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Exponential Stable Behavior of a Class of Impulsive Partial Stochastic Differential Equations Driven by Lévy Noise

Fangxia Lu

Abstract. In this paper, using exponential stable property, the stochastic analysis techniques and a fixed-point theorem for condensing maps, we obtain the existence results of piecewise weighted pseudo almost periodic in distribution mild solutions for a class of impulsive partial stochastic differential equations driven by Lévy noise in Hilbert spaces under non-Lipschitz conditions. Furthermore, the exponential stable of mild solution in mean square is investigated. Finally, an example is presented to illustrate the results.

1. Introduction

Weighted pseudo almost periodic functions are more general than pseudo almost periodic functions, which thus can better meet the demands of practical application. Weighted pseudo almost periodic functions were introduced by Diagana [10, 11]. Recently Agarwal et al. [1], Ezzinbi [14], Chen and Hu [6] have studied basic properties of weighted pseudo almost periodic functions and then used these results to study the existence and uniqueness of weighted pseudo almost periodic mild solutions for some abstract differential equations.

On the other hand, the stochastic differential equations have received a lot of attention due to its applications in various fields of science and engineering. Almost periodic and pseudo almost periodic solutions of stochastic differential equations were studied by several authors; see [4, 5, 8, 12, 33]. Chen and Lin [7] studied the mean square weighted pseudo almost automorphic solutions for a general class of non-autonomous stochastic evolution equations. Bedouhene et al. [3] obtained the existence and uniqueness of weighted pseudo almost periodic and Stepanov-like weighted pseudo almost periodic mild solution to some stochastic evolution equations. Most of the studies on almost periodic and almost automorphic solutions for stochastic differential equations are concerned with equations perturbed by Brownian motion. However, many real models involve jump perturbations or more general Lévy noise. Wang and Liu [29] first introduced the concept of Poisson square-mean almost periodicity and studied the existence, uniqueness and stability of the

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mean square almost period solutions for stochastic evolution equations driven by Lévy noise. Li [19] established the existence and uniqueness of mean-square weighted pseudo almost automorphic solutions for some linear and semilinear nonautonomous stochastic differential equations driven by Lévy noise. Indeed, as indicated in [18, 22], it seems that almost periodicity in distribution sense is a more appropriate concept compared with solutions of stochastic differential equations. Recently, Liu and Sun [21] studied almost automorphic in distribution solutions of stochastic differential equations with Lévy noise. Diop et al. [13] introduced the concept of μ-pseudo almost automorphic processes in distribution and studied the existence, the uniqueness and the stability of the square-mean μ-pseudo almost automorphic solutions in distribution to a class of abstract stochastic evolution equations driven by Lévy noise.

The theory of impulsive differential equations is an important branch of differential equations, which has an extensive physical background [26]. Therefore, it seems interesting to study the asymptotic properties of solutions of impulsive differential equations (see [20, 28, 30]). Song et al. [27], Gu et al. [15] investigated piecewise weighted pseudo almost periodic functions and its applications in impulsive differential and integro-differential equations under the Lipschitz conditions. In addition to impulsive effects, stochastic effects likewise exist in real systems. A lot of dynamical systems have variable structures subject to stochastic abrupt changes, which may result from abrupt phenomena such as stochastic failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, etc. So, it is necessary and important to consider the qualitative properties of stochastic systems with impulsive effects. In recent years, several interesting results on impulsive partial stochastic systems have been reported in [16, 25, 32] and the references therein. Recently, Yan and Lu [31] studied the existence and exponential stability of pseudo almost periodic solutions for impulsive nonautonomous partial stochastic evolution equations with delay. Up to now, to the best of the author' knowledge, there are no results available in the literature concerning the weighted pseudo almost periodicity in distribution to partial stochastic differential systems with impulsive effects and Lévy noise are available in the literature.This paper focuses on the existence and exponential stable behavior of piecewise weighted pseudo almost periodic in distribution mild solutions for the following nonlinear impulsive partial stochastic differential equations driven by Lévy noise under non-Lipschitz conditions:

$$\begin{aligned} dx(t) &= [Ax(t) + g(t, x(t))]\, dt + f(t, x(t))\, dW(t) + \int_{|y|_V<1} G(t, x(t-), y)\widetilde{N}(dt, dy) \\ &\quad + \int_{|y|_V>1} F(t, x(t-), y)N(dt, dy), \quad t \in \mathbb{R},\ t \neq t_i,\ i \in \mathbb{Z}, \end{aligned} \tag{1.1}$$

$$\Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \quad i \in \mathbb{Z}, \tag{1.2}$$

where A is the generator of an exponentially stable C_0 semigroup $\{T(t)\}_{t\geq 0}$ on $L^p(\mathbb{P},\mathbb{H})$ and W, N and $\widetilde{N}$ are the Lévy–Itô decomposition components of the two-sided Lévy process L defined on the filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_t)$, where $\mathcal{F}_t=\sigma\{W(s)-W(\tau): s,\tau\leq t\}$. The functions g, f, G, F, I_i, t_i satisfy suitable conditions which will be established later. The notations $x(t_i^+)$, $x(t_i^-)$ represent the right-hand side and the left-hand side limits of $x(\cdot)$ at t_i, respectively.

Two major contributions are made in this paper. For one thing, a new concept, concerning p-mean piecewise weighted pseudo almost periodic in distribution for stochastic processes driven by Lévy noise, is proposed, which naturally extends the concept of weighted pseudo almost periodicity in distribution to dynamical systems represented by these impulsive systems. For another, using exponential stable property and a fixed-point theorem for condensing maps with stochastic analysis theory, we study and obtain the existence and exponential stable behavior of mean-square piecewise weighted pseudo almost periodic in distribution mild solutions to system (1.1)–(1.2) under non-Lipschitz conditions. Such a result generalizes most of known results on the existence of almost periodic in distribution solutions of type system (1.1). Some results of almost periodic and pseudo almost periodic in distribution solutions to stochastic differential equations without impulse are included in it as special cases.

The rest of this paper is arranged as follows. In the next section, some notations and a preliminary lemma are given. In Section 3, it is shown that equation (1.1)–(1.2) exist piecewise weighted pseudo almost periodic in distribution mild solutions. In Section 4, the exponential stable behavior of piecewise weighted pseudo almost periodic in distribution mild solutions for (1.1)–(1.1) are obtained. An example is given to illustrate our results in Section 5. A natural conclusion is drawn in the last section.

2. Preliminaries

Throughout the paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{R}^+$ stand for the set of natural numbers, integers, real numbers, positive real numbers, respectively. We assume that $(\mathbb{H},\|\cdot\|)$, $(V,|\cdot|_V)$ are real separable Hilbert spaces and $(\Omega,\mathcal{F},\mathbb{P})$ is supposed to be a filtered complete probability space. The notation $L^p(\mathbb{P},\mathbb{H})$, for $p\geq 1$ stands for the space of all $\mathbb{H}$-valued random variables x such that $E\|x\|^p=\int_\Omega \|x\|^p\,d\mathbb{P}<\infty$. Then $L^p(\mathbb{P},\mathbb{H})$ is a Hilbert space when it is equipped with its natural norm $\|\cdot\|_p$ defined by $\|x\|_p=\big(\int_\Omega E\|x\|^p\,d\mathbb{P}\big)^{1/p}<\infty$ for each $x\in L^p(\mathbb{P},\mathbb{H})$. Let $C(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$, $BC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$ stand for the collection of all continuous functions from $\mathbb{R}$ into $L^p(\mathbb{P},\mathbb{H})$, the Banach space of all bounded continuous functions from $\mathbb{R}$ into $L^p(\mathbb{P},\mathbb{H})$, equipped with the sup norm, respectively. We let $L(V,\mathbb{H})$ be the space of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, equipped with the usual operator norm $\|\cdot\|_{L(V,\mathbb{H})}$; in particular, this is simply denoted by $L(\mathbb{H})$ when $V=\mathbb{H}$. The filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ with covariance operator Q, that is $E\langle W(t), x\rangle_{\mathbb{H}}\langle W(s), y\rangle_{\mathbb{H}} = (t \wedge s)\langle Qx, y\rangle_{\mathbb{H}}$ for all $x, y \in \mathbb{H}$, where Q is a positive, self-adjoint, trace class operator on $\mathbb{H}$. Furthermore, $L_2^0(V, \mathbb{H})$ denotes the space of all Q-Hilbert-Schmidt operators from V to $\mathbb{H}$ with the norm $\|\psi\|^2_{L_2^0} = \mathrm{Tr}(\psi Q \psi^*) < \infty$ for any $\psi \in L(V, \mathbb{H})$.

2.1. Lévy process

Definition 2.1. [2] A V-valued stochastic process $L = (L(t), t \geq 0)$ is called Lévy process if

(i) $L(0) = 0$ almost surely;

(ii) L has independent and stationary increments;

(iii) L is stochastically continuous, i.e., for all $\varepsilon > 0$ and for all $s > 0$

$$\lim_{t \to s} P(|L(t) - L(s)|_V > \varepsilon) = 0.$$

Definition 2.2. [2]

(i) B is Borel set in $V - \{0\}$ that is bounded below.

(ii) $v(\cdot) = E(N(1, \cdot))$ is called the intensity measure associated with L, where $\Delta L(t) = L(t) - L(t_-)$ for each $t \geq 0$ and

$$N(t, B)(\omega) := \sharp\{0 \leq s \leq t : \Delta L(t)(\omega) \in B\} = \sum_{0 \leq s \leq t} \chi_B(\Delta L(t)(\omega))$$

with χ_B being the indicator function for any Borel set B in $V - \{0\}$.

(iii) $N(t, B)$ is called Poisson random measure if B is bounded below, for each $t \geq 0$.

(iv) For each $t \geq 0$ and B bounded below, we define the compensated Poisson random measure by $\widetilde{N}(t, B) = N(t, B) - tv(B)$.

Proposition 2.3. (Lévy–Itô decomposition [23]) *If L is a V-valued Lévy process, then there exist $a \in V$, V-valued Wiener process W with covariance operator Q, so called Q-Wiener process, and an independent Poisson random measure on $\mathbb{R}^+ \times (V - \{0\})$ such that for each $t \geq 0$,*

$$L(t) = at + W(t) + \int_{|x|_V < 1} x\widetilde{N}(t, dx) + \int_{|x|_V > 1} xN(t, dx).$$

Here the Poisson random measure N has the intensity measure v which satisfies $\int_V (|y|_V^2 \vee 1)\, v(dy) < \infty$ and $\widetilde{N}$ is the compensated Poisson random measure of N.

In this paper, Wiener processes we consider are Q-Wiener processes; see [9] for details. For simplicity, we assume that the covariance operator Q of W is of trace class, i.e., $\operatorname{Tr} Q < \infty$. Assume that L_1 and L_2 are two independent, identically distributed Lévy processes with decompositions as in Proposition 2.3 with a, Q, W, N. Let

$$L(t) = \begin{cases} L_1(t) & \text{for } t \geq 0, \\ -L_2(-t) & \text{for } t \leq 0. \end{cases}$$

Then L is a two-sided Lévy process. We assume that the two sided Lévy process L is defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}})$. By the Lévy–Itô decomposition, it follows that $\int_{|x|_V \geq 1} v(dx) < \infty$. Then we denote $b := \int_{|x|_V \geq 1} v(dx)$ throughout this paper. We note that the process $\widetilde{L} := (\widetilde{L}(t) = L(t+s) - L(s))$ for some $s \in \mathbb{R}$ is also a two-sided Lévy process with the same law as L.

