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Bregman Projections and Parallel Extragradient Methods for Solving Multiple-sets Split Problems

Fridoun Moradlou, Zeynab Jouymandi and Fahimeh Akhavan Ghassabzade*

Abstract. In this paper, utilizing Bregman projections which are different from the sunny generalized nonexpansive retractions and generalized metric projection in Banach spaces, we introduce some new parallel extragradient methods for finding the solution of the multiple-sets split equilibrium problem and the solution of the multiple-sets split variational inequality problem in p-uniformly convex and uniformly smooth Banach spaces. Moreover, we introduce a Δ-Lipschitz-type condition on the equilibrium bifunctions to prove strongly convergent of the generated iterates in parallel extragradient methods. To illustrate the usability of our results and also to show the efficiency of the proposed methods, we present some comparative examples with several existing schemes in the literature in finite and infinite dimensional spaces.

1. Introduction

Assume that C_i $(i = 1, 2, \ldots, r)$ and Q_j $(j = 1, 2, \ldots, s)$ are nonempty, convex and closed subsets of real Banach spaces E_1 and E_2, respectively and also, assume that $C = \bigcap_{i=1}^r C_i$ and $Q = \bigcap_{j=1}^s Q_j$. Suppose that $A\colon E_1 \to E_2$ is a bounded linear operator and $f_i\colon C_i \times C_i \to \mathbb{R}$ $(i = 1, 2, \ldots, r)$ and $g_j\colon Q_j \times Q_j \to \mathbb{R}$ $(j = 1, 2, \ldots, s)$ are equilibrium bifunctions. The multiple-sets split equilibrium problem (MSSEP) [16] is formulated as follows:

$$\text{find } x^* \in C = \bigcap_{i=1}^{r} C_i \text{ such that } f_i(x^*, y) \geq 0,\ \forall\, y \in C_i,\ \forall\, i = 1, 2, \ldots, r,$$

where

$$Ax^* \in Q = \bigcap_{j=1}^{s} Q_j \text{ such that } g_j(Ax^*, w) \geq 0,\ \forall\, w \in Q_j,\ \forall\, j = 1, 2, \ldots, s.$$

We denote the solution set of (MSSEP) by ME. In the case of $r = s = 1$, (MSSEP) reduces to well known split equilibrium problem (SEP), which was first introduced by Moudafi [19].

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*Corresponding author.

Assume that $A_i\colon C_i \to 2^{E_1^*}$ and $B_j\colon Q_j \to 2^{E_2^*}$ are given mappings. The multiple-sets split variational inequality problem (MSSVIP) [16] is formulated as follows:

$$\text{find } x^* \in C = \bigcap_{i=1}^{r} C_i \text{ such that } \langle y - x^*, A_i x^* \rangle \geq 0,\ \forall\, y \in C_i,\ \forall\, i = 1, 2, \ldots, r,$$

where

$$Ax^* \in Q = \bigcap_{j=1}^{s} Q_j \text{ such that } \langle w - Ax^*, B_j Ax^* \rangle \geq 0,\ \forall\, w \in Q_j,\ \forall\, j = 1, 2, \ldots, s.$$

We denote the solution set of (MSSVIP) by $\mathtt{MS}$. If $r = s = 1$, then (MSSVIP) reduces to split variational inequality problem (SVIP), which was first proposed by Censor et al. [6].

It is well known that the equilibrium problem [20] covers many problems in mathematics and sciences. Some of these problems are the optimization problem, the variational inequality problem, the fixed point problem, the generalized Nash equilibrium problem in game theory, the saddle point problem and others; (see [2, 12, 21, 22, 24]). So, it's theory and applications has been extensively studied by many researchers.

Auxiliary problem principle as a useful tool for solving optimization problem has been introduced by Cohen [9] and then it has been extended to variational inequality [10]. Recently, the auxiliary problem principle has been extended to equilibrium problems by Mastroeni [18] under the assumption that the equilibrium bifunction f satisfies the following Lipschitz-type condition:

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|y - x\|^2 - c_2\|z - y\|^2, \quad \forall\, x, y, z \in C, \tag{1.1}$$

where $c_1, c_2 > 0$. In fact, this method is the famous classical extragradient method which has been introduced by Korpelevich [17], if equilibrium problem is replaced by variational inequality. The efficiency of the extragradient method has caused a great deal of attention among researchers. So, many authors introduced extragradient algorithms for solving various problems in Hilbert spaces [13, 16, 24, 25, 27–29] and this method was first introduced by Z. Jouymandi and F. Moradlou [14, 15] for solving equilibrium problem, variational inequality problem and fixed point problem, utilizing sunny generalized nonexpansive retractions and generalized metric projection in Banach spaces where the convergence of the proposed algorithms was required f to satisfy in a ϕ_*-Lipschitz-type condition or ϕ-Lipschitz-type condition [14, 15].

In this paper, motivated by D. S. Kim and B. V. Dinh [16], we present some new parallel extragradient algorithms (i.e., the algorithms have two extragradient steps) for finding an element $\mathtt{ME}$ and $\mathtt{MS}$, utilizing Bregman projections. Using these algorithms, we present and establish some strong convergence theorems under suitable conditions. Moreover, we give some numerical examples related to our algorithms and compare the results of our schemes with several existing ones in the literature. The number of iterations and also CPU times to get the solution for the generated sequence by our methods to show that our algorithms reach to a solution element faster than the other schemes.

2. Preliminaries

Suppose that E^* is the dual of a Banach space E and $S(E)$ is the unit sphere centered at the origin of E. Let $p \geq 1$, the Banach space E is called to be p-uniformly convex [26] if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in [0,2]$, where $\delta_E(\epsilon)$ is the modulus of convexity of E. It is well known that for $1 < p \leq 2$, L_p is 2-uniformly convex and for $p \geq 2$, L_p is p-uniformly convex. The Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$, where $\rho_E(t)$ is the smoothness modules of E. For $p > 1$, L_p is uniformly smooth.

The mapping J_{p_E} from E to 2^{E^*} defined by

$$J_{p_E}x = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x^*\| = \|x\|^{p-1}\right\}, \quad \forall\, x \in E \tag{2.1}$$

is called the generalized duality mapping, where $p > 1$ is a real number.

Let $1 < q \leq 2 \leq p$ with $1/p + 1/q = 1$. If E is p-uniformly convex and uniformly smooth, then J_{p_E} is monotone, one-to-one, onto and $J_{p_E}^{-1} = J_{q_E}^*$ (the generalized duality mapping on E^*) [1, 7].

If E is uniformly smooth, then J_{p_E} is uniformly norm-to-norm continuous on bounded sets of E [8] i.e., for a bounded set M of E and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|J_{p_E}x - J_{p_E}y\| < \varepsilon$$

for all $x, y \in M$ such that $\|x - y\| < \delta$. Let E be a p-uniformly convex, uniformly smooth real Banach space, we define the Bregman distance $\Delta_p^E \colon E \times E \to \mathbb{R}$ by

$$\Delta_p^E(x,y) = \frac{1}{q}\|x\|^p - \langle y, J_{p_E}x\rangle + \frac{1}{p}\|y\|^p \tag{2.2}$$

for all $x, y \in E$. It is worth noting that, if $p = q = 2$, then $\Delta_2^E(x,y) = \frac{1}{2}\phi(y,x)$ [14] for all $x, y \in E$ and in a Hilbert space H, $\Delta_2^H(x,y) = \frac{1}{2}\|x-y\|^2$ for all $x, y \in H$. Using the definition of Δ_p^E, we can easily conclude that

$$\Delta_p^E(x,y) = \Delta_p^E(x,z) + \Delta_p^E(z,y) + \langle z - y, J_{p_E}x - J_{p_E}z\rangle, \quad \forall\, x, y, z \in E \tag{2.3}$$

and

$$\Delta_p^E(x,y) + \Delta_p^E(y,x) = \langle x - y, J_{p_E}x - J_{p_E}y\rangle, \quad \forall\, x, y \in E. \tag{2.4}$$

Moreover, in a p-uniformly convex space X, we have the following property [23]:

$$\tau\|x-y\|^p \leq \Delta_p^E(x,y) \tag{2.5}$$

for all $x, y \in X$ and for some fixed number $\tau > 0$.

Lemma 2.1. [23] *Let C be a nonempty closed convex subset of a p-uniformly convex and uniformly smooth real Banach space E and let $(x,z) \in E \times C$. Then the following results hold:*

(i) $z = \Pi_C x$ *if and only if* $\langle y - z, J_{p_E}x - J_{p_E}z\rangle \leq 0$ *for all* $y \in C$,

(ii) $\Delta_p^E(\Pi_C x, z) + \Delta_p^E(x, \Pi_C x) \leq \Delta_p^E(x,z)$,

(iii) $\Delta_p^E(x, \Pi_C x) = \min_{y\in C} \Delta_p^E(x,y)$.

Thus $\Pi_C \colon E \to C$ *is Bregman projection and is defined by* $\Pi_C x = \arg\min_{y\in C} \Delta_p^E(x,y)$.

For a convex subset C of a Banach space E, we denote by $N_C(\nu)$ the normal cone for C at a point $\nu \in C$, i.e.,

$$N_C(\nu) := \{x^* \in E^* : \langle \nu - y, x^*\rangle \geq 0, \forall\, y \in C\}.$$

Let $f \colon E \to (-\infty, +\infty]$ be a proper function. For $x_0 \in \mathrm{Dom}(f)$, the subdifferential of f at x_0 as the subset of E^* given by

$$\partial f(x_0) = \big\{x^* \in E^* : f(x) \geq f(x_0) + \langle x^*, x - x_0\rangle, \forall\, x \in E\big\}.$$

If $\partial f(x_0) \neq \emptyset$, then f is called subdifferentiable at x_0. If $\partial f(x_0)$ is single valued, then f is said to be Gâteaux differentiable at x_0 which is denoted by $\nabla f(x_0)$.

