

Continuous Orbit Equivalence for Automorphism Systems of Equivalence Relations

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Abstract. We introduce notions of continuous orbit equivalence and strong (respective, weak) continuous orbit equivalence for automorphism systems of étale equivalence relations, and characterize them in terms of the semi-direct product groupoids, as well as their reduced groupoid C^* -algebras and the associated C^* -automorphism systems of group actions or coactions on them. In particular, we study topological rigidity of expansive automorphism actions on compact (connected) metrizable groups.

1. Introduction

The interplay between orbit equivalence of topological dynamical systems and C^* -algebras has been studied by many authors. An early celebrated result in this direction is the work on strong orbit equivalence of minimal homeomorphisms on Cantor sets given by Giordano, Putnam and Skau [8]. Later, Tomiyama and Boyle–Tomiyama studied a generalization of GPS’s result to the case of topologically free homeomorphisms on compact Hausdorff spaces (see [3, 31]). In [13], Matsumoto introduced the notion of continuous orbit equivalence of one-sided topological Markov shifts and characterized them in terms of the existence of diagonal preserving $*$ -isomorphisms between the associated Cuntz–Krieger algebras. In [18], Matui and Matsumoto gave a classification result of two-sided irreducible topological Markov shifts in the sense of flow equivalence by means of continuous orbit equivalence of one-sided topological Markov shifts. We can refer to [5, 6] for some generalizations on flow equivalence and study on the relation between topological conjugacy of two-sided shifts of finite type and the associated stabilized Cuntz–Krieger algebras with the canonical Cartan subalgebras and gauge actions. More recently, in [14, 15], Matsumoto introduced notions of asymptotic continuous orbit equivalence, asymptotic conjugacy and asymptotic flip conjugacy in Smale spaces and characterized them in terms of their groupoids and asymptotic Ruelle algebras with their dual actions. He also characterized topological conjugacy classes of one-sided topological Markov shifts in terms of the associated Cuntz–Krieger algebras and its gauge actions with potentials in [17].

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Our interests lie in group actions. As a topological analogue of the classification results on the probability measure preserving actions in the sense of orbit equivalence, Li introduced the notion of continuous orbit equivalence for continuous group actions and proved that two topologically free systems are continuously orbit equivalent if and only if their associated transformation groupoids are isomorphic (see [11]). By Renault's result in [24], these conditions are also equivalent to the existence of a C^* -isomorphism preserving the canonical Cartan subalgebras between the corresponding crossed product algebras. In [7], Li's rigidity result has been generalized to the case of group actions with torsion-free and abelian essential stabilisers.

The local conjugacy relations from expansive group action systems are generalizations of asymptotic equivalence relations of Smale spaces (see [21, 29]). In [9], we characterized continuous orbit equivalence of expansive systems up to local conjugacy relations and showed that two expansive actions are asymptotically continuous orbit equivalent if and only if the associated semi-direct product groupoids of local conjugacy relations are isomorphic.

In this paper, we consider continuous orbit equivalence between automorphism systems of étale equivalence relations. Given an étale equivalence relation \mathcal{R} on a compact metrizable space X , let $G \curvearrowright_\alpha (X, \mathcal{R})$ be a dynamical system arising from an automorphism action of a countable group G on \mathcal{R} in the sense that each α_g is an automorphism of \mathcal{R} as an étale groupoid. Denote by $\mathcal{R} \rtimes_\alpha G$ the associated semi-direct product groupoid. We say that two systems $G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ *conjugate* if there exist an isomorphism $\tilde{\varphi}: \mathcal{R} \rightarrow \mathcal{S}$ as étale groupoids and a group isomorphism $\theta: G \rightarrow H$ such that $\tilde{\varphi}(g\gamma) = \theta(g)\tilde{\varphi}(\gamma)$ for $\gamma \in \mathcal{R}$ and $g \in G$. We call the set $[x]_{G, \mathcal{R}} = \{y \in X : (gx, y) \in \mathcal{R} \text{ for some } g \in G\}$ *the bi-orbit* of x . Motivated by the notion of usual orbit equivalence of dynamical systems, we say that $G \curvearrowright (X, \mathcal{R})$ and $H \curvearrowright (Y, \mathcal{S})$ are *orbit equivalent* if there exists a homeomorphism $\varphi: X \rightarrow Y$ such that $\varphi([x]_{G, \mathcal{R}}) = [\varphi(x)]_{H, \mathcal{S}}$ for $x \in X$. We say they are *continuously orbit equivalent* if there exist a homeomorphism $\varphi: X \rightarrow Y$, continuous maps $a: \mathcal{R} \times G \rightarrow H$ and $b: \mathcal{S} \times H \rightarrow G$ such that both the maps $((x, y), g) \in \mathcal{R} \times G \rightarrow (\varphi(x), a((x, y), g)\varphi(g^{-1}y)) \in \mathcal{S}$ and $((x, y), g) \in \mathcal{S} \times H \rightarrow (\varphi^{-1}(x), b((x, y), g)\varphi^{-1}(g^{-1}y)) \in \mathcal{R}$ are well-defined and continuous. The following are the main results.

Theorem 1.1 (Theorem 4.12). *Assume that $G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ are essentially free. Then the following statements are equivalent.*

- (i) $G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ are continuously orbit equivalent;
- (ii) $\mathcal{R} \rtimes_\alpha G$ and $\mathcal{S} \rtimes_\beta H$ are isomorphic as étale groupoids;
- (iii) there exists a C^* -isomorphism Φ from $C_r^*(\mathcal{R} \rtimes_\alpha G)$ onto $C_r^*(\mathcal{S} \rtimes_\beta H)$ such that

$$\Phi(C(X)) = C(Y).$$

Here the notion of essential freeness for $G \curvearrowright_\alpha (X, \mathcal{R})$ is a generalization and analogue of topological freeness of dynamical systems. When $\mathcal{R} = \{(x, x) : x \in X\}$ is a trivial étale equivalence relation, or \mathcal{R} is the local conjugacy relation or asymptotic equivalence relation arising from an expansive system $G \curvearrowright_\alpha X$ or an irreducible Smale space (X, φ) , this result is reduced to Theorem 1.2 in [11], Theorem 3.4 in [14] and Theorem 4.2 in [9]. In addition, we also introduce two stronger versions of continuous orbit equivalence for automorphism systems, namely strong continuous orbit equivalence and weak continuous orbit equivalence. The properties of these two stronger versions correspond to two special orbit equivalence with some uniform conditions, and are also analogues of asymptotic flip conjugacy in [14] and (strong) asymptotic conjugacy in [9]. Let ρ_α be the canonical cocycle from $\mathcal{R} \rtimes_\alpha G$ onto G . It follows from [7, Lemma 6.1] that ρ_α gives us a C^* -coaction system $(C_r^*(\mathcal{R} \rtimes_\alpha G), G; \delta_\alpha)$.

Theorem 1.2 (Corollary 4.16). *Assume that $G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ are essentially free. Then*

- (i) *$G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ are weakly continuously orbit equivalent if and only if there is an isomorphism $\Lambda: \mathcal{R} \rtimes_\alpha G \rightarrow \mathcal{S} \rtimes_\beta H$ such that $\Lambda(\mathcal{R}) = \mathcal{S}$. Moreover, if these conditions hold, then there is a C^* -isomorphism $\Phi: C_r^*(\mathcal{R} \rtimes_\alpha G) \rightarrow C_r^*(\mathcal{S} \rtimes_\beta H)$ such that $\Phi(C(X)) = C(Y)$ and $\Phi(C_r^*(\mathcal{R})) = C_r^*(\mathcal{S})$.*
- (ii) *$G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ are strongly continuously orbit equivalent if and only if there exist an étale groupoid isomorphism $\Lambda: \mathcal{R} \rtimes_\alpha G \rightarrow \mathcal{S} \rtimes_\beta H$ and a group isomorphism $\theta: G \rightarrow H$ such that $\theta\rho_\alpha = \rho_\beta\Lambda$.*

These two statements are also equivalent to the existence of a conjugacy ϕ from the coaction system $(C_r^(\mathcal{R} \rtimes_\alpha G), G; \delta_\alpha)$ onto $(C_r^*(\mathcal{S} \rtimes_\beta H), H; \delta_\beta)$ with $\phi(C(X)) = C(Y)$.*

Furthermore, when \mathcal{R} and \mathcal{S} are minimal or X and Y are connected, these two notions of strong continuous orbit equivalence and weak continuous orbit equivalence are equivalent.

The assumption of essential freeness in the above theorems is necessary. Automorphism systems on local conjugacy relations from expansive actions are typical examples. The automorphism systems of local conjugacy relations from a full shift $G \curvearrowright A^G$ over a finite set A and an irreducible Smale space (X, ψ) are essentially free (see [9, 14]). The following result generalizes Matsumoto's result.

Theorem 1.3 (Theorem 5.2). *Let $\mathbb{Z} \curvearrowright_\alpha X$ be an expansive system generated by a homeomorphism φ on X , and let \mathcal{R}_α be the local conjugacy relation associated to $\mathbb{Z} \curvearrowright_\alpha X$. Assume that X is infinite and has no isolated points. Then $\mathbb{Z} \curvearrowright_\alpha (X, \mathcal{R}_\alpha)$ is essentially free.*

In [2], Bhattacharya proved that topological conjugacy and algebraic conjugacy between two automorphism actions on compact abelian connected metrizable spaces are in agreement. We have a rigidity result for automorphism actions on nonabelian groups.

Proposition 1.4 (Proposition 6.5, Proposition 6.8). *Let $G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ be two systems on local conjugacy relations from topologically free, expansive automorphism actions on compact and connected metrizable groups X and Y , respectively. Assume that the homoclinic group Δ_α associated to $G \curvearrowright_\alpha X$ is dense in X . Then the following statements are equivalent:*

- (i) $G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ are conjugate;
- (ii) $G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ are weakly continuously orbit equivalent;
- (iii) $G \curvearrowright_\alpha X$ and $H \curvearrowright_\beta Y$ are conjugate;
- (iv) $G \curvearrowright_\alpha X$ and $H \curvearrowright_\beta Y$ are algebraically conjugate.

In particular, two hyperbolic toral automorphisms on $\mathbb{R}^n/\mathbb{Z}^n$ are flip conjugate if and only if the \mathbb{Z} -actions they generate are continuously orbit equivalent up to the associated local conjugacy relations.

This paper is organized as follows. Section 3 characterizes conjugacy of automorphism systems of étale equivalence relations and the reduced C^* -algebra of the associated semi-direct product groupoid of equivalence relations. In Section 4, we introduce notions of continuous orbit equivalence, strong- and weak- continuous orbit equivalence for automorphism systems, and characterize them in terms of the semi-direct product groupoids and the corresponding C^* -algebras. In Section 5, we discuss essential freeness of automorphism systems on local conjugacy equivalence relations arising from expansive actions, and in Section 6, we study topological rigidity of expansive automorphism actions on compact (connected) metrizable groups. As an example, we characterize the structure of the local conjugacy relation from a hyperbolic toral automorphism on the n -torus.

2. Preliminaries

Unless otherwise specified, all our groups are discrete and countable, their identity elements are denoted by the same symbol e , and all topological groupoids are second countable, locally compact and Hausdorff. We refer to [23, 28] for more details on topological groupoids and their C^* -algebras, and refer to [19, 32] for C^* -dynamical systems.

For a topological groupoid \mathcal{G} , let $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(2)}$ be the unit space and the set of composable pairs, respectively. The range and domain maps r, d from \mathcal{G} onto $\mathcal{G}^{(0)}$ are defined by

$r(g) = gg^{-1}$ and $d(g) = g^{-1}g$, respectively. If r and d are local homeomorphisms then \mathcal{G} is called to be *étale*. For $u, v \in \mathcal{G}^{(0)}$, we write $\mathcal{G}^u = r^{-1}(u)$, $\mathcal{G}_u = d^{-1}(u)$ and $\mathcal{G}_u^v = \mathcal{G}^v \cap \mathcal{G}_u$. When \mathcal{G} is étale, these sets are discrete and countable, and $\mathcal{G}^{(0)}$ is open and closed in \mathcal{G} . Recall that \mathcal{G} is *topologically principle* if $\{u \in \mathcal{G}^{(0)} : \mathcal{G}_u^u = \{u\}\}$ is dense in $\mathcal{G}^{(0)}$.

Each equivalence relation $\mathcal{R} \subseteq X \times X$ on a topological space X is a groupoid with multiplication $(x, y)(w, z) = (x, z)$ if $y = w$ and inverse $(x, y)^{-1} = (y, x)$. If we identify (x, x) with x , then the unit space $\mathcal{R}^{(0)}$ equals X and the range (resp. domain) map is defined by $r(x, y) = x$ (resp. $d(x, y) = y$). If there exists a topology on \mathcal{R} (not necessarily the relative product topology from $X \times X$) for which \mathcal{R} is an étale groupoid, then \mathcal{R} is called an *étale equivalence relation* on X . In this case, if every \mathcal{R} -equivalence class is dense in X then \mathcal{R} is *minimal*.

By a *dynamical system*, denoted by $G \curvearrowright_\alpha X$ (or simply by $G \curvearrowright X$), we mean an action α of a group G on a second countable, locally compact and Hausdorff space X by homeomorphisms. The action α is usually expressed as $(g, x) \in G \times X \rightarrow gx \in X$. The associated transformation groupoid $X \rtimes G$ is given by the set $X \times G$ with the product topology, multiplication $(x, g)(y, h) = (x, gh)$ if $y = g^{-1}x$, and inverse $(x, g)^{-1} = (g^{-1}x, g^{-1})$. Clearly, $X \rtimes G$ is étale, and if (x, e) is identified with x then its unit space equals X , its range map is given by $r(x, g) = x$, and its domain map by $d(x, g) = g^{-1}x$. A system $G \curvearrowright X$ is said to be *topologically free* if for every $e \neq g \in G$, $\{x \in X : gx \neq x\}$ is dense in X . From [11, Corollary 2.3], $G \curvearrowright X$ is topologically free if and only if $X \rtimes G$ is topologically principal. Two systems $G \curvearrowright X$ and $H \curvearrowright Y$ are *conjugate* if there exist a homeomorphism $\varphi: X \rightarrow Y$ and a group isomorphism $\theta: G \rightarrow H$ such that $\varphi(gx) = \theta(g)\varphi(x)$ for $x \in X$ and $g \in G$.

A map $\Phi: \mathcal{G} \rightarrow \mathcal{H}$ between étale groupoids \mathcal{G} and \mathcal{H} is a homomorphism if it is continuous and, for all $(\gamma, \gamma') \in \mathcal{G}^{(2)}$, we have $(\Phi(\gamma), \Phi(\gamma')) \in \mathcal{H}^{(2)}$ and $\Phi(\gamma\gamma') = \Phi(\gamma)\Phi(\gamma')$. Moreover, if Φ is a homeomorphism such that Φ and Φ^{-1} are homomorphisms, then it is called an *isomorphism*. In this case, the restriction, $\Phi|_{\mathcal{G}^{(0)}}$, of Φ to the unit space $\mathcal{G}^{(0)}$ is a homeomorphism from $\mathcal{G}^{(0)}$ onto $\mathcal{H}^{(0)}$. A homomorphism from \mathcal{G} into a group Γ is also called a *cocycle* on \mathcal{G} . Two étale equivalence relations $\mathcal{R} \subseteq X \times X$ and $\mathcal{S} \subseteq Y \times Y$ are isomorphic if and only if there exists a homeomorphism $\varphi: X \rightarrow Y$ such that $\varphi \times \varphi: (x, y) \in \mathcal{R} \rightarrow (\varphi(x), \varphi(y)) \in \mathcal{S}$ is an isomorphism.

