $B(f;r) \cap \Gamma_g$ is not λ-porous at elements of $B(f;r) \cap \Gamma_h$.

Since $(|f| + \beta^{-1}g\chi_A)^p \in L^1(\Omega,\mu)$ and μ is a Radon measure, the mapping ν defined by

$$\nu(B) := \int_B (|f| + \beta^{-1}g\chi_A)^p \, d\mu \quad \text{for every Borel set } B \subseteq \Omega$$

is a Radon measure [9]. Hence, there are some $0 < \epsilon < 1$, a function $k \in B(f;r) \cap \Gamma_g$ and a compact subset D of Ω with $\mu(D) > 0$ such that

$$\|k - f\|_p < \epsilon^{1/p} r \quad \text{and} \quad \int_{D^c} (|f| + \beta^{-1}g\chi_A)^p \, d\mu < (1-\epsilon) r^p.$$

Pick some α with

$$\|k - f\|_p < \alpha < \epsilon^{1/p} r,$$

and denote

$$\delta := \frac{\epsilon^{1/p} r - \alpha}{2\mu(D)^{1/p}}.$$

Now, we define two functions $h, \xi \colon \Omega \to \mathbb{C}$ by

$$h := (g\chi_A + \delta)\chi_D + \beta^{-1} g\chi_A\chi_{\Omega\setminus D} \quad \text{and} \quad \xi := (|k| + \delta)\eta\chi_D + h\chi_{\Omega\setminus D},$$

where

$$\eta(x) := \begin{cases} \frac{k(x)}{|k(x)|} & \text{if } k(x) \neq 0, \\ 1 & \text{if } k(x) = 0 \end{cases}$$

for all $x \in \Omega$. Since D is compact, we have $h\chi_A \in L^p(\Omega, \mu)$. Also, for each $x \in D$,

$$|k(x) - \xi(x)| = \big|k(x) - (|k(x)| + \delta)\eta(x)\big| = \big|k(x) - k(x) - \delta\eta(x)\big| = \delta,$$

and therefore

$$\|(\xi - k)\chi_D\|_p = \delta\mu(D)^{1/p} = \frac{\epsilon^{1/p} r - \alpha}{2}.$$

This implies that

$$\|(\xi - f)\chi_D\|_p \le \|(\xi - k)\chi_D\|_p + \|(k - f)\chi_D\|_p \le \frac{\epsilon^{1/p} r - \alpha}{2} + \alpha < \epsilon^{1/p} r.$$

Hence,

$$\begin{aligned} \|\xi - f\|_p^p &= \int_D |\xi - f|^p \, d\mu + \int_{\Omega\setminus D} |\xi - f|^p \, d\mu < \epsilon r^p + \int_{\Omega\setminus D} |\beta^{-1} g\chi_A - f|^p \, d\mu \\ &\le \epsilon r^p + \int_{\Omega\setminus D} (\beta^{-1} g\chi_A + |f|)^p \, d\mu < \epsilon r^p + (1-\epsilon) r^p = r^p, \end{aligned}$$

and so, $\xi \in B(f; r)$. Moreover,

$$|\xi(x)| = |k(x)| + \delta \ge g(x) + \delta = h(x) \quad \text{a.e. on } D \cap A,$$

and for each $x \in (\Omega \setminus D) \cap A$ we have $|\xi(x)| = h(x)$. This shows that $\xi \in \Gamma_h$, and so

$$\emptyset \ne B(f; r) \cap \Gamma_h \subseteq B(f; r) \cap \Gamma_g$$

because $h \ge g$.

Next, we recall that by the condition (2.1), since D is compact, there exists a constant C_K such that

$$|f|\chi_{A\cap K} \le C_K \|f\|_p \ \text{ a.e.} \quad (f \in L^p(\Omega, \mu)).$$

Now, let $u \in B(f; r) \cap \Gamma_h$ and put $r' := \min\big\{\frac{\delta}{C_K}, \lambda(r - \|f - u\|_p)\big\}$. Let $v \in B(u; r')$. We define the function $\gamma\colon \Omega \to \mathbb{C}$ by

$$\gamma(x) := \begin{cases} v(x) & \text{if } x \in D, \\ (|v(x)| + \beta|u(x) - v(x)|)\theta(x) & \text{if } x \in \Omega \setminus D, \end{cases}$$

where

$$\theta(x) := \begin{cases} \frac{v(x)}{|v(x)|} & \text{if } v(x) \ne 0, \\ 1 & \text{if } v(x) = 0. \end{cases}$$