Suppose that C is a nonempty subset of a Banach space E. The bifunction $f \colon C \times C \to \mathbb{R}$ is called

(i) pseudomonotone on C, i.e., $f(x,y) \geq 0 \;\Rightarrow\; f(y,x) \leq 0$, $\forall\, x, y \in C$,

(ii) jointly continuous, i.e., if $x, y \in C$ and $\{x_n\}$ and $\{y_n\}$ are two sequences in C such that $x_n \to x$ and $y_n \to y$, then $f(x_n, y_n) \to f(x,y)$.

Lemma 2.2. [3,14] *Let E be reflexive Banach space. If $f \colon E \to \mathbb{R} \cup \{+\infty\}$ and $g \colon E \to \mathbb{R} \cup \{+\infty\}$ are nontrivial, convex, and lower semicontinuous functions and if $0 \in \mathrm{Int}(\mathrm{Dom}\, f - \mathrm{Dom}\, g)$, then*

$$\partial(f+g)(x) = \partial(f) + \partial(g).$$

Lemma 2.3. [3,14] *Let C be a nonempty convex subset of a Banach space E and $f \colon E \to \mathbb{R}$ be a convex and subdifferentiable function, then f is minimized at $x \in E$ if and only if*

$$0 \in \partial f(x) + N_C(x).$$

3. Parallel extragradient methods

Throughout this paper, let C_i $(i = 1, 2, \dots, r)$ and Q_j $(j = 1, 2, \dots, s)$ be nonempty, convex and closed subsets of p-uniformly convex and uniformly smooth real Banach spaces E_1 and E_2 respectively. Assume that $C = \bigcap_{i=1}^{r} C_i$ and $Q = \bigcap_{j=1}^{s} Q_j$. Also, we suppose that $A\colon E_1 \to E_2$ is a bounded linear operator and the bifunctions $f_i\colon C_i \times C_i \to \mathbb{R}$ $(i = 1, 2, \dots, r)$ and $g_j\colon Q_j \times Q_j \to \mathbb{R}$ $(j = 1, 2, \dots, s)$ satisfy in following conditions which are defined as follows:

(A1) $f_i(x, x) = 0$ for all $x \in C_i$ $(i = 1, 2, \dots, r)$ and $g_j(x', x') = 0$ for all $x' \in Q_j$ $(j = 1, 2, \dots, s)$.

(A2) f_i $(i = 1, 2, \dots, r)$ and g_j $(j = 1, 2, \dots, s)$ are pseudomonotone on C_i and Q_j, respectively.

(A3) f_i $(i = 1, 2, \dots, r)$ and g_j $(j = 1, 2, \dots, s)$ are jointly continuous on $C_i \times C_i$ and $Q_j \times Q_j$, respectively.

(A4) $f_i(x, \cdot\,)$ $(i = 1, 2, \dots, r)$ and $g_j(x', \cdot\,)$ $(j = 1, 2, \dots, s)$ are convex, lower semicontinuous and subdifferentiable on C_i and Q_j, respectively, for all $x \in C_i$ and all $x' \in Q_j$.

(A5) f_i $(i = 1, 2, \dots, r)$ and g_j $(j = 1, 2, \dots, s)$ satisfy Δ-Lipschitz-type condition:

$$
\begin{aligned}
&\exists\, c_1^i, c_2^i \ \text{ s.t. } \ f_i(x, y) + f_i(y, z) \ge f_i(x, z) - c_1^i \Delta_p^{E_1}(x, y) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - c_2^i \Delta_p^{E_1}(y, z), \quad \forall\, x, y, z \in C_i, \\
&\exists\, c_1'^j, c_2'^j \ \text{ s.t. } \ g_j(x', y') + g_j(y', z') \ge g_j(x', z') - c'^j{}_1 \Delta_p^{E_2}(x', y') \\
&\qquad\qquad\qquad\qquad\qquad\qquad - c_2'^j \Delta_p^{E_2}(y', z'), \quad \forall\, x', y', z' \in Q_j.
\end{aligned}
\tag{3.1}
$$

Example 3.1. Assume that $f\colon C \times C \to \mathbb{R}$ is a bifunction defined by $f(x, y) = \frac{5}{q}\|y\|^p + 3\langle y, J_{p_{E_1}} x\rangle - \frac{8}{q}\|x\|^p$. Since $p \ge 2$, we obtain

$$
\begin{aligned}
&f(x, y) + f(y, z) \\
&= f(x, z) - 3\left(\frac{1}{p}\|x\|^p - \langle y, J_{p_{E_1}} x\rangle + \frac{1}{q}\|y\|^p\right) - 3\left(\frac{1}{p}\|y\|^p - \langle z, J_{p_{E_1}} y\rangle + \frac{1}{q}\|z\|^p\right) \\
&\quad + 3\left(\frac{1}{p}\|x\|^p - \langle z, J_{p_{E_1}} x\rangle + \frac{1}{q}\|z\|^p\right) + \left(8 - \frac{13}{p}\right)\|y\|^2 \\
&\ge f(x, z) - 3\Delta_p^{E_1}(x, y) - 3\Delta_p^{E_1}(y, z),
\end{aligned}
$$

i.e., f satisfies in the Δ-Lipschitz-type condition with $c_1 = c_2 = 3$.

Remark 3.2. If $p = q = 2$, then Δ-Lipschitz-type condition reduces to ϕ-Lipschitz-type condition and also If E_1 and E_2 are Hilbert spaces, then Δ-Lipschitz-type condition reduces to Lipschitz-type condition (1.1).

Algorithm 3.3 (Parallel Extragradient Method)**.**

Step 0: *For all* $i = 1, 2, \dots, r$ *and all* $j = 1, 2, \dots, s$, *choose the sequences* $\{\lambda_n^i\}, \{\gamma_n^j\} \subseteq (0, 1]$ *such that* $0 < a \leq \lambda_n^i, \gamma_n^j \leq b < \min\left\{\frac{1}{c}, \frac{1}{c'}\right\}$, *where* $c = \max\{c_1^i, c_1'^j\}$ *and* $c' = \max\{c_2^i, c_2'^j\}$. *Also,* $\{\alpha_n^i\}, \{\beta_n^j\} \subseteq [e, d]$ *such that* $0 < e \leq d < 1$ *and*

$$\sum_{i=1}^{r} \alpha_n^i = \sum_{j=1}^{s} \beta_n^j = 1, \quad \lambda_{n,r} = \frac{1}{r} \sum_{i=1}^{r} \lambda_n^i \quad \text{and} \quad \gamma_{n,s} = \frac{1}{s} \sum_{j=1}^{s} \gamma_n^j.$$

Step 1: *Let* $x_0 \in C$. *Set* $n = 0$.

Step 2: *Compute* y_n^i *and* z_n^i $(i = 1, 2, \dots, r)$ *such that*

$$y_n^i \in \arg\min_{y \in C_i} \left\{\lambda_n^i f_i(x_n, y) + \Delta_p^{E_1}(x_n, y)\right\}, \tag{3.2}$$

and

$$z_n^i \in \arg\min_{y \in C_i} \left\{\lambda_n^i f_i(y_n^i, y) + \Delta_p^{E_1}(x_n, y)\right\}.$$

Step 3: *Put* $z_n = J_{q_{E_1}}^{*}\left(\lambda_{n,r} J_{p_{E_1}} x_n + (1 - \lambda_{n,r}) \sum_{i=1}^{r} \alpha_n^i J_{p_{E_1}} z_n^i\right)$ *and* $t_n = \Pi_Q(Az_n)$.

Step 4: *Compute* v_n^j *and* u_n^j $(j = 1, 2, \dots, s)$ *such that*

$$v_n^j \in \arg\min_{w \in Q_j} \left\{\gamma_n^j g_j(t_n, w) + \Delta_p^{E_2}(t_n, w)\right\},$$

and

$$u_n^j \in \arg\min_{w \in Q_j} \left\{\gamma_n^j g_j(v_n^j, w) + \Delta_p^{E_2}(t_n, w)\right\}.$$

Step 5: *Put* $u_n = J_{q_{E_2}}^{*}\left(\gamma_{n,s} J_{p_{E_2}} t_n + (1 - \gamma_{n,s}) \sum_{j=1}^{s} \beta_n^j J_{p_{E_2}} u_n^j\right)$.

Step 6: *Compute* $x_{n+1} = \Pi_{D_n} x_n$, *where* $D_n = \bigcap_{k=0}^{n} O_k \cap A^{-1}(U_k)$ *such that* $O_0 = C$, $U_0 = Q$ *and*

$$O_n = \left\{x \in C : \Delta_p^{E_1}(z_n, x) \leq \Delta_p^{E_1}(x_n, x)\right\},$$

and

$$U_n = \left\{w \in Q : \Delta_p^{E_2}(u_n, w) \leq \Delta_p^{E_2}(t_n, w)\right\}.$$

Step 7: *Set* $n := n + 1$ *and go to Step* 2.