Given an étale groupoid \mathcal{G} , the linear space, $C_c(\mathcal{G})$, of continuous complex functions with compact support on \mathcal{G} is a $*$ -algebra under the operations: $f^*(\gamma) = \overline{f(\gamma^{-1})}$ and $f * g(\gamma) = \sum_{\gamma' \in \mathcal{G}_{d(\gamma)}} f(\gamma\gamma'^{-1})g(\gamma')$ for $f, g \in C_c(\mathcal{G})$ and $\gamma \in \mathcal{G}$. For each $u \in \mathcal{G}^{(0)}$, there is a $*$ -representation Ind_u of $C_c(\mathcal{G})$ on the Hilbert space $l^2(\mathcal{G}_u)$ of square summable functions on \mathcal{G}_u by $\text{Ind}_u(f)(\xi)(\gamma) = \sum_{\gamma' \in \mathcal{G}_u} f(\gamma\gamma'^{-1})\xi(\gamma')$ for $f \in C_c(\mathcal{G})$, $\xi \in l^2(\mathcal{G}_u)$ and $\gamma \in \mathcal{G}_u$. The reduced C^* -algebra $C_r^*(\mathcal{G})$ of \mathcal{G} is the completion of $C_c(\mathcal{G})$ with respect to the norm

$\|f\|_{\text{red}} = \sup_{u \in \mathcal{G}^{(0)}} \|\text{Ind}_u(f)\|$ for $f \in C_c(\mathcal{G})$. Since $\mathcal{G}^{(0)}$ is clopen in \mathcal{G} , $C_c(\mathcal{G}^{(0)})$ is contained in $C_c(\mathcal{G})$ in the canonical way, and this extends to an injection $C_0(\mathcal{G}^{(0)}) \hookrightarrow C_r^*(\mathcal{G})$. For an open subgroupoid \mathcal{H} of \mathcal{G} , $C_c(\mathcal{H})$ can be embedded into $C_c(\mathcal{G})$ as a $*$ -subalgebra, so $C_r^*(\mathcal{H})$ is embedded into $C_r^*(\mathcal{G})$ as a C^* -subalgebra in the canonical way. The C^* -algebra $C_r^*(X \rtimes G)$ of the transformation groupoid is isomorphic to the reduced crossed product $C_0(X) \rtimes_{\alpha, r} G$ (see [28]).

Given two groups N, H and a homomorphism φ from H into the automorphism group $\text{Aut}(N)$ of N , the semi-direct product, denoted by $N \rtimes_{\varphi} H$, of N by H is defined as the set $N \times H$ with group law given by the formulas $(n, h)(n_1, h_1) = (n\varphi_h(n_1), hh_1)$ and $(n, h)^{-1} = (\varphi_{h^{-1}}(n^{-1}), h^{-1})$.

3. Automorphism systems of étale equivalence relations and the associated semi-direct product groupoids

Given an étale equivalence relation \mathcal{R} on a compact metrizable space X , we call a dynamical system $G \curvearrowright_{\alpha} \mathcal{R}$ an *automorphism system* if each α_g is an automorphism of \mathcal{R} as an étale groupoid. Clearly, this system induces an action, also denoted by α , of G on X by homeomorphisms such that $g(x, y) = (gx, gy)$ for $g \in G$ and $(x, y) \in \mathcal{R}$. We use the notation $G \curvearrowright_{\alpha} (X, \mathcal{R})$ (or $G \curvearrowright (X, \mathcal{R})$ for short) to denote such an automorphism system.

The semi-direct product groupoid, $\mathcal{R} \rtimes_{\alpha} G$, attached to $G \curvearrowright_{\alpha} (X, \mathcal{R})$, is the set $\mathcal{R} \times G$ with multiplication $((x, y), g)((u, v), h) = ((x, gv), gh)$ if $u = g^{-1}y$, and inverse $((x, y), g)^{-1} = ((g^{-1}y, g^{-1}x), g^{-1})$. The unit space identifies with X by identifying $((x, x), e)$ with x . Then $r((x, y), g) = x$ and $d((x, y), g) = g^{-1}y$. Endowed with the relative product topology from $\mathcal{R} \times G$, the groupoid $\mathcal{R} \rtimes_{\alpha} G$ is étale (see [23]). The following is another characterization of the semi-direct product groupoid.

Definition 3.1. Let

$$\mathcal{R} \rtimes_{\alpha} G = \{(x, g, y) \mid g \in G, x, y \in X, (x, gy) \in \mathcal{R}\}.$$

Then, under the following multiplication and inverse,

$$(x, g, y)(y, h, v) = (x, gh, v) \quad \text{and} \quad (x, g, y)^{-1} = (y, g^{-1}, x),$$

$\mathcal{R} \rtimes_{\alpha} G$ is a groupoid. Define a map $\gamma_0: \mathcal{R} \rtimes_{\alpha} G \rightarrow \mathcal{R} \times_{\alpha} G$, by $\gamma_0(x, g, y) = ((x, gy), g)$, which is a bijection with inverse $\gamma_0^{-1}((x, y), g) = (x, g, g^{-1}y)$. We transfer the product topology from $\mathcal{R} \times_{\alpha} G$ over to $\mathcal{R} \rtimes_{\alpha} G$. Then $\mathcal{R} \rtimes_{\alpha} G$ is an étale groupoid and γ_0 is an étale groupoid isomorphism.

Remark 3.2. If we identify the unit space $(\mathcal{R} \rtimes_{\alpha} G)^{(0)}$ with X as topological spaces by identifying (x, e, x) with x , then $r(x, g, y) = x$ and $d(x, g, y) = y$. The equivalence relation \mathcal{R} and the transformation groupoid $X \rtimes G$ can be embedded into $\mathcal{R} \rtimes_{\alpha} G$ as étale subgroupoids through the identifications $(x, y) \in \mathcal{R} \rightarrow (x, e, y) \in \mathcal{R} \rtimes_{\alpha} G$ and $(x, g) \in X \rtimes G \rightarrow (x, g, g^{-1}x) \in \mathcal{R} \rtimes_{\alpha} G$.

One can check that the map $\rho_{\alpha}: \mathcal{R} \rtimes_{\alpha} G \rightarrow G$, defined by $\rho_{\alpha}(x, g, y) = g$, is a cocycle.

We call two automorphism systems $G \curvearrowright_{\alpha} (X, \mathcal{R})$ and $H \curvearrowright_{\beta} (Y, \mathcal{S})$ on compact metrizable spaces *conjugate* if there exist an isomorphism $\tilde{\varphi}: \mathcal{R} \rightarrow \mathcal{S}$ and a group isomorphism $\theta: G \rightarrow H$ such that $\tilde{\varphi}(g\gamma) = \theta(g)\tilde{\varphi}(\gamma)$ for $\gamma \in \mathcal{R}$ and $g \in G$. Clearly, this is equivalent to there being a homeomorphism $\varphi: X \rightarrow Y$ and a group isomorphism $\theta: G \rightarrow H$ such that $\varphi \times \varphi: (x, y) \in \mathcal{R} \rightarrow (\varphi(x), \varphi(y)) \in \mathcal{S}$ is an isomorphism and $\varphi(gx) = \theta(g)\varphi(x)$ for $x \in X$ and $g \in G$. In particular, the two systems $G \curvearrowright_{\alpha} X$ and $H \curvearrowright_{\beta} Y$ are conjugate.

Proposition 3.3. *If $G \curvearrowright_{\alpha} (X, \mathcal{R})$ and $H \curvearrowright_{\beta} (Y, \mathcal{S})$ are conjugate, then there is an isomorphism, $\Lambda: \mathcal{R} \rtimes_{\alpha} G \rightarrow \mathcal{S} \rtimes_{\beta} H$, such that $\Lambda(\mathcal{R}) = \mathcal{S}$ and $\Lambda(X \rtimes G) = Y \rtimes H$.*

Assume that one of the following statements holds:

- (i) X and Y are connected.
- (ii) \mathcal{R} and \mathcal{S} are minimal.

Then the above converse holds, i.e., $G \curvearrowright_{\alpha} (X, \mathcal{R})$ and $H \curvearrowright_{\beta} (Y, \mathcal{S})$ are conjugate if and only if there is an isomorphism, $\Lambda: \mathcal{R} \rtimes_{\alpha} G \rightarrow \mathcal{S} \rtimes_{\beta} H$, such that $\Lambda(\mathcal{R}) = \mathcal{S}$ and $\Lambda(X \rtimes G) = Y \rtimes H$.

Proof. Assume that $G \curvearrowright_{\alpha} (X, \mathcal{R})$ and $H \curvearrowright_{\beta} (Y, \mathcal{S})$ are conjugate by a homeomorphism φ from X onto Y and a group isomorphism θ from G onto H . Define the map Λ from $\mathcal{R} \rtimes_{\alpha} G$ into $\mathcal{S} \rtimes_{\beta} H$ by $\Lambda(x, g, y) = (\varphi(x), \theta(g), \varphi(y))$. Then Λ is an isomorphism with inverse $\Lambda^{-1}(u, h, v) = (\varphi^{-1}(u), \theta^{-1}(h), \varphi^{-1}(v))$ and $\Lambda(\mathcal{R}) = \mathcal{S}$ and $\Lambda(X \rtimes G) = Y \rtimes H$.

For the converse, let Λ be an isomorphism from $\mathcal{R} \rtimes_{\alpha} G$ onto $\mathcal{S} \rtimes_{\beta} H$ such that $\Lambda(\mathcal{R}) = \mathcal{S}$ and $\Lambda(X \rtimes G) = Y \rtimes H$. Let φ be the restriction of Λ to X , and let $a = \rho_{\beta}\Lambda$ and $b = \rho_{\alpha}\Lambda^{-1}$. Then φ is a homeomorphism from X onto Y , and a and b are continuous cocycles on $\mathcal{R} \rtimes_{\alpha} G$ and $\mathcal{S} \rtimes_{\beta} H$, respectively. Moreover, $\Lambda(x, g, y) = (\varphi(x), a(x, g, y), \varphi(y))$, and its inverse $\Lambda^{-1}(u, h, v) = (\varphi^{-1}(u), b(u, h, v), \varphi^{-1}(v))$. The fact that $\Lambda(\mathcal{R}) = \mathcal{S}$ implies that $a(x, e, y) = e$ and $\varphi \times \varphi: (x, y) \in \mathcal{R} \rightarrow (\varphi(x), \varphi(y)) \in \mathcal{S}$ is an isomorphism. The requirement that $\Lambda(X \rtimes G) = Y \rtimes H$ gives us that

$$(3.1) \quad \varphi(x) = a(x, g, g^{-1}x)\varphi(g^{-1}x).$$

Also since $(x, g, g^{-1}x)(g^{-1}x, e, g^{-1}y)(g^{-1}y, g^{-1}, y) = (x, e, y)$ for $(x, y) \in \mathcal{R}$ and $g \in G$, we have $a(x, g, g^{-1}x) = a(y, g, g^{-1}y)$. By symmetry, b has a similar property to a .

Assume that X and Y are connected. Since the restricted map $a|_{X \rtimes G}: X \rtimes G \rightarrow H$ is continuous, we have, for every $g \in G$, the map $a|_{X \times \{g\}}$ is a constant, and thus $a(x, g, g^{-1}x) = a(y, g, g^{-1}y)$ for all $x, y \in X$ and $g \in G$. Similarly, we have $b(u, h, h^{-1}u) = b(v, h, h^{-1}v)$ for all $u, v \in Y$ and $h \in H$.

Assume that \mathcal{R} and \mathcal{S} are minimal. For $x, y \in X$ and $g \in G$, we choose a sequence $\{x_n\}$ in X converging to y and satisfying $(x_n, x) \in \mathcal{R}$ for each n . From the above proof, $a(x_n, g, g^{-1}x_n) = a(x, g, g^{-1}x)$ for each n , which implies that $a(x, g, g^{-1}x) = a(y, g, g^{-1}y)$ from the continuity of a . Similarly, we have $b(u, h, h^{-1}u) = b(v, h, h^{-1}v)$ for all $u, v \in Y$ and $h \in H$.

Consequently, under the hypothesis of (i) or (ii), there exist two maps $\theta: G \rightarrow H$ and $\vartheta: H \rightarrow G$ such that $a(x, g, g^{-1}x) = \theta(g)$ and $b(u, h, h^{-1}u) = \vartheta(h)$ for every $x \in X$, $u \in Y$, $g \in G$ and $h \in H$. Since Λ is an isomorphism with inverse Λ^{-1} , θ is a group isomorphism with inverse ϑ . Moreover, (3.1) implies that $\varphi(gx) = \theta(g)\varphi(x)$ for $x \in X$ and $g \in G$. Hence $G \curvearrowright_{\alpha} (X, \mathcal{R})$ and $H \curvearrowright_{\beta} (Y, \mathcal{S})$ are conjugate. \square

Given an automorphism system $G \curvearrowright_{\alpha} (X, \mathcal{R})$, one can check that the map

$$\alpha_g(f)(x, y) = f(g^{-1}x, g^{-1}y)$$

for $f \in C_c(\mathcal{R})$, $(x, y) \in \mathcal{R}$ and $g \in G$ gives a C^* -dynamical system $(C_r^*(\mathcal{R}), G, \alpha)$. Let $C_c(G, C_r^*(\mathcal{R}))$ be the set of all continuous complex functions from G to $C_r^*(\mathcal{R})$ with compact support. Then it is a $*$ -algebra over \mathbb{C} under the following multiplication and involution:

$$(\xi * \eta)(g) = \sum_{h \in G} \xi(h) \alpha_h(\eta(h^{-1}g)), \quad \xi^*(g) = \alpha_g(\xi(g^{-1})^*)$$

for $\xi, \eta \in C_c(G, C_r^*(\mathcal{R}))$. The reduced crossed product C^* -algebra, denoted by $C_r^*(\mathcal{R}) \rtimes_{\alpha, r} G$, associated to the C^* -dynamical system is defined to be the closure of $C_c(G, C_r^*(\mathcal{R}))$ under the reduced crossed norm (see [19, 32]). By identifying an element $a \in C_r^*(\mathcal{R})$ with the element $\xi_a \in C_c(G, C_r^*(\mathcal{R}))$ defined by $\xi_a(e) = a$ and $\xi_a(g) = 0$ for $g \neq e$, $C_r^*(\mathcal{R})$ can be embedded into $C_r^*(\mathcal{R}) \rtimes_{\alpha, r} G$ as a unital C^* -subalgebra. When G is abelian, we let $(C_r^*(\mathcal{R}) \rtimes_{\alpha, r} G, \widehat{G}, \widehat{\alpha})$ be the (dual) C^* -automorphism system of the dual group \widehat{G} , defined by

$$\widehat{\alpha}_{\xi}(f)(g) = \langle \xi, g \rangle f(g)$$

for $\xi \in \widehat{G}$, $f \in C_c(G, C_r^*(\mathcal{R}))$ and $g \in G$, where $\langle \xi, g \rangle$ is the value of the character $\xi \in \widehat{G}$ at $g \in G$.