Therefore, for each $x \in \Omega \setminus D$ we have

$$|\gamma(x) - v(x)| = \beta|u(x) - v(x)| \quad \text{and} \quad |\gamma(x)| \ge \beta|u(x)|.$$

It is easy to see that

$$\begin{aligned}\|\gamma - v\|_p^p &= \|(\gamma - v)\chi_D\|_p^p + \|(\gamma - v)\chi_{\Omega\setminus D}\|_p^p = \|(\gamma - v)\chi_{\Omega\setminus D}\|_p^p \\ &= \beta^p\|(u - v)\chi_{\Omega\setminus D}\|_p^p \leq \beta^p\|u - v\|_p^p < \lambda^p\|u - v\|_p^p,\end{aligned}$$

and hence,

$$\gamma \in B(v; \lambda\|u - v\|_p) \subseteq B(f; r).$$

In addition,

$$|\gamma(x)| \geq \beta|u(x)| \geq \beta h(x) = g(x) \quad \text{for a.e. } x \in (\Omega \setminus D) \cap A$$

and

$$|\gamma(x)| = |v(x)| \geq |u(x)| - \delta \geq g(x) \quad \text{for a.e. } x \in D \cap A,$$

because $|u(x)| - |v(x)| \leq |u(x) - v(x)| \leq C_D\|u - v\|_p \leq \delta$ for a.e. $x \in D \cap A$ and also $|u| \geq h$. Therefore,

$$B(v; \lambda\|u - v\|_p) \cap B(f; r) \cap \Gamma_g \neq \emptyset,$$

and this completes the proof. □

Remark 2.4. If in the condition (2.1) we set $A := \Omega$, then this implies that $L^p(\Omega, \mu) \subseteq L^\infty(\Omega, \mu)$, and this inclusion is equivalent to

$$\alpha := \inf\{\mu(E) : \mu(E) > 0\} > 0, \tag{2.2}$$

and equivalently, for each $q > p$, $L^p(\Omega, \mu) \subseteq L^q(\Omega, \mu)$; see [18]. If in addition, $\operatorname{supp}\mu = \Omega$, then the condition (2.2) implies that for each $x \in \Omega$,

$$\mu(\{x\}) = \inf\{\mu(F) : F \text{ is a compact neighborhood of } x\} > 0.$$

Specially, if Ω is a locally compact group (or hypergroup) and μ is a left Haar measure of it, then the condition (2.1) implies that Ω is a discrete topological space.

The next result is a direct conclusion of Theorem 2.3.

Corollary 2.5. *Let Ω be a discrete topological space and $\varphi := (\varphi_j)_{j\in\Omega} \subseteq (0, \infty)$. Put $\mu_\varphi := \sum_{j\in\Omega} \varphi_j\delta_j$, where δ_j is the point-mass measure at j. Then, for each $g \in L^p(\Omega, \mu_\varphi)$, the set*

$$\Gamma_g := \big\{f \in L^p(\Omega, \mu_\varphi) : |f| \geq |g|\big\}$$

is not σ-porous in $L^p(\Omega, \mu_\varphi)$.

Proof. Just note that for each finite subset D of Ω, $k \in D$ and $f \in L^p(\Omega, \mu_\varphi)$,

$$\|f\|_p^p = \sum_{j\in\Omega} |f(j)|^p\mu_\varphi(\{j\}) \geq \sum_{j\in D} |f(j)|^p\mu_\varphi(\{j\}) \geq |f(k)|^p\varphi_k \geq |f(k)|^pM_D,$$

where $M_D = \min\{\varphi_j : j \in D\}$. □

In particular, if a set is endowed with the counting measure, we get the fact.

Corollary 2.6. *Let $p \geq 1$ and A be a nonempty set. Then, for each $g \in \ell^p(A)$, the set*

$$\Gamma_g := \{f \in \ell^p(A) : |f| \geq |g|\}$$

is not σ-porous in $\ell^p(A)$.

Remark 2.7. The main Theorem 2.3 is valid also for the sequence space c_0, (that is the space of sequences vanishing at infinity equipped with sup norm), because the sequences with finitely many non-zero coefficients approximate sequences in c_0.

At the end of this section, we give a class of non-σ-porous subsets of the L^p-space on real line. In the proof of this result, which is also based on Lemma 2.2, we apply some functions defined in the proof of Theorem 2.3.

Theorem 2.8. *Let $p \geq 1$, and τ be the Lebesgue measure on $\mathbb{R}$. For each $g \in L^p(\mathbb{R}, \tau)$, put*

$$\Theta_g := \big\{f \in L^p(\mathbb{R}, \tau) : \|f\chi_{[m,m+1]}\|_p \geq \|g\chi_{[m,m+1]}\|_p \textit{ for all } m \in \mathbb{Z}\big\}.$$

Then, Θ_g is not σ-porous in $L^p(\mathbb{R}, \tau)$.