We replace Step 2 and Step 4 in Algorithm 3.3 by the following steps:

Algorithm 3.4 (Parallel Extragradient Algorithm or CQ Algorithm)**.**

Step 2a: *Compute y_n^i and z_n^i $(i = 1, 2, \dots, r)$ such that*

$$\text{(3.3)} \qquad y_n^i = \Pi_{C_i} J_{q_{E_1}}^* (J_{p_{E_1}} x_n - \lambda_n^i A_i x_n), \quad z_n^i = \Pi_{C_i} J_{q_{E_1}}^* (J_{p_{E_1}} x_n - \lambda_n^i A_i y_n^i).$$

Step 4a: *Compute v_n^j and u_n^j $(j = 1, 2, \dots, s)$ such that*

$$\text{(3.4)} \qquad v_n^j = \Pi_{Q_j} J_{q_{E_2}}^* (J_{p_{E_2}} t_n - \gamma_n^j B_j t_n), \quad u_n^j = \Pi_{Q_j} J_{q_{E_2}}^* (J_{p_{E_2}} t_n - \gamma_n^j B_j v_n^j).$$

Lemma 3.5. *If f_i $(i = 1, 2, \dots, r)$ and g_j $(j = 1, 2, \dots, s)$ satisfy in the properties* (A1)–(A4)*, then the solution set* ME *is closed and convex.*

Proof. To show the closedness of ME, suppose that $x_n \in \mathtt{ME}$ such that $x_n \to x^*$. This implies that $x_n \in C = \bigcap_{i=1}^r C_i$ such that $f_i(x_n, y) \geq 0$ for all $y \in C_i$ and $Ax_n \in Q = \bigcap_{j=1}^s Q_j$ such that $g_j(Ax_n, t) \geq 0$ for all $t \in Q_j$. Also, $Ax_n \to Ax^*$, because of A is bounded linear. Since C and Q are closed, we have $x^* \in C$ and $Ax^* \in Q$ and from the condition (A3), we conclude $f_i(x^*, y) \geq 0$ for all $y \in C_i$ and $g_j(Ax^*, y) \geq 0$ for all $t \in Q_j$, therefore $x^* \in \mathtt{ME}$.

For proving convexity of ME, assume that $x^*, x' \in \mathtt{ME}$ and $0 \leq \lambda \leq 1$. So, $\lambda x^* + (1 - \lambda)x' \in C = \bigcap_{i=1}^r C_i$ and $\lambda Ax^* + (1-\lambda)Ax' \in Q = \bigcap_{j=1}^s Q_j$, because of C and Q are convex. Also, from conditions (A2) and (A4), we get

$$\text{(3.5)} \qquad f_i(y, \lambda x^* + (1-\lambda)x') \leq \lambda f_i(y, x^*) + (1-\lambda) f_i(y, x') \leq 0$$

for all $y \in C_i$ $(i = 1, 2, \dots, r)$ and

$$\text{(3.6)} \qquad g_j(z, \lambda Ax^* + (1-\lambda)Ax') \leq \lambda g_j(z, Ax^*) + (1-\lambda) g_j(z, Ax') \leq 0$$

for all $z \in Q_j$ $(j = 1, 2, \dots, s)$. Let $y_t = ty + (1-t)(\lambda x^* + (1-\lambda)x')$ for all $y \in C_i$, and $z_t = tz + (1-t)(\lambda Ax^* + (1-\lambda)Ax')$ all $z \in Q_j$ and all $0 < t < 1$. Therefore, utilizing the conditions (A1), (A4) and inequalities (3.5) and (3.6), we obtain

$$0 = f_i(y_t, y_t) \leq t f_i(y_t, y) + (1-t) f_i(y_t, \lambda x^* + (1-\lambda)x') \leq t f_i(y_t, y)$$

and

$$0 = g_j(z_t, z_t) \leq t g_j(z_t, y) + (1-t) g_j(z_t, \lambda Ax^* + (1-\lambda)Ax') \leq t g_j(z_t, z).$$

Hence, $f_i(y_t, y) \geq 0$ and $g_j(z_t, z)$ for all $y \in C_i$ and all $z \in Q_j$. Letting $t \to 0$ and using the condition (A3), we derive that $f_i(\lambda x^* + (1-\lambda)x', y) \geq 0$ for all $y \in C_i$ $(i = 1, 2, \dots, r)$ and $g_j(\lambda Ax^* + (1-\lambda)Ax'), z) \geq 0$ for all $z \in Q_j$ $(j = 1, 2, \dots, s)$, so $\lambda x^* + (1-\lambda)x' \in \mathtt{ME}$. □

Lemma 3.6. *For every $x^* \in \mathtt{ME}$, every $i = 1, 2, \dots, r$, every $j = 1, 2, \dots, s$ and every $n \in \mathbb{N} \cup \{0\}$, we obtain*

(i) $\langle y - y_n^i, J_{p_{E_1}} x_n - J_{p_{E_1}} y_n^i \rangle \le \lambda_n^i f_i(x_n, y) - \lambda_n^i f_i(x_n, y_n^i)$, $\forall\, y \in C_i$,

(ii) $\Delta_p^{E_1}(z_n^i, x^*) \le \Delta_p^{E_1}(x_n, x^*) - (1 - c\lambda_n^i)\Delta_p^{E_1}(x_n, y_n^i) - (1 - c'\lambda_n^i)\Delta_p^{E_1}(y_n^i, z_n^i)$,

(iii) $\langle w - v_n^j, J_{p_{E_2}} v_n^j - J_{p_{E_2}} t_n \rangle \le \gamma_n^j g_j(t_n, w) - \gamma_n^j g_j(t_n, v_n^j)$, $\forall\, w \in Q_j$,

(iv) $\Delta_p^{E_2}(u_n^j, Ax^*) \le \Delta_p^{E_2}(t_n, Ax^*) - (1 - c\gamma_n^j)\Delta_p^{E_2}(t_n, v_n^j) - (1 - c'\gamma_n^j)\Delta_p^{E_2}(v_n^j, u_n^j)$.

Proof. Utilizing the condition (A4) and Lemmas 2.2 and 2.3, we get

$$\begin{aligned} & z_n^i \in \arg\min_{y \in C_i} \{\lambda_n^i f_i(y_n^i, y) + \Delta_p^{E_1}(x_n, y)\} \\ \iff \quad & 0 \in \lambda_n^i \partial_2 f_i(y_n^i, z_n^i) + \nabla_2 \Delta_p^{E_1}(x_n, z_n^i) + N_{C_i}(z_n^i). \end{aligned}$$

Using Proposition 4.9 of [7], we have $\partial\big(\frac{\|\cdot\|_E^p}{p}\big) = J_{p_E}\cdot$ and from (2.2), we deduce that $\nabla_2 \Delta_p^{E_1}(x_n, z_n^i) = J_{p_{E_1}} z_n^i - J_{p_{E_1}} x_n$. Thus $w_i \in \partial_2 f_i(y_n^i, z_n^i)$ and $\overline{w}_i \in N_{C_i}(z_n^i)$ exist such that

$$0 = \lambda_n^i w_i + J_{p_{E_1}} z_n^i - J_{p_{E_1}} x_n + \overline{w}_i, \tag{3.7}$$

therefore, using the definition of $\partial_2 f_i(y_n^i, z_n^i)$, we get

$$\langle y - z_n^i, w_i \rangle \le f(y_n^i, y) - f(y_n^i, z_n^i), \quad \forall\, y \in C_i. \tag{3.8}$$

Let $x^* \in \mathtt{ME}$. Letting $y = x^*$ in inequality (3.8), we conclude

$$\langle x^* - z_n^i, w_i \rangle \le f(y_n^i, x^*) - f(y_n^i, z_n^i).$$

So, using the definition of $N_{C_i}(z_n^i)$ and equality (3.7), we obtain

$$\lambda_n^i \langle z_n^i - y, w_i \rangle \le \langle y - z_n^i, J_{p_{E_1}} z_n^i - J_{p_{E_1}} x_n \rangle, \quad \forall\, y \in C_i. \tag{3.9}$$

Setting $y = x^*$ in inequality (3.9), we have

$$\langle x^* - z_n^i, J_{p_{E_1}} z_n^i - J_{p_{E_1}} x_n \rangle \ge \lambda_n^i \{f(y_n^i, z_n^i) - f(y_n^i, x^*)\} \ge \lambda_n^i f(y_n^i, z_n^i) \tag{3.10}$$

since $f(x^*, y_n^i) \ge 0$ and f_i is pseudomonotone on C_i. Replacing x, y and z by x_n, y_n^i and z_n^i in inequality (3.1), respectively, we get

$$f(y_n^i, z_n^i) \ge f(x_n, z_n^i) - f(x_n, y_n^i) - c_1^i \Delta_p^{E_1}(x_n, y_n^i) - c_2^i \Delta_p^{E_1}(y_n^i, z_n^i). \tag{3.11}$$

Using a similar argument, since $y_n^i = \arg\min_{y \in C_i} \big\{\lambda_n^i f(x_n, y) + \Delta_p^{E_1}(x_n, y)\big\}$, we have

$$\langle y - y_n^i, J_{p_{E_1}} x_n - J_{p_{E_1}} y_n^i \rangle \le \lambda_n^i \{f(x_n, y) - f(x_n, y_n^i)\}, \quad \forall\, y \in C_i. \tag{3.12}$$

Hence (i) is proved and by the same argument as (i), inequality (iii) can be proved.