Recall that a conjugacy between two C^* -dynamical systems (\mathcal{A}, G, α) and (\mathcal{B}, H, β) is a $*$ -isomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ that is α - β equivariant in the sense that there exists a group isomorphism $\theta: G \rightarrow H$ satisfying that $\phi \alpha_g = \beta_{\theta(g)} \phi$ for each $g \in G$. If such a ϕ exists, we call two systems conjugate. Note that the existence of an isomorphism between two

étale equivalence relations \mathcal{R} on X and \mathcal{S} on Y is equivalent to the existence of a C^* -isomorphism between their associated reduced groupoid C^* -algebras $C_r^*(\mathcal{R})$ and $C_r^*(\mathcal{S})$ preserving the canonical subalgebras $C(X)$ and $C(Y)$ (see [23]). Thus one can check the following proposition by definitions.

Proposition 3.4. *Two automorphism systems $G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ are conjugate if and only if there is a conjugacy ϕ between $(C_r^*(\mathcal{R}), G, \alpha)$ and $(C_r^*(\mathcal{S}), H, \beta)$ such that $\phi(C(X)) = C(Y)$.*

In this case, there exists a $$ -isomorphism $\Lambda: C_r^*(\mathcal{R}) \rtimes_{\alpha, r} G \rightarrow C_r^*(\mathcal{S}) \rtimes_{\beta, r} H$ such that $\Lambda(C_r^*(\mathcal{R})) = C_r^*(\mathcal{S})$ and $\Lambda(C(X)) = C(Y)$.*

From [23, Proposition II.5.1], when G is abelian, the canonical cocycle ρ_α on $\mathcal{R} \rtimes_\alpha G$ induces the dual action, denoted by $\widehat{\rho}_\alpha$, of the dual group \widehat{G} on $C_r^*(\mathcal{R} \rtimes_\alpha G)$, defined by

$$(\widehat{\rho}_\alpha \xi f)(x, g, y) = \langle \xi, g \rangle f(x, g, y)$$

for $\xi \in \widehat{G}$, $f \in C_c(\mathcal{R} \rtimes_\alpha G)$, $(x, g, y) \in \mathcal{R} \rtimes_\alpha G$. Thus, this forms a C^* -dynamical system $(C_r^*(\mathcal{R} \rtimes_\alpha G), \widehat{G}, \widehat{\rho}_\alpha)$. Moreover, if $G = \mathbb{Z}$, then the fixed point algebra of $\widehat{\rho}_\alpha$ is isomorphic to $C_r^*(\mathcal{R})$ (see [25, Proposition 3.3.7]). The following theorem characterizes the reduced groupoid C^* -algebra of $\mathcal{R} \rtimes_\alpha G$ by the crossed product construction, which is perhaps a well-known fact, as we were unable to find an explicit reference, we provide a proof.

Theorem 3.5. *Let $G \curvearrowright_\alpha (X, \mathcal{R})$ be an automorphism system. Then $C_r^*(\mathcal{R} \rtimes_\alpha G)$ is isomorphic to $C_r^*(\mathcal{R}) \rtimes_{\alpha, r} G$. Moreover, if G is abelian, then the two C^* -dynamical systems $(C_r^*(\mathcal{R} \rtimes_\alpha G), \widehat{G}, \widehat{\rho}_\alpha)$ and $(C_r^*(\mathcal{R}) \rtimes_{\alpha, r} G, \widehat{G}, \widehat{\alpha})$ are conjugate.*

Proof. To simplify symbols, let $\mathcal{G} = \mathcal{R} \rtimes_\alpha G$. Define

$$\Phi(\xi)(x, g, y) = \xi(g)(x, gy) \quad \text{for } \xi \in C_c(G, C_c(\mathcal{R})) \text{ and } (x, g, y) \in \mathcal{G}$$

and

$$\Psi(\eta)(g)(x, y) = \eta(x, g, g^{-1}y) \quad \text{for } \eta \in C_c(\mathcal{G}) \text{ and } g \in G, (x, y) \in \mathcal{R}.$$

One can check that $\Phi: C_c(G, C_c(\mathcal{R})) \rightarrow C_c(\mathcal{G})$ and $\Psi: C_c(\mathcal{G}) \rightarrow C_c(G, C_c(\mathcal{R}))$ are $*$ -isomorphisms such that Φ and Ψ are inverse to each other.

Given $x \in X$, let $l^2(\mathcal{R}_x)$ be the Hilbert space of all square-summable complex-valued functions on the \mathcal{R} -equivalent class \mathcal{R}_x of x . We consider two Hilbert spaces $l^2(G, l^2(\mathcal{R}_x)) = \{\varphi: G \rightarrow l^2(\mathcal{R}_x) \mid \sum_{g \in G} \|\varphi(g)\|^2 < +\infty\}$ and $l^2(\mathcal{G}_x) = \{\psi: \mathcal{G}_x \rightarrow \mathbb{C} \mid \sum_{\gamma \in \mathcal{G}_x} \|\psi(\gamma)\|^2 < +\infty\}$. Then the map U_x , defined by $(U_x \varphi)(y, g, x) = \varphi(g)(g^{-1}y, x)$ for $\varphi \in l^2(G, l^2(\mathcal{R}_x))$ and $(y, g, x) \in \mathcal{G}_x$, is a unitary operator from $l^2(G, l^2(\mathcal{R}_x))$ onto $l^2(\mathcal{G}_x)$.

Let π_x and λ_x be the regular representations of $C_c(\mathcal{G})$ on $l^2(\mathcal{G}_x)$ and $C_c(\mathcal{R})$ on $l^2(\mathcal{R}_x)$ associated to x , respectively. Then we have the direct sums of representations

$$\pi = \bigoplus_{x \in X} \pi_x: C_c(\mathcal{G}) \rightarrow \bigoplus_{x \in X} B(l^2(\mathcal{G}_x)), \quad \lambda = \bigoplus_{x \in X} \lambda_x: C_c(\mathcal{R}) \rightarrow \bigoplus_{x \in X} B(l^2(\mathcal{R}_x)).$$

Then π_x , λ_x , π and λ can be extended to their corresponding reduced groupoid C^* -algebras and we use the same symbols to denote their extensions. Moreover, π and λ are faithful representations of $C_r^*(\mathcal{G})$ and $C_r^*(\mathcal{R})$, respectively.

The representation λ induces a faithful representation

$$\tilde{\lambda}: \xi \in C_c(G, C_r^*(\mathcal{R})) \rightarrow \bigoplus_{x \in X} \tilde{\lambda}_x(\xi) \in \bigoplus_{x \in X} B(l^2(G, l^2(\mathcal{R}_x))),$$

where, for each $x \in X$, $\tilde{\lambda}_x$ is the representation of $C_c(G, C_r^*(\mathcal{R}))$ on the Hilbert space $l^2(G, l^2(\mathcal{R}_x))$, given by $(\tilde{\lambda}_x(\xi)\varphi)(g) = \sum_{h \in G} \lambda_x(\alpha_{g^{-1}}(\xi(h)))\varphi(h^{-1}g)$ for $\xi \in C_c(G, C_r^*(\mathcal{R}))$, $\varphi \in l^2(G, l^2(\mathcal{R}_x))$. Let $\hat{\lambda}_x(\xi) = U_x \tilde{\lambda}_x(\xi) U_x^*$ for $x \in X$ and $\xi \in C_c(G, C_r^*(\mathcal{R}))$. Then

$$\hat{\lambda}: \xi \in C_c(G, C_r^*(\mathcal{R})) \rightarrow \bigoplus_{x \in X} \hat{\lambda}_x(\xi) \in \bigoplus_{x \in X} B(l^2(\mathcal{G}_x))$$

is a faithful representation. We can check that $\pi_x \Phi(\xi) = \hat{\lambda}_x(\xi)$ for each $x \in X$, thus $\pi \Phi(\xi) = \hat{\lambda}(\xi)$ for all $\xi \in C_c(G, C_c(\mathcal{R}))$.

In fact, for each φ in $l^2(G, l^2(\mathcal{R}_x))$, (y, g, x) in \mathcal{G}_x , we have

$$\begin{aligned} (\pi_x \Phi(\xi) U_x)(\varphi)(y, g, x) &= \sum_{\substack{h \in G \\ (u, h^{-1}gx) \in \mathcal{R}}} [\Phi(\xi)(y, h, u)] [U_x \varphi(u, h^{-1}g, x)] \\ &= \sum_{\substack{h \in G \\ (g^{-1}hu, x) \in \mathcal{R}_x}} [\xi(h)(y, hu)] [\varphi(h^{-1}g)(g^{-1}hu, x)] \\ &= \sum_{\substack{h \in G \\ (v, x) \in \mathcal{R}_x}} \xi(h)(y, gv) \varphi(h^{-1}g)(v, x) \end{aligned}$$

and

$$\begin{aligned} U_x(\tilde{\lambda}_x(\xi)(\varphi))(y, g, x) &= (\tilde{\lambda}_x(\xi)\varphi)(g)(g^{-1}y, x) \\ &= \sum_{h \in G} \sum_{(u, x) \in \mathcal{R}_x} \xi(h)(y, gu) \varphi(h^{-1}g)(u, x). \end{aligned}$$

Then, for each $\xi \in C_c(G, C_c(\mathcal{R}))$, we have

$$\|\Phi(\xi)\|_{\text{red}} = \sup_{x \in X} \|\pi_x(\Phi(\xi))\|_{B(l^2(\mathcal{G}_x))} = \sup_{x \in X} \|\hat{\lambda}_x(\xi)\|_{B(l^2(\mathcal{G}_x))}.$$

Thus $\|\Phi(\xi)\|_{\text{red}} = \|\xi\|_{\text{red}}$ for $\xi \in C_c(G, C_c(\mathcal{R}))$, and Φ is an isomorphism.

The conjugacy of two C^* -systems follows from the definitions of dual actions and the construction of Ψ . \square

Remark 3.6. For a countable discrete group Γ , let $\lambda: g \in \Gamma \rightarrow \lambda_g \in B(l^2(\Gamma))$ be the left regular representation of Γ , and $C_r^*(\Gamma)$ be the reduced group C^* -algebra of Γ . Let $\delta_\Gamma: C_r^*(\Gamma) \rightarrow C_r^*(\Gamma) \otimes C_r^*(\Gamma)$ (where we use the minimal tensor product) be the C^* -homomorphism defined by $\delta_\Gamma(\lambda_g) = \lambda_g \otimes \lambda_g$ for each $g \in \Gamma$. Given a unital C^* -algebra \mathcal{A} , we recall that a coaction of Γ on \mathcal{A} is a nondegenerate homomorphism $\delta: \mathcal{A} \rightarrow \mathcal{A} \otimes C_r^*(\Gamma)$ satisfying the coaction identity $(\delta \otimes \text{id}) \circ \delta = (\text{id} \otimes \delta_\Gamma) \circ \delta$, where id is the identity map. We call $(\mathcal{A}, \Gamma; \delta)$ a C^* -coaction system. Recall that two C^* -coaction systems $(\mathcal{A}, G; \delta)$ and $(\mathcal{B}, H; \varrho)$ are called conjugate if there exists a conjugacy ϕ between two systems, that is, ϕ is a C^* -isomorphism from \mathcal{A} onto \mathcal{B} such that there exists an isomorphism $\theta: G \rightarrow H$ satisfying $(\phi \otimes \tilde{\theta}) \circ \delta = \varrho \circ \phi$, where $\tilde{\theta}: C_r^*(G) \rightarrow C_r^*(H)$ is the C^* -isomorphism induced by θ .

For an automorphism system $G \curvearrowright_\alpha (X, \mathcal{R})$, it follows from [7, Lemma 6.1] that the canonical cocycle $\rho_\alpha: \mathcal{R} \rtimes_\alpha G \rightarrow G$ induces a coaction $\delta_\alpha: C_r^*(\mathcal{R} \rtimes_\alpha G) \rightarrow C_r^*(\mathcal{R} \rtimes_\alpha G) \otimes C_r^*(G)$, of G on $C_r^*(\mathcal{R} \rtimes_\alpha G)$ such that $\delta_\alpha(f) = f \otimes \lambda_g$ when $g \in G$ and $f \in C_c(\mathcal{R} \rtimes_\alpha G)$ satisfy that $\text{supp}(f) \subseteq \rho_\alpha^{-1}(g)$. This gives us a C^* -coaction system $(C_r^*(\mathcal{R} \rtimes_\alpha G), G; \delta_\alpha)$. On the other hand, for the C^* -system $(C_r^*(\mathcal{R}), G, \alpha)$, there is a canonical dual coaction $\hat{\alpha}: C_r^*(\mathcal{R}) \rtimes_{\alpha, r} G \rightarrow (C_r^*(\mathcal{R}) \rtimes_{\alpha, r} G) \otimes C^*(G)$ of G on $C_r^*(\mathcal{R}) \rtimes_{\alpha, r} G$, defined by $\hat{\alpha}(a) = a \otimes I$ and $\hat{\alpha}(u_g) = u_g \otimes v_g$, where $\{a : a \in C_r^*(\mathcal{R})\} \cup \{u_g : g \in G\}$ is the canonical generators of $C_r^*(\mathcal{R}) \rtimes_{\alpha, r} G$, and $C^*(G)$ is the full group C^* -algebra with generators $\{v_g : g \in G\}$ (see [10]). Thus when G is amenable, we have the other C^* -coaction system $(C_r^*(\mathcal{R}) \rtimes_{\alpha, r} G, G; \hat{\alpha})$. Considering the above theorem, we conjecture that the two systems $(C_r^*(\mathcal{R} \rtimes_\alpha G), G; \delta_\alpha)$ and $(C_r^*(\mathcal{R}) \rtimes_{\alpha, r} G, G; \hat{\alpha})$ are conjugate when G is amenable.

4. Continuous orbit equivalence of automorphism systems

Given an automorphism system $G \curvearrowright_\alpha (X, \mathcal{R})$ on a compact metrizable space X , for $x \in X$, we let $[x]_G := \{gx : g \in G\}$ and $[x]_\mathcal{R} := \{y \in X : (x, y) \in \mathcal{R}\}$ be the orbits of x under the action α and the relation \mathcal{R} , respectively. We call the set $[x]_{G, \mathcal{R}} = \{y \in X : (gx, y) \in \mathcal{R} \text{ for some } g \in G\}$ the *bi-orbit* of x . Clearly, $[x]_{G, \mathcal{R}} = \bigcup_{y \in [x]_G} [y]_\mathcal{R} = \bigcup_{y \in [x]_\mathcal{R}} [y]_G = d((\mathcal{R} \rtimes_\alpha G)^x) = r((\mathcal{R} \rtimes_\alpha G)_x)$.