2.2. Weighted pseudo almost periodic processes

Definition 2.4. [5] A stochastic process $x\colon \mathbb{R} \to L^p(\mathbb{P}, \mathbb{H})$ is said to be bounded if there exists a constant $M_0 > 0$ such that

$$E\|x(t)\|^p \leq M_0, \quad t \in \mathbb{R}.$$

Definition 2.5. [5] A stochastic process $x\colon \mathbb{R} \to L^p(\mathbb{P}, \mathbb{H})$ is said to be continuous provided that for any $s \in \mathbb{R}$,

$$\lim_{t\to s} E\|x(t) - x(s)\|^p = 0.$$

Let $\mathbb{T}$ be the set consisting of all real sequences $\{t_i\}_{i\in\mathbb{Z}}$ such that $\alpha = \inf_{i\in\mathbb{Z}}(t_{i+1} - t_i) > 0$, $\lim_{i\to\infty} t_i = \infty$ and $\lim_{i\to-\infty} t_i = -\infty$. For $\{t_i\}_{i\in\mathbb{Z}} \in \mathbb{T}$, let $PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ be the space consisting of all bounded piecewise continuous processes $f\colon \mathbb{R} \to L^p(\mathbb{P}, \mathbb{H})$ such that $f(\cdot)$ is continuous at t for any $t \notin \{t_i\}_{i\in\mathbb{Z}}$ and $f(t_i) = f(t_i^-)$ for all $i \in \mathbb{Z}$; let $PC(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}), L^p(\mathbb{P}, \mathbb{H}))$ be the space formed by all piecewise continuous processes $f\colon \mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}) \to L^p(\mathbb{P}, \mathbb{H})$ such that for any $x \in L^p(\mathbb{P}, \mathbb{H})$, $f(\,\cdot\,, x) \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ and for any $t \in \mathbb{R}$, $f(t, \cdot\,)$ is continuous at $x \in L^p(\mathbb{P}, \mathbb{H})$.

Definition 2.6. (Cf. [26]) For $\{t_i\}_{i\in\mathbb{Z}} \in \mathbb{T}$, the stochastic process $f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is said to be p-mean piecewise almost periodic if the following conditions are fulfilled:

(i) $\{t_i^j = t_{i+j} - t_i\}$, $j \in \mathbb{Z}$, is equipotentially almost periodic, that is, for every sequence of integer numbers $\{\alpha_n'\}$, there exist a subsequence $\{\alpha_n\}$ and a sequence $\{\widetilde{t}_i\}$ such that $\lim_{n\to\infty} |t_{i+\alpha_n} - t_i - \widetilde{t}_i| = 0$ for all $i \in \mathbb{Z}$.

(ii) For any $\varepsilon > 0$, there exists a positive number $\widetilde{\delta} = \widetilde{\delta}(\varepsilon)$ such that if the points t' and t'' belong to a same interval of continuity of f and $|t' - t''| < \widetilde{\delta}$, then $E\|f(t') - f(t'')\|^p < \varepsilon$.

(iii) For every sequence of real numbers $\{s'_n\}$, there exist a subsequence $\{s_n\}$ and a process $\widetilde{f} \in C(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ such that

$$\lim_{n\to\infty} E\|f(t+s_n) - \widetilde{f}(t)\|^p = 0$$

for all $t \in \mathbb{R}$ satisfying the condition $|t - t_i| > \varepsilon$ for any $\varepsilon > 0$, $i \in \mathbb{Z}$.

We denote by $AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ the collection of all the p-mean piecewise almost periodic functions. Obviously, the space $AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ endowed with the sup norm defined by $\|f\|_\infty = \left(\sup_{t\in\mathbb{R}} E\|f(t)\|^p\right)^{1/p}$ for any $f \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is a Banach space. Throughout the rest of this paper, we always assume that $\{t_i^j\}$ are equipotentially almost periodic. Let $UPC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ be the space of all stochastic functions $f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ such that f satisfies the condition (ii) in Definition 2.6.

Definition 2.7. (Cf. [26]) The stochastic process $f \in PC(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}), L^p(\mathbb{P}, \mathbb{H}))$ is said to be p-mean piecewise almost periodic in $t \in \mathbb{R}$ uniform in $x \in L^p(\mathbb{P}, \mathbb{H})$ if for every sequence of real numbers $\{s'_n\}$, there exist a subsequence $\{s_n\}$ and a process $\widetilde{f} \in C(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}), L^p(\mathbb{P}, \mathbb{H}))$ such that

$$\lim_{n\to\infty} E\|f(t+s_n, x) - \widetilde{f}(t, x)\|^p = 0$$

for every bounded or compact set $K \subset L^p(\mathbb{P}, \mathbb{H})$, $x \in K$, and $t \in \mathbb{R}$ satisfying $|t - t_i| > \varepsilon$ for any $\varepsilon > 0$, $i \in \mathbb{Z}$. Denote by $AP_T(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}), L^p(\mathbb{P}, \mathbb{H}))$ the set of all such processes.

Let U_d denote the collection of functions (weights) $\rho\colon \mathbb{Z} \to (0, +\infty)$. For $\rho \in U_d$, and $m \in Z^+ = \{n \in \mathbb{Z}, n \geq 0\}$, set $\mu(m, \rho) := \sum_{k=-m}^{m} \rho_k$, $\rho_k \in U_d$. Denote $U_{d,\infty} := \{\rho \in U_d : \lim_{m\to\infty} \mu(m, \rho) = \infty\}$. For $\rho \in U_{d,\infty}$, define $l^\infty(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H})) = \big\{x\colon \mathbb{Z} \to L^p(\mathbb{P}, \mathbb{H}) : \|x\| = \sup_{n\in\mathbb{Z}} (E\|x(n)\|^p)^{1/p} < \infty\big\}$, and

$$WPAP_0(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}), \rho) = \left\{x \in l^\infty(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H})) : \lim_{m\to\infty} \frac{1}{\mu(m, \rho)} \sum_{k=-m}^{m} E\|x_k\|^p \rho_k = 0\right\}.$$

Definition 2.8. (Cf. [27]) Let $\rho \in U_{d,\infty}$. A sequence $\{x_n\}_{n\in Z} \in l^\infty(\mathbb{Z}, X)$ is called p-mean weighted pseudo almost periodic if $x_n = x_n^1 + x_n^2$, where $x_n^1 \in AP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}))$, $x_n^2 \in WPAP_0(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}), \rho)$. Denote by $WPAP(\mathbb{Z}, L^p(\mathbb{P}, \mathbb{H}), \rho)$ the set of such sequences.

Let U be the set of all functions $\rho\colon \mathbb{R} \to (0, \infty)$ which are positive and locally integrable over $\mathbb{R}$. For a given $r > 0$ and each $\rho \in U$, set $\mu(r, \rho) := \int_{-r}^{r} \rho(t)\, dt$. Define $U_\infty := \{\rho \in U : \lim_{r\to\infty} \mu(r, \rho) = \infty\}$, $U_B := \{\rho \in U_\infty : \rho$ is bounded and $\inf_{x\in\mathbb{R}} \rho(x) > 0\}$. It is clear that $U_B \subset U_\infty \subset U$.

Definition 2.9. [11] Let $\rho_1, \rho_2 \in U_\infty$, ρ_1 is said to be equivalent to ρ_2 (i.e., $\rho_1 \sim \rho_2$) if $\frac{\rho_1}{\rho_2} \in U_B$.

It is trivial to show that "$\sim$" is a binary equivalence relation on U_∞. The equivalence class of a given weight $\rho \in U_\infty$ is denoted by $cl(\rho) = \{\vartheta \in U_\infty : \rho \sim \vartheta\}$. It is clear that $U_\infty = \bigcup_{\rho \in U_\infty} cl(\rho)$.

For $\rho \in U_\infty$, we need to introduce the new space of functions defined by

$$PC_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) = \left\{ f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) : \lim_{t\to\infty} E\|f(t)\|^p = 0 \right\},$$

$$WPAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), \rho) = \left\{ f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) : \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} E\|f(t)\|^p \rho(t)\, dt = 0 \right\},$$

$$WPAP_T^0(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}), L^p(\mathbb{P}, \mathbb{H}), \rho) = \Big\{ f \in PC(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}), L^p(\mathbb{P}, \mathbb{H})) : \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} E\|f(t,x)\|^p \rho(t)\, dt = 0 \text{ uniformly}$$
$$\text{with respect to } x \in K, \text{ where } K \text{ is an arbitrary compact subset of } L^p(\mathbb{P}, \mathbb{H}) \Big\}.$$

Definition 2.10. A function $f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is said to be p-mean piecewise weighted pseudo almost periodic if it can be decomposed as $f = h + \varphi$, where $h \in AP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ and $\varphi \in WPAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), \rho)$. Denote by $WPAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), \rho)$ the set of all such functions.

Let $\rho \in U_\infty$, $\tau \in \mathbb{R}$, and define ρ^τ by $\rho^\tau(t) = \rho(t+\tau)$ for $t \in \mathbb{R}$. Define $U_T = \{\rho \in U_\infty : \rho \sim \rho^\tau \text{ for each } t \in \mathbb{R}\}$. It is easy to see that U_T contains many of weights. $WPAP_T(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), \rho)$, $\rho \in U_T$, is a Banach space with the sup norm $\|\cdot\|_\infty$.

Similar to [27], one has

Remark 2.11. (i) For $\rho \in U_T$, $WPAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), \rho)$ is a translation invariant set of $PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$.

(ii) $PC_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) \subset WPAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), \rho)$.

Let us now establish a useful lemma which we will need in the section below.

Lemma 2.12. *Let $\{f_n\}_{n\in\mathbb{N}} \subset WPAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), \rho)$ be a sequence of functions. If f_n converges uniformly to f, then $f \in WPAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), \rho)$.*

The proof is similar to that of Lemma 3.6 in [27].

Definition 2.13. A stochastic process $f \in PC(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}), L^p(\mathbb{P}, \mathbb{H}))$ is said to be p-mean piecewise weighted pseudo almost periodic if it can be decomposed as $f = h + \varphi$, where $h \in AP_T(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}), L^p(\mathbb{P}, \mathbb{H}))$ and $\varphi \in WPAP_T^0(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}), L^p(\mathbb{P}, \mathbb{H}), \rho)$. Denote by $WPAP_T(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}), L^p(\mathbb{P}, \mathbb{H}), \rho)$ the set of all such processes.