Proof. Let $0 < \lambda \leq 1/2$ and $0 < \beta < \lambda$. Denote

$$\mathcal{F} := \{\Theta_g : g \in L^p(\mathbb{R}, \tau)\}.$$

We prove that the collection $\mathcal{F}$ satisfies the conditions of Lemma 2.2. Let $0 \leq g \in L^p(\mathbb{R}, \tau)$. Then, easily $\Theta_g \neq \emptyset$ and it is closed in $L^p(\mathbb{R}, \tau)$. Now, assume that $f \in L^p(\mathbb{R}, \tau)$ and $r > 0$ with $B(f; r) \cap \Theta_g \neq \emptyset$. Then, there exist a large enough number $N \in \mathbb{N}$, some $0 < \epsilon < 1$ and a function $k \in B(f; r) \cap \Theta_g$ such that

$$\|k - f\|_p < \epsilon^{1/p} r \quad \text{and} \quad \int_{[-N,N]^c} (|f| + \beta^{-1} g)^p \, d\tau < (1 - \epsilon) r^p.$$

Pick some α with $\|k - f\|_p < \alpha < \epsilon^{1/p} r$, and denote $\delta := \frac{\epsilon^{1/p} r - \alpha}{2(2N)^{1/p}}$. Put

$$A_1 := \{m \in [N] : g = 0 \text{ a.e. on } [m, m+1]\}, \quad A_2 := [N] \setminus A_1$$

and

$$B_1 := \{m \in [N] : k = 0 \text{ a.e. on } [m, m+1]\}, \quad B_2 := [N] \setminus B_1,$$

where $[N] := \{-N, \ldots, N-1\}$, and then define

$$\rho := \sum_{m \in A_1} \chi_{[m,m+1]} + \sum_{m \in A_2} \frac{g\chi_{[m,m+1]}}{\|g\chi_{[m,m+1]}\|_p}$$

and

$$\eta := \sum_{m\in B_1} \chi_{[m,m+1]} + \sum_{m\in B_2} \frac{k\chi_{[m,m+1]}}{\|k\chi_{[m,m+1]}\|_p}.$$

Now, we define $h, \xi\colon \mathbb{R} \to \mathbb{C}$ by

$$h := g\chi_{[-N,N]} + \delta\rho + \beta^{-1} g\chi_{[-N,N]^c} \quad \text{and} \quad \xi := (|k|\chi_{[-N,N]} + \delta)\eta + h\chi_{[-N,N]^c}.$$

Clearly, $h \in L^p(\mathbb{R}, \tau)$. For each $x \in [-N, N]$ we have $|k(x) - \xi(x)| = \delta|\eta(x)|$, and so

$$\|(k-\xi)\chi_{[-N,N]}\|_p^p = \delta^p \|\eta\chi_{[-N,N]}\|_p^p = \delta^p \sum_{m\in[N]} \|\eta\chi_{[m,m+1]}\|_p^p = \delta^p 2N.$$

Hence, $\|(k-\xi)\chi_{[-N,N]}\|_p = \delta(2N)^{1/p}$. Now, similar to the proof of Theorem 2.3 we have $\xi \in B(f; r)$. Moreover,

$$\|\xi\chi_{[m,m+1]}\|_p = \|k\chi_{[m,m+1]}\|_p + \delta \geq \|g\chi_{[m,m+1]}\|_p + \delta = \|h\chi_{[m,m+1]}\|_p$$

for all $m \in [N]$. And also for each $m \notin [N]$,

$$\|\xi\chi_{[m,m+1]}\|_p = \|h\chi_{[m,m+1]}\|_p \geq \|g\chi_{[m,m+1]}\|_p.$$

So,

$$\xi \in B(f; r) \cap \Theta_h \subseteq B(f; r) \cap \Theta_g.$$

Now, let $u \in B(f; r) \cap \Theta_h$ and put $r' := \min\{\delta, \lambda(r - \|f - u\|_p)\}$. Assume that $v \in B(u; r')$. We define the function $\gamma\colon \mathbb{R} \to \mathbb{C}$ by

$$\gamma(x) := \begin{cases} v(x) & \text{if } x \in [-N, N], \\ (|v(x)| + \beta|u(x) - v(x)|)\theta(x) & \text{if } x \in [-N, N]^c, \end{cases}$$

where

$$\theta(x) := \begin{cases} \frac{v(x)}{|v(x)|} & \text{if } v(x) \neq 0, \\ 1 & \text{if } v(x) = 0. \end{cases}$$