Recall that $G \curvearrowright X$ and $H \curvearrowright Y$ are *orbit equivalent* if there exists a homeomorphism $\varphi: X \rightarrow Y$ such that $\varphi([x]_G) = [\varphi(x)]_H$ for $x \in X$. They are said to be *continuously orbit equivalent* if there exist a homeomorphism $\varphi: X \rightarrow Y$ and continuous maps $a: G \times X \rightarrow H$ and $b: H \times Y \rightarrow G$ such that $\varphi(gx) = a(g, x)\varphi(x)$ for $x \in X$ and $g \in G$, and $\varphi^{-1}(hy) = b(h, y)\varphi^{-1}(y)$ for $y \in Y$ and $h \in H$ (see [11]). Motivated by these notions, we introduce the following definitions.

Definition 4.1. Two systems $G \curvearrowright (X, \mathcal{R})$ and $H \curvearrowright (Y, \mathcal{S})$ are *orbit equivalent* if there exists a homeomorphism $\varphi: X \rightarrow Y$ such that $\varphi([x]_{G, \mathcal{R}}) = [\varphi(x)]_{H, \mathcal{S}}$ for $x \in X$.

In this case, for $x, y \in X$ and $g \in G$ with $(gx, y) \in \mathcal{R}$, there exists h in H such that $(h\varphi(x), \varphi(y)) \in \mathcal{S}$. Similarly, for $u, v \in Y$ and $h \in H$ with $(hu, v) \in \mathcal{S}$, there exists g in G such that $(g\varphi^{-1}(u), \varphi^{-1}(v)) \in \mathcal{R}$. Thus, we have the following notion.

Definition 4.2. Two systems $G \curvearrowright (X, \mathcal{R})$ and $H \curvearrowright (Y, \mathcal{S})$ are *continuously orbit equivalent* and we write $G \curvearrowright (X, \mathcal{R}) \sim_{\text{coe}} H \curvearrowright (Y, \mathcal{S})$, if there exist a homeomorphism $\varphi: X \rightarrow Y$ and continuous maps $a: \mathcal{R} \times G \rightarrow H$ and $b: \mathcal{S} \times H \rightarrow G$ such that the following maps:

$$((x, y), g) \in \mathcal{R} \times G \rightarrow (\varphi(x), a((x, y), g)\varphi(g^{-1}y)) \in \mathcal{S}$$

and

$$((x, y), g) \in \mathcal{S} \times H \rightarrow (\varphi^{-1}(x), b((x, y), g)\varphi^{-1}(g^{-1}y)) \in \mathcal{R}$$

are continuous.

Clearly, continuous orbit equivalence implies orbit equivalence for automorphism systems. Assume a system $G \curvearrowright_{\alpha} X$ is free in the sense that, for $g \in G$ and $x \in X$, $gx = x$ only if $g = e$. We consider two automorphism systems $G \curvearrowright (X, \mathcal{R}_1)$ and $G \curvearrowright (X, \mathcal{R}_2)$, where $\mathcal{R}_1 = \{(x, x) : x \in X\}$ is the trivial étale equivalence relation on X under the relative product topology and $\mathcal{R}_2 = \{(x, gx) : x \in X, g \in G\}$ is the orbit equivalence relation under α . Noticing that the map $(x, g) \in X \rtimes G \rightarrow (x, g^{-1}x) \in \mathcal{R}_2$ is a bijection, we transfer the product topology on $X \rtimes G$ over to \mathcal{R}_2 via this map. Then \mathcal{R}_2 is an étale equivalence relation on X .

Proposition 4.3. *Assume that $G \curvearrowright X$ is free. Then $G \curvearrowright (X, \mathcal{R}_1)$ and $G \curvearrowright (X, \mathcal{R}_2)$ are continuously orbit equivalent, but not conjugate.*

Proof. Let φ be the identity map on X , and let $a((x, x), g) = g$ for $((x, x), g) \in \mathcal{R}_1 \times G$. For each $(x, y) \in \mathcal{R}_2$, there exists unique an element in G , denoted by $k(x, y)$, such that $y = k(x, y)x$. Let $b((x, y), g) = k(x, y)^{-1}g$ for $((x, y), g) \in \mathcal{R}_2 \times G$. Then φ , a and b satisfy the requirements in Definition 4.2, thus $G \curvearrowright (X, \mathcal{R}_1)$ and $G \curvearrowright (X, \mathcal{R}_2)$ are continuously orbit equivalent.

Since \mathcal{R}_1 and \mathcal{R}_2 are never isomorphic, $G \curvearrowright (X, \mathcal{R}_1)$ and $G \curvearrowright (X, \mathcal{R}_2)$ are not conjugate. \square

Using the semi-direct product groupoid $\mathcal{R} \rtimes_{\alpha} G$ and the canonical homeomorphism γ_0 , one can check the following lemma.

Lemma 4.4. *Two systems $G \curvearrowright (X, \mathcal{R}) \sim_{\text{coe}} H \curvearrowright (Y, \mathcal{S})$ if and only if there exist a homeomorphism $\varphi: X \rightarrow Y$ and continuous maps $a: \mathcal{R} \rtimes_{\alpha} G \rightarrow H$ and $b: \mathcal{S} \rtimes_{\beta} H \rightarrow G$ such that the following maps:*

$$(4.1) \quad \Psi: (x, g, y) \in \mathcal{R} \rtimes_{\alpha} G \rightarrow (\varphi(x), a(x, g, y), \varphi(y)) \in \mathcal{S} \rtimes_{\beta} H$$

and

$$(4.2) \quad \tilde{\Psi}: (u, h, v) \in \mathcal{S} \rtimes_{\beta} H \rightarrow (\varphi^{-1}(u), b(u, h, v), \varphi^{-1}(v)) \in \mathcal{R} \rtimes_{\alpha} G$$

are continuous.

Recall that an étale groupoid \mathcal{G} is *topologically principal* if $\{u \in \mathcal{G}^{(0)} : \mathcal{G}_u^u = \{u\}\}$ is dense in $\mathcal{G}^{(0)}$. Since \mathcal{G} is assumed to be second countable, it follows from [4, 24] that it is topologically principal if and only if the interior of \mathcal{G}' is $\mathcal{G}^{(0)}$, where $\mathcal{G}' = \bigcup_{u \in \mathcal{G}^{(0)}} \mathcal{G}_u^u$ is the isotropy bundle of \mathcal{G} . For $G \curvearrowright_{\alpha} (X, \mathcal{R})$, we have

$$(\mathcal{R} \rtimes_{\alpha} G)' = \{(x, g, x) : x \in X, g \in G, (x, gx) \in \mathcal{R}\}$$

and

$$(\mathcal{R} \times_{\alpha} G)' = \{((x, gx), g) : x \in X, g \in G, (x, gx) \in \mathcal{R}\}.$$

Moreover, we have that $\gamma_0((\mathcal{R} \rtimes_{\alpha} G)') = (\mathcal{R} \times_{\alpha} G)'$. Motivated by [9, 14], we have the following notion.

Definition 4.5. A system $G \curvearrowright (X, \mathcal{R})$ is said to be *essentially free* if for every $e \neq g \in G$, $\{x \in X : (x, gx) \notin \mathcal{R}\}$ is dense in X .

One can easily see that $G \curvearrowright (X, \mathcal{R})$ is essentially free, if and only if the interior of $\{x \in X : g[x]_{\mathcal{R}} = [x]_{\mathcal{R}}\}$ in X is empty for every $g \neq e$.

Lemma 4.6. A system $G \curvearrowright_{\alpha} (X, \mathcal{R})$ is essentially free if and only if $\mathcal{R} \times_{\alpha} G$ (or $\mathcal{R} \rtimes_{\alpha} G$) is topologically principal.

Moreover, one of these two equivalent conditions implies that both of the systems $G \curvearrowright X$ and $G \curvearrowright \mathcal{R}$ are topologically free.

Proof. It follows from the definitions that the topological principality of $\mathcal{R} \times_{\alpha} G$ implies the essential freeness of $G \curvearrowright (X, \mathcal{R})$, thus implies the topological freeness of $G \curvearrowright X$. To see that the essential freeness of $G \curvearrowright_{\alpha} (X, \mathcal{R})$ implies the topological principality of $\mathcal{R} \times_{\alpha} G$, we only need to show that $((x, gx), g)$ is not in the interior of $(\mathcal{R} \times_{\alpha} G)'$ in $(\mathcal{R} \times_{\alpha} G)$ for each $e \neq g \in G$ and $x \in X$ with $(x, gx) \in \mathcal{R}$.

In fact, for otherwise, choose $e \neq g_0 \in G$ and $x_0 \in X$ such that $(x_0, g_0 x_0) \in \mathcal{R}$ and $((x_0, g_0 x_0), g_0)$ is an interior point of $(\mathcal{R} \times_{\alpha} G)'$. Then there exists an open neighbourhood \tilde{U} of $(x_0, g_0 x_0)$ in \mathcal{R} such that

$$((x_0, g_0 x_0), g_0) \in \tilde{U} \times \{g_0\} \subseteq (\mathcal{R} \times_{\alpha} G)'.$$

The last inclusion implies that $y = g_0 x$ for each $(x, y) \in \tilde{U}$. Hence $\{x \in X : (x, g_0 x) \in \mathcal{R}\}$ contains the non-empty open subset $r(\tilde{U})$ of X , which is contrast to the essential freeness of $G \curvearrowright (X, \mathcal{R})$.

Assume $G \curvearrowright (X, \mathcal{R})$ is essentially free. Given $e \neq g \in G$ and a non-empty open subset $U \subseteq \mathcal{R}$, it follows from the openness of $r(U)$ that there exists $x_0 \in r(U)$ with $(x_0, gx_0) \notin \mathcal{R}$, thus $x_0 \neq gx_0$. Choose $(x_0, y_0) \in U$. Then $g(x_0, y_0) \neq (x_0, y_0)$, which implies that $\{(x, y) \in \mathcal{R} : g(x, y) \neq (x, y)\}$ is dense in \mathcal{R} . Hence $G \curvearrowright \mathcal{R}$ is topologically free. \square

Remark 4.7. The topological freeness of neither $G \curvearrowright X$ nor $G \curvearrowright \mathcal{R}$ can imply the essential freeness of $G \curvearrowright (X, \mathcal{R})$. To see this, if $G \curvearrowright X$ is free, then both systems $G \curvearrowright \mathcal{R}_1$ and $G \curvearrowright \mathcal{R}_2$ in Proposition 4.3 are free, and $G \curvearrowright (X, \mathcal{R}_1)$ is essentially free, but $G \curvearrowright (X, \mathcal{R}_2)$ is not.

If $G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ are essentially free, then the mappings a and b in Lemma 4.4 (or in Definition 4.2) are uniquely determined by (4.1) and (4.2). In fact, suppose that $a' : \mathcal{R} \rtimes_\alpha G \rightarrow H$ is another continuous map such that $\Psi' : (x, g, y) \in \mathcal{R} \rtimes_\alpha G \rightarrow (\varphi(x), a'(x, g, y), \varphi(y)) \in \mathcal{S} \rtimes_\beta H$ is continuous. Then

$$(x, g, y) \in \mathcal{R} \rtimes_\alpha G \rightarrow (a(x, g, y)\varphi(y), a'(x, g, y)\varphi(y)) \in \mathcal{S}$$

is continuous. Hence, from the continuity of a , a' and ρ_α , for $(x, g, y) \in \mathcal{R} \rtimes_\alpha G$, there exists an open neighbourhood \tilde{U} of (x, g, y) such that the map $d|_{\tilde{U}} : \tilde{U} \rightarrow d(\tilde{U})$ is a homeomorphism, $\rho_\alpha(\gamma) = g$, $a(\gamma) = a(x, g, y)$, and $a'(\gamma) = a'(x, g, y)$ for each $\gamma \in \tilde{U}$. For each $z \in \varphi(d(\tilde{U}))$, choose $\gamma \in \tilde{U}$ such that $z = \varphi(d(\gamma))$. The choice of \tilde{U} implies that we can assume that $\gamma = (u, g, v)$, thus $z = \varphi(v)$. Note that $(\varphi(u), a(\gamma)z)$ and $(\varphi(u), a'(\gamma)z)$, thus $(a(\gamma)z, a'(\gamma)z)$ are in \mathcal{S} . Hence $(a(x, g, y)z, a'(x, g, y)z) \in \mathcal{S}$ for each $z \in \varphi(d(\tilde{U}))$. The essential freeness of $H \curvearrowright_\beta (Y, \mathcal{S})$ implies that $a(x, g, y) = a'(x, g, y)$. By symmetry, b is uniquely determined by (4.2).

Lemma 4.8. *In Definition 4.2, if $G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ are essentially free, then the mappings a and b are cocycles on $\mathcal{R} \rtimes_\alpha G$ and $\mathcal{S} \rtimes_\beta H$, respectively.*

Proof. We only need to show that the mappings a and b in Lemma 4.4 are cocycles. Let $\gamma_1 = (x, g, y), \gamma_2 = (y, h, z) \in \mathcal{R} \rtimes_\alpha G$ be arbitrary, and write $\gamma' = \gamma_1\gamma_2 = (x, gh, z)$. From the continuity of a and ρ_α , choose open neighbourhoods U, V and W of γ_1, γ_2 and γ' in $\mathcal{R} \rtimes_\alpha G$, respectively, such that $a(\gamma) = a(\gamma_1)$, $\rho_\alpha(\gamma) = g$ for each $\gamma \in U$, $a(\eta) = a(\gamma_2)$, $\rho_\alpha(\eta) = h$ for each $\eta \in V$, and $a(\sigma) = a(\gamma')$, $\rho_\alpha(\sigma) = gh$ for each $\sigma \in W$. Since the multiplication on $(\mathcal{R} \rtimes_\alpha G)^{(2)}$ is continuous at (γ_1, γ_2) , we can assume that $\gamma\eta \in W$ when $\gamma \in U, \eta \in V$ and $(\gamma, \eta) \in (\mathcal{R} \rtimes_\alpha G)^{(2)}$. Also since the range r and domain d are local homeomorphisms and $d(\gamma_1) = r(\gamma_2) = y$, we can assume that the restrictions $d|_U$ and $r|_V$ are homeomorphisms onto their respective ranges and $d(U) = r(V)$.

For each $\tilde{y} \in \varphi(d(V))$, choose $\eta \in V$ such that $\tilde{y} = \varphi(d(\eta))$. The choice of V permits us to assume that $\eta = (v, h, w)$ and $a(\eta) = a(\gamma_2)$. Hence $\tilde{y} = \varphi(w)$. Since $v \in r(V) =$

$d(U)$, it follows from the choice of U that we have $\gamma = (u, g, v) \in U$ and $a(\gamma) = a(\gamma_1)$. Hence $\gamma\eta = (u, gh, w) \in W$ and $a(\gamma\eta) = a(\gamma')$. The hypothesis on Ψ in Lemma 4.4 implies that $(\varphi(u), a(\gamma)\varphi(v))$, $(\varphi(v), a(\eta)\varphi(w))$ and $(\varphi(u), a(\gamma\eta)\varphi(w))$ are all in \mathcal{S} . Thus, $(a(x, g, y)a(y, h, z)\tilde{y}, a(x, gh, z)\tilde{y})$ is in \mathcal{S} for every $\tilde{y} \in \varphi(d(V))$. The essential freeness of $H \curvearrowright_\beta (Y, \mathcal{S})$ implies that $a(x, g, y)a(y, h, z) = a(x, gh, z)$, thus a is a cocycle. By a similar way, we can show that b is a cocycle. \square

Lemma 4.9. *In Definition 4.2, if $G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ are essentially free, then*

$$\begin{aligned} b((\varphi(x), a((x, y), g)\varphi(g^{-1}y)), a((x, y), g)) &= g, \\ a((\varphi^{-1}(u), b((u, v), h)\varphi^{-1}(h^{-1}v)), b((u, v), h)) &= h \end{aligned}$$

for every $((x, y), g) \in \mathcal{R} \times G$ and $((u, v), h) \in \mathcal{S} \times H$.