Definition 2.14. [23] A stochastic process $J\colon \mathbb{R}\times V \to L^p(\mathbb{P},\mathbb{H})$ is said to be Poisson stochastically bounded if there exists a constant $M_1 > 0$ such that

$$\int_V E\|J(t)\|^p\, v(dx) \le M_1, \quad t\in\mathbb{R}.$$

Definition 2.15. [23] A stochastic process $J\colon \mathbb{R}\times V \to L^p(\mathbb{P},\mathbb{H})$ is said to be Poisson stochastically continuous provided that for any $s\in\mathbb{R}$,

$$\lim_{t\to s}\int_V E\|J(t)-J(s)\|^p\, v(dx) = 0.$$

Definition 2.16. (Cf. [29]) For $\{t_i\}_{i\in\mathbb{Z}}\in\mathbb{T}$, the stochastic process $F\in PC(\mathbb{R}\times V, L^p(\mathbb{P},\mathbb{H}))$ is said to be piecewise uniformly Poisson almost periodic if the following conditions are fulfilled:

(i) $\{t_i^j : j\in\mathbb{Z}\}$ is equipotentially almost periodic.

(ii) $F\in UPC(\mathbb{R}\times V, L^p(\mathbb{P},\mathbb{H}))$ for each $y\in V$.

(iii) For every sequence of real numbers $\{s_n'\}$ and every bounded or compact set $K\subset L^p(\mathbb{P},\mathbb{H}))$, there exists a subsequence $\{s_n\}$, for some stochastic process $\widetilde{F}\colon \mathbb{R}\times V\to L^p(\mathbb{P},\mathbb{H}))$ with $\int_V E\|\widetilde{F}(t,x)\|^p\, v(dx)<\infty$ such that

$$\lim_{n\to\infty}\sup_{t\in\mathbb{R},x\in K}\int_V E\|F(t+s_n,x)-\widetilde{F}(t,x)\|^p\, v(dx) = 0.$$

Denote by $PAP_T(\mathbb{R}\times V, L^p(\mathbb{P},\mathbb{H}))$ the set of all such processes.

Definition 2.17. (Cf. [29]) A stochastic process $F\in PC(\mathbb{R}\times L^p(\mathbb{P},\mathbb{H})\times V, L^p(\mathbb{P},\mathbb{H}))$ is said to be uniformly Poisson almost periodic if F is Poisson continuous and that for every sequence of real numbers $\{s_n'\}$, there exists a subsequence $\{s_n\}$, for some stochastic process $\widetilde{F}\colon \mathbb{R}\times L^p(\mathbb{P},\mathbb{H})\times V\to L^p(\mathbb{P},\mathbb{H})$ with $\int_V E\|\widetilde{F}(t,x,y)\|^p\, v(dx)<\infty$ such that

$$\lim_{n\to\infty}\sup_{t\in\mathbb{R},x\in K}\int_V E\|F(t+s_n,x,y)-\widetilde{F}(t,x,y)\|^p\, v(dx) = 0$$

for every bounded or compact set $K\subset L^p(\mathbb{P},\mathbb{H})$, $x\in K$, and $t\in\mathbb{R}$ satisfying $|t-t_i|>\varepsilon$ for any $\varepsilon>0$, $i\in\mathbb{Z}$. Denote by $PAP_T(\mathbb{R}\times L^p(\mathbb{P},\mathbb{H})\times V, L^p(\mathbb{P},\mathbb{H}))$ the set of all such processes.

For $\rho\in U_\infty$, we need to introduce the new space of functions defined by

$$\begin{aligned}
&PWPAP_T^0(\mathbb{R}\times V, L^p(\mathbb{P},\mathbb{H}),\rho)\\
&= \left\{F\in PC(\mathbb{R}\times V, L^p(\mathbb{P},\mathbb{H})) : \lim_{r\to\infty}\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\int_V E\|F(t,y)\|^p\rho(t)\, v(dy)\, dt = 0\right\},
\end{aligned}$$

$$
\begin{aligned}
&PWPAP_T^0(\mathbb{R}\times L^p(\mathbb{P},\mathbb{H})\times V, L^p(\mathbb{P},\mathbb{H}),\rho)\\
&=\Big\{F\in PC(\mathbb{R}\times L^p(\mathbb{P},\mathbb{H})\times V, L^p(\mathbb{P},\mathbb{H})):\\
&\quad \lim_{r\to\infty}\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\int_V E\|F(t,x,y)\|^p\rho(t)\,v(dy)\,dt=0 \text{ uniformly with respect to}\\
&\quad x\in K, \text{ where } K \text{ is an arbitrary compact subset of } L^p(\mathbb{P},\mathbb{H})\Big\}.
\end{aligned}
$$

Definition 2.18. A stochastic process $F\in PC(\mathbb{R}\times L^p(\mathbb{P},\mathbb{H})\times V, L^p(\mathbb{P},\mathbb{H}))$ is said to be Poisson p-mean piecewise weighted pseudo almost periodic if it can be decomposed as $F=h+\varphi$, where $h\in PAP_T(\mathbb{R}\times L^p(\mathbb{P},\mathbb{H})\times V, L^p(\mathbb{P},\mathbb{H}))$ and $\varphi\in PWPAP_T^0(\mathbb{R}\times L^p(\mathbb{P},\mathbb{H})\times V, L^p(\mathbb{P},\mathbb{H}),\rho)$. Denote by $PWPAP_T(\mathbb{R}\times L^p(\mathbb{P},\mathbb{H})\times V, L^p(\mathbb{P},\mathbb{H}),\rho)$ the set of all such processes.

2.3. Weighted pseudo almost periodicity in distribution

Let $\mathcal{P}(\mathbb{H})$ be the space of all Borel probability measures on $\mathbb{H}$ endowed with the metric:

$$
d_{BL}(\mu,\nu):=\sup\left\{\left|\int f\,d\mu-\int f\,d\nu\right|:\|f\|_{BL}\le 1\right\},\quad \mu,\nu\in\mathcal{P}(\mathbb{H}),
$$

where f is Lipschitz continuous real-valued function on $\mathbb{H}$ with the norm

$$
\|f\|_{BL}=\max\{\|f\|_L,\|f\|_\infty\},\quad \|f\|_L=\sup_{x\ne y}\frac{|f(x)-f(y)|}{\|x-y\|},\quad \|f\|_\infty=\sup_{x\in\mathbb{H}}|f(x)|.
$$

We denote by $\mathrm{law}(x(t))$ the distribution of the random variable $x(t)$. We say that x has almost periodic in one-dimensional distribution if the mapping $t\to \mathrm{law}(x(t))$ from $\mathbb{R}$ to $(\mathcal{P}(\mathbb{H}),d_{BL})$ is almost periodic.

Definition 2.19. (Cf. [18]) For $\{t_i\}_{i\in\mathbb{Z}}\in\mathbb{T}$, the stochastic process $f\in PC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$ is said to be piecewise almost periodic in distribution if the following conditions are fulfilled:

(i) $\{t_i^j:j\in\mathbb{Z}\}$ is equipotentially almost periodic.

(ii) $f\in UPC(\mathbb{R},L^p(\mathbb{P},\mathbb{H}))$.

(iii) The law $\mu(t)$ of $f(t)$ is a $\mathcal{P}(\mathbb{H})$-valued almost periodic mapping, i.e., for every sequence of real numbers $\{s_n'\}$ there exist a subsequence $\{s_n\}$ and a $\mathcal{P}(\mathbb{H})$-valued continuous mapping $\widetilde{\mu}(t)$ such that

$$
\lim_{n\to\infty} d_{BL}(\mu(t+s_n)-\widetilde{\mu}(t))=0
$$

hold for all $t\in\mathbb{R}$ satisfying the condition $|t-t_i|>\varepsilon$ for any $\varepsilon>0$, $i\in\mathbb{Z}$.

Definition 2.20. A stochastic process $f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is said to be piecewise weighted pseudo almost periodic in distribution if it can be decomposed as $f = f_1 + f_2$, where f_1 is piecewise weighted almost periodic in distribution and $f_2 \in WPAP_T^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}), \rho)$.

Next, we introduce a useful compactness criterion on $PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$.

Let $l\colon \mathbb{R} \to \mathbb{R}^+$ be a continuous function such that $l(t) \geq 1$ for all $t \in \mathbb{R}$ and $l(t) \to \infty$ as $|t| \to \infty$. Define

$$PC_l^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) = \left\{ f \in PC(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H})) : \lim_{|t| \to \infty} \frac{E\|f(t)\|^p}{l(t)} = 0 \right\}$$

endowed with the norm $\|f\|_l = \sup_{t \in \mathbb{R}} \frac{E\|f(t)\|^p}{l(t)}$, it is a Banach space.

Lemma 2.21. *A set $B \subseteq PC_l^0(\mathbb{R}, L^p(\mathbb{P}, \mathbb{H}))$ is relatively compact if and only if it satisfies the following conditions:*

(i) $\lim_{|t| \to \infty} \frac{E\|f(t)\|^p}{l(t)} = 0$ *uniformly for $f \in B$.*

(ii) $B(t) = \{f(t) : f \in B\}$ *is relatively compact in $L^p(\mathbb{P}, \mathbb{H})$ for every $t \in \mathbb{R}$.*

(iii) *The set B is equicontinuous on each interval (t_i, t_{i+1}), $i \in Z$.*

One can refer to Lemma 4.2 in [20] for the proof of Lemma 2.21.

Lemma 2.22. [24] *Let D be a convex, bounded and closed subset of a Banach space X and let $\Psi\colon D \to D$ be a condensing map. Then, Ψ has a fixed-point in D.*

3. Existence

In this section, we investigate the existence of mean-square weighted pseudo almost periodic in distribution mild solution for system (1.1)–(1.2). Throughout the rest of the paper, we assume $\rho \in U_\infty$ is chosen so that the following assumptions hold:

$$\limsup_{s \to \infty} \frac{\rho(s+\tau)}{\rho(s)} < \infty, \quad \limsup_{r \to \infty} \frac{m(r+\tau, \rho)}{m(r, \rho)} < \infty$$

for every $\tau \in \mathbb{R}$. We begin introducing the followings concepts of mild solutions.

Definition 3.1. An $\mathcal{F}_t$-progressively measurable process $x\colon \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$, $\sigma > 0$, is called a mild solution of system (1.1)–(1.2), if for every $t \geq \sigma$, $\sigma \in \mathbb{R}$ and $\sigma \neq t_i$, $i \in \mathbb{Z}$, it satisfies the corresponding stochastic integral equation

$$
\begin{aligned}
x(t) &= T(t-\sigma)x(\sigma) + \int_{\sigma}^{t} T(t-s)g(s,x(s))\,ds + \int_{\sigma}^{t} T(t-s)f(s,x(s))\,dW(s) \\
&\quad + \int_{\sigma}^{t}\int_{|y|_V<1} T(t-s)G(s,x(s-),y)\widetilde{N}(ds,dy) \qquad (3.1)\\
&\quad + \int_{\sigma}^{t}\int_{|y|_V\geq 1} T(t-s)F(s,x(s-),y)N(ds,dy) + \sum_{\sigma<t_i<t} T(t-t_i)I_i(x(t_i)).
\end{aligned}
$$

Additionally, we make the following hypotheses:

(H1) A is the infinitesimal generator of an exponentially stable C_0 semigroup $(T(t))_{t\geq 0}$ on $L^p(\mathbb{P},\mathbb{H})$ such that for all $t\geq 0$, $\|T(t)\|\leq Me^{-\delta t}$ with $M,\delta>0$. Moreover, $T(t)$ is compact for $t>0$.