Putting $y = z_n^i$ in inequality (3.12), we deduce

$$\langle z_n^i - y_n^i, J_{p_{E_1}} x_n - J_{p_{E_1}} y_n^i \rangle \leq \lambda_n^i \{ f(x_n, z_n^i) - f(x_n, y_n^i) \}. \tag{3.13}$$

Combining inequalities (3.10), (3.11) and (3.13), we get

$$\begin{aligned} &\langle x^* - z_n^i, J_{p_{E_1}} z_n^i - J_{p_{E_1}} x_n \rangle \\ &\geq \langle y_n^i - z_n^i, J_{p_{E_1}} y_n^i - J_{p_{E_1}} x_n \rangle - c_1^i \lambda_n^i \Delta_p^{E_1}(x_n, y_n^i) - c_2^i \lambda_n^i \Delta_p^{E_1}(y_n^i, z_n^i). \end{aligned}$$

Using above inequality and equality (2.3), we have

$$\begin{aligned} &\Delta_p^{E_1}(x_n, x^*) - \Delta_p^{E_1}(z_n^i, x^*) \\ &\geq \Delta_p^{E_1}(x_n, y_n^i) + \Delta_p^{E_1}(y_n^i, z_n^i) - c_1^i \lambda_n^i \Delta_p^{E_1}(x_n, y_n^i) - c_2^i \lambda_n^i \Delta_p^{E_1}(y_n^i, z_n^i) \\ &\geq \Delta_p^{E_1}(x_n, y_n^i) + \Delta_p^{E_1}(y_n^i, z_n^i) - c \lambda_n^i \Delta_p^{E_1}(x_n, y_n^i) - c' \lambda_n^i \Delta_p^{E_1}(y_n^i, z_n^i). \end{aligned}$$

Hence, (ii) is proved and by a similar way, (iv) can be proved. □

Lemma 3.7. *For all $x^* \in \mathtt{ME}$ and all $n \in \mathbb{N} \cup \{0\}$, we get*

(i) $\Delta_p^{E_1}(z_n, x^*) \leq \Delta_p^{E_1}(x_n, x^*)$,

(ii) $\Delta_p^{E_2}(u_n, Ax^*) \leq \Delta_p^{E_2}(t_n, Ax^*)$.

Proof. Assume that $x^* \in \mathtt{ME}$. Using Lemma 3.6(ii), equalities (2.1) and (2.2), and since $J_{p_{E_1}}^{-1} = J_{q_{E_1}}^*$, we obtain

$$\begin{aligned} \Delta_p^{E_1}(z_n, x^*) &= \frac{1}{q}\|z_n\|^p - \langle x^*, J_p z_n \rangle + \frac{1}{p}\|x^*\|^p \\ &= \frac{1}{q} \left\| J_{q_{E_1}}^* \left(\lambda_{n,r} J_{p_{E_1}} x_n + (1 - \lambda_{n,r}) \sum_{i=1}^r \alpha_n^i J_{p_{E_1}} z_n^i \right) \right\|^p - \lambda_{n,r} \langle x^*, J_{p_{E_1}} x_n \rangle \\ &\quad - (1 - \lambda_{n,r}) \sum_{i=1}^r \alpha_n^i \langle x^*, J_{p_{E_1}} z_n^i \rangle + \frac{1}{p}\|x^*\|^p \\ &= \frac{1}{q} \left\| J_{p_{E_1}} J_{q_{E_1}}^* \left(\lambda_{n,r} J_{p_{E_1}} x_n + (1 - \lambda_{n,r}) \sum_{i=1}^r \alpha_n^i J_{p_{E_1}} z_n^i \right) \right\|^q - \lambda_{n,r} \langle x^*, J_{p_{E_1}} x_n \rangle \\ &\quad - (1 - \lambda_{n,r}) \sum_{i=1}^r \alpha_n^i \langle x^*, J_{p_{E_1}} z_n^i \rangle + \frac{1}{p}\|x^*\|^p \\ &\leq \frac{1}{q}\lambda_{n,r}\|J_{p_{E_1}} x_n\|^q + \frac{1}{q}(1 - \lambda_{n,r}) \sum_{i=1}^r \alpha_n^i \|J_{p_{E_1}} z_n^i\|^q - \lambda_{n,r} \langle x^*, J_{p_{E_1}} x_n \rangle \\ &\quad - (1 - \lambda_{n,r}) \sum_{i=1}^r \alpha_n^i \langle x^*, J_{p_{E_1}} z_n^i \rangle + \frac{1}{p}\|x^*\|^p \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{q}\lambda_{n,r}\|x_n\|^p + \frac{1}{q}(1-\lambda_{n,r})\sum_{i=1}^{r}\alpha_n^i\|z_n^i\|^p - \lambda_{n,r}\langle x^*, J_{p_{E_1}}x_n\rangle \\
&\quad - (1-\lambda_{n,r})\sum_{i=1}^{r}\alpha_n^i\langle x^*, J_{p_{E_1}}z_n^i\rangle + \frac{1}{p}\|x^*\|^p \\
&= \lambda_{n,r}\Delta_p^{E_1}(x_n, x^*) + (1-\lambda_{n,r})\sum_{i=1}^{r}\alpha_n^i\Delta_p^{E_1}(z_n^i, x^*) \\
&\leq \lambda_{n,r}\Delta_p^{E_1}(x_n, x^*) + (1-\lambda_{n,r})\sum_{i=1}^{r}\alpha_n^i\Delta_p^{E_1}(x_n, x^*) \\
&= \Delta_p^{E_1}(x_n, x^*).
\end{aligned}
$$

Applying a similar manner, we can prove that $\Delta_p^{E_2}(u_n, Ax^*) \leq \Delta_p^{E_2}(t_n, Ax^*)$. □

Lemma 3.8. *For all $n \in \mathbb{N} \cup \{0\}$, D_n is nonempty, closed and convex.*

Proof. Assume that $D_\infty = \bigcap_{n=0}^{\infty} D_n \subseteq D_n$ and $x^* \in \mathtt{ME}$. Using Lemma 3.7 for all $k = 0, 1, 2, \dots, n$, we obtain $x^* \in O_k$ and $Ax^* \in U_k$. Therefore $x^* \in O_k \cap A^{-1}(U_k)$, i.e., $x^* \in D_n$ for all $n \in \mathbb{N} \cup \{0\}$. Hence $x^* \in D_\infty$. On the other hand, sine

$$
\Delta_p^{E_1}(z_n, x) \leq \Delta_p^{E_1}(x_n, x) \iff \langle x, J_{p_{E_1}}x_n - J_{p_{E_1}}z_n\rangle \leq \frac{1}{q}(\|x_n\|^p - \|z_n\|^p), \tag{3.14}
$$

and also, since A is bounded linear and

$$
\Delta_p^{E_2}(u_n, w) \leq \Delta_p^{E_2}(t_n, w) \iff \langle w, J_{p_{E_2}}t_n - J_{p_{E_2}}u_n\rangle \leq \frac{1}{q}(\|t_n\|^p - \|u_n\|^p). \tag{3.15}
$$

We can conclude that for all $k = 0, 1, 2, \dots, n$, $O_k \cap A^{-1}(U_k)$ is closed and convex and cosequently for all $n \in \mathbb{N} \cup \{0\}$, D_n is closed and convex. □

Lemma 3.9. *The optimal solutions y_n^i and z_n^i ($i = 1, 2, \dots, r$) and v_n^j and u_n^j ($j = 1, 2, \dots, s$) in Algorithm* 3.3*, are uniquely determined.*

Proof. Assume that $y_n^i, \acute{y}_n^i \in \arg\min_{y \in C_i}\left\{\lambda_n^i f_i(x_n, y) + \Delta_p^{E_1}(x_n, y)\right\}$ for all $i = 1, 2, \dots, r$. Utilizing of Lemma 3.6(i), we have

$$
\langle y - y_n^i, J_{p_{E_1}}x_n - J_{p_{E_1}}y_n^i\rangle \leq \lambda_n^i f_i(x_n, y) - \lambda_n^i f_i(x_n, y_n^i), \quad \forall\, y \in C_i \tag{3.16}
$$

and

$$
\langle y - \acute{y}_n^i, J_{p_{E_1}}x_n - J_{p_{E_1}}\acute{y}_n^i\rangle \leq \lambda_n^i f_i(x_n, y) - \lambda_n^i f_i(x_n, \acute{y}_n^i), \quad \forall\, y \in C_i. \tag{3.17}
$$

Putting $y = \acute{y}_n^i$ in (3.16) and $y = y_n^i$ in (3.17) and adding them, we obtain

$$
\langle \acute{y}_n^i - y_n^i, J_{p_{E_1}}\acute{y}_n^i - J_{p_{E_1}}y_n^i\rangle \leq 0. \tag{3.18}
$$

Since $J_{p_{E_1}}$ is monotone so,

$$\langle \acute{y}_n^i - y_n^i, J_{p_{E_1}} \acute{y}_n^i - J_{p_{E_1}} y_n^i \rangle \geq 0. \tag{3.19}$$

Thus, using inequalities (3.18) and (3.19), we obtain $\langle \acute{y}_n^i - y_n^i, J_{p_{E_1}} \acute{y}_n^i - J_{p_{E_1}} y_n^i \rangle = 0$ and since $J_{p_{E_1}}$ is one to one, we get $y_n^i = \acute{y}_n^i$ $(i = 1, 2, \ldots, r)$. Using the similar argument, we can prove that z_n^i, v_n^j and u_n^j $(j = 1, 2, \ldots, s)$ are uniquely determined. □

Theorem 3.10. *Suppose that* $\mathtt{ME} \neq \emptyset$, *then the sequence* $\{x_n\}_{n=0}^{\infty}$ *generated by Algorithm* 3.3 *strong converges to the some solution* $\varphi \in \mathtt{ME}$, *where* $\varphi = \lim_{n\to\infty} \Pi_{\mathtt{ME}} x_n$.