Proof. We only show that the maps a and b in Lemma 4.4 satisfy that

$$b(\varphi(x), a(x, g, y), \varphi(y)) = g, \quad a(\varphi^{-1}(u), b(u, h, v), \varphi^{-1}(v)) = h$$

for every $(x, g, y) \in \mathcal{R} \rtimes_\alpha G$ and $(u, h, v) \in \mathcal{S} \rtimes_\beta H$.

As before, let ρ_α and ρ_β be the canonical cocycles on $\mathcal{R} \rtimes_\alpha G$ and $\mathcal{S} \rtimes_\beta H$, respectively. For an arbitrary $(x, g, y) \in \mathcal{R} \rtimes_\alpha G$, we have $(\varphi(x), h, \varphi(y)) \in \mathcal{S} \rtimes_\beta H$, where $h = a(x, g, y)$. From the continuity of b and ρ_β , there exists an open neighbourhood U of $(\varphi(x), h, \varphi(y))$ in $\mathcal{S} \rtimes_\beta H$ such that $\rho_\beta(\gamma) = h$, $b(\gamma) = b(\varphi(x), h, \varphi(y))$ for every $\gamma \in U$, and $r|_U, d|_U$ are homeomorphisms from U onto $r(U)$ and $d(U)$, respectively.

By the continuity of ρ_α , Ψ and a at (x, g, y) , as well as that of φ at x and y , there is an open neighbourhood V of (x, g, y) in $\mathcal{R} \rtimes_\alpha G$ such that

- (i) $\rho_\alpha(\gamma) = g$, $a(\gamma) = h$ and $\Psi(\gamma) \in U$ for every $\gamma \in V$;
- (ii) $r|_V$ and $d|_V$ are homeomorphisms from V onto $r(V)$ and $d(V)$, respectively;
- (iii) $\varphi(r(V)) \subseteq r(U)$ and $\varphi(d(V)) \subseteq d(U)$.

For each $v \in d(V)$, let $\gamma \in V$ such that $d(\gamma) = v$. The above condition (i) implies that we can let $\gamma = (u, g, v)$ and have $a(\gamma) = h$, thus $\Psi(\gamma) = (\varphi(u), h, \varphi(v)) \in U$. The map $\tilde{\Psi}$ gives that $(u, b(\varphi(u), h, \varphi(v))v) \in \mathcal{R}$. From the choice of U , $b(\varphi(u), h, \varphi(v)) = b(\varphi(x), h, \varphi(y))$. It follows that $(u, b(\varphi(x), h, \varphi(y))v) \in \mathcal{R}$. Also since $(u, gv) \in \mathcal{R}$, we have $(b(\varphi(x), h, \varphi(y))v, gv) \in \mathcal{R}$. The essential freeness of $G \curvearrowright_\alpha (X, \mathcal{R})$ implies that $b(\varphi(x), a(x, g, y), \varphi(y)) = b(\varphi(x), h, \varphi(y)) = g$.

By a similar way, we can show that $a(\varphi^{-1}(u), b(u, h, v), \varphi^{-1}(v)) = h$ for each $(u, h, v) \in \mathcal{S} \rtimes_\beta H$. \square

The following definition comes from [9, Definition 4.1].

Definition 4.10. For two étale equivalence relations \mathcal{R} and \mathcal{S} on X and Y , let $G \curvearrowright X$ and $H \curvearrowright Y$ be two systems generating two automorphism systems $G \curvearrowright (X, \mathcal{R})$ and $H \curvearrowright (Y, \mathcal{S})$. We say $G \curvearrowright X$ and $H \curvearrowright Y$ are *continuously orbit equivalent up to \mathcal{R} and \mathcal{S}* , if there exist a homeomorphism $\varphi: X \rightarrow Y$, continuous cocycles $a: X \rtimes G \rightarrow H$, $b: Y \rtimes H \rightarrow G$, $\sigma: \mathcal{R} \rightarrow H$, and $\tau: \mathcal{S} \rightarrow G$ satisfying the following conditions:

- (i) $\sigma(x, y)a(y, g) = a(x, g)\sigma(g^{-1}x, g^{-1}y)$ for $(x, y) \in \mathcal{R}$ and $g \in G$;
- (ii) $\tau(x, y)b(y, g) = b(x, g)\tau(g^{-1}x, g^{-1}y)$ for $(x, y) \in \mathcal{S}$ and $g \in H$;
- (iii) The map $\xi_1: (x, g) \in X \times G \rightarrow (a(x, g)^{-1}\varphi(x), \varphi(g^{-1}x)) \in \mathcal{S}$ is well-defined and continuous. Moreover,

$$b(\varphi(x), a(x, g))\tau(\xi_1(x, g)) = g \quad \text{for } x \in X \text{ and } g \in G.$$

- (iv) The map $\xi_2: (x, g) \in Y \times H \rightarrow (b(x, g)^{-1}\varphi^{-1}(x), \varphi^{-1}(g^{-1}x)) \in \mathcal{R}$ is well-defined and continuous. Moreover,

$$a(\varphi^{-1}(x), b(x, g))\sigma(\xi_2(x, g)) = g \quad \text{for } x \in Y \text{ and } g \in H.$$

- (v) The map $\eta_1: (x, y) \in \mathcal{R} \rightarrow (\sigma(x, y)^{-1}\varphi(x), \varphi(y)) \in \mathcal{S}$ is well-defined and continuous. Moreover,

$$b(\varphi(x), \sigma(x, y))\tau(\eta_1(x, y)) = e \quad \text{for } (x, y) \in \mathcal{R}.$$

- (vi) The map $\eta_2: (x, y) \in \mathcal{S} \rightarrow (\tau(x, y)^{-1}\varphi^{-1}(x), \varphi^{-1}(y)) \in \mathcal{R}$ is well-defined and continuous. Moreover,

$$a(\varphi^{-1}(x), \tau(x, y))\sigma(\eta_2(x, y)) = e \quad \text{for } (x, y) \in \mathcal{S}.$$

Proposition 4.11. *Let $G \curvearrowright_{\alpha} (X, \mathcal{R})$ and $H \curvearrowright_{\beta} (Y, \mathcal{S})$ be two automorphism systems. Then $G \curvearrowright X$ and $H \curvearrowright Y$ are continuously orbit equivalent up to \mathcal{R} and \mathcal{S} if and only if $\mathcal{R} \rtimes_{\alpha} G$ and $\mathcal{S} \rtimes_{\beta} H$ are isomorphic as étale groupoids.*

The proof of this proposition is the same as that of [9, Theorem 4.2] in which the local conjugacy is not necessary. We only provide a brief proof. For details, see [9, Theorem 4.2].

Proof of Proposition 4.11. Assume that $\Lambda: \mathcal{R} \rtimes_{\alpha} G \rightarrow \mathcal{S} \rtimes_{\beta} H$ is an isomorphism. Let φ be the restriction of Λ to the unit space $(\mathcal{R} \rtimes G)^{(0)}$ and let $a(x, g) = \rho_{\beta}\Lambda(x, g, g^{-1}x)$, $\sigma(x, y) = \rho_{\beta}\Lambda(x, e, y)$, and $b(u, h) = \rho_{\alpha}\Lambda^{-1}(u, h, h^{-1}u)$, $\tau(u, v) = \rho_{\alpha}\Lambda^{-1}(u, e, v)$. Then φ , a , b , σ and τ satisfy the requirements in Definition 4.10, thus $G \curvearrowright X$ and $H \curvearrowright Y$ are continuously orbit equivalent up to \mathcal{R} and \mathcal{S} .

Conversely, assume that there are maps φ , a , b , σ and τ satisfying the requirements in Definition 4.10. Define

$$\Lambda(x, g, y) = (\varphi(x), a(x, g)\sigma(g^{-1}x, y), \varphi(y)) \quad \text{for } (x, g, y) \in \mathcal{R} \rtimes_{\alpha} G.$$

Then Λ is an isomorphism from $\mathcal{R} \rtimes_{\alpha} G$ onto $\mathcal{S} \rtimes_{\beta} H$, whose inverse Λ^{-1} is defined by $\Lambda^{-1}(u, h, v) = (\varphi^{-1}(u), b(u, h)\tau(h^{-1}u, v), \varphi^{-1}(v))$. \square

Theorem 4.12. *Assume that $G \curvearrowright_{\alpha} (X, \mathcal{R})$ and $H \curvearrowright_{\beta} (Y, \mathcal{S})$ are essentially free. Then the following statements are equivalent.*

- (i) $G \curvearrowright_{\alpha} (X, \mathcal{R}) \sim_{\text{coe}} H \curvearrowright_{\beta} (Y, \mathcal{S})$;
- (ii) $G \curvearrowright_{\alpha} X$ and $H \curvearrowright_{\beta} Y$ are continuously orbit equivalent up to \mathcal{R} and \mathcal{S} ;
- (iii) $\mathcal{R} \rtimes_{\alpha} G$ and $\mathcal{S} \rtimes_{\beta} H$ are isomorphic as étale groupoids;
- (iv) there exists a C^* -isomorphism Φ from $C_r^*(\mathcal{R} \rtimes_{\alpha} G)$ onto $C_r^*(\mathcal{S} \rtimes_{\beta} H)$ such that $\Phi(C(X)) = C(Y)$.

Proof. The equivalence of (ii) and (iii) follows from Proposition 4.11. From Lemma 4.6, $\mathcal{R} \rtimes_{\alpha} G$ and $\mathcal{S} \rtimes_{\beta} H$ are topological principal, thus the equivalence of (iii) and (iv) follows from [7, 24].

Assume (iii) holds, i.e., there is an isomorphism Λ from $\mathcal{R} \rtimes_{\alpha} G$ onto $\mathcal{S} \rtimes_{\beta} H$. Let φ be the restriction of Λ to the unit space X , and let $a(x, g, y) = \rho_{\beta}\Lambda(x, g, y)$ for $(x, g, y) \in \mathcal{R} \rtimes_{\alpha} G$, $b(u, h, v) = \rho_{\alpha}\Lambda^{-1}(u, h, v)$ for $(u, h, v) \in \mathcal{S} \rtimes_{\beta} H$. Then φ is a homeomorphism from X onto Y , and $\Lambda(x, g, y) = (\varphi(x), a(x, g, y), \varphi(y))$ and $\Lambda^{-1}(u, h, v) = (\varphi^{-1}(u), b(u, h, v), \varphi^{-1}(v))$. So φ , a and b satisfy the requirements in Lemma 4.4, thus $G \curvearrowright_{\alpha} (X, \mathcal{R}) \sim_{\text{coe}} H \curvearrowright_{\beta} (Y, \mathcal{S})$, i.e., (i) holds.

Assume (i) holds. From Lemma 4.4, there exist mappings φ , a and b such that the mappings $\Psi: (x, g, y) \in \mathcal{R} \rtimes_{\alpha} G \rightarrow (\varphi(x), a(x, g, y), \varphi(y)) \in \mathcal{S} \rtimes_{\beta} H$ and $\tilde{\Psi}: (u, h, v) \in \mathcal{S} \rtimes_{\beta} H \rightarrow (\varphi^{-1}(u), b(u, h, v), \varphi^{-1}(v)) \in \mathcal{R} \rtimes_{\alpha} G$ are continuous. From Lemmas 4.8 and 4.9, Ψ and $\tilde{\Psi}$ are étale groupoid isomorphisms and inverse to each other, thus (iii) holds. \square

Remark 4.13. For $G \curvearrowright_{\alpha} X$, let $\mathcal{R}_1 = \{(x, x) : x \in X\}$ be as in Proposition 4.3. Then $\mathcal{R}_1 \rtimes_{\alpha} G$ is isomorphic to the transformation groupoid $X \rtimes G$, and the notions of continuous orbit equivalence for $G \curvearrowright (X, \mathcal{R}_1)$ and $G \curvearrowright X$ in Li's sense are equivalent. Hence Theorem 4.12 is a generalization of Theorem 1.2 in [11].

There are two special cases for orbit equivalence of two systems $G \curvearrowright_{\alpha} (X, \mathcal{R})$ and $H \curvearrowright_{\beta} (Y, \mathcal{S})$ via a homeomorphism $\varphi: X \rightarrow Y$. One is, for each $g \in G$, there is $h \in H$ such that $(h\varphi(x), \varphi(y)) \in \mathcal{S}$ for each $x, y \in X$ with $(gx, y) \in \mathcal{R}$, and by symmetry, for each

$h \in H$, there is $g \in G$ such that $(g\varphi^{-1}(x), \varphi^{-1}(y)) \in \mathcal{R}$ for each $x, y \in Y$ with $(hx, y) \in \mathcal{S}$. The other is, for each $g \in G$ and $x \in X$, there is $h \in H$ such that $(h\varphi(x), \varphi(z)) \in \mathcal{S}$ for each $(gx, z) \in \mathcal{R}$, and by symmetry, for each $h \in H$ and $y \in Y$, there is $g \in G$ such that $(g\varphi^{-1}(y), \varphi^{-1}(z)) \in \mathcal{R}$ for each $(hy, z) \in \mathcal{S}$. Inspired by these ideas, we have the following notions, comparing with those of (strong) asymptotic conjugacy in [9, Definition 4.4].

Definition 4.14. We say $G \curvearrowright (X, \mathcal{R})$ and $H \curvearrowright (Y, \mathcal{S})$ are *strongly continuously orbit equivalent*, and write $G \curvearrowright (X, \mathcal{R}) \sim_{scoe} H \curvearrowright (Y, \mathcal{S})$, if they are continuously orbit equivalent and in Definition 4.2 we can take the maps a to have $a(\gamma, g) = a(\gamma', g)$ for all $\gamma, \gamma' \in \mathcal{R}$ and b to have $b(\nu, h) = b(\nu', h)$ for all $\nu, \nu' \in \mathcal{S}$.

We say these two systems are *weakly continuously orbit equivalent*, and write $G \curvearrowright (X, \mathcal{R}) \sim_{wcoe} H \curvearrowright (Y, \mathcal{S})$, if they are continuously orbit equivalent and in Definition 4.2 we can take the maps a to have $a(\gamma, g) = a(\gamma', g)$ for $\gamma, \gamma' \in \mathcal{R}$ with $d(\gamma) = d(\gamma')$, and b to have $b(\nu, h) = b(\nu', h)$ for $\nu, \nu' \in \mathcal{S}$ with $d(\nu) = d(\nu')$.