(H2) $g=g_1+g_2\in WPAP_T(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}),L^2(\mathbb{P},\mathbb{H}),\rho)$, where $g_1\in AP_T(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}), L^2(\mathbb{P},\mathbb{H}))$ and $g_2\in WPAP_T^0(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}),L^2(\mathbb{P},\mathbb{H}),\rho)$. $f=f_1+f_2\in WPAP_T(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}),L^2(\mathbb{P},L_2^0),\rho)$, where $f_1\in AP_T(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}),L^2(\mathbb{P},L_2^0))$ and $f_2\in WPAP_T^0(\mathbb{R}\times L^p(\mathbb{P},\mathbb{H}),L^2(\mathbb{P},L_2^0),\rho)$. $G=G_1+G_2, F=F_1+F_2\in PWPAP_T(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\times V,L^2(\mathbb{P},\mathbb{H}),\rho)$, where $G_1,F_1\in PAP_T(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\times V,L^2(\mathbb{P},\mathbb{H}))$ and $G_2,F_2\in PWPAP_T^0(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\times V,L^2(\mathbb{P},\mathbb{H}),\rho)$. $I_i=I_{i,1}+I_{i,2}\in WPAP(\mathbb{Z},L^2(\mathbb{P},\mathbb{H}),\rho)$, where $I_{i,1}\in AP(\mathbb{Z},L^2(\mathbb{P},\mathbb{H}))$ and $I_{i,2}\in WPAP_0(\mathbb{Z},L^2(\mathbb{P},\mathbb{H}),\rho)$.

(H3) The functions $g\colon \mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\to L^2(\mathbb{P},\mathbb{H})$, $f\colon\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\to L^2(\mathbb{P},L_2^0)$ are continuous with respect to x. For any $\varsigma>0$, there exist a constant $\varrho>0$ and nondecreasing continuous function $\Theta_1\colon[0,\infty)\to[0,\infty)$ such that, for all $t\in\mathbb{R}$, and $x\in L^2(\mathbb{P},\mathbb{H})$ with $E\|x\|^2>\varrho$,

$$\sup_{t\in\mathbb{R}}\left[E\|g(t,x)\|^2+E\|f(t,x)\|_{L_2^0}^2\right]\leq\varsigma\Theta_1(\|x\|^2),\quad (t,x)\in\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}).$$

(H4) The functions $G,F\colon\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\times V\to L^2(\mathbb{P},\mathbb{H})$ are continuous in y uniformly with respect to x. For any $\varsigma>0$, there exist a constant $\varrho>0$ and nondecreasing continuous function $\Theta_2\colon[0,\infty)\to[0,\infty)$ such that, for all $t\in\mathbb{R}$, and $x\in L^2(\mathbb{P},\mathbb{H})$ with $E\|x\|^2>\varrho$,

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}}\left[\int_{|y|_V<1}E\|G(t,x,y)\|^2\,v(dy)+\int_{|y|_V>1}E\|F(t,x,y)\|^2\,v(dy)\right]\\
&\leq\varsigma\Theta_2(\|x\|^2),\quad (t,x,y)\in\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\times V.
\end{aligned}
$$

(H5) The functions $I_i\colon L^2(\mathbb{P},\mathbb{H})\to L^2(\mathbb{P},\mathbb{H})$ are continuous with respect to x. For any $\varsigma>0$, there exist a constant $\varrho>0$ and nondecreasing continuous functions $\widetilde{\Theta}_i\colon[0,\infty)\to(0,\infty)$, $i\in\mathbb{Z}$, such that, for all $t\in\mathbb{R}$, and $x\in L^2(\mathbb{P},\mathbb{H})$ with $E\|x\|^2>\varrho$,

$$E\|I_i(x)\|^2\leq\varsigma\widetilde{\Theta}_i(E\|x\|^2),\quad x\in L^2(\mathbb{P},\mathbb{H}).$$

(H6) The functions $g_1(t,\cdot)$, $f_1(t,\cdot)$ are uniformly continuous in each bounded subset of $L^2(\mathbb{P},\mathbb{H})$ uniformly in $t\in\mathbb{R}$. $G_1(t,\cdot,y)$, $F_1(t,\cdot,y)$ are uniformly Poisson stochastically continuous in each bounded subset of $L^2(\mathbb{P},\mathbb{H})$ uniformly in $t\in\mathbb{R}$ and $y\in V$. $I_{i,1}(\cdot)$ are uniformly continuous in $x\in L^2(\mathbb{P},\mathbb{H})$ uniformly in $i\in\mathbb{Z}$.

Theorem 3.2. *Assume that assumptions* (H1)–(H6) *are satisfied. Then system* (1.1)–(1.2) *has a mean square weighted pseudo almost periodic in distribution mild solution on* $\mathbb{R}$.

Proof. Let $\mathbb{Y}=UPC(\mathbb{R},L^2(\mathbb{P},\mathbb{H}))$. Consider the operator $\Psi\colon\mathbb{Y}\to PC(\mathbb{R},L^2(\mathbb{P},\mathbb{H}))$ defined by

$$\begin{aligned}(\Psi x)(t)&=\int_{-\infty}^{t}T(t-s)g(s,x(s))\,ds+\int_{-\infty}^{t}T(t-s)f(s,x(s))\,dW(s)\\&\quad+\int_{-\infty}^{t}\int_{|y|_V<1}T(t-s)G(s,x(s-),y)\widetilde{N}(ds,dy)\\&\quad+\int_{-\infty}^{t}\int_{|y|_V\geq 1}T(t-s)F(s,x(s-),y)N(ds,dy)+\sum_{t_i<t}T(t-t_i)I_i(x(t_i)),\quad t\in\mathbb{R}\end{aligned}$$

is well defined and satisfies (3.1). To prove which we shall employ Lemma 2.22, we divide the proof into several steps.

Step 1. $\Psi x\in\mathbb{Y}$. Let $t',t''\in(t_i,t_{i+1})$, $i\in\mathbb{Z}$, $t''<t'$. By (H1), for any $\varepsilon>0$, there exists $0<\xi<\min\big\{\frac{\varepsilon}{10\overline{g}},\frac{\varepsilon}{10\overline{f}},\frac{\varepsilon}{10\overline{G}},\frac{\varepsilon}{10(1+\frac{\delta}{b})\overline{F}}\big\}$ such that $0<t'-t''<\xi$ and

$$\|T(t'-t'')-I\|^2\leq\min\left\{\frac{\overline{\delta}_1\varepsilon}{10\overline{g}},\frac{\overline{\delta}_2\varepsilon}{10\overline{f}},\frac{\overline{\delta}_3\varepsilon}{10\overline{G}},\frac{\overline{\delta}_4\varepsilon}{10\overline{F}},\frac{(1-e^{-\delta\alpha})^2\varepsilon}{\overline{\gamma}}\right\},$$

where $\overline{g}=10M^2\|g\|_\infty^2$, $\overline{\delta}_1=\delta^2$, $\overline{f}=10M^2\|f\|_\infty^2$, $\overline{\delta}_2=2\delta$, $\overline{\delta}_1=\delta^2$, $\overline{G}=10M^2\sup_{s\in\mathbb{R}}\int_{|y|_V<1}E\|G(s,x(s-),y)\|^2\,v(dy)$, $\overline{\delta}_3=2\delta$, $\overline{F}=10M^2\sup_{s\in\mathbb{R}}\int_{|y|_V>1}E\|F(s,x(s-),y)\|^2\,v(dy)$, $\overline{\delta}_4=2\delta+\frac{\delta^2}{b}$, $\overline{\gamma}=10M^2\sup_{i\in\mathbb{Z}}\|I_i\|_\infty^2$. Then,

$$\begin{aligned}&E\|(\Psi x)(t')-(\Psi x)(t'')\|^2\\&\leq 5E\left\|\int_{-\infty}^{t'}T(t'-s)g(s,x(s))\,ds-\int_{-\infty}^{t''}T(t''-s)g(s,x(s))\,ds\right\|^2\\&\quad+5E\left\|\int_{-\infty}^{t'}T(t'-s)f(s,x(s))\,dW(s)-\int_{-\infty}^{t''}T(t''-s)f(s,x(s))\,dW(s)\right\|^2\\&\quad+5E\left\|\int_{-\infty}^{t'}\int_{|y|_V<1}T(t'-s)G(s,x(s-),y)\widetilde{N}(ds,dy)\right.\\&\quad\left.-\int_{-\infty}^{t''}\int_{|y|_V<1}T(t''-s)G(s,x(s-),y)\widetilde{N}(ds,dy)\right\|^2\end{aligned}$$

$$
\begin{aligned}
&+ 5E\bigg\| \int_{-\infty}^{t'} \int_{|y|_V<1} T(t'-s)F(s,x(s-),y)N(ds,dy) \\
&- \int_{-\infty}^{t''} \int_{|y|_V<1} T(t''-s)F(s,x(s-),y)N(ds,dy)\bigg\|^2 \\
&+ 5E\bigg\| \sum_{t_i<t'} T(t'-t_i)I_i(x(t_i)) - \sum_{t_i<t''} T(t''-t_i)I_i(x(t_i))\bigg\|^2 \\
=:& \sum_{j=1}^{5} \Xi_j.
\end{aligned}
$$

By (H2) and Hölder's inequality, we have

$$
\begin{aligned}
\Xi_1 &\le 10E\bigg\| \int_{-\infty}^{t''} T(t''-s)[T(t'-t'')-I]g(s,x(s))\,ds\bigg\|^2 \\
&\quad + 10E\bigg\| \int_{t''}^{t'} T(t'-s)g(s,x(s))\,ds\bigg\|^2 \\
&\le 10M^2\|T(t'-t'')-I\|^2\bigg(\int_{-\infty}^{t''} e^{-\delta(t''-s)}\,ds\bigg)\bigg(\int_{-\infty}^{t''} e^{-\delta(t''-s)}E\|g(s,x(s))\|^2\,ds\bigg) \\
&\quad + 10M^2\bigg(\int_{t''}^{t'} e^{-\delta(t'-s)}\,ds\bigg)\bigg(\int_{t'}^{t'} e^{-\delta(t'-s)}E\|g(s,x(s))\|^2\,ds\bigg) \\
&\le 10M^2\|T(t'-t'')-I\|^2\frac{1}{\delta^2}\|g\|_\infty^2 + 10M^2(t'-t'')^2\|g\|_\infty^2 < \frac{\varepsilon}{5}.
\end{aligned}
$$

By (H2) and the Itô integral [17], we have

$$
\begin{aligned}
\Xi_2 &\le 10E\bigg\| \int_{-\infty}^{t''} T(t''-s)[T(t'-t'')-I]f(s,x(s))\,dW(s)\bigg\|^2 \\
&\quad + 10E\bigg\| \int_{t''}^{t'} T(t'-s)f(s,x(s))\,dW(s)\bigg\|^2 \\
&\le 10M^2\|T(t'-t'')-I\|^2\bigg(\int_{-\infty}^{t''} e^{-2\delta(t''-s)}\,ds\bigg)\|f\|_\infty^2 \\
&\quad + 10M^2\bigg(\int_{t''}^{t'} e^{-2\delta(t'-s)}\,ds\bigg)\|f\|_\infty^2 \\
&\le 10M^2\|T(t'-t'')-I\|^2\|f\|_\infty^2\frac{1}{2\delta} + 10M^2\|f\|_\infty^2(t'-t'') < \frac{\varepsilon}{5}.
\end{aligned}
$$