Proof. Suppose that $x^* \in \mathtt{ME} \subseteq D_\infty$. It follows from Lemma 3.8 that, $\{x_n\}$ is well defined. Also, since $x_{n+1} = \Pi_{D_n} x_n$ and $x^* \in D_n$ for all $n \in \mathbb{N} \cup \{0\}$, utilizing Lemma 2.1(ii), we conclude

$$0 \leq \Delta_p^{E_1}(x_{n+1}, x^*) \leq \Delta_p^{E_1}(x_n, x^*) - \Delta_p^{E_1}(x_n, x_{n+1}) \leq \Delta_p^{E_1}(x_n, x^*). \tag{3.20}$$

This implies that $\lim_{n\to\infty} \Delta_p^{E_1}(x_n, x^*)$ exists and so $\{\Delta_p^{E_1}(x_n, x^*)\}$ is bounded. Therefore, $\lim_{n\to\infty} \Delta_p^{E_1}(x_n, x_{n+1}) = 0$ since $\Delta_p^{E_1}(x_n, x_{n+1}) \leq \Delta_p^{E_1}(x_n, x^*) - \Delta_p^{E_1}(x_{n+1}, x^*) \to 0$. So, from inequality (2.5) for some fixed number $\tau > 0$,

$$\tau \|x_n - x_{n+1}\| \leq \Delta_p^{E_1}(x_n, x^*) - \Delta_p^{E_1}(x_{n+1}, x^*), \tag{3.21}$$

so $\{x_n\}$ is bounded. Also, for all $n, m \in \mathbb{N} \cup \{0\}$ and $m \geq n$, from inequalities (3.20) and (3.21), we get

$$\begin{aligned}
\tau\|x_n - x_m\| &\leq \tau\big(\|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \|x_{n+2} - x_{n+3}\| + \cdots + \|x_{m-1} - x_m\|\big) \\
&\leq \Delta_p^{E_1}(x_n, x^*) - \Delta_p^{E_1}(x_{n+1}, x^*) + \Delta_p^{E_1}(x_{n+1}, x^*) - \Delta_p^{E_1}(x_{n+2}, x^*) \\
&\quad + \Delta_p^{E_1}(x_{n+2}, x^*) - \Delta_p^{E_1}(x_{n+3}, x^*) + \cdots + \Delta_p^{E_1}(x_{m-1}, x^*) - \Delta_p^{E_1}(x_m, x^*) \\
&\leq \Delta_p^{E_1}(x_n, x^*) - \Delta_p^{E_1}(x_m, x^*).
\end{aligned}$$

Since $\lim_{n\to\infty} \Delta_p^{E_1}(x_n, x^*)$ exists, $\{x_n\} \subseteq D_{n-1}$ is a Cauchy sequence and consequently from Lemma 3.8, $\{x_n\}$ converges strongly to u. On the other hand, $D_n \subseteq D_{n-1} \subseteq D_{n-2} \subseteq \cdots \subseteq D_0$ so $\{x_n\} \subseteq D_0 = C \cap A^{-1}Q$. Also, $C \cap A^{-1}Q = \bigcap_{i=1}^r C_i \cap A^{-1}\big(\bigcap_{j=1}^s Q_j\big)$ is closed. Hence, we can conclude that $u \in C$ and $Au \in Q$.

Since $x_{n+1} \in O_n$ and from inequalities (3.20) and (2.5), $\tau_1 > 0$ exists such that

$$\lim_{n\to\infty} \tau_1 \|z_n - x_{n+1}\|^p \leq \lim_{n\to\infty} \Delta_p^{E_1}(z_n, x_{n+1}) \leq \lim_{n\to\infty} \Delta_p^{E_1}(x_n, x_{n+1}) = 0.$$

Thus $\{z_n\}$ is bounded and $z_n \to u$ and consequently $Az_n \to Au$ because A is bounded linear. Utilizing Lemma 2.1(ii) and uniformly norm-to-norm continuity of $J_{p_{E_2}}$ on bounded sets, we get

$$
\begin{aligned}
\lim_{n\to\infty} \Delta_p^{E_2}(t_n, Au) &\leq \lim_{n\to\infty} \Delta_p^{E_2}(Az_n, Au) \\
&= \lim_{n\to\infty} \left(\frac{1}{q}\|Az_n\|^2 - \langle Au, J_{p_{E_2}} Az_n \rangle + \frac{1}{p}\|Au\|^p \right) \qquad (3.22) \\
&= 0
\end{aligned}
$$

and

$$
\lim_{n\to\infty} \Delta_p^{E_1}(z_n, x^*) = \lim_{n\to\infty} \Delta_p^{E_1}(x_n, x^*). \qquad (3.23)
$$

Therefore $\{t_n\}$ is bounded and $t_n \to Au$. On the other hand, because of $Ax_{n+1} \in U_n$ and using a similar technique as (3.22), $\tau_2 > 0$ exists such that

$$
\lim_{n\to\infty} \tau_2 \|u_n - Ax_{n+1}\|^p \leq \lim_{n\to\infty} \Delta_p^{E_2}(t_n, Ax_{n+1}) = 0.
$$

Hence, $\{u_n\}$ is bounded and $u_n \to Au$ and also utilizing a similar technique as (3.23), we get

$$
\lim_{n\to\infty} \Delta_p^{E_2}(u_n, Ax^*) = \lim_{n\to\infty} \Delta_p^{E_2}(t_n, Ax^*).
$$

In addition, it follows from the proof of Lemma 3.7 that for all $i = 1, 2, \ldots, r$ and all $j = 1, 2, \ldots, s$, we have

$$
\Delta_p^{E_1}(z_n^i, x^*) \leq \Delta_p^{E_1}(x_n, x^*) \quad \Longrightarrow \quad \Delta_p^{E_1}(z_n, x^*) \leq \Delta_p^{E_1}(x_n, x^*)
$$

and

$$
\Delta_p^{E_2}(u_n^j, Ax^*) \leq \Delta_p^{E_2}(t_n, Ax^*) \quad \Longrightarrow \quad \Delta_p^{E_2}(u_n, Ax^*) \leq \Delta_p^{E_2}(t_n, Ax^*).
$$

Therefore, $\Delta_p^{E_1}(z_n, x^*) \leq \Delta_p^{E_1}(z_n^i, x^*)$ and $\Delta_p^{E_2}(u_n, Ax^*) \leq \Delta_p^{E_2}(u_n^j, Ax^*)$. Thus, from Lemma 3.6(ii) and (iv) and inequality (2.5), $\tau_3' > 0$ and $\tau_4 > 0$ exist such that, for all $i = 1, 2, \ldots, r$ and all $j = 1, 2, \ldots, s$, we can obtain that

$$
\begin{aligned}
&\lim_{n\to\infty} \tau_3'(1 - cb)\|x_n - y_n^i\|^p \\
&\leq \lim_{n\to\infty} (1 - cb)\Delta_p^{E_1}(x_n, y_n^i) \leq \lim_{n\to\infty} (1 - c\lambda_n^i)\Delta_p^{E_1}(x_n, y_n^i) \qquad (3.24) \\
&\leq \lim_{n\to\infty} \left(\Delta_p^{E_1}(x_n, x^*) - \Delta_p^{E_1}(z_n^i, x^*)\right) \leq \lim_{n\to\infty} \left(\Delta_p^{E_1}(x_n, x^*) - \Delta_p^{E_1}(z_n, x^*)\right) \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{n\to\infty} \tau_4(1 - cb)\|t_n - v_n^j\|^p \\
&\leq \lim_{n\to\infty} (1 - cb)\Delta_p^{E_2}(t_n, v_n^j) \leq \lim_{n\to\infty} (1 - c\gamma_n^j)\Delta_p^{E_2}(t_n, v_n^j) \qquad (3.25) \\
&\leq \lim_{n\to\infty} \left(\Delta_p^{E_2}(t_n, Ax^*) - \Delta_p^{E_2}(u_n^j, Ax^*)\right) \leq \lim_{n\to\infty} \left(\Delta_p^{E_2}(t_n, Ax^*) - \Delta_p^{E_2}(u_n, Ax^*)\right) \\
&= 0.
\end{aligned}
$$

So, using inequalities (3.24) and (3.25) and the condition $(1-cb)>0$, we obtain $y_n^i \to u$ and $v_n^j \to Au$, due to $x_n \to u$ and $t_n \to Au$. It follows from (i) and (iii) of Lemma 3.6 and the condition $a \le \lambda_n^i, \gamma_n^j \le 1$ that

$$\langle y - y_n^i, J_{p_{E_1}} x_n - J_{p_{E_1}} y_n^i \rangle \le f_i(x_n, y) - a f_i(x_n, y_n^i), \quad \forall\, y \in C_i,\ \forall\, 1 \le i \le r \tag{3.26}$$

and

$$\langle w - v_n^j, J_{p_{E_2}} v_n^j - J_{p_{E_2}} t_n \rangle \le g_j(t_n, w) - a g_j(t_n, v_n^j), \quad \forall\, w \in Q_j,\ \forall\, 1 \le j \le s. \tag{3.27}$$

By letting $n \to \infty$ in inequalities (3.26) and (3.27), it follows from the conditions (A1) and (A3) and uniformly norm-to-norm continuity of $J_{p_{E_1}}$ and $J_{p_{E_2}}$ on bounded sets that

$$u \in C = \bigcap_{i=1}^{r} C_i \quad \text{such that} \quad f_i(u, y) \ge 0, \quad \forall\, y \in C_i$$

and

$$Au \in Q = \bigcap_{i=1}^{s} Q_j \quad \text{such that} \quad g_j(Au, w) \ge 0, \quad \forall\, w \in Q_j,$$

so $u \in \mathtt{ME}$.