Remark 4.15. Clearly, the strong continuous orbit equivalence implies the weak one. If $G \curvearrowright_{\alpha} X$ is free, then $G \curvearrowright (X, \mathcal{R}_1)$ and $G \curvearrowright (X, \mathcal{R}_2)$ in Proposition 4.3 are continuously orbit equivalent, but not weakly continuously orbit equivalent, because they do not satisfy the second special case.

The following corollary is an analogy to [9, Proposition 4.5].

Corollary 4.16. *Assume that $G \curvearrowright_{\alpha} (X, \mathcal{R})$ and $H \curvearrowright_{\beta} (Y, \mathcal{S})$ are essentially free. Then*

- (i) $G \curvearrowright_{\alpha} (X, \mathcal{R}) \sim_{wcoe} H \curvearrowright_{\beta} (Y, \mathcal{S})$ if and only if there is an isomorphism $\Lambda: \mathcal{R} \rtimes_{\alpha} G \rightarrow \mathcal{S} \rtimes_{\beta} H$ such that $\Lambda(\mathcal{R}) = \mathcal{S}$.

Moreover, if these equivalent conditions hold, then there is a C^* -isomorphism $\Phi: C_r^*(\mathcal{R} \rtimes_{\alpha} G) \rightarrow C_r^*(\mathcal{S} \rtimes_{\beta} H)$ such that $\Phi(C(X)) = C(Y)$ and $\Phi(C_r^*(\mathcal{R})) = C_r^*(\mathcal{S})$.

- (ii) $G \curvearrowright_{\alpha} (X, \mathcal{R}) \sim_{scoe} H \curvearrowright_{\beta} (Y, \mathcal{S})$ if and only if there exist a homeomorphism $\varphi: X \rightarrow Y$ and a group isomorphism $\theta: G \rightarrow H$ such that $\Lambda: (x, g, y) \in \mathcal{R} \rtimes_{\alpha} G \rightarrow (\varphi(x), \theta(g), \varphi(y)) \in \mathcal{S} \rtimes_{\beta} H$ is an isomorphism if and only if there exist an étale groupoid isomorphism $\Lambda: \mathcal{R} \rtimes_{\alpha} G \rightarrow \mathcal{S} \rtimes_{\beta} H$ and a group isomorphism $\theta: G \rightarrow H$ such that $\theta\rho_{\alpha} = \rho_{\beta}\Lambda$.

The above equivalent statements are also equivalent to the two coaction systems $(C_r^*(\mathcal{R} \rtimes_{\alpha} G), G; \delta_{\alpha})$ and $(C_r^*(\mathcal{S} \rtimes_{\beta} H), H; \delta_{\beta})$ being conjugate by a conjugacy ϕ with $\phi(C(X)) = C(Y)$.

Furthermore, when \mathcal{R} and \mathcal{S} are minimal or X and Y are connected, these two notions of strong continuous orbit equivalence and weak continuous orbit equivalence are equivalent.

Proof. One can check that if the map a in Definition 4.2 is a cocycle on $\mathcal{R} \times_\alpha G$, then $a(\gamma, g) = a(\gamma', g)$ for $\gamma, \gamma' \in \mathcal{R}$ with $d(\gamma) = d(\gamma')$ if and only if $a(\gamma, e) = e$ for all $\gamma \in \mathcal{R}$. By symmetry, b has a similar characterization when it is a cocycle. From Lemma 4.8, Theorem 4.12 and its proof, we can obtain (i) and (ii), where the equivalence of the last two statements in (ii) comes from [7, Theorem 6.2].

We now show that the weak continuous orbit equivalence of $G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ implies the strong one when \mathcal{R} and \mathcal{S} are minimal or X and Y are connected. To see this, by assumption and the first paragraph of this proof, we have a homeomorphism φ and two continuous cocycles a, b with $a(x, e, y) = e$ for all $(x, y) \in \mathcal{R}$ and $b(u, e, v) = e$ for $(u, v) \in \mathcal{S}$, satisfying Lemma 4.4.

Assume that X and Y are connected. For each $g \in G$, the map $x \in X \rightarrow a(x, g, g^{-1}x) \in H$ is continuous, thus it is a constant. Hence $a(x, g, g^{-1}x) = a(y, g, g^{-1}y)$ for all $x, y \in X$. By symmetry, b has a similar property.

Assume that \mathcal{R} and \mathcal{S} are minimal. For $(x, y) \in \mathcal{R}$ and $g \in G$, since

$$(x, g, g^{-1}x)(g^{-1}x, e, g^{-1}y)(g^{-1}y, g^{-1}, y) = (x, e, y),$$

we have $a(x, g, g^{-1}x) = a(g^{-1}y, g^{-1}, y)^{-1} = a(y, g, g^{-1}y)$. Given arbitrary $x, y \in X$ and $g \in G$, we choose a sequence $\{x_n\}$ in $[x]\mathcal{R}$ converging to y in X . Then $\{(x_n, g, g^{-1}x_n)\}$ converges to $(y, g, g^{-1}y)$ in $\mathcal{R} \times_\alpha G$, thus the continuity of a implies that $a(x, g, g^{-1}x) = a(y, g, g^{-1}y)$.

We remark that $a(x, g, y) = a(x, g, g^{-1}x)a(g^{-1}x, e, y) = a(x, g, g^{-1}x)$ for $(x, g, y) \in \mathcal{R} \times_\alpha G$. Consequently, if one of the above two assumptions holds, then $a(x, g, y) = a(u, g, v)$ for $(x, g, y), (u, g, v) \in \mathcal{R} \times_\alpha G$. By a similar way, we can show that b satisfies a similar requirement. Hence $G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ are strongly continuously orbit equivalent. \square

5. Local conjugacy relations from expansive systems

The condition of essential freeness of automorphism systems in Theorem 4.12 and Corollary 4.16 is necessary. In this section, we give some examples satisfying the requirement. Recall that a system $G \curvearrowright_\alpha X$ is called *expansive* if the action α is expansive, which means for a metric d on X compatible with the topology, there exists a constant $\delta > 0$ such that, for $x, y \in X$, if $d(gx, gy) < \delta$ for all $g \in G$ then $x = y$. For convenience, given a real-valued function ψ on G , the notation $\lim_{g \rightarrow \infty} \psi(g) = 0$ means that, for any $\epsilon > 0$, there exists a finite subset F of G such that $|\psi(g)| < \epsilon$ for all $g \notin F$.

A triple (U, V, γ) , consisting of open subsets U, V of X and a homeomorphism $\gamma: U \rightarrow V$, is called a local conjugacy, if $\lim_{g \rightarrow \infty} \sup_{z \in U} d(gz, g\gamma(z)) = 0$. Two points x and y in X are said to be *locally conjugate*, if there exists a local conjugacy (U, V, γ) such that

$x \in U, y \in V$ and $\gamma(x) = y$. Let

$$\mathcal{R}_\alpha = \{(x, y) \in X \times X : x \text{ and } y \text{ are locally conjugate}\}$$

be the local conjugacy relation on X . From [27] (also see [29]), \mathcal{R}_α is an étale equivalence relation on X under the topology whose base consists of the sets of the form

$$\{(x, \gamma(x)) : x \in U\},$$

where (U, V, γ) is a local conjugacy. Moreover, $G \curvearrowright_\alpha X$ induces an automorphism system $G \curvearrowright_\alpha \mathcal{R}_\alpha$: $g(x, y) = (gx, gy)$ for $g \in G$ and $(x, y) \in \mathcal{R}_\alpha$. Thus we have an automorphism system $G \curvearrowright_\alpha (X, \mathcal{R}_\alpha)$.

Remark 5.1. If two expansive systems $G \curvearrowright_\alpha X$ and $H \curvearrowright_\beta Y$ are conjugate by a homeomorphism φ from X onto Y and a group isomorphism ρ from G onto H , then $(\varphi(U), \varphi(V), \varphi\gamma\varphi^{-1}|_{\varphi(U)})$ is a local conjugacy for each local conjugacy (U, V, γ) , thus $\varphi \times \varphi: (x, y) \in \mathcal{R}_\alpha \rightarrow (\varphi(x), \varphi(y)) \in \mathcal{R}_\beta$ is an isomorphism. Hence $G \curvearrowright_\alpha (X, \mathcal{R}_\alpha)$ and $H \curvearrowright_\beta (Y, \mathcal{R}_\beta)$ are conjugate, thus two notions of conjugacy for $G \curvearrowright_\alpha (X, \mathcal{R}_\alpha)$ and $G \curvearrowright_\alpha X$ are equivalent.

From [9, 14], the automorphism systems of local conjugacy relations associated to a full shift $G \curvearrowright A^G$ over a finite set A and an irreducible Smale space (X, ψ) are essentially free. The following result generalizes Matsumoto's result in the Smale space case to the \mathbb{Z} -expansive system case.

Theorem 5.2. *Let $\mathbb{Z} \curvearrowright_\alpha X$ be an expansive system generated by a homeomorphism φ on X , and let \mathcal{R}_α be the local conjugacy relation associated to $\mathbb{Z} \curvearrowright_\alpha X$. Assume that X is infinite and has no isolated points. Then $\mathbb{Z} \curvearrowright_\alpha (X, \mathcal{R}_\alpha)$ is essentially free.*

Proof. For an arbitrary integer $p \geq 1$, we first claim that the set

$$X_p = \left\{ x \in X : \lim_{n \rightarrow \infty} \varphi^{pn}(x) \text{ and } \lim_{n \rightarrow \infty} \varphi^{-pn}(x) \text{ exist} \right\}$$

is countable.

In fact, when $p = 1$, it follows from [1, Theorem 2.2.22] that X_1 is countable. Moreover, the expansiveness of φ^p implies that X_p is also countable for every p . For completeness, we provide a proof for the claim. Since φ is expansive, it follows that the p -periodic point set $F_p(\varphi) = \{x \in X : \varphi^p(x) = x\}$ is finite, say $F_p(\varphi) = \{y_1, y_2, \dots, y_k\}$. For each $x \in X_p$, let $\lim_{n \rightarrow \infty} \varphi^{pn}(x) = y$ and $\lim_{n \rightarrow \infty} \varphi^{-pn}(x) = z$. One can see that $y, z \in F_p(\varphi)$. For $1 \leq i, j \leq k$, set $X_p(i, j) = \{x \in X : \lim_{n \rightarrow \infty} \varphi^{pn}(x) = y_i, \lim_{n \rightarrow \infty} \varphi^{-pn}(x) = y_j\}$. Given $x \in X_p(i, j)$, we have $\lim_{n \rightarrow \infty} \varphi^{pn+r}(x) = \varphi^r(y_i)$ and $\lim_{n \rightarrow \infty} \varphi^{-pn-r}(x) = \varphi^{-r}(y_j)$ for each $0 \leq r \leq p-1$. Hence there exists an integer $N \geq 2$ such that $d(\varphi^n(x), \varphi^n(y_i)) < c/2$

for all $n \geq N$ and $d(\varphi^{-n}(x), \varphi^{-n}(y_j)) < c/2$ for all $n \leq -N$, where c is an expansive constant for φ . Set

$$X_{p,N}(i, j) = \{x \in X : d(\varphi^n(x), \varphi^n(y_i)) < c/2 \text{ for all } n \geq N \\ d(\varphi^{-n}(x), \varphi^{-n}(y_j)) < c/2 \text{ for all } n \leq -N\}.$$

Thus $X_p(i, j) = \bigcup_{N \geq 2} X_{p,N}(i, j)$ and $X = \bigcup_{1 \leq i, j \leq k} X_p(i, j)$. To finish the claim, we show, for each $N \geq 2$ and $1 \leq i, j \leq k$, the set $X_{p,N}(i, j)$ is finite.

For otherwise, $X_{p,N}(i, j)$ is infinite for some i, j, N . Choose $\delta < c/2$ such that if $d(y, z) \leq \delta$ for $y, z \in X$ then $d(\varphi^l(y), \varphi^l(z)) < c/2$ for each integer l with $|l| \leq N - 1$. Since $X_{p,N}(i, j)$ is infinite, there are two different y, z in $X_{p,N}(i, j)$ such that $d(y, z) < \delta$. Thus $d(\varphi^l(y), \varphi^l(z)) < c$ for every integer l , which implies that $y = z$ by expansiveness of φ and is a contradiction. We have established the claim.

For each $p \geq 1$, we next claim that if $x \in X$ with $(x, \varphi^p(x)) \in \mathcal{R}_\alpha$, then $x \in X_p$.

We use the method in [22, Lemma 5.3] to prove the claim. Assume that z is a limit point of $\{\varphi^{pn}(x) \mid n \geq 1\}$. Choose a subsequence $\{m_n\}$ of positive integers such that $\lim_{n \rightarrow \infty} d(z, \varphi^{pm_n}(x)) = 0$. Thus $\lim_{n \rightarrow \infty} d(\varphi^p(z), \varphi^{pm_n}(\varphi^p(x))) = 0$. Since $(x, \varphi^p(x)) \in \mathcal{R}_\alpha$, we have $\lim_{|n| \rightarrow \infty} d(\varphi^n(x), \varphi^n(\varphi^p(x))) = 0$. Consequently, $\varphi^p(z) = z$, which implies that each limit point of $\{\varphi^{pn}(x) \mid n \geq 1\}$ is in $F_p(\varphi) = \{y_1, y_2, \dots, y_k\}$. Choose an open neighbourhood U_i of y_i such that $\overline{U_i} \cap \overline{U_j} = \emptyset$ and $\varphi^p(U_i) \cap U_j = \emptyset$ for $i \neq j$, where $\overline{U_i}$ is the closure of U_i . The limit point property of $\{\varphi^{pn}(x) \mid n \geq 1\}$ shows that there exists $N \geq 1$ such that $\varphi^{pn}(x) \in \bigcup_{i=1}^k U_i$ for $n \geq N$. If $\varphi^{pN}(x) \in U_{i_0}$ for some i_0 , then, by the choice of U_i 's, $\varphi^{pn}(x) \in U_{i_0}$ for all $n \geq N$. Hence the sequence $\{\varphi^{pn}(x) \mid n \geq 1\}$ has a unique limit point z_{i_0} , thus it converges.