By (H2) and the properties of the integral for the Poisson random measure, we have

$$
\begin{aligned}
\Xi_3 &\le 10E\bigg\| \int_{-\infty}^{t''} \int_{|y|_V<1} T(t''-s)[T(t'-t'')-I]G(s,x(s-),y)\widetilde{N}(ds,dy)\bigg\|^2 \\
&\quad + 10E\bigg\| \int_{t''}^{t'} \int_{|y|_V<1} T(t'-s)G(s,x(s-),y)\widetilde{N}(ds,dy)\bigg\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq 10M^2 \int_{-\infty}^{t''} \int_{|y|_V<1} e^{-2\delta(t''-s)} \|T(t'-t'') - I\|^2 E\|G(s, x(s-), y)\|^2 \, v(dy)\, ds \\
&\quad + 10M^2 \int_{t''}^{t'} \int_{|y|_V<1} e^{-2\delta(t'-s)} E\|G(s, x(s-), y)\|^2 \, v(dy)\, ds \\
&\leq 10M^2 \|T(t'-t'') - I\|^2 \frac{1}{2\delta} \sup_{s\in\mathbb{R}} \int_{|y|_V<1} E\|G(s, x(s-), y)\|^2 \, v(dy) \\
&\quad + 10M^2 \sup_{s\in\mathbb{R}} \int_{|y|_V<1} E\|G(s, x(s-), y)\|^2 \, v(dy)(t'-t'') < \frac{\varepsilon}{5},
\end{aligned}
$$

similarly, we have

$$
\begin{aligned}
\Xi_4 &\leq 10E \left\| \int_{-\infty}^{t''} \int_{|y|_V<1} T(t''-s)[T(t'-t'') - I] F(s, x(s-), y) N(ds, dy) \right\|^2 \\
&\quad + 10E \left\| \int_{t''}^{t'} \int_{|y|_V<1} T(t'-s) F(s, x(s-), y) N(ds, dy) \right\|^2 \\
&\leq 20E \left\| \int_{-\infty}^{t''} \int_{|y|_V<1} T(t''-s)[T(t'-t'') - I] F(s, x(s-), y) \widetilde{N}(ds, dy) \right\|^2 \\
&\quad + 20E \left\| \int_{-\infty}^{t''} \int_{|y|_V<1} T(t''-s)[T(t'-t'') - I] F(s, x(s-), y) \, v(dy)\, ds \right\|^2 \\
&\quad + 20E \left\| \int_{t''}^{t'} \int_{|y|_V<1} T(t'-s) F(s, x(s-), y) \widetilde{N}(ds, dy) \right\|^2 \\
&\quad + 20E \left\| \int_{t''}^{t'} \int_{|y|_V<1} T(t'-s) F(s, x(s-), y) \, v(dy)\, ds \right\|^2 \\
&\leq 20M^2 \int_{-\infty}^{t''} \int_{|y|_V<1} e^{-2\delta(t''-s)} \|T(t'-t'') - I\|^2 E\|F(s, x(s-), y)\|^2 \, v(dy)\, ds \\
&\quad + 20M^2 b \int_{-\infty}^{t''} e^{-\delta(t''-s)} \, ds \\
&\qquad \times \int_{-\infty}^{t''} \int_{|y|_V<1} e^{-\delta(t''-s)} \|T(t'-t'') - I\|^2 E\|F(s, x(s-), y)\|^2 \, v(dy)\, ds \\
&\quad + 20M^2 \int_{t''}^{t'} \int_{|y|_V<1} e^{-2\delta(t'-s)} E\|F(s, x(s-), y)\|^2 \, v(dy)\, ds \\
&\quad + 20M^2 b \int_{t''}^{t'} e^{-\delta(t'-s)} \, ds \int_{t''}^{t'} \int_{|y|_V<1} e^{-\delta(t'-s)} E\|F(s, x(s-), y)\|^2 \, v(dy)\, ds \\
&\leq 20M^2 \|T(t'-t'') - I\|^2 \frac{1}{2\delta} \sup_{s\in\mathbb{R}} \int_{|y|_V<1} E\|F(s, x(s-), y)\|^2 \, v(dy) \\
&\quad + 20M^2 \|T(t'-t'') - I\|^2 \frac{b}{\delta^2} \sup_{s\in\mathbb{R}} \int_{|y|_V<1} E\|F(s, x(s-), y)\|^2 \, v(dy) \\
&\quad + 20M^2 \sup_{s\in\mathbb{R}} \int_{|y|_V<1} E\|F(s, x(s-), y)\|^2 \, v(dy)(t'-t'')
\end{aligned}
$$

$$+ 20M^2\frac{b}{\delta}\sup_{s\in\mathbb{R}}\int_{|y|_V<1} E\|F(s,x(s-),y)\|^2\,v(dy)(t'-t'') < \frac{\varepsilon}{5}.$$

By (H2) and Hölder's inequality again, we have

$$\begin{aligned}
\Xi_5 &= 5E\bigg\|\sum_{t_i<t''} T(t''-t_i)[T(t'-t'')-I]I_i(x(t_i))\bigg\|^2\\
&\le 5M^2\|T(t'-t'')-I\|^2\bigg(\sum_{t_i<t''} e^{-\delta(t''-t_i)}\bigg)\bigg(\sum_{t_i<t''} e^{-\delta(t''-t_i)}E\|I_i(x(t_i))\|^2\bigg)\\
&\le 5M^2\|T(t'-t'')-I\|^2\bigg(\sum_{t_i<t''} e^{-\delta(t''-t_i)}\bigg)^2\sup_{i\in\mathbb{Z}}\|I_i\|_\infty^2 < \frac{\varepsilon}{5}.
\end{aligned}$$

By the above discussion, one has

$$E\|(\Psi x)(t')-(\Psi x)(t'')\|^2 < \varepsilon.$$

Consequently, $\Psi x \in \mathbb{Y}$.

Step 2. *Ψ has a fixed point in $\mathbb{Y}$.*

(1) There exists $r^*>0$ such that $\Psi(B_{r^*})\subset B_{r^*}$, where $B_{r^*}=\{x\in\mathbb{Y}: E\|x\|^p\le r^*\}$.

For each $r^*>0$, B_{r^*} is clearly a bounded closed convex subset in $\mathbb{Y}$. We claim that there exists $r^*>0$ such that $\Psi(B_{r^*})\subset B_{r^*}$. In fact, if this is not true, then for each $r^*>0$ there exists $x^{r^*}\in B_{r^*}$ and $t^{r^*}\in R$ such that $r^*<E\|(\Psi x^{r^*})(t^{r^*})\|^2$. On the other hand, let $\varsigma>0$ be fixed. By (H3)–(H5), it follows that there exist a positive constant ϱ such that, for all $t\in\mathbb{R}$ and $x\in L^2(\mathbb{P},\mathbb{H})$, $y\in V$ with $\|x\|^2>\varrho$, $i\in\mathbb{Z}$,

$$E\|g(t,x)\|^2+E\|f(t,\psi)\|^2_{L_2^0}\le\varsigma\Theta_1(\|x\|^2),$$

$$\int_{|y|_V<1}E\|G(t,x,y)\|^2\,v(dy)+\int_{|y|_V>1}E\|F(t,x,y)\|^2\,v(dy)\le\varsigma\Theta_2(\|x\|^2),$$

$$E\|I_i(x)\|^2\le\varsigma\widetilde{\Theta}_i(\|x\|^2).$$

Let

$$\begin{aligned}
\nu_1 &= \sup_{t\in\mathbb{R}}\Big\{E\|g(t,x)\|^2+E\|f(t,x)\|^2_{L_2^0}: E\|x\|^2\le\varrho\Big\},\\
\nu_2 &= \sup_{t\in\mathbb{R}}\bigg\{\int_{|y|_V<1}E\|G(t,x,y)\|^2\,v(dy)+\int_{|y|_V>1}E\|F(t,x,y)\|^2\,v(dy): E\|x\|^2\le\varrho\bigg\},\\
\nu_3 &= \sup_{t\in\mathbb{R},i\in Z}\big\{E\|I_i(x)\|^2: E\|x\|^2\le\varrho\big\}.
\end{aligned}$$

Thus, for all $t\in\mathbb{R}$ and $x\in L^2(\mathbb{P},\mathbb{H})$, $y\in V$, we have

$$E\|g(t,x)\|^2+E\|f(t,x)\|^2_{L_2^0}\le\varsigma\Theta_1(E\|x\|^2)+\nu_1, \tag{3.2}$$

$$\int_{|y|_V<1}E\|G(t,x,y)\|^2\,v(dy)+\int_{|y|_V>1}E\|F(t,x,y)\|^2\,v(dy)\le\varsigma\Theta_2(E\|x\|^2)+\nu_2, \tag{3.3}$$

$$E\|I_i(x)\|^2\le\varsigma\widetilde{\Theta}_i(E\|x\|^2)+\nu_3,\quad i\in\mathbb{Z}. \tag{3.4}$$

Note that, for ς sufficiently small, we have

$$1 \le 3M^2\varsigma\Bigg[\left(\frac{1}{\delta^2}+\frac{1}{2\delta}\right)\liminf_{r^*\to\infty}\frac{\Theta_1(r^*)}{r^*}+\left(\frac{3}{2\delta}+\frac{2b}{\delta^2}\right)\liminf_{r^*\to\infty}\frac{\Theta_2(r^*)}{r^*} + \frac{1}{(1-e^{-\delta\alpha})^2}\sup_{i\in Z}\liminf_{r^*\to\infty}\frac{\widetilde{\Theta}_i(r^*)}{r^*}\Bigg]. \tag{3.5}$$