Similar to the proof of Theorem 2.3, we have $\gamma \in B(v; \lambda\|u - v\|_p)$. Now, for each $m \notin [N]$,

$$|\gamma|\chi_{(m,m+1)} = (|v| + \beta|u - v|)\chi_{(m,m+1)} \geq \beta|u|\chi_{(m,m+1)}.$$

Hence,

$$\|\gamma\chi_{[m,m+1]}\|_p \geq \beta\|u\chi_{[m,m+1]}\|_p \geq \beta\|h\chi_{[m,m+1]}\|_p$$

since $u \in B(f; r) \cap \Theta_h$. However, in this case we have $(m, m+1) \in [-N, N]^c$, so $h\chi_{(m,m+1)} = \beta^{-1} g\chi_{(m,m+1)}$. Thus, $\beta\|h\chi_{[m,m+1]}\|_p = \|g\chi_{[m,m+1]}\|_p$. If $m \in [N]$, we have $\gamma\chi_{[m,m+1]} = v\chi_{[m,m+1]}$ because $\gamma\chi_{[-N,N]} = v\chi_{[-N,N]}$ and $[m, m+1] \subseteq [-N, N]$. We get

$$\big|\|u\chi_{[m,m+1]}\|_p - \|v\chi_{[m,m+1]}\|_p\big| \leq \|(u - v)\chi_{[m,m+1]}\|_p \leq \|u - v\|_p < \delta$$

because $v \in B(u; r')$, hence

$$\|\gamma\chi_{[m,m+1]}\|_p = \|v\chi_{[m,m+1]}\|_p \geq \|u\chi_{[m,m+1]}\|_p - \delta \geq \|h\chi_{[m,m+1]}\|_p - \delta = \|g\chi_{[m,m+1]}\|_p.$$

Therefore,

$$\gamma \in B(v; \lambda\|u - v\|_p) \cap B(f; r) \cap \Theta_g,$$

and the proof is complete. □

3. Applications

In this section, we will apply the results of the previous section, to prove that the set of all non-hypercyclic vectors of some sequences of weighted translation operators is non-σ-porous.

Definition 3.1. Let $\mathcal{X}$ be a Banach space. A sequence $(T_n)_{n\in\mathbb{N}_0}$ of operators in $B(\mathcal{X})$ is called *hypercyclic* if there is an element $x \in \mathcal{X}$ (called *hypercyclic vector*) such that the orbit $\{T_n(x) : n \in \mathbb{N}_0\}$ is dense in $\mathcal{X}$. The set of all hypercyclic vectors of a sequence $(T_n)_{n\in\mathbb{N}_0}$ is denoted by $\mathrm{HC}((T_n)_{n\in\mathbb{N}_0})$. An operator $T \in B(\mathcal{X})$ is called *hypercyclic* if the sequence $(T^n)_{n\in\mathbb{N}_0}$ is hypercyclic.

Let G be a locally compact group and $a \in G$. Then, for each function $f\colon G \to \mathbb{C}$ we define $L_a f\colon G \to \mathbb{C}$ by $L_a f(x) := f(a^{-1}x)$ for all $x \in G$. Note that if $p \geq 1$, then the left translation operator

$$L_a\colon L^p(G) \to L^p(G), \quad f \mapsto L_a f$$

is not hypercyclic because $\|L_a\| \leq 1$. Hypercyclicity of *weighted* translation operators on $L^p(G)$ and regarding an aperiodic element a was studied in [5] (an element $a \in G$ is called *aperiodic* if the closed subgroup of G generated by a is not compact).

Definition 3.2. Let G be a locally compact group with a left Haar measure μ. Fix $p \geq 1$. We denote $L^p(G) := L^p(G, \mu)$. Assume that $w\colon G \to (0, \infty)$ is a bounded measurable function (called a *weight*) and $a \in G$. Then, the weighted translation operator $T_{a,w,p}\colon L^p(G) \to L^p(G)$ is defined by

$$T_{a,w,p}(f) := wL_a f, \quad f \in L^p(G).$$

For each $n \in \mathbb{N}$ we denote $\varphi_n := wL_a w \cdots L_{a^{n-1}} w$, where $a^0 := e$, the identity element of G.