By a similar argument, one can obtain that $\{\varphi^{-pn}(x) \mid n \geq 1\}$ converges. Thus $x \in X_p$ and the claim is established. So for each nonzero integer p , we have $\{x \in X : (x, \varphi^p(x)) \in \mathcal{R}_\alpha\} = \{x \in X : (x, \varphi^{|p|}(x)) \in \mathcal{R}_\alpha\} \subseteq X_{|p|}$, thus $\{x \in X : (x, \varphi^p(x)) \in \mathcal{R}_\alpha\}$ is countable. Since X is infinite and has no isolated points, it follows that $\{x \in X : (x, \varphi^p(x)) \notin \mathcal{R}_\alpha\}$ is dense in X for each nonzero integer p . Consequently, $\mathbb{Z} \curvearrow_\alpha (X, \mathcal{R}_\alpha)$ is essentially free. \square

6. Expansive automorphism actions on compact groups

Let X be a compact metrizable group with an invariant compatible metric d , i.e., $d(xy, xz) = d(yx, zx) = d(y, z)$ for $x, y, z \in X$. Assume that $G \curvearrow_\alpha X$ is an expansive automorphism system in the sense that it is expansive and each α_g is a continuous automorphism on X . Let

$$\Delta_\alpha = \left\{ x \in X : \lim_{g \rightarrow \infty} d(\alpha_g(x), \alpha_g(e)) = 0 \right\}$$

be the associated *homoclinic group*, which is an α -invariant countable subgroup of X in the sense that $\alpha_g(a) \in \Delta_\alpha$ for every $a \in \Delta_\alpha$ and $g \in G$ (see [29]). Denote by σ the

left-multiplication action of Δ_α on X :

$$\sigma_u(x) = ux \quad \text{for } u \in \Delta_\alpha \text{ and } x \in X,$$

and by $X \rtimes_\sigma \Delta_\alpha$ the associated transformation groupoid. Let $G \curvearrowright_\alpha (X, \mathcal{R})$ be the automorphism system associated to the local conjugacy equivalence relation as in Section 5. Refer to [29, Lemma 3.7] for the following facts.

Lemma 6.1. *Let $G \curvearrowright_\alpha X$ be an expansive automorphism system. Then*

- (i) *two elements x and y in X are locally conjugate, if and only if they are homoclinic, i.e., $\lim_{g \rightarrow \infty} d(gx, gy) = 0$, if and only if $xy^{-1} \in \Delta_\alpha$, if and only if $x^{-1}y \in \Delta_\alpha$, if and only if x^{-1} and y^{-1} are locally conjugate.*
- (ii) *The map $\Lambda: (x, y) \in \mathcal{R} \rightarrow (x, xy^{-1}) \in X \rtimes_\sigma \Delta_\alpha$ is an étale groupoid isomorphism.*

Proof. We only give a proof for (ii). One can see that Λ is an algebraic isomorphism from \mathcal{R} onto $X \rtimes_\sigma \Delta_\alpha$ with inverse map Λ^{-1} , defined by $\Lambda^{-1}(x, u) = (x, u^{-1}x)$ for $(x, u) \in X \rtimes_\sigma \Delta_\alpha$. Given $(x, y) \in \mathcal{R}$, for $S \subseteq \Delta_\alpha$ and an open subset $U \subseteq X$ with $x \in U$ and $xy^{-1} \in S$, we define $\gamma(z) = yx^{-1}z$ for $z \in U$. Then $(U, \gamma(U), \gamma)$ is a local conjugacy from x to y , and $\Lambda(\{(z, \gamma(z)) : z \in U\}) \subseteq U \times S$, thus Λ is continuous at (x, y) . By a similar way, we show that Λ^{-1} is continuous, thus Λ is a homeomorphism. \square

Definition 6.2. Let $\Gamma = \Delta_\alpha \rtimes G$ be the semi-direct product of Δ_α by G . Define the action $\tilde{\alpha}$ of Γ on X as follows. For $(a, g) \in \Gamma$ and $x \in X$,

$$\tilde{\alpha}_{(a,g)}(x) = a\alpha_g(x).$$

One can check that $\Gamma \curvearrowright_{\tilde{\alpha}} X$ is an expansive affine system. We remark that Δ_α and G can be thought of as subgroups of Γ by identifying $a \in \Delta_\alpha$ with $(a, e) \in \Gamma$, and $g \in G$ with $(e, g) \in \Gamma$, thus the restrictions of $\tilde{\alpha}$ to Δ_α and G are the same as σ and α , respectively. Hence the transformation groupoid $X \rtimes_{\tilde{\alpha}} \Gamma$ contains $X \rtimes_\sigma \Delta_\alpha$ and $X \rtimes_\alpha G$ as open subgroupoids.

Proposition 6.3. *The map $\Lambda: (x, g, y) \mapsto (x, (x\alpha_g(y^{-1}), g))$ is an isomorphism of $\mathcal{R} \rtimes_\alpha G$ onto $X \rtimes_{\tilde{\alpha}} \Gamma$ as étale groupoids. Moreover, $\Lambda(\mathcal{R}) = X \rtimes_\sigma \Delta_\alpha$, and $\Lambda(X \rtimes_\alpha G) = X \rtimes_{\tilde{\alpha}} G$.*

Proof. From Lemma 6.1, Λ is well-defined and injective. For each $(x, (a, g))$ in $X \rtimes_{\tilde{\alpha}} \Gamma$, we have $(x, g, \alpha_{g^{-1}}(a^{-1}x)) \in \mathcal{R} \rtimes_\alpha G$ and $\Lambda(x, g, \alpha_{g^{-1}}(a^{-1}x)) = (x, (a, g))$, thus Λ is bijective. For $(x, g, y), (u, h, z) \in \mathcal{R} \rtimes_\alpha G$, we have (x, g, y) and (u, h, z) are composable in $\mathcal{R} \rtimes_\alpha G$, if and only if $u = y$, if and only if $\Lambda(x, g, y)$ and $\Lambda(y, h, z)$ are composable in $X \rtimes_{\tilde{\alpha}} \Gamma$. Moreover,

$$\Lambda(x, g, y)\Lambda(y, h, z) = (x, (x\alpha_g(y)^{-1}, g))(y, (y\alpha_h(z)^{-1}, h))$$

$$\begin{aligned}
&= (x, (x\alpha_{gh}(z))^{-1}, gh)) \\
&= \Lambda((x, g, y)(y, h, z)).
\end{aligned}$$

The continuity of Λ is implied by Lemma 6.1 and the canonical homeomorphism γ_0 from $\mathcal{R} \rtimes_\alpha G$ onto $\mathcal{R} \times G$. Hence Λ is an étale groupoid isomorphism. \square

Proposition 6.4. (i) *The system $G \curvearrowright_\alpha X$ is topologically free, if and only if $\Gamma \curvearrowright_{\tilde{\alpha}} X$ is topologically free, if and only if $\mathcal{R} \rtimes_\alpha G$ is topologically principal, if and only if $G \curvearrowright_\alpha (X, \mathcal{R})$ is essentially free.*

(ii) *If G is torsion-free and Δ_α is dense in X , then $G \curvearrowright_\alpha X$ is topologically free.*

Proof. (i) It follows from [11, Corollary 2.3], Lemma 4.6 and Proposition 6.3 that we only need to show that the topological freeness for α and $\tilde{\alpha}$ is equivalent. Since G can be embedded into Γ as a subgroup and the restriction of $\tilde{\alpha}$ to G is the same as the action α , the topological freeness of $\tilde{\alpha}$ implies that of α .

To see the contrary, it is sufficient to show that, for arbitrary $(e, e) \neq (a, g) \in \Gamma$ and non-empty open subset U of X , there exists $x \in U$ such that $a\alpha_g(x) \neq x$.

In fact, since the restriction of $\tilde{\alpha}$ to Δ_α is free, we can assume that $g \neq e$ and $a \neq e$. Clearly, we can also assume that there exists $y \in U$ such that $a\alpha_g(y) = y$. The topological freeness of α implies there is $z \in y^{-1}U$ such that $\alpha_g(z) \neq z$. Let $z = y^{-1}x$ for $x \in U$. Then $a\alpha_g(x) \neq x$.

(ii) Given $g \in G$, assume that there exists an open subset U of X such that $\alpha_g(z) = z$ for every $z \in U$. We can let $e \notin U$. Since Δ_α is dense in X , there is $x_0 \in U \cap \Delta_\alpha$, thus $\lim_{h \rightarrow \infty} d(\alpha_h(x_0), e) = 0$. If $g \neq e$, then, from the torsion-freeness of G , the set $\{g^n : n \in \mathbb{Z}\}$ is infinite, we have $\lim_{n \rightarrow \infty} d(\alpha_{g^n}(x_0), e) = 0$, which contradicts the fact $x_0 \neq e$ and $\alpha_{g^n}(x_0) = x_0$ for all $n \in \mathbb{Z}$. Consequently, $g = e$, thus α is topologically free. \square

Recall that two automorphism systems $G \curvearrowright_\alpha X$ and $H \curvearrowright_\beta Y$ on compact metrizable groups are said to be *algebraically conjugate* if there exist a continuous isomorphism $\varphi: X \rightarrow Y$ and an isomorphism $\rho: G \rightarrow H$ such that $\varphi(\alpha_g(x)) = \beta_{\rho(g)}(\varphi(x))$ for $g \in G$ and $x \in X$. From [2], when X and Y are abelian, the notions of algebraic conjugacy and conjugacy for automorphism systems are equivalent. In the following we have a similar result for automorphism actions on nonabelian groups.

Proposition 6.5. *Let $G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ be two automorphism systems on local conjugacy relations from topologically free, expansive automorphism actions on compact and connected metrizable groups X and Y , respectively. Then the following statements are equivalent:*

- (i) $G \curvearrowright_\alpha (X, \mathcal{R})$ and $H \curvearrowright_\beta (Y, \mathcal{S})$ are conjugate;
- (ii) $G \curvearrowright_\alpha (X, \mathcal{R}) \sim_{\text{wcoe}} H \curvearrowright_\beta (Y, \mathcal{S})$;
- (iii) $G \curvearrowright_\alpha X$ and $H \curvearrowright_\beta Y$ are continuously orbit equivalent;
- (iv) $G \curvearrowright_\alpha X$ and $H \curvearrowright_\beta Y$ are conjugate.

Moreover, if Δ_α is dense in X , then the above conditions are equivalent to the following statement.

- (v) $G \curvearrowright_\alpha X$ and $H \curvearrowright_\beta Y$ are algebraically conjugate.

Proof. Since X and Y are connected, the continuous orbit equivalence and conjugacy of $G \curvearrowright_\alpha X$ and $H \curvearrowright_\beta Y$ are equivalent. To complete the proof, we only need to prove that (ii) \Rightarrow (iv) and (ii) \Rightarrow (v) when Δ_α is dense in X . From Corollary 4.16 and Proposition 6.3, there is an étale groupoid isomorphism $\Lambda: X \rtimes_{\tilde{\alpha}} (\Delta_\alpha \rtimes G) \rightarrow Y \rtimes_{\tilde{\beta}} (\Delta_\beta \rtimes H)$ such that $\Lambda(X \rtimes_\sigma \Delta_\alpha) = Y \rtimes_{\sigma'} \Delta_\beta$, where σ and σ' are the left-multiplication actions, and $\tilde{\alpha}$ and $\tilde{\beta}$ are as in Definition 6.2. Since X and Y are connected, there exist a homeomorphism $\varphi: X \rightarrow Y$ and a group isomorphism $\theta: \Delta_\alpha \rtimes G \rightarrow \Delta_\beta \rtimes H$ such that

$$(6.1) \quad \varphi(a\alpha_g(x)) = \tilde{\beta}_{\theta(a,g)}(\varphi(x)) \quad \text{for every } (a, g) \in \Delta_\alpha \rtimes G \text{ and } x \in X,$$

and $\theta(\Delta_\alpha) = \Delta_\beta$, where Δ_α and Δ_β are subgroups of the semi-direct product groups as before. Define two maps $\xi: G \rightarrow \Delta_\beta$ and $\rho: G \rightarrow H$ by $\theta(e, g) = (\xi(g), \rho(g))$ for $g \in G$. One can check that ρ is a group isomorphism by considering the inverse isomorphism θ^{-1} .

Letting $a = e$, the identity of X , in (6.1), we have $\varphi(\alpha_g(x)) = \xi(g)\beta_{\rho(g)}(\varphi(x))$ for every $g \in G$ and $x \in X$. In particular, $\varphi(e) = \xi(g)\beta_{\rho(g)}(\varphi(e))$. Thus $\varphi(\alpha_g(x)) = \varphi(e)\beta_{\rho(g)}(\varphi(e)^{-1}\varphi(x))$ for $x \in X$ and $g \in G$. Define $\tilde{\varphi}(x) = \varphi(e)^{-1}\varphi(x)$ for $x \in X$. Then $\tilde{\varphi}: X \rightarrow Y$ is a homeomorphism and

$$\tilde{\varphi}(\alpha_g(x)) = \beta_{\rho(g)}(\tilde{\varphi}(x)) \quad \text{for } x \in X \text{ and } g \in G.$$

Consequently, $G \curvearrowright_\alpha X$ and $H \curvearrowright_\beta Y$ are conjugate.

Assume that Δ_α is dense in X . We remark that $\theta(a, e) \in \Delta_\beta$, thus $\tilde{\beta}_{\theta(a,e)}(y) = \theta(a, e)y$ for $a \in \Delta_\alpha$ and $y \in Y$. Letting $g = e$, the identity of G , and letting $x = e$, the identity of X , in (6.1), one can see that $\varphi(a) = \theta(a, e)\varphi(e)$ for $a \in \Delta_\alpha$. Thus, by putting $g = e$ in (6.1), we have $\varphi(ax) = \theta(a, e)\varphi(x) = (\varphi(a)\varphi(e)^{-1})\varphi(x)$, which implies that $\tilde{\varphi}(ax) = \tilde{\varphi}(a)\tilde{\varphi}(x)$ for every $a \in \Delta_\alpha$ and $x \in X$. From the density of Δ_α in X , the map $\tilde{\varphi}: X \rightarrow Y$ is a continuous isomorphism. So $G \curvearrowright_\alpha X$ and $H \curvearrowright_\beta Y$ are algebraically conjugate. \square

Proposition 6.6. *Let $G \curvearrowright_\alpha (X, \mathcal{R})$ be an automorphism system on the local conjugacy relation from a topologically free, expansive automorphism action. Then the following statements are equivalent.*

- (i) $C_r^*(\mathcal{R})$ is simple;
- (ii) $C_r^*(\mathcal{R})$ has a unique tracial state;
- (iii) Δ_α is dense;
- (iv) $C_r^*(\mathcal{R} \rtimes_\alpha G)$ is simple;
- (v) $C_r^*(\mathcal{R} \rtimes_\alpha G)$ has a unique tracial state.

Proof. For the equivalence of (i), (ii) and (iii), we refer to [29, Corollary 3.9]. From Proposition 6.3, $C_r^*(\mathcal{R} \rtimes_\alpha G)$ is isomorphic to $C(X) \rtimes_{r, \tilde{\alpha}} \Gamma$, thus they have the same simplicity and the uniqueness of tracial states. From Proposition 6.4 and [11], $X \rtimes_{\tilde{\alpha}} \Gamma$ is topologically principal, thus there is a one-to-one correspondence between the family of ideals of $C(X) \rtimes_{r, \tilde{\alpha}} \Gamma$ and that of $\tilde{\alpha}$ -invariant open subsets of X (see [23]).

Assume (iii) holds. Since each non-empty $\tilde{\alpha}$ -invariant open subset U in X is invariant under the left-multiplication by elements in Δ_α , we have $U = X$. Hence $C(X) \rtimes_{r, \tilde{\alpha}} \Gamma$ is simple, thus (iv) holds. On the contrary, if (iv) holds, then $C(X) \rtimes_{r, \tilde{\alpha}} \Gamma$ is simple, which implies that the $\tilde{\alpha}$ -invariant open subset $X \setminus \overline{\Delta_\alpha}$ of X is empty, where $\overline{\Delta_\alpha}$ is the closure of Δ_α in X . Thus $\overline{\Delta_\alpha} = X$, i.e., (iii) holds.