Then, by using (3.2)–(3.4), Hölder's inequality, the Itô integral and the properties of the integral for the Poisson random measure, we have

$$\begin{aligned}
&E\|(\Psi_1 x^{r^*})(t^{r^*})\|^2 \\
&\le 5E\left\|\int_{-\infty}^{t^{r^*}} T(t^{r^*}-s)g(s,x^{r^*}(s))\,ds\right\|^2 \\
&\quad + 5E\left\|\int_{-\infty}^{t^{r^*}} T(t^{r^*}-s)f(s,x^{r^*}(s))\,dW(s)\right\|^2 \\
&\quad + 5E\left\|\int_{-\infty}^{t^{r^*}}\int_{|y|_V<1} T(t^{r^*}-s)G(s,x^{r^*}(s-),y)\widetilde{N}(ds,dy)\right\|^2 \\
&\quad + 5E\left\|\int_{-\infty}^{t^{r^*}}\int_{|y|_V>1} T(t^{r^*}-s)F(s,x^{r^*}(s-),y)N(ds,dy)\right\|^2 \\
&\quad + 5E\left\|\sum_{t_i<t^{r^*}} T(t^{r^*}-t_i)I_i(x^{r^*}(t_i))\right\|^2 \\
&\le 5M^2\left(\int_{-\infty}^{t^{r^*}} e^{-\delta(t^{r^*}-s)}\,ds\right)\left(\int_{-\infty}^{t^{r^*}} e^{-\delta(t^{r^*}-s)}E\|g(s,x^{r^*}(s))\|^2\,ds\right) \\
&\quad + 5M^2\int_{-\infty}^{t^{r^*}} e^{-2\delta(t^{r^*}-s)}E\|f(s,x^{r^*}(s))\|^2_{L_2^0}\,ds \\
&\quad + 5M^2\int_{-\infty}^{t^{r^*}}\int_{|y|_V<1} e^{-2\delta(t^{r^*}-s)}E\|G(s,x^{r^*}(s-),y)\|^2\,v(dy)\,ds \\
&\quad + 10M^2\int_{-\infty}^{t^{r^*}}\int_{|y|_V>1} e^{-2\delta(t^{r^*}-s)}E\|F(s,x^{r^*}(s-),y)\|^2\,v(dy)\,ds \\
&\quad + 10M^2 b\int_{-\infty}^{t^{r^*}} e^{-\delta(t^{r^*}-s)}\,ds\int_{-\infty}^{t^{r^*}}\int_{|y|_V>1} e^{-\delta(t^{r^*}-s)}E\|F(s,x^{r^*}(s-),y)\|^2\,v(dy)\,ds \\
&\quad + 5M^2\left(\sum_{t_i<t^{r^*}} e^{-\delta(t^{r^*}-t_i)}\right)\left(\sum_{t_i<t^{r^*}} e^{-\delta(t^{r^*}-t_i)}E\|I_i(x^{r^*}(t_i))\|^2\right) \\
&\le 5M^2\frac{1}{\delta}\int_{-\infty}^{t^{r^*}} e^{-\delta(t^{r^*}-s)}[\varsigma\Theta_1(E\|x^{r^*}(s)\|^2)+\nu_1]\,ds \\
&\quad + 5M^2\int_{-\infty}^{t^{r^*}} e^{-2\delta(t^{r^*}-s)}[\varsigma\Theta_1(E\|x^{r^*}(s)\|^2)+\nu_1]\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ 15M^2 \int_{-\infty}^{t^{r^*}} e^{-2\delta(t^{r^*}-s)}[\varsigma\Theta_2(E\|x^{r^*}(s)\|^2) + \nu_2]\, ds \\
&+ 10M^2 \frac{b}{\delta} \int_{-\infty}^{t^{r^*}} e^{-\delta(t^{r^*}-s)}[\varsigma\Theta_2(E\|x^{r^*}(s)\|^2) + \nu_2]\, ds \\
&+ 5M^2 \frac{1}{1-e^{-\delta\alpha}} \left(\sum_{t_i < t^{r^*}} e^{-\delta(t^{r^*}-t_i)}[\varsigma\widetilde{\Theta}_i(E\|x^{r^*}(t_i)\|^2) + \nu_3] \right) \\
\le{}& 5M^2 \left[\left(\frac{1}{\delta^2} + \frac{1}{2\delta} \right) [\varsigma\Theta_1(r^*) + \nu_1] + \left(\frac{3}{2\delta} + \frac{2b}{\delta^2} \right) [\varsigma\Theta_2(r^*) + \nu_2] \right. \\
&\left. + \frac{1}{(1-e^{-\delta\alpha})^2} \left[\varsigma \sup_{i\in\mathbb{Z}} \widetilde{\Theta}_i(r^*) + \nu_3 \right] \right].
\end{aligned}
$$

Then we have for all $t \in \mathbb{R}$,

$$
\begin{aligned}
r^* &< E\|(\Psi x^{r^*})(t^{r^*})\|^2 \\
&\le 5M^2 \varsigma \left[\left(\frac{1}{\delta^2} + \frac{1}{2\delta} \right) [\varsigma\Theta_1(r^*) + \nu_1] + \left(\frac{3}{2\delta} + \frac{2b}{\delta^2} \right) [\varsigma\Theta_2(r^*) + \nu_2] \right. \\
&\qquad \left. + \frac{1}{(1-e^{-\delta\alpha})^2} \left[\sup_{i\in\mathbb{Z}} \widetilde{\Theta}_i(r^*) + \nu_3 \right] \right].
\end{aligned}
$$

Dividing both sides by r^* and taking the lower limit as $r^* \to \infty$, we obtain

$$
\begin{aligned}
1 \le 5M^2 &\left[\left(\frac{1}{\delta^2} + \frac{1}{2\delta} \right) \liminf_{r^*\to\infty} \frac{\Theta_1(r^*)}{r^*} + \left(\frac{3}{2\delta} + \frac{2b}{\delta^2} \right) \liminf_{r^*\to\infty} \frac{\Theta_2(r^*)}{r^*} \right. \\
&\left. + \frac{1}{(1-e^{-\delta\alpha})^2} \sup_{i\in\mathbb{Z}} \liminf_{r^*\to\infty} \frac{\widetilde{\Theta}_i(r^*)}{r^*} \right].
\end{aligned}
$$

This contradicts condition (3.5). Thus, for some positive number r^*, $\Psi(B_{r^*}) \subset B_{r^*}$,

(2) Ψ is continuous on B_{r^*}.

Let $\{x^{(n)}\} \subseteq B_{r^*}$ with $x^{(n)} \to x$ $(n \to \infty)$ in $\mathbb{Y}$, then there exists a bounded subset $K \subseteq L^p(\mathbb{P}, \mathbb{H})$ such that $R(x) \subseteq K$, $R(x^n) \subseteq K$, $n \in \mathbb{N}$. By the assumption (H3)–(H5), for any $\varepsilon > 0$, there exists $\xi > 0$ such that $x, \overline{x} \in K$ and $E\|x - \overline{x}\|^2 < \xi$ implies that

$$
\begin{gathered}
E\|g(s, x(s)) - g(s, \overline{x}(s))\|^2 < \varepsilon, \\
E\|f(s, x(s)) - f(s, \overline{x}(s))\|_{L_2^0}^2 < \varepsilon, \\
\int_{|y|_V<1} E\|G(s, x(s-), y) - G(s, \overline{x}(s-), y)\|^2\, v(dy) < \varepsilon, \\
\int_{|y|_V>1} E\|F(s, x(s-), y) - F(s, \overline{x}(s-), y)\|^2\, v(dy) < \varepsilon
\end{gathered}
$$

for all $t \in R$, and

$$
E\|I_i(x) - I_i(\overline{x})\|^2 < \varepsilon
$$

for all $i \in \mathbb{Z}$. For the above ξ there exists n_0 such that $E\|x^{(n)}(t) - x(t)\|^2 < \varepsilon$ for $n > n_0$ and $t \in \mathbb{R}$, then for $n > n_0$, we have

$$
\begin{gathered}
E\|g(s, x^{(n)}(s)) - g(s, x(s))\|^2 < \varepsilon, \\
E\|f(s, x^{(n)}(s)) - f(s, x(s))\|_{L_2^0}^2 < \varepsilon, \\
\int_{|y|_V<1} E\|G(s, x^{(n)}(s-), y) - G(s, x(s-), y)\|^2 \, v(dy) < \varepsilon, \\
\int_{|y|_V>1} E\|F(s, x^{(n)}(s-), y) - F(s, x(s-), y)\|^2 \, v(dy) < \varepsilon
\end{gathered}
$$

for all $t \in \mathbb{R}$, and

$$E\|I_i(x^{(n)}) - I_i(x)\|^2 < \varepsilon$$

for all $i \in \mathbb{Z}$. Then, by Hölder's inequality, the Itô integral and the properties of the integral for the Poisson random measure, we have

$$
\begin{aligned}
&E\|(\Psi x^{(n)})(t) - (\Psi x)(t)\|^2 \\
&\le 5E\left\| \int_{-\infty}^{t} T(t-s)[g(s, x^{(n)}(s)) - g(s, x(s))] \, ds \right\|^2 \\
&\quad + 5E\left\| \int_{-\infty}^{t} T(t-s)[f(s, x^{(n)}(s)) - f(s, x(s))] \, dW(s) \right\|^2 \\
&\quad + 5E\left\| \int_{-\infty}^{t} \int_{|y|_V<1} T(t-s)[G(s, x^{(n)}(s-), y) - G(s, x(s-), y)] \widetilde{N}(ds, dy) \right\|^2 \\
&\quad + 5E\left\| \int_{-\infty}^{t} \int_{|y|_V>1} T(t-s)[F(s, x^{(n)}(s-), y) - F(s, x(s-), y)] N(ds, dy) \right\|^2 \\
&\quad + 5E\left\| \sum_{t_i<t} T(t-t_i)[I_i(x^{(n)}(t_i)) - I_i(x(t_i))] \right\|^2 \\
&\le 5M^2 \left(\int_{-\infty}^{t} e^{-\delta(t-s)} \, ds \right) \left(\int_{-\infty}^{t} e^{-\delta(t-s)} E\|g(s, x^{(n)}(s)) - g(s, x(s))\|^2 \, ds \right) \\
&\quad + 5M^2 \int_{-\infty}^{t} e^{-2\delta(t-s)} E\|f(s, x^{(n)}(s)) - f(s, x(s))\|_{L_2^0}^2 \, ds \\
&\quad + 5M^2 \int_{-\infty}^{t} \int_{|y|_V<1} e^{-2\delta(t-s)} E\|G(s, x^{(n)}(s-), y) - G(s, x(s-), y)\|^2 \, v(dy) \, ds \\
&\quad + 10M^2 \int_{-\infty}^{t} e^{-2\delta(t-s)} \int_{|y|_V>1} E\|F(s, x^{(n)}(s-), y) - F(s, x(s-), y)\|^2 \, v(dy) \, ds \\
&\quad + 10M^2 \frac{b}{\delta} \int_{-\infty}^{t} \int_{|y|_V>1} e^{-\delta(t-s)} E\|F(s, x^{(n)}(s-), y) - F(s, x(s-), y)\|^2 \, v(dy) \, ds \\
&\quad + 5M^2 \left(\sum_{t_i<t} e^{-\delta(t-t_i)} \right) \left(\sum_{t_i<t} e^{-\delta(t-t_i)} E\|I_i(x^{(n)}(t_i)) - I_i(x(t_i))\|^2 \right)
\end{aligned}
$$

$$
\begin{aligned}
&\le 5M^2\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\,ds\right)^2 \varepsilon + 20M^2\left(\int_{-\infty}^{t} e^{-2\delta(t-s)}\,ds\right)\varepsilon \\
&\quad + 10M^2\frac{b}{\delta}\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\,ds\right)\varepsilon + 5M^2\frac{1}{1-e^{-\delta\alpha}}\left(\sum_{t_i<t} e^{-\delta(t-t_i)}\right)\varepsilon \\
&\le 5M^2\left[\frac{1+2b}{\delta^2} + \frac{3}{\delta} + \frac{1}{(1-e^{-\delta\alpha})^2}\right]\varepsilon.
\end{aligned}
$$

Thus Ψ is continuous on B_{r^*}.

(3) $\Psi(B_{r^*}) = \{\Psi x : x \in B_{r^*}\}$ is equicontinuous.