Theorem 3.3. *Let $p \geq 1$, G be a discrete group and $a \in G$. Let μ be a left Haar measure on G and $(\gamma_n)_n$ be an unbounded sequence of nonnegative integers. Let $w\colon G \to (0, \infty)$ be* *a bounded function such that for some finite nonempty set $F \subseteq G$ and some $N > 0$ we have*

$$a^{\gamma_n} F \cap F = \emptyset, \quad n \geq N, \quad \textit{and} \quad \beta := \inf\left\{\prod_{k=1}^{\gamma_n} w(a^k t) : n \geq N, t \in F\right\} > 0.$$

Then, the set

$$\Lambda := \left\{f \in L^p(G,\mu) : \|T_{a,w,p}^{\gamma_n} f - \chi_F\|_p \geq \mu(F)^{1/p} \textit{ for all } n \geq N\right\}$$

is non-σ-porous.

Proof. Let $\Gamma := \left\{f \in L^p(G,\mu) : |f| \geq \frac{1}{\beta}\chi_F\right\}$. Then, Γ is not σ-porous in $L^p(G,\mu)$ thanks to Theorem 2.3. Also, for each $f \in \Gamma$ and $n \geq N$ we have

$$\begin{aligned}
\|T_{a,w,p}^{\gamma_n} f - \chi_F\|_p^p &= \int_G \left|\prod_{k=1}^{n} w(a^{-\gamma_n+k}x) f(a^{-\gamma_n}x) - \chi_F(x)\right|^p d\mu(x) \\
&= \int_G \left|\prod_{k=1}^{\gamma_n} w(a^k x) f(x) - \chi_F(a^{\gamma_n}x)\right|^p d\mu(x) \\
&= \int_G \left|\prod_{k=1}^{\gamma_n} w(a^k x) f(x) - \chi_{a^{-\gamma_n}F}(x)\right|^p d\mu(x) \\
&\geq \int_F \left|\prod_{k=1}^{\gamma_n} w(a^k x) f(x) - \chi_{a^{-\gamma_n}F}(x)\right|^p d\mu(x) \\
&= \int_F \left|\prod_{k=1}^{\gamma_n} w(a^k x) f(x)\right|^p d\mu(x) \geq \int_F \left|\beta\frac{1}{\beta}\right|^p d\mu(x) = \mu(F).
\end{aligned}$$

This completes the proof. $\square$

Example 3.4. Let G be the additive group $\mathbb{Z}$ with the counting measure. Let F be a finite nonempty subset of $\mathbb{Z}$. Put $N := \max\{|j| : j \in F\}$. If $w := (w_n)_{n\in\mathbb{Z}} \subseteq (0,\infty)$ is a bounded sequence. Then the required conditions in the previous theorem hold with respect to F and $a := 1$.

The following fact is a direct conclusion of the previous theorem.

Corollary 3.5. *Let $p \geq 1$, G be a discrete group and $a \in G$ with infinite order. Let μ be the counting measure on G and $(\gamma_n)_n$ be an unbounded sequence of nonnegative integers. Let $w\colon G \to (0,\infty)$ be a bounded function such that for some $t \in G$,*

$$\inf\left\{\prod_{k=1}^{\gamma_n} w(a^k t) : n \in \mathbb{N}\right\} > 0.$$

Then, the set

$$\left\{f \in L^p(G,\mu) : \|T_{a,w}^{\gamma_n} f - \chi_{\{t\}}\|_p \geq 1 \textit{ for all } n\right\}$$

is non-σ-porous.

Theorem 3.6. *Let $p \geq 1$, G be a discrete group, and $a \in G$. Let μ be a left Haar measure on G. Let $(\gamma_n)_n$ be an unbounded sequence of nonnegative integers and let $w \colon G \to (0, \infty)$ be a bounded function such that*

$$\inf_{n \in \mathbb{N}} \prod_{k=1}^{\gamma_n} w(a^k) > 0.$$

Then, the set

$$\Gamma := \left\{ f \in L^p(G, \mu) : |f(e)| \inf_{n \in \mathbb{N}} \prod_{k=1}^{\gamma_n} w(a^k) \geq 1 \right\}$$

is non-σ-porous. In particular, setting $T_n := T_{a,w,p}^{\gamma_n}$ for all n, the set of all non-hypercyclic vectors of the sequence $(T_n)_n$ is not σ-porous in $L^p(G, \mu)$.