Now, assume that $h_n = \Pi_{\mathtt{ME}} x_n$, using inequality (3.20), we have

$$\Delta_p^{E_1}(x_{n+1}, h_n) \le \Delta_p^{E_1}(x_n, h_n). \tag{3.28}$$

Thus, utilizing Lemma 2.1(iii) and inequality (3.28), we get

$$\Delta_p^{E_1}(x_{n+1}, h_{n+1}) = \Delta_p^{E_1}(x_{n+1}, \Pi_{\mathtt{ME}} x_{n+1}) \le \Delta_p^{E_1}(x_n, h_n).$$

Hence, $\lim_{n\to\infty} \Delta_p^{E_1}(x_n, h_n)$ exists and $\left\{\Delta_p^{E_1}(x_n, h_n)\right\}$ is bounded and so (2.5) implies $\{h_n\}$ is bounded. Since $h_{n+m} = \Pi_{\mathtt{ME}} x_{n+m}$ for all $m \in \mathbb{N}$, utilizing Lemma 2.1(ii) and (3.28), we get

$$\Delta_p^{E_1}(x_{n+m}, h_n) + \Delta_p^{E_1}(x_{n+m}, h_{n+m}) \le \Delta_p^{E_1}(x_{n+m}, h_n) \le \Delta_p^{E_1}(x_n, h_n),$$

thus we can conclude that

$$\Delta_p^{E_1}(x_{n+m}, h_n) \le \Delta_p^{E_1}(x_n, h_n) - \Delta_p^{E_1}(x_{n+m}, h_{n+m}).$$

So, using (2.5) and the above inequality, there exists $\tau_3 > 0$ such that

$$\tau_3 \|h_{n+m} - h_n\|^p \le \Delta_p^{E_1}(x_{n+m}, h_n) \le \Delta_p^{E_1}(x_n, h_n) - \Delta_p^{E_1}(x_{n+m}, h_{n+m}).$$

Hence,

$$\tau_3 \|h_{n+m} - h_n\|^p \le \Delta_p^{E_1}(x_n, h_n) - \Delta_p^{E_1}(x_{n+m}, h_{n+m}) \to 0.$$

So, $\tau_3\|h_{n+m} - h_n\|^p \to 0$. This implies that $\{h_n\} \subseteq \mathtt{ME}$ is a Cauchy sequence and so converges strongly to $\varphi \in \mathtt{ME}$, due to Lemma 3.5 implies that $\mathtt{ME}$ is closed. Using Lemma 2.1(i), we get $\langle u - h_n, J_{p_E}x_n - J_{p_E}h_n\rangle \leq 0$, taking the limits as $n \to \infty$, we have $\langle u - \varphi, J_{p_E}u - J_{p_E}\varphi\rangle \leq 0$, because $J_{p_{E_1}}$ is uniformly norm-to-norm continuity of on bounded sets. Also, $\langle u - \varphi, J_{p_E}u - J_{p_E}\varphi\rangle \geq 0$, due to $J_{p_{E_1}}$ is monotone. Consequently, $u = \varphi$, since $J_{p_{E_1}}$ is one to one. Therefore $x_n \to \varphi$, where $\varphi = \lim_{n\to\infty} \Pi_{\mathtt{ME}}x_n$. □

Theorem 3.11. *Suppose that* $\mathtt{MS} \neq \emptyset$*, then the generated sequence* $\{x_n\}_{n=0}^{\infty}$ *by Algorithm* 3.4 *strong converges to the some solution* $\varphi \in \mathtt{MS}$*, where* $\varphi = \lim_{n\to\infty} \Pi_{\mathtt{MS}}x_n$.

Proof. In Algorithm 3.3, putting $f_i(x, y) = \langle y - x, A_ix\rangle$ for all $x, y \in C_i$ $(i = 1, 2, \ldots, r)$. Also, setting $g_j(x', y') = \langle y' - x', B_jx'\rangle$ for all $x', y' \in Q_j$ $(j = 1, 2, \ldots, s)$, where $A_i\colon C_i \to E_1^*$ and $B_j\colon Q_j \to E_2^*$ are two continuous mappings. Then, $\mathtt{ME} = \mathtt{MS}$ and using (2.4), (3.2) and Lemma 3.9, we have

$$
\begin{aligned}
y_n^i &= \arg\min_{y\in C_i} \big\{\langle y - x_n, \lambda_n^i A_ix_n\rangle + \Delta_p^{E_1}(x_n, y)\big\} \\
&= \arg\min_{y\in C_i} \big\{\langle y - x_n, \lambda_n^i A_ix_n\rangle + \langle y - x_n, J_{p_{E_1}}y - J_{p_{E_1}}x_n\rangle - \Delta_p^{E_1}(y, x_n)\big\} \qquad (3.29)\\
&= \arg\min_{y\in C_i} \big\{\big\langle y - x_n, J_{p_{E_1}}y - (J_{p_{E_1}}x_n - \lambda_n^i A_ix_n)\big\rangle - \Delta_p^{E_1}(y, x_n)\big\}.
\end{aligned}
$$

The minimum amount of (3.29) occurs when $\langle y_n^i - x_n, J_{p_{E_1}}y_n^i - (J_{p_{E_1}}x_n - \lambda_n^i A_ix_n)\rangle \leq 0$. Consequently, utilizing Lemma 2.1(i), we get $y_n^i = \Pi_{C_i}J_{q_{E_1}}^*(J_{p_{E_1}}x_n - \lambda_n^i A_ix_n)$. Using the same argument for z_n^i, we have

$$
\begin{aligned}
z_n^i &= \arg\min_{y\in C_i} \big\{\langle y - y_n^i, \lambda_n^i A_iy_n^i\rangle + \Delta_p^{E_1}(x_n, y)\big\} \\
&= \arg\min_{y\in C_i} \big\{\langle y - y_n^i, \lambda_n^i A_iy_n^i\rangle + \langle y - y_n^i, J_{p_{E_1}}y - J_{p_{E_1}}x_n\rangle \\
&\qquad\qquad + \langle y_n^i - x_n, J_{p_{E_1}}y - J_{p_{E_1}}x_n\rangle - \Delta_p^{E_1}(y, x_n)\big\} \qquad (3.30)\\
&= \arg\min_{y\in C_i} \big\{\big\langle y - y_n^i, J_{p_{E_1}}y - (J_{p_{E_1}}x_n - \lambda_n^i A_iy_n^i)\big\rangle \\
&\qquad\qquad + \langle y_n^i - x_n, J_{p_{E_1}}y - J_{p_{E_1}}x_n\rangle - \Delta_p^{E_1}(y, x_n)\big\} \\
&= \arg\min_{y\in C_i} \big\{\big\langle y - y_n^i, J_{p_{E_1}}y - (J_{p_{E_1}}x_n - \lambda_n^i A_iy_n^i)\big\rangle + \Delta_p^{E_1}(x_n, y_n^i) - \Delta_p^{E_1}(y, y_n^i)\big\}.
\end{aligned}
$$

So, the minimum amount of (3.30) occurs when

$$\langle z_n^i - x_n, J_{p_{E_1}}z_n^i - (J_{p_{E_1}}x_n - \lambda_n^i A_iy_n^i)\rangle \leq 0.$$

Consequently, utilizing Lemma 2.1(i), we get $z_n^i = \Pi_{C_i}J_{q_{E_1}}^*(J_{p_{E_1}}x_n - \lambda_n^i A_iy_n^i)$.

Using the same argument as above, we can obtain Step 4a of Algorithm 3.4 from Algorithm 3.3. Hence, utilizing the analogous argument such as Theorem 3.10, we can conclude that $\{x_n\}$ converge strongly to the some solution $\varphi \in \mathtt{MS}$, where $\varphi = \lim_{n\to\infty} \Pi_{\mathtt{MS}}x_n$. □

Remark 3.12. In a real Hilbert space, if $r = s = 1$, then (3.3) and (3.4) reduce to the classical extragradient method which has been introduced by Korpelevich [17].

4. Numerical illustrations

In this section, to investigate the behavior of our algorithms, we present some numerical examples in finite and infinite dimensional spaces. Moreover, to show the efficiency of our algorithms, we compare the numerical results of them with other ones in the literature. In finite dimensional case, the optimization subproblems have been solved by FMINCON optimization toolbox in MATLAB R2015a. For more details about the notations which are used in this section, we refer readers to [4, 5].

4.1. Finite dimensional case

Example 4.1. In Algorithm 3.3, put $r = s = 2$, $E_1 = \mathbb{R}^2$ and $E_2 = \mathbb{R}^3$. Also, assume that $C_1 = \{x \in \mathbb{R}^2 : \langle x, p\rangle \geq -2\}$ and $C_2 = \{x \in \mathbb{R}^2 : \langle x, p\rangle \leq 4\}$, where $p = (1, 2)$. Therefore, $C = C_1 \cap C_2 = \{x \in \mathbb{R}^2 : -2 \leq \langle x, p\rangle \leq 4\}$. Also, take

$$A = \begin{bmatrix} 1 & -1 \\ -0.5 & 2 \\ 3 & -5 \end{bmatrix}, \quad Q_1 = \{x \in \mathbb{R}^3 : \langle x, q\rangle \geq -3\} \quad \text{and} \quad Q_2 = \{x \in \mathbb{R}^3 : \langle x, q\rangle \leq 2\},$$

where $q = (-1, 2, 5)$. It easy to see that $Q = \{x \in \mathbb{R}^3 : -3 \leq \langle x, q\rangle \leq 2\}$ and

$$\Pi_Q x = \frac{1}{2} P_Q x = \frac{1}{2} \begin{cases} x - \frac{\langle x,q\rangle - 2}{\|q\|^2} q & \text{if } \langle x, q\rangle > 2, \\ x & \text{if } -3 \leq \langle x, q\rangle \leq 2, \\ x - \frac{\langle x,q\rangle + 3}{\|q\|^2} q & \text{if } \langle x, q\rangle < -3. \end{cases}$$

Now, suppose that

$$f_1(x, y) = 3\|y\|^2 + 2\langle y, x\rangle - 5\|x\|^2, \qquad f_2(x, y) = 4\|y\|^2 + 4\langle y, x\rangle - 8\|x\|^2,$$
$$g_1(x, y) = \|y\|^2 + 3\langle y, x\rangle - 4\|x\|^2, \qquad g_2(x, y) = \|y\|^2 + \langle y, x\rangle - 2\|x\|^2.$$

So, f_1, f_2, g_1 and g_2 satisfy in the conditions (A1)–(A5) with $c_1^1 = c_2^1 = 2$, $c_1^2 = c_2^2 = 4$, $c_1'^1 = c_2'^1 = 3$ and $c_1'^2 = c_2'^2 = 1$, respectively. Consequently, $\min\left\{\frac{1}{c}, \frac{1}{c'}\right\} = \frac{1}{4}$. Hence $\mathtt{ME} = \{0\}$.