Let $\tau_1, \tau_2 \in (t_i, t_{i+1})$, $i \in \mathbb{Z}$, $\tau_1 < \tau_2$, and $x \in B_{r^*}$. Then, by (3.2)–(3.4), Hölder's inequality, the Itô integral and the properties of the integral for the Poisson random measure, we have

$$
\begin{aligned}
&E\|(\Psi x)(\tau_2) - (\Psi x)(\tau_1)\|^2 \\
&\le 10E\left\|\int_{-\infty}^{\tau_1} [T(\tau_2-\tau_1) - I]T(\tau_1 - s)g(s,x(s))\,ds\right\|^2 \\
&\quad + 10E\left\|\int_{\tau_1}^{\tau_2} T(\tau_2 - s)g(s,x(s))\,ds\right\|^2 \\
&\quad + 10E\left\|\int_{-\infty}^{\tau_1} [T(\tau_2-\tau_1) - I]T(\tau_1 - s)f(s,x(s))\,dW(s)\right\|^2 \\
&\quad + 10E\left\|\int_{\tau_1}^{\tau_2} T(\tau_2 - s)f(s,x(s))\,dW(s)\right\|^2 \\
&\quad + 10E\left\|\int_{-\infty}^{\tau_1}\int_{|y|_V<1} [T(\tau_2-\tau_1) - I]T(\tau_1 - s)G(s,x(s-),y)\widetilde{N}(ds,dy)\right\|^2 \\
&\quad + 10E\left\|\int_{\tau_1}^{\tau_2}\int_{|y|_V<1} T(\tau_2 - s)G(s,x(s-),y)\widetilde{N}(ds,dy)\right\|^2 \\
&\quad + 10E\left\|\int_{-\infty}^{\tau_1}\int_{|y|_V>1} [T(\tau_2-\tau_1) - I]T(\tau_1 - s)F(s,x(s-),y)N(ds,dy)\right\|^2 \\
&\quad + 10E\left\|\int_{\tau_1}^{\tau_2}\int_{|y|_V<1} T(\tau_2 - s)F(s,x(s-),y)N(ds,dy)\right\|^2 \\
&\quad + 5E\left\|\sum_{t_i<\tau_1} [T(\tau_2-\tau_1) - I]T(\tau_1 - t_i)PI_i(x(t_i))\right\|^2 \\
&\le 10M^2\|T(\tau_2-\tau_1) - I\|^2\left(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1-s)}\,ds\right)\left(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1-s)}E\|g(s,x(s))\|^2\,ds\right) \\
&\quad + 10M^2\left(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}\,ds\right)\left(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}E\|g(s,x(s))\|^2\,ds\right) \\
&\quad + 10M^2\int_{-\infty}^{\tau_1} e^{-2\delta(\tau_1-s)}\|T(\tau_2-\tau_1) - I\|^2 E\|f(s,x(s))\|^2_{L_2^0}\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ 10M^2 \int_{\tau_1}^{\tau_2} e^{-2\delta(\tau_2 - s)} E\|f(s, x(s))\|_{L_2^0}^2 \, ds \\
&+ 10M^2 \int_{-\infty}^{\tau_1} \int_{|y|_V<1} e^{-2\delta(\tau_1 - s)} \|T(\tau_2 - \tau_1) - I\|^2 E\|G(s, x(s-), y)\|^2 \, v(dy) \, ds \\
&+ 10M^2 \int_{\tau_1}^{\tau_2} \int_{|y|_V<1} e^{-2\delta(\tau_2 - s)} E\|G(s, x(s-), y)\|^2 \, v(dy) \, ds \\
&+ 20M^2 \int_{-\infty}^{\tau_1} \int_{|y|_V>1} e^{-2\delta(\tau_1 - s)} \|T(\tau_2 - \tau_1) - I\|^2 E\|F(s, x(s-), y)\|^2 \, v(dy) \, ds \\
&+ 20M^2 \int_{\tau_1}^{\tau_2} \int_{|y|_V>1} e^{-2\delta(\tau_2 - s)} E\|F(s, x(s-), y)\|^2 \, v(dy) \, ds \\
&+ 20M^2 \frac{b}{\delta} \int_{-\infty}^{\tau_1} \int_{|y|_V>1} e^{-\delta(\tau_1 - s)} \|T(\tau_2 - \tau_1) - I\|^2 E\|F(s, x(s-), y)\|^2 \, v(dy) \, ds \\
&+ 20M^2 \frac{b}{\delta} \int_{\tau_1}^{\tau_2} \int_{|y|_V>1} e^{-\delta(\tau_2 - s)} E\|F(s, x(s), y)\|^2 \, v(dy) \, ds \\
&+ 5M^2 \|T(\tau_2 - \tau_1) - I\|^2 \bigg(\sum_{t_i<\tau_1} e^{-\delta(\tau_1 - t_i)} \bigg) \bigg(\sum_{t_i<\tau_1} e^{-\delta(\tau_1 - t_i)} E\|I_i(x(t_i))\|^2 \bigg) \\
\le{}& 10M^2 \|T(\tau_2 - \tau_1) - I\|^2 \bigg(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1 - s)} \, ds \bigg) \bigg(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1 - s)} [\varsigma \Theta_1(E\|x(s)\|^2) + \nu_1] \, ds \bigg) \\
&+ 10M^2 \bigg(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2 - s)} \, ds \bigg) \bigg(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2 - s)} [\varsigma \Theta_1(E\|x(s)\|^2) + \nu_1] \, ds \bigg) \\
&+ 10M^2 \|T(\tau_2 - \tau_1) - I\|^2 \bigg(\int_{-\infty}^{\tau_1} e^{-2\delta(\tau_1 - s)} [\varsigma \Theta_1(E\|x(s)\|^2) + \nu_1] \, ds \bigg) \\
&+ 10M^2 \bigg(\int_{\tau_1}^{\tau_2} e^{-2\delta(\tau_2 - s)} [\varsigma \Theta_1(E\|x(s)\|^2) + \nu_1] \, ds \bigg) \\
&+ 30M^2 \|T(\tau_2 - \tau_1) - I\|^2 \bigg(\int_{-\infty}^{\tau_1} e^{-2\delta(\tau_1 - s)} [\varsigma \Theta_2(E\|x(s)\|^2) + \nu_2] \, ds \bigg) \\
&+ 30M^2 \bigg(\int_{\tau_1}^{\tau_2} e^{-2\delta(\tau_2 - s)} [\varsigma \Theta_2(E\|x(s)\|^2) + \nu_2] \, ds \bigg) \\
&+ 20M^2 \frac{b}{\delta} \|T(\tau_2 - \tau_1) - I\|^2 \bigg(\int_{-\infty}^{\tau_1} e^{-\delta(\tau_1 - s)} [\varsigma \Theta_2(E\|x(s)\|^2) + \nu_2] \, ds \bigg) \\
&+ 20M^2 \frac{b}{\delta} \bigg(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2 - s)} [\varsigma \Theta_2(E\|x(s)\|^2) + \nu_2] \, ds \bigg) \\
&+ 5M^2 \|T(\tau_2 - \tau_1) - I\|^p \bigg(\sum_{t_i<\tau_1} e^{-\delta(\tau_1 - t_i)} \bigg) \bigg(\sum_{t_i<\tau_1} e^{-\delta(\tau_1 - t_i)} [\varsigma \widetilde{\Theta}_i(E\|x(t_i)\|^2) + \nu_3] \bigg) \\
\le{}& 10M^2 \bigg(\frac{1}{\delta^2} + \frac{1}{2\delta} \bigg) \|T(\tau_2 - \tau_1) - I\|^2 [\varsigma \Theta_1(r^*) + \nu_1] \\
&+ 10M^2 \bigg[\bigg(\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2 - s)} \, ds \bigg)^2 + \int_{\tau_1}^{\tau_2} e^{-2\delta(\tau_2 - s)} \, ds \bigg] [\varsigma \Theta_1(r^*) + \nu_1]
\end{aligned}
$$

$$
\begin{aligned}
&+ 10M^2\|T(\tau_2-\tau_1)-I\|^2\left(\frac{3}{2\delta}+\frac{2b}{\delta^2}\right)[\varsigma\Theta_2(r^*)+\nu_2]\\
&+ 10M^2\left[3\int_{\tau_1}^{\tau_2} e^{-2\delta(\tau_2-s)}\,ds+\frac{b}{\delta}\int_{\tau_1}^{\tau_2} e^{-\delta(\tau_2-s)}\,ds\right][\varsigma\Theta_2(r^*)+\nu_2]\\
&+ 5^{p-1}M^p\|T(\tau_2-\tau_1)-I\|^2\frac{1}{(1-e^{-\delta\alpha})^2}\Big[\varsigma\sup_{i\in\mathbb{Z}}\widetilde{\Theta}_i(r^*)+\nu_3\Big].
\end{aligned}
$$

The right-hand side of the above inequality is independent of $x \in B_{r^*}$ and tends to zero as $\tau_2 \to \tau_1$, since the compactness of $T(t)$ for $t > 0$ implies imply the continuity in the uniform operator topology. Thus, Ψ maps B_{r^*} into an equicontinuous family of functions.

(4) $U(t) = \{(\Psi x)(t) : x \in B_{r^*}\}$ is relatively compact in $L^2(\mathbb{P}, \mathbb{H})$ for each $t \in \mathbb{R}$. For each $t \in \mathbb{R}$, and let ε be a real number satisfying $0 < \varepsilon < 1$. For $x \in B_{r^*}$, we define

$$
\begin{aligned}
(\Psi^\varepsilon x)(t) &= \int_{-\infty}^{t-\varepsilon} T(t-s)g(s,x(s))\,ds+\int_{-\infty}^{t-\varepsilon} T(t-s)f(s,x(s))\,dW(s)\\
&\quad+\sum_{t_i<t-\varepsilon} T(t-t_i)I_i(x(t_i))\\
&= T(\varepsilon)[(\Psi x)(t-\varepsilon)].
\end{aligned}
$$