For the implication (v) \Rightarrow (iii), assume that $C_r^*(\mathcal{R} \rtimes_\alpha G)$, thus $C(X) \rtimes_{r, \tilde{\alpha}} \Gamma$, has a unique tracial state. If Δ_α is not dense in X , then the Haar measure ν on $\overline{\Delta_\alpha}$ extends a Borel probability measure $\hat{\nu}$ on X different from the Haar measure μ_0 on X . Since μ_0 is invariant under the actions σ , α and $\tilde{\alpha}$, for a Borel subset E of X and $(a, g) \in \Gamma$, we have $\hat{\nu}(\tilde{\alpha}_{(a, g)}(E)) = \nu(\tilde{\alpha}_{(a, g)}(E \cap \overline{\Delta_\alpha})) = \hat{\nu}(E)$, thus $\hat{\nu}$ is $\tilde{\alpha}$ -invariant. The probability measures $\hat{\nu}$ and μ_0 produce two different tracial states on $C(X) \rtimes_{r, \tilde{\alpha}} \Gamma$, which is a contradiction.

For the implication (iii) \Rightarrow (v), assume that Δ_α is dense in X . Then the Haar measure μ_0 on X is the unique $\tilde{\alpha}$ -invariant Borel probability measure on X . From [30, Proposition 3.2.4], $C(X) \rtimes_{r, \tilde{\alpha}} \Gamma$, and thus $C_r^*(\mathcal{R} \rtimes_\alpha G)$ has a unique tracial state. \square

Example 6.7 (Hyperbolic toral automorphisms). For $n \geq 2$, we consider an expansive \mathbb{Z} -action on the n -dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$ generated by a single hyperbolic toral automorphism α . Let $\pi: \mathbb{R}^n \rightarrow \mathbb{R}^n/\mathbb{Z}^n$ be the usual quotient map. Recall that $\mathbb{R}^n/\mathbb{Z}^n$ is a compact and connected additive group under the following metric compatible with the quotient topology:

$$d(\pi(x), \pi(y)) = \inf_{z \in \mathbb{Z}^n} \|x - y - z\| \quad \text{for } x, y \in \mathbb{R}^n,$$

where $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^n . The elements in \mathbb{R}^n are denoted by column vectors or row vectors.

It is well-known that, for such a hyperbolic toral automorphism α on $\mathbb{R}^n/\mathbb{Z}^n$, there is a matrix $A \in \text{GL}(n, \mathbb{Z})$ with $\det(A) = \pm 1$ and having no eigenvalues of modules 1 such that

$$\alpha(\pi(x)) = \pi(Ax) \quad \text{for } x \in \mathbb{R}^n.$$

Then $\mathbb{R}^n = E^s \oplus E^u$, where $E^s = \{x \in \mathbb{R}^n : \lim_{k \rightarrow +\infty} A^k x = 0\}$ and $E^u = \{w \in \mathbb{R}^n : \lim_{k \rightarrow +\infty} A^{-k} w = 0\}$ are two invariant subspaces of the linear map on \mathbb{R}^n determined by A . We remark that $E^s \cap \mathbb{Z}^n = \{0\}$, $E^u \cap \mathbb{Z}^n = \{0\}$, and both subgroups $\pi(E^s)$ and $\pi(E^u)$, as well as the homoclinic group $\Delta_\alpha = \pi(E^s) \cap \pi(E^u)$ induced by α , are dense in $\mathbb{R}^n/\mathbb{Z}^n$. Moreover, the system $\mathbb{Z} \curvearrowright_\alpha \mathbb{R}^n/\mathbb{Z}^n$ generated by α is topologically free (see [12]).

Each $\mathbf{m} \in \mathbb{Z}^n$ has the unique decomposition $\mathbf{m} = \mathbf{m}_s - \mathbf{m}_u \in E^s \oplus E^u$. Then the map $\theta: \mathbb{Z}^n \rightarrow \Delta_\alpha$ by $\theta(\mathbf{m}) = \pi(\mathbf{m}_s)$ ($= \pi(\mathbf{m}_u)$) is a group isomorphism. As before, we let σ be the translation action of Δ_α on $\mathbb{R}^n/\mathbb{Z}^n$:

$$\sigma_u(x) = u + x \quad \text{for } u \in \Delta_\alpha, x \in \mathbb{R}^n/\mathbb{Z}^n.$$

Let τ be the action of \mathbb{Z}^n on $\mathbb{R}^n/\mathbb{Z}^n$ by homeomorphisms:

$$\tau_{\mathbf{n}}(x) = \theta(\mathbf{n}) + x \quad \text{for } \mathbf{n} \in \mathbb{Z}^n, x \in \mathbb{R}^n/\mathbb{Z}^n.$$

Then $\mathbb{Z}^n \curvearrowright_\tau \mathbb{R}^n/\mathbb{Z}^n$ and $\Delta_\alpha \curvearrowright_\sigma \mathbb{R}^n/\mathbb{Z}^n$ are conjugate.

Denote by $\mathbb{Z}^n \rtimes \mathbb{Z}$ the semi-direct product of \mathbb{Z}^n by the automorphism given by A : $\mathbf{m} \in \mathbb{Z}^n \rightarrow A\mathbf{m} \in \mathbb{Z}^n$. Let γ be the action of $\mathbb{Z}^n \rtimes \mathbb{Z}$ on $\mathbb{R}^n/\mathbb{Z}^n$:

$$\gamma_{(\mathbf{m}, k)}(x) = \theta(\mathbf{m}) + \alpha^k(x) \quad \text{for } (\mathbf{m}, k) \in \mathbb{Z}^n \rtimes \mathbb{Z} \text{ and } x \in \mathbb{R}^n/\mathbb{Z}^n.$$

So $\mathbb{Z}^n \rtimes \mathbb{Z} \curvearrowright_\gamma \mathbb{R}^n/\mathbb{Z}^n$ and $\Delta_\alpha \rtimes \mathbb{Z} \curvearrowright_{\tilde{\alpha}} \mathbb{R}^n/\mathbb{Z}^n$ are conjugate, where $\tilde{\alpha}$ is given by Definition 6.2.

We consider the multiplicative coordinate system on the n -dimensional torus by $\mathbb{T}^n := \{(z_1, z_2, \dots, z_n) : z_i \in \mathbb{C}, |z_i| = 1 \text{ for } 1 \leq i \leq n\}$. The correspondence

$$\varphi: [(x_1, x_2, \dots, x_n)] \in \mathbb{R}^n/\mathbb{Z}^n \rightarrow (e^{2\pi i x_1}, e^{2\pi i x_2}, \dots, e^{2\pi i x_n}) \in \mathbb{T}^n$$

is an isomorphism between two representations, where $[(x_1, x_2, \dots, x_n)] = \pi(x_1, x_2, \dots, x_n)$ for $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Using this coordinate system, we can rewrite the above toral automorphism α and the actions τ, γ as follows. Let $A = (a_{ij})$ and $A^{-1} = (b_{ij})$. Define the automorphism β of \mathbb{T}^n by

$$\beta(z_1, z_2, \dots, z_n) = (z_1^{a_{11}} z_2^{a_{12}} \dots z_n^{a_{1n}}, z_1^{a_{21}} z_2^{a_{22}} \dots z_n^{a_{2n}}, \dots, z_1^{a_{n1}} z_2^{a_{n2}} \dots z_n^{a_{nn}})$$

for $(z_1, z_2, \dots, z_n) \in \mathbb{T}^n$, the rotation action ρ of \mathbb{Z}^n on \mathbb{T}^n by

$$\rho_{\mathbf{m}}(v) = \varphi(\theta(\mathbf{m}))v \quad \text{for } \mathbf{m} \in \mathbb{Z}^n \text{ and } v \in \mathbb{T}^n,$$

and the action $\tilde{\gamma}$ of $\mathbb{Z}^n \rtimes \mathbb{Z}$ on \mathbb{T}^n by

$$\tilde{\gamma}_{(\mathbf{m},k)}(v) = \varphi(\theta(\mathbf{m}))\beta^k(v) \quad \text{for } (\mathbf{m},k) \in \mathbb{Z}^n \rtimes \mathbb{Z} \text{ and } v \in \mathbb{T}^n.$$

Then $\mathbb{Z}^n \curvearrow_{\tau} \mathbb{R}^n/\mathbb{Z}^n$ and $\mathbb{Z}^n \curvearrow_{\rho} \mathbb{T}^n$ are conjugate, and $\mathbb{Z}^n \rtimes \mathbb{Z} \curvearrow_{\tilde{\gamma}} \mathbb{T}^n$ and $\mathbb{Z}^n \rtimes \mathbb{Z} \curvearrow_{\gamma} \mathbb{R}^n/\mathbb{Z}^n$ are conjugate.

From Lemma 6.1, Proposition 6.3 and the above, the local conjugacy relation \mathcal{R} given by α and the associated semi-direct product $\mathcal{R} \rtimes_{\alpha} \mathbb{Z}$ are isomorphic to the transformation groupoids $\mathbb{T}^n \rtimes_{\rho} \mathbb{Z}^n$ and $\mathbb{T}^n \rtimes_{\tilde{\gamma}} (\mathbb{Z}^n \rtimes \mathbb{Z})$, respectively.

We still denote by ρ the automorphism action of \mathbb{Z}^n on $C(\mathbb{T}^n)$ induced by the system $\mathbb{Z}^n \curvearrow_{\rho} \mathbb{T}^n$:

$$\rho_{\mathbf{m}}(f)(v) = f(\varphi(\theta(\mathbf{m}))^{-1}v)$$

for $\mathbf{m} \in \mathbb{Z}^n$, $f \in C(\mathbb{T}^n)$ and $v \in \mathbb{T}^n$. Let \mathbf{e}_k , $1 \leq k \leq n$, be the canonical basis of \mathbb{Z}^n and $\theta(\mathbf{e}_k) = [(\theta_{k1}, \dots, \theta_{kn})] \in \Delta_{\alpha}$, where $\theta_{kj} \in [0, 1]$. Let U_j , $1 \leq j \leq n$, be the unitaries in $C(\mathbb{T}^n)$ defined by $U_j(z_1, \dots, z_n) = z_j$ for $(z_1, \dots, z_n) \in \mathbb{T}^n$, and let V_k , $1 \leq k \leq n$, be the unitaries implementing the C^* -automorphism $\rho_{\mathbf{e}_k}$ on $C(\mathbb{T}^n)$. One can check that

$$(6.2) \quad U_j U_k = U_k U_j, \quad V_j V_k = V_k V_j, \quad U_j V_k = e^{2\pi i \theta_{kj}} V_k U_j$$

for $1 \leq j, k \leq n$. From Propositions 6.4 and 6.6, $C_r^*(\mathcal{R})$, thus $C(\mathbb{T}^n) \rtimes_{\rho} \mathbb{Z}^n$ are simple and have unique tracial states. Hence $C(\mathbb{T}^n) \rtimes_{\rho} \mathbb{Z}^n$ is generated by U_j , V_j , $1 \leq j \leq n$, thus is the $2n$ -dimensional noncommutative torus A_{Θ} for a $2n \times 2n$ real skew-symmetric matrix $\Theta = (\tilde{\theta}_{kl})$ defined by $\tilde{\theta}_{kl} = 0$ for $1 \leq k, l \leq n$ or $n+1 \leq k, l \leq 2n$, $\tilde{\theta}_{kl} = \theta_{k(l-n)}$ for $1 \leq k \leq n$ and $n+1 \leq l \leq 2n$, and $\tilde{\theta}_{kl} = -\theta_{l(k-n)}$ for $n+1 \leq k \leq 2n$ and $1 \leq l \leq n$ (see [26]). From [20], $C_r^*(\mathcal{R})$ is an AT-algebra with real rank zero and the range of the unique tracial state acting on $K_0(C_r^*(\mathcal{R}))$ is an isomorphism invariant.

Similarly, we also denote by $\tilde{\gamma}$ the automorphism action of $\mathbb{Z}^n \rtimes \mathbb{Z}$ on $C(\mathbb{T}^n)$ induced by the system $\mathbb{Z}^n \rtimes \mathbb{Z} \curvearrow_{\tilde{\gamma}} \mathbb{T}^n$:

$$\tilde{\gamma}_{(\mathbf{m},k)}(f)(v) = f((\varphi\theta(A^{-k}\mathbf{m}))^{-1} \cdot \beta^{-k}(v))$$

for $(\mathbf{m},k) \in \mathbb{Z}^n \rtimes \mathbb{Z}$, $f \in C(\mathbb{T}^n)$ and $v \in \mathbb{T}^n$. Let U_j , $j = 1, 2, \dots, n$, be the generating set of $C(\mathbb{T}^n)$ as above, and let V'_j , $j = 1, 2, \dots, n$ and W be the unitaries implementing the automorphisms $\tilde{\gamma}_{(\mathbf{e}_j,0)}$ and $\tilde{\gamma}_{(0,1)}$ associated to the generating set $(\mathbf{e}_j, 0)$ for $1 \leq j \leq n$ and $(0, 1)$ of $\mathbb{Z}^n \rtimes \mathbb{Z}$. Then we have

$$(6.3) \quad \begin{aligned} U_j U_k &= U_k U_j, & V'_j V'_k &= V'_k V'_j, & U_j V'_k &= e^{2\pi i \theta_{kj}} V'_k U_j, \\ W U_j W^* &= \prod_{l=1}^n U_l^{b_{jl}}, & W V'_j W^* &= \prod_{l=1}^n V_l'^{a_{lj}} \end{aligned}$$

for $1 \leq j, k \leq n$. Since $C(\mathbb{T}^n) \rtimes_{\tilde{\gamma}} (\mathbb{Z}^n \rtimes \mathbb{Z})$ is simple from Propositions 6.4 and 6.6, it is generated by the unitaries U_j , V'_j , $1 \leq j \leq n$, and W satisfying the above relations.

From the above arguments and [29, Theorems 3.33 and 3.36], we have the following results which generalize [16, Theorem 2.9 and Proposition 6.1].

Proposition 6.8. *Let α be a hyperbolic toral automorphism on $\mathbb{R}^n/\mathbb{Z}^n$ defined by a hyperbolic matrix A . Let \mathcal{R} be the local conjugacy relation induced by α . Then*

- (i) $C_r^*(\mathcal{R})$ is generated by the unitaries $U_j, V_j, 1 \leq j \leq n$, satisfying the relations (6.2), thus is isomorphic to a simple $2n$ -dimensional noncommutative torus and is an AT-algebra with real rank zero.
- (ii) $C_r^*(\mathcal{R} \rtimes_{\alpha} \mathbb{Z})$ is generated by unitaries $U_j, V_j^!, 1 \leq j \leq n$, and W satisfying the relations (6.3).

Moreover, two hyperbolic toral automorphisms on $\mathbb{R}^n/\mathbb{Z}^n$ are flip conjugate if and only if the \mathbb{Z} -actions they generate are continuously orbit equivalent up to the associated local conjugacy relations.

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