Proof. Since $\mu(\{e\}) > 0$, applying Theorem 2.3 the set Γ is non-σ-porous, because

$$\left[\inf_{n \in \mathbb{N}} \prod_{k=1}^{\gamma_n} w(a^k) \right]^{-1} \chi_{\{e\}} \in L^p(G, \mu).$$

Let $f \in \Gamma$. If n is a nonnegative integer, then for every x in G we have

$$\|T_n f\|_p \geq \left| \varphi_{\gamma_n}(x) L_{a^{\gamma_n}} f(x) \right|,$$

and so setting $x = a^{\gamma_n}$ we have

$$\|T_n f\|_p \geq \left| \varphi_n(a^{\gamma_n}) L_{a^{\gamma_n}} f(a^{\gamma_n}) \right| = \left[\prod_{k=1}^{\gamma_n} w(a^k) \right] |f(e)| \geq |f(e)| \inf_{m \in \mathbb{N}} \prod_{k=1}^{\gamma_m} w(a^k) \geq 1.$$

This implies that the set $\{T_n f : n \in \mathbb{N}\}$ is not dense in $L^p(G, \mu)$, and so Γ is a subset of the set of all non-hypercyclic vectors of T. This completes the proof. $\square$

Now, we recall the definition of hypergroups which are generalizations of locally compact groups; see the monograph [4] and the basic paper [12] for more details. In locally compact hypergroups the convolution of two Dirac measures is not necessarily a Dirac measure. Let K be a locally compact Hausdorff space. We denote by $\mathbb{M}(K)$ the space of all regular complex Borel measures on K, and by δ_x the Dirac measure at the point x. The support of a measure $\mu \in \mathbb{M}(K)$ is denoted by $\mathrm{supp}(\mu)$.

Definition 3.7. Suppose that K is a locally compact Hausdorff space, $(\mu, \nu) \mapsto \mu * \nu$ is a bilinear positive-continuous mapping from $\mathbb{M}(K) \times \mathbb{M}(K)$ into $\mathbb{M}(K)$ (called *convolution*), and $x \mapsto x^-$ is an involutive homeomorphism on K (called *involution*) with the following properties:

(i) $\mathbb{M}(K)$ with $*$ is a complex associative algebra;

(ii) if $x, y \in K$, then $\delta_x * \delta_y$ is a probability measure with compact support;

(iii) the mapping $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ from $K \times K$ into $\mathbf{C}(K)$ is continuous, where $\mathbf{C}(K)$ is the set of all nonempty compact subsets of K equipped with Michael topology;

(iv) there exists a (necessarily unique) element $e \in K$ (called identity) such that for all $x \in K$, $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$;

(v) for all $x, y \in K$, $e \in \operatorname{supp}(\delta_x * \delta_y)$ if and only if $x = y^-$.

Then, $K \equiv (K, *, ^-, e)$ is called a locally compact *hypergroup*.

A nonzero nonnegative regular Borel measure m on K is called the (left) *Haar measure* if for each $x \in K$, $\delta_x * m = m$. For each $x, y \in K$ and measurable function $f\colon K \to \mathbb{C}$ we denote

$$f(x * y) := \int_K f \, d(\delta_x * \delta_y),$$

while this integral exists.

Definition 3.8. Suppose that $a := (a_n)_{n \in \mathbb{N}_0}$ is a sequence in a hypergroup K, and w is a weight function on K. For each $n \in \mathbb{N}_0$ we define the bounded linear operator Λ_{n+1} on $L^p(K)$ by

$$\Lambda_{n+1} f(x) := w(a_0 * x) w(a_1 * x) \cdots w(a_n * x) f(a_{n+1} * x), \quad f \in L^p(K)$$

for all $x \in K$. Also, we assume that Λ_0 is the identity operator on $L^p(K)$.

Some linear dynamical properties of this sequence of operators were studied in [13]. The sequence $\{\Lambda_n\}_n$ is a generalization of the usual powers of a single weighted translation operator on $L^p(G)$, where G is a locally compact group. In fact, any locally compact group G with the mapping

$$\mu * \nu \mapsto \int_G \int_G \delta_{xy} \, d\mu(x) d\nu(y), \quad \mu, \nu \in \mathbb{M}(G)$$

as convolution, and $x \mapsto x^{-1}$ from G onto G as involution is a locally compact hypergroup. Let $\eta := (a_n)_{n \in \mathbb{N}_0}$ be a sequence in G, and w be a weight on G. Then for each $f \in L^p(G)$, $n \in \mathbb{N}_0$ and $x \in G$, we have

$$\Lambda_{n+1} f(x) = w(a_0 x) w(a_1 x) \cdots w(a_n x) f(a_{n+1} x).$$

In particular, let $a \in G$ and for each $n \in \mathbb{N}_0$, put $a_n := a^{-n}$. Then, $\Lambda_n = T^n_{a,w,p}$ for all $n \in \mathbb{N}$. In this case, the operator $T_{a,w,p}$ is hypercyclic if and only if the sequence $(\Lambda_n)_n$ is hypercyclic.