Now, assume that $\lambda_n^1 = \left(\frac{1}{6}\right)^{n+1}$, $\lambda_n^2 = \frac{1}{5} - \frac{1}{2n+10}$, $\gamma_n^1 = \left(\frac{1}{5}\right)^{n+1}$, $\gamma_n^2 = \frac{1}{4} - \frac{1}{n+6}$. Also,

$$\lambda_n^i f_i(x_n, y_n^i) + \Delta_p^{E_1}(x_n, y_n^i) = \min_{y \in C_i} \left\{\lambda_n^i f_i(x_n, y) + \Delta_p^{E_1}(x_n, y)\right\}, \quad i = 1, 2,$$
$$\lambda_n^i f_i(y_n^i, z_n^i) + \Delta_p^{E_1}(x_n, z_n^i) = \min_{y \in C_i} \left\{\lambda_n^i f_i(y_n^i, y) + \Delta_p^{E_1}(x_n, y)\right\}, \quad i = 1, 2,$$

$$\gamma_n^j g_j(t_n, v_n^j) + \Delta_p^{E_2}(t_n, v_n^j) = \min_{w \in Q_j} \left\{\gamma_n^j g_j(t_n, w) + \Delta_p^{E_2}(t_n, w)\right\}, \quad j = 1, 2,$$

$$\gamma_n^j g_j(v_n^j, u_n^j) + \Delta_p^{E_2}(t_n, u_n^j) = \min_{w \in Q_j} \left\{\gamma_n^j g_j(v_n^j, w) + \Delta_p^{E_2}(t_n, w)\right\}, \quad j = 1, 2.$$

Furthermore, $z_n = \frac{\lambda_n^1+\lambda_n^2}{2} x_n + \left(1 - \frac{\lambda_n^1+\lambda_n^2}{2}\right)(\alpha_n^1 z_n^1 + \alpha_n^2 z_n^2)$, $u_n = \frac{\gamma_n^1+\gamma_n^2}{2} t_n + \left(1 - \frac{\gamma_n^1+\gamma_n^2}{2}\right)(\beta_n^1 u_n^1 + \beta_n^2 u_n^2)$ and $t_n = \Pi_Q(Az_n)$. Now, assume that $\alpha_n^1 = \frac{5}{8} - \frac{1}{n+9}$, $\alpha_n^2 = \frac{3}{8} + \frac{1}{n+9}$, $\beta_n^1 = \frac{3}{7} - \frac{1}{2n+8}$, $\beta_n^2 = \frac{4}{7} + \frac{1}{2n+8}$. Also, $x_{n+1} = \Pi_{D_n} x_n = \frac{1}{2} P_{D_n} x_n = \frac{1}{2} \arg\min_{x \in D_n} \|x - x_n\|^2$ which $D_n = \bigcap_{k=0}^n O_k \cap \bigcap_{k=0}^n A^{-1}(U_k)$ and therefore using (3.14) and (3.15), we get

$$\bigcap_{k=0}^{n} O_k = \bigcap_{k=0}^{n} \left\{x \in C : \langle x, x_k - z_k \rangle \leq \frac{1}{2}(\|x_k\|^2 - \|z_k\|^2)\right\}$$

and

$$\bigcap_{k=0}^{n} A^{-1}(U_k) = \bigcap_{k=0}^{n} \left\{A^{-1}w : w \in Q \text{ and } \langle w, t_k - u_k \rangle \leq \frac{1}{2}(\|t_k\|^2 - \|u_k\|^2)\right\}.$$

Table 4.1: Obtained results for convergence behavior of the sequence $\{x_n\}$ generated by algorithms (M1)–(M4) with starting point $x_0 = (-5, 2)$.

n	(M1)	(M2)	(M3)	(M4)
0	5.3852	5.3852	5.3852	5.3852
1	0.5590	0.9224	1.7145	0.2310
2	0.2365	0.5605	1.1923	0.7404
3	0.1035	0.3977	1.0275	2.5730
4	0.0471	0.2881	0.8909	8.2476
5	0.0210	0.2094	4.3479	0.0145
⋮	⋮	⋮	⋮	⋮
STOP	11	40	33	500
CPU(s)	2.600s	13.921s	11.467s	71.343s

Now, to get some comparison results, we consider the following algorithms and theorems with above conditions, starting point $x_0 = (-5, 2)$ and stopping criteria $\|x_n\| < 10^{-3}$:

(M1) Our Algorithm 3.3 and Theorem 3.10.

(M2) Algorithm 3.3 and Theorem 3.1 of [16] with $L_1 = c$, $L_2 = c'$, $\rho_k^i = \lambda_n^i$, $r_k^j = \gamma_n^j$ and $\mu = 0.1$.

(M3) Algorithm 3.4 and Theorem 3.2 of [16] with the considered condition for (M2).

(M4) Algorithm 3.3 and Theorem 1 of [11] with $\lambda = \frac{1}{9}$, $\mu = \frac{1}{3}$, $F_1 = g_1$, $F_2 = g_2$ and $r_n = \gamma_n^1$.

The numerical results for the sequence $\{x_n\}$ generated by algorithms (M1)–(M4), show that $\{\|x_n\|\}$ and consequently, $\{x_n\}$ converges strongly to 0, see Table 4.1 and Figure 4.1. Also, the number of iterations and CPU times to get the solution imply that our parallel extragradient method for solving (MSSEP), i.e., (M1) is faster than the other three algorithms.

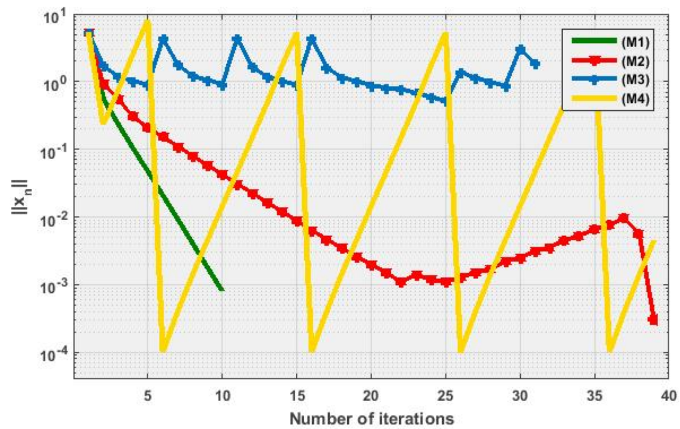

Figure 4.1: Comparison results for the sequence $\{x_n\}$ generated by methods (M1)–(M4) with starting point $x_0 = (-5, 2)$.

Example 4.2. In Algorithm 3.4, suppose that C_i, Q_j, λ_n^i, γ_n^j, α_n^i, β_n^j for all $i, j = 1, 2$ and A, t_n, x_{n+1}, p, q are the same as in Example 4.1. Also, let

$$A_1 = \begin{bmatrix} 3 & -2 \\ -4 & 5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 \\ -3 & -4 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 4 & -1 & 2 \\ 3 & -4 & 7 \\ 2 & -3 & -1 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} -3 & -2 & 1 \\ 6 & 0 & 2 \\ 8 & 5 & -3 \end{bmatrix}.$$

Therefore, $\mathtt{MS} = \{0\}$. It is straightforward to calculate that

$$\Pi_{C_1} x = \frac{1}{2} P_{C_1} x = \frac{1}{2}\left(\max\left\{0, \frac{-2 - \langle x, p\rangle}{\|p\|^2}\right\} p + x\right),$$
$$\Pi_{C_2} x = \frac{1}{2} P_{C_2} x = \frac{1}{2}\left(\min\left\{0, \frac{4 - \langle x, p\rangle}{\|p\|^2}\right\} p + x\right),$$
$$\Pi_{Q_1} x = \frac{1}{2} P_{Q_1} x = \frac{1}{2}\left(\max\left\{0, \frac{-3 - \langle x, q\rangle}{\|q\|^2}\right\} q + x\right),$$
$$\Pi_{Q_2} x = \frac{1}{2} P_{Q_2} x = \frac{1}{2}\left(\min\left\{0, \frac{2 - \langle x, q\rangle}{\|q\|^2}\right\} q + x\right),$$

and also

$$\begin{aligned} y_n^1 &= \frac{1}{2}P_{C_1}(x_n - \lambda_n^1 A_1 x_n), & y_n^2 &= \frac{1}{2}P_{C_2}(x_n - \lambda_n^2 A_2 x_n),\\ z_n^1 &= \frac{1}{2}P_{C_1}(x_n - \lambda_n^1 A_1 y_n^1), & z_n^2 &= \frac{1}{2}P_{C_2}(x_n - \lambda_n^2 A_2 y_n^2),\\ v_n^1 &= \frac{1}{2}P_{Q_1}(t_n - \gamma_n^1 B_1 t_n), & v_n^2 &= \frac{1}{2}P_{Q_2}(t_n - \gamma_n^2 B_2 t_n),\\ u_n^1 &= P_{Q_1}(t_n - \gamma_n^1 B_1 v_n^1), & u_n^2 &= P_{Q_2}(t_n - \gamma_n^2 B_2 v_n^2). \end{aligned}$$