Let K be a discrete hypergroup with the convolution $*$ between Radon measures of K and the involution $\cdot^{-}\colon K \to K$. Then, by [12, Theorem 7.1A], the measure μ on K given by

$$\mu(\{x\}) := \frac{1}{\delta_x * \delta_{x^-}(\{e\})}, \quad x \in K \tag{3.1}$$

is a left Haar measure on K.

Proposition 3.9. *Let K be a discrete hypergroup, μ be the Haar measure* (3.1)*, and $p \geq 1$. Then for each $g \in L^p(K,\mu)$, the set*

$$\{f \in L^p(K,\mu) : |f| \geq |g|\}$$

is not σ-porous in $L^p(K,\mu)$.

Proof. Just note that for each $x \in K$ we have $\mu(\{x\}) \geq 1$ because

$$1 = \delta_x * \delta_{x^-}(K) \geq \delta_x * \delta_{x^-}(\{e\}).$$

Hence, the measure space (K,μ) satisfies the condition of Corollary 2.5. □

Let $a := (a_n)_{n\in\mathbb{N}}$ be a sequence in a discrete hypergroup K such that $a_n \neq a_m$ for each $m \neq n$, and let $w\colon K \to (0,\infty)$ be bounded. We define $h_{a,w}\colon K \to \mathbb{C}$ by

$$h_{a,w} := \sum_{n\in\mathbb{N}_0} \frac{1}{w(a_0)w(a_1)\cdots w(a_n)}\chi_{\{a_{n+1}\}}.$$

Theorem 3.10. *Let $p \geq 1$, and K be a discrete hypergroup endowed with the left Haar measure* (3.1)*. Let $a := (a_n)_{n\in\mathbb{N}_0} \subseteq K$ with distinct terms, and w be a weight on K such that $h_{a,w} \in L^p(K)$. Then, the set of all non-hypercyclic vectors of the sequence $(\Lambda_n)_n$ is not σ-porous.*

Proof. First, thanks to Proposition 3.9, the set

$$E := \left\{ f \in L^p(K) : |f(a_{n+1})| \geq \frac{1}{w(a_0)w(a_1)\cdots w(a_n)} \text{ for all } n \right\}$$

is not σ-porous because it equals to the set $\{f \in L^p(K) : |f| \geq h_{a,w}\}$. Now, for each $f \in E$,

$$\begin{aligned}\|\Lambda_{n+1} f\|_p &\geq \sup_{x\in K} w(a_0 * x)w(a_1 * x)\cdots w(a_n * x)|f(a_{n+1} * x)| \\ &\geq w(a_0)w(a_1)\cdots w(a_n)|f(a_{n+1})| \geq 1\end{aligned}$$

for all $n \in \mathbb{N}_0$. This implies that 0 does not belong to the closure of $\{\Lambda_n f : n \in \mathbb{N}\}$ in $L^p(K)$, and so $E \subseteq [\mathrm{HC}((\Lambda_n)_n)]^c$. This completes the proof. □

Since any group is a hypergroup, we can give the fact below.

Corollary 3.11. *Let $p \geq 1$, and G be a discrete group. Let $a \in G$ be of infinite order, $(\gamma_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{N}$ be with distinct terms and $w\colon G \to (0,\infty)$ be a weight such that*

$$\left(\frac{1}{w(a^{\gamma_0})w(a^{\gamma_1})\cdots w(a^{\gamma_n})}\right)_n \in \ell^p(G).$$

Then, the set of all non-hypercyclic vectors of the sequence $(T^{\gamma_n}_{a,w,p})_n$ is not σ-porous in $\ell^p(G)$.

Now, we can write the next corollary which is a generalization of [1, Theorem 1].

Corollary 3.12. *Let $p \geq 1$, $(\gamma_n)_n \subseteq \mathbb{N}$ be strictly increasing and $(w_n)_{n\in\mathbb{Z}}$ be a bounded sequence in $(0,\infty)$ such that*

$$\left(\frac{1}{w_{\gamma_0}w_{\gamma_1}w_{\gamma_2}\cdots w_{\gamma_n}}\right)_n \in \ell^p(\mathbb{Z}).$$

Then, the set of all non-hypercyclic vectors of the sequence $(T_n)_n$ is not σ-porous, where

$$(T_{n+1}a)_k := w_{\gamma_0}w_{\gamma_1}w_{\gamma_2}\cdots w_{\gamma_n}a_{k+\gamma_{n+1}}, \quad k \in \mathbb{N}_0$$

for all $a := (a_j)_j \in \ell^p(\mathbb{Z})$.