Table 4.2: The convergence behavior of the sequence $\{x_n\}$ generated by algorithms (M5)–(M7) with starting point $x_0 = (-4, 3)$.

n	(M5)	(M6)	(M7)
0	1.0022	5.000	5.000
1	0.7975	5.7989	5.5544
2	0.5979	6.1424	5.4915
3	0.4276	7.1049	5.6778
4	0.2811	8.5045	5.4214
5	0.1582	10.3512	6.1779
$\vdots$	$\vdots$	$\vdots$	$\vdots$
STOP	21	2130	No
CPU(s)	1.360s	64.021s	No

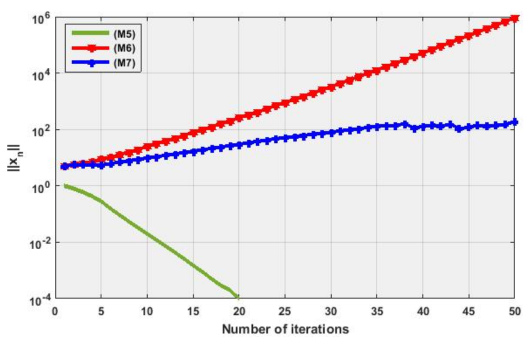

Figure 4.2: Obtained results for the sequence $\{x_n\}$ generated by algorithms (M5)–(M7) with starting point $x_0 = (-4, 3)$.

Now, we consider the following algorithms and theorems with the considered conditions in Example 4.1, starting point $x_0 = (-4, 3)$ and stopping condition $\|x_n\| < 10^{-4}$ and compare their results with each other.

(M5) Our Algorithm 3.4 and Theorem 3.11.

(M6) Corollary 3.2 of [16] with $F_i = A_i$, $G_j = B_j$, $\mu = 0.02$ and L_1, L_2, ρ_k^i, r_k^j are the same as in (M2).

(M7) Corollary 3.4 of [16] with the considered conditions in (M6).

The obtained numerical results for the sequence $\{x_n\}$ generated by the above methods, i.e., (M5)–(M7), show that $\{x_n\}$ converges strongly to 0, see Table 4.2 and Figure 4.2. The number of iterations and also CPU times to get the solution for the sequence $\{x_n\}$ show that our algorithm, (M5), reaches the stopping condition faster than the other schemes.

4.2. Infinite dimensional case

Example 4.3. In Algorithm 3.4, put $r = s = 2$ and $E_1 = E_2 = L_2[1, 2]$ with

$$\langle x, y\rangle = \int_1^2 x(t)y(t)\,dt \quad \text{and} \quad \|x\|^2 = \int_1^2 x^2(t)\,dt,$$

and also assume that

$$C_1 = \{x \in L_2[1, 2] : \langle x, t^2\rangle \geq -2\} \quad \text{and} \quad C_2 = \{x \in L_2[1, 2] : \langle x, t^2\rangle \leq 4\}.$$

Therefore,

$$C = C_1 \cap C_2 = \{x \in L_2[1, 2] : -2 \leq \langle x, t^2\rangle \leq 4\}.$$

Furthermore, take $A = 2I$, consequently $A^* = 2I$, $A^{-1} = \frac{1}{2}I$ and $\|A\| = 2$. Also, set

$$Q_1 = \{x \in L_2[1, 2] : \langle x, t^3\rangle \geq -3\} \quad \text{and} \quad Q_2 = \{x \in L_2[1, 2] : \langle x, t^3\rangle \leq 2\}.$$

Thus, it is readily seen that $Q = \{x \in L_2[1, 2] : -3 \leq \langle x, t^3\rangle \leq 2\}$ and

$$\Pi_Q x = \frac{1}{2}P_Q x = \frac{1}{2}\begin{cases} x - \frac{\langle x,t^3\rangle - 2}{\|t^3\|^2}t^3, & \langle x, t^3\rangle > 2,\\ x, & -3 \leq \langle x, t^3\rangle \leq 2,\\ x - \frac{\langle x,t^3\rangle + 3}{\|t^3\|^2}t^3, & \langle x, t^3\rangle < -3.\end{cases}$$

Moreover, $A_1 = 2I$, $A_2 = 4I$, $B_1 = 3I$ and $B_2 = I$. Therefore, we can conclude that

$$\begin{aligned}
\Pi_{C_1} x &= \frac{1}{2}P_{C_1} x = \frac{1}{2}\max\left\{0, \frac{-2 - \langle x, t^2\rangle}{\|t^2\|^2}\right\} t^2 + x,\\
\Pi_{C_2} x &= \frac{1}{2}P_{C_2} x = \frac{1}{2}\min\left\{0, \frac{4 - \langle x, t^2\rangle}{\|t^2\|^2}\right\} t^2 + x,\\
\Pi_{Q_1} x &= \frac{1}{2}P_{Q_1} x = \frac{1}{2}\max\left\{0, \frac{-3 - \langle x, t^3\rangle}{\|t^3\|^2}\right\} t^3 + x,\\
\Pi_{Q_2} x &= \frac{1}{2}P_{Q_2} x = \frac{1}{2}\min\left\{0, \frac{2 - \langle x, t^3\rangle}{\|t^3\|^2}\right\} t^3 + x,
\end{aligned}$$

where λ_n^i, γ_n^j, α_n^i, β_n^j, y_n^i, z_n^i, v_n^j, u_n^j for all $i, j = 1, 2$ are the same as in Example 4.1 and $t_n = \Pi_Q(2z_n) = \frac{1}{2}P_Q(2z_n)$. Also, using Proposition 29.23 of [4], we get $x_{n+1} = \Pi_{D_n}x_n = \frac{1}{2}P_{D_n}x_n$, consequently $\mathtt{MS} = \{0\}$.

Table 4.3: The convergence behavior of the sequence $\{x_n\}$ generated by algorithms (M8)–(M10) with starting point $x_0 = \frac{1}{2}\ln(t)$.

n	(M8)	(M9)	(M10)
0	0.2170	0.2170	0.2170
1	0.0216	0.1644	0.1728
2	0.0021	0.1387	0.1489
3	0.0002	0.1207	0.1316
4	0.0001	0.1059	0.1173
$\vdots$	$\vdots$	$\vdots$	$\vdots$
STOP	6	82	102
CPU(s)	0.382s	1.136s	80.467s

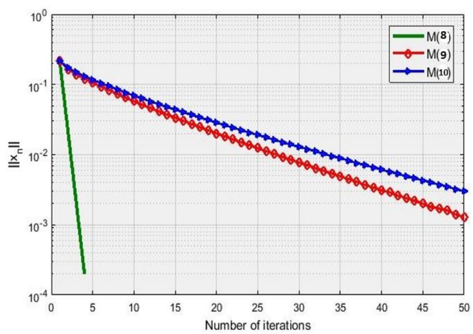

Figure 4.3: Obtained results for the sequence $\{x_n\}$ generated by algorithms (M8)–(M10) with starting point $x_0 = \frac{1}{2}\ln(t)$.

Now, we consider following algorithms and theorems with the conditions of Example 4.3, starting point $x_0 = \frac{1}{2}\ln(t)$ and the stopping condition $\|x_n\| < 10^{-4}$ and compare their results with each other.

(M8) Our Algorithm 3.4 and Theorem 3.11.

(M9) Corollary 3.2 of [16] with $F_i = A_i$, $G_j = B_j$, $\mu = 0.02$ and L_1, L_2, ρ_k^i, r_k^j are the same as in (M2).

(M10) Corollary 3.4 of [16] with the considered conditions in (M9).

Comparison results which are reported in Table 4.3 and Figure 4.3 for the sequence $\{x_n\}$ generated by methods (M8)–(M10), show that $\{x_n\}$ converges strongly to 0. The number of iterations and also CPU times to get the solution for the sequence $\{x_n\}$ show that our algorithm, (M8), reaches the stopping condition faster than the other two algorithms.

5. Conclusions

Utilizing Bregman projections, we have proposed a new parallel extragradient and some new CQ algorithms for solving the multiple-sets split equilibrium problem and the multiple-sets split variational inequality problem in p-uniformly convex and uniformly smooth Banach spaces. We have proven some theorems to show that the generated iterates by our schemes are strongly convergent. To show the efficiency of our algorithm, we have presented some comparative numerical examples between our algorithms and some other ones in the literature. The number of iterations and also CPU times have shown that convergence of the generated sequences by parallel extragradient and CQ algorithms are faster than some existing schemes in the literature.

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Fridoun Moradlou
Department of Mathematics, Sahand University of Technology, Tabriz, Iran
E-mail addresses: moradlou@sut.ac.ir, fridoun.moradlou@gmail.com

Zeynab Jouymandi and Fahimeh Akhavan Ghassabzade
Department of Mathematics, Faculty of sciences, University of Gonabad, Gonabad, Iran
E-mail addresses: z_jouymandi@sut.ac.ir, zjouymandi@gmail.com,
dakhavan@gonabad.ac.ir, akhavan_gh@yahoo.com