In sequel, we find some application for Theorem 2.8 regarding hypercyclicity of weighted composition operators on $L^p(\mathbb{R},\tau)$.

Theorem 3.13. *Consider the weighted translation operator $T_{\alpha,w}$ on $L^p(\mathbb{R},\tau)$ given by $T_{\alpha,w}f := w\cdot(f\circ\alpha)$, where $0 < w$, $w^{-1} \in C_b(\mathbb{R})$ and $\alpha(t) = t+1$. For each $n \in \mathbb{N}$ put $A_n := [n, n+1] = \alpha^n([0,1])$. Set*

$$y_{\alpha,w} := \sum_{n\in\mathbb{N}} \frac{1}{\inf_{t\in A_n}\prod_{k=1}^{n}(w\circ\alpha^{-k})(t)}\chi_{A_n}$$

and assume that $y_{\alpha,w} \in L^p(\mathbb{R},\tau)$. Then, the set

$$\{f \in L^p(\mathbb{R},\tau) : \|T^n_{\alpha,w}(f)\|_p \geq 1 \text{ for all } n \in \mathbb{N}\}$$

is not σ-porous.

Proof. By Theorem 2.8, the set

$$E := \{f \in L^p(\mathbb{R},\tau) : \|f\chi_{A_n}\|_p \geq \|y_{\alpha,w}\chi_{A_n}\|_p \text{ for all } n \in \mathbb{N}\}$$

is not σ-porous, because it equals to

$$\{f \in L^p(\mathbb{R},\tau) : \|f\chi_{[m,m+1]}\|_p \geq \|y_{\alpha,w}\chi_{[m,m+1]}\|_p \text{ for all } m \in \mathbb{Z}\},$$

as $y_{\alpha,w}\chi_{[m,m+1]} = 0$ for all $m \in \mathbb{Z}$ with $m \leq 0$. Now, note that for each $f \in E$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\|T^n_{\alpha,w}(f)\|_p^p \\
&= \int_{\mathbb{R}} \left[\prod_{k=1}^{n}(w\circ\alpha^{n-k})(t)\right]^p |(f\circ\alpha^n)(t)|^p\, d\tau = \int_{\mathbb{R}} \left[\prod_{k=1}^{n}(w\circ\alpha^{-k})(t)\right]^p |f(t)|^p\, d\tau \\
&\geq \int_{A_n} \left[\prod_{k=1}^{n}(w\circ\alpha^{-k})(t)\right]^p |f(t)|^p\, d\tau \geq \inf_{t\in A_n}\left[\prod_{k=1}^{n}(w\circ\alpha^{-k})(t)\right]^p \|y_{\alpha,w}\chi_{A_n}\|_p^p \\
&= \inf_{t\in A_n}\left[\prod_{k=1}^{n}(w\circ\alpha^{-k})(t)\right]^p \frac{1}{\inf_{t\in A_n}\left[\prod_{k=1}^{n}(w\circ\alpha^{-k})(t)\right]^p}\tau(A_n) \\
&= 1.
\end{aligned}$$ □

Assume now that there exists some $l \in \mathbb{Z}$ such that

$$\beta := \inf\left\{\prod_{k=1}^{n}(w\circ\alpha^{-k})(t) : t\in[l,l+1], n\in\mathbb{N}\right\} > 0.$$

Put

$$F := \left\{ f \in L^p(\mathbb{R},\tau) : \|f\chi_{[m,m+1]}\|_p \geq \left\|\frac{1}{\beta}\chi_{[l,l+1]}\chi_{[m,m+1]}\right\|_p \text{ for all } m\in\mathbb{Z}\right\}.$$

So by Theorem 2.8, F is not σ-porous. For every $f \in F$, we have

$$\begin{aligned}
\|T^n_{\alpha,w}(f)\|_p^p &= \int_{\mathbb{R}} \left[\prod_{k=1}^{n}(w\circ\alpha^{n-k})(t)\right]^p |(f\circ\alpha^n)(t)|^p\, d\tau = \int_{\mathbb{R}} \left[\prod_{k=1}^{n}(w\circ\alpha^{-k})(t)\right]^p |f(t)|^p\, d\tau \\
&\geq \int_{[l,l+1]} \left[\prod_{k=1}^{n}(w\circ\alpha^{-k})(t)\right]^p |f(t)|^p\, d\tau \geq 1.
\end{aligned}$$

Hence, the set

$$\{f\in L^p(\mathbb{R},\tau) : \|T^n_{\alpha,w}(f)\|_p \geq 1 \text{ for all } n\in\mathbb{N}\}$$

is not σ-porous.

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