Since $T(t)$ $(t > 0)$ is compact, then the set $U_\varepsilon(t) = \{(\Psi^\varepsilon x)(t) : x \in B_{r^*}\}$ is relatively compact in $L^2(\mathbb{P}, \mathbb{H})$ for each $t \in \mathbb{R}$. Moreover, for every $x \in B_{r^*}$, we have

$$
\begin{aligned}
&E\|(\Psi x)(t)-(\Psi^\varepsilon x)(t)\|^2\\
&\le 5E\left\|\int_{t-\varepsilon}^{t} T(t-s)g(s,x(s))\,ds\right\|^2+5E\left\|\int_{t-\varepsilon}^{t} T(t-s)f(s,x(s))\,dW(s)\right\|^2\\
&\quad+5E\left\|\int_{t-\varepsilon}^{t}\int_{|y|_V<1} T(t-s)G(s,x(s-),y)\widetilde{N}(ds,dy)\right\|^2\\
&\quad+5E\left\|\int_{t-\varepsilon}^{t}\int_{|y|_V>1} T(t-s)F(s,x(s-),y)N(ds,dy)\right\|^2\\
&\quad+5E\left\|\sum_{t-\varepsilon<t_i<t} T(t-t_i)I_i(x(t_i))\right\|^2\\
&\le 5M^2\left(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}\,ds\right)\left(\int_{t-\varepsilon}^{t} e^{-\delta(t-s)}E\|g(s,x(s))\|^2\,ds\right)\\
&\quad+5M^2\int_{t-\varepsilon}^{t} e^{-2\delta(t-s)}E\|f(s,x(s))\|^2_{L_2^0}\,ds\\
&\quad+5M^2\int_{t-\varepsilon}^{t}\int_{|y|_V<1} e^{-2\delta(t-s)}E\|G(s,x(s),y)\|^2\,v(dy)\,ds\\
&\quad+10M^2\int_{t-\varepsilon}^{t}\int_{|y|_V>1} e^{-2\delta(t-s)}E\|F(s,x(s-),y)\|^2\,v(dy)\,ds\\
&\quad+10M^2\frac{b}{\delta}\int_{t-\varepsilon}^{t}\int_{|y|_V>1} e^{-\delta(t-s)}E\|F(s,x(s),y)\|^2\,v(dy)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 5M^2\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\Big)\Big(\sum_{t-\epsilon<t_i<t} e^{-\delta(t-t_i)}\|I_i(x(t_i))\|^2\Big)\\
&\le 5M^2\Big(\int_{t-\varepsilon}^t e^{-\delta(t-s)}\,ds\Big)\Big(\int_{t-\varepsilon}^t e^{-\delta(t-s)}[\varsigma\Theta_1(E\|x(s)\|^2)+\nu_1]\,ds\Big)\\
&\quad + 5M^2\int_{t-\varepsilon}^t e^{-2\delta(t-s)}[\varsigma\Theta_1(E\|x(s)\|^2)+\nu_1]\,ds\\
&\quad + 15M^2\int_{t-\varepsilon}^t e^{-2\delta(t-s)}[\varsigma\Theta_2(E\|x(s)\|^2)+\nu_2]\,ds\\
&\quad + 10M^2\frac{b}{\delta}\int_{t-\varepsilon}^t e^{-\delta(t-s)}[\varsigma\Theta_2(E\|x(s)\|^2)+\nu_2]\,ds\\
&\quad + 5M^2\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\Big)\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}[\varsigma\widetilde{\Theta}_i(E\|x_i(t_i)\|^2)+\nu_3]\Big)\\
&\le 5M^2\Big[\Big(\int_{t-\varepsilon}^t e^{-\delta(t-s)}\,ds\Big)^2+\int_{t-\varepsilon}^t e^{-2\delta(t-s)}\,ds\Big][\varsigma\Theta_1(r^*)+\nu_1]\\
&\quad + 5M^2\Big[3\int_{t-\varepsilon}^t e^{-2\delta(t-s)}\,ds+\frac{2b}{\delta}\int_{t-\varepsilon}^t e^{-\delta(t-s)}\,ds\Big][\varsigma\Theta_2(r^*)+\nu_2]\\
&\quad + 5M^2\Big(\sum_{t-\varepsilon<t_i<t} e^{-\delta(t-t_i)}\Big)^2\Big[\varsigma\sup_{i\in\mathbb{Z}}\widetilde{\Theta}_i(r^*)+\nu_3\Big].
\end{aligned}
$$

Therefore, letting $\varepsilon\to 0$, it follows that there are relatively compact sets $U_\varepsilon(t)$ arbitrarily close to $U(t)$ and hence $U(t)$ is also relatively compact in $L^2(\mathbb{P},\mathbb{H})$ for each $t\in\mathbb{R}$. Since $\{\Psi x : x\in B_{r^*}\}\subset PC_l^0(\mathbb{R},L^2(\mathbb{P},\mathbb{H}))$, then $\{\Psi x : x\in B_{r^*}\}$ is a relatively compact set by Lemma 2.21, then Ψ is a compact operator.

Therefore, Ψ is a condensing map from B_{r^*} into B_{r^*}. Consequently, by Lemma 2.22, we deduce that Ψ has a fixed point $x\in B_{r^*}$, which is in turn a mild solution of the system (1.1)–(1.2).

Step 3. *Weighted pseudo almost periodic in distribution of mild solution.* For given $x\in WPAP_T(\mathbb{R},L^2(\mathbb{P},\mathbb{H}),\rho)$, by the definition of the mapping Ψ, we have

$$(\Psi x)(t)=(\Phi x)(t)+(\Upsilon x)(t),$$

where

$$
\begin{aligned}
(\Phi x)(t) &= \int_{-\infty}^t T(t-s)g_1(s,x(s))\,ds+\int_{-\infty}^t T(t-s)f_1(s,x(s))\,dW(s)\\
&\quad+\int_{-\infty}^t\int_{|y|_V<1} T(t-s)G_1(s,x(s-),y)\widetilde{N}(ds,dy)\\
&\quad+\int_{-\infty}^t\int_{|y|_V>1} T(t-s)F_1(s,x(s-),y)N(ds,dy)+\sum_{t_i<t}T(t-t_i)I_{i,1}(x(t_i))
\end{aligned}
$$

and

$$(\Upsilon x)(t) = \int_{-\infty}^{t} T(t-s)g_2(s,x(s))\,ds + \int_{-\infty}^{t} T(t-s)f_2(s,x(s))\,dW(s)$$
$$+ \int_{-\infty}^{t}\int_{|y|_V<1} T(t-s)G_2(s,x(s-),y)\widetilde{N}(ds,dy)$$
$$+ \int_{-\infty}^{t}\int_{|y|_V>1} T(t-s)F_2(s,x(s-),y)N(ds,dy) + \sum_{t_i<t} T(t-t_i)I_{i,2}(x(t_i)).$$

(1) Φx is almost periodic in distribution.

Let $t_i < t \le t_{i+1}$. For $\varepsilon > 0$ and $0 < \eta < \min\{\varepsilon, \alpha/2\}$, since $g_1 \in AP_T(\mathbb{R} \times L^2(\mathbb{R},\mathbb{H}), L^2(\mathbb{P},\mathbb{H}))$, $f_1 \in AP_T(\mathbb{R} \times L^2(\mathbb{P},\mathbb{H}), L^2(\mathbb{P},L_2^0))$, $G_1, F_1 \in PAP_T(\mathbb{R} \times L^2(\mathbb{P} \times \mathbb{H}) \times V, L^2(\mathbb{P},\mathbb{H}))$, thus for every sequence of real numbers $\{s'_n\}$, there exist a subsequence $\{s_n\}$ and a stochastic processes $\widetilde{g}_1 \in AP_T(\mathbb{R} \times L^2(\mathbb{P},\mathbb{H}), L^2(\mathbb{P},\mathbb{H}))$, $\widetilde{f}_1 \in AP_T(\mathbb{R} \times L^2(\mathbb{P},\mathbb{H}), L^2(\mathbb{P},L_2^0))$, $\widetilde{G}_1, \widetilde{F}_1 \in PAP_T(\mathbb{R} \times L^2(\mathbb{P},\mathbb{H}) \times V, L^2(\mathbb{P},\mathbb{H}))$, such that

$$\lim_{n\to\infty} E\|g_1(t+s_n,x) - \widetilde{g}_1(t,x)\|^2 = 0,$$
$$\lim_{n\to\infty} E\|f_1(t+s_n,x) - \widetilde{f}_1(t,x)\|^2_{L_2^0} = 0,$$
$$\lim_{n\to\infty}\int_{|y|_V<1} E\|G_1(t+s_n,x,y) - \widetilde{G}_1(t,x,y)\|^2\,v(dy) = 0, \tag{3.6}$$
$$\lim_{n\to\infty}\int_{|y|_V>1} E\|F_1(t+s_n,x,y) - \widetilde{F}_1(t,x,y)\|^2\,v(dy) = 0 \tag{3.7}$$

for each $t \in \mathbb{R}$, $x \in K$, where K is any bounded subset in $L^2(\mathbb{P},\mathbb{H})$. Also, since $I_{i,1} \in AP_T(\mathbb{Z}, L^2(\mathbb{P},\mathbb{H}))$, thus for every sequence of integer numbers $\{\alpha'_n\}$, there exist a subsequence $\{\alpha_n\}$ and a stochastic processes $\widetilde{I}_{i,1} \in AP_T(\mathbb{Z}, L^2(\mathbb{P},\mathbb{H}))$, such that

$$\lim_{n\to\infty} E\|I_{i+\alpha_n,1}(x) - \widetilde{I}_{i,1}(x)\|^2 = 0 \tag{3.8}$$

for each $x \in \overline{B}$, where $\overline{B}$ is any bounded subset in $L^2(\mathbb{P},\mathbb{H})$.

Let $\widetilde{W}_n(s) := W(s+s_n) - W(s_n)$, for each $s \in \mathbb{R}$. It is easy to show that $\widetilde{W}_n$ is a Q-Wiener process with the same distribution as W, then

$$(\Phi x)(t+s_n) = \int_{-\infty}^{t+s_n} T(t+s_n-s)g_1(s,x(s))\,ds$$
$$+ \int_{-\infty}^{t+s_n} T(t+s_n-s)f_1(s,x(s))\,dW(s)$$
$$+ \int_{-\infty}^{t+s_n}\int_{|y|_V<1} T(t+s_n-s)G_1(s,x(s-),y)\widetilde{N}(ds,dy)$$
$$+ \int_{-\infty}^{t+s_n}\int_{|y|_V>1} T(t+s_n-s)F_1(s,x(s-),y)N(ds,dy)$$

$$
\begin{aligned}
&+ \sum_{t_i<t+s_n} T(t+s_n-t_i)I_{i,1}(x(t_i))\\
&= \int_{-\infty}^{t} T(t-s)g_1(s+s_n, x(s+s_n))\,ds\\
&+ \int_{-\infty}^{t} T(t-s)f_1(s+s_n, x(s+s_n))\,d\widetilde{W}_n(s)\\
&+ \int_{-\infty}^{t}\int_{|y|_V<1} T(t-s)G_1(s+s_n, x(s+s_n-), y)\widetilde{N}(ds,dy)\\
&+ \int_{-\infty}^{t}\int_{|y|_V>1} T(t-s)F_1(s+s_n, x(s+s_n-), y)N(ds,dy)\\
&+ \sum_{t_i<t+s_n} T(t+s_n-t_i)I_{i,1}(x(t_i)).
\end{aligned}
$$

Consider the process

$$
\begin{aligned}
x_n(t) &= \int_{-\infty}^{t} T(t-s)g_1(s+s_n, x_n(s))\,ds + \int_{-\infty}^{t} T(t-s)f_1(s+s_n, x_n(s))\,dW(s)\\
&+ \int_{-\infty}^{t}\int_{|y|_V<1} T(t-s)G_1(s+s_n, x_n(s-), y)\widetilde{N}(ds,dy)\\
&+ \int_{-\infty}^{t}\int_{|y|_V>1} T(t-s)F_1(s+s_n, x_n(s-), y)N(ds,dy)\\
&+ \sum_{t_i<t+s_n} T(t+s_n-t_i)I_{i,1}(x_n(t_i)).
\end{aligned}
$$

It is easy to see that $(\Phi x)(t+s_n)$ has the same distribution as $x_n(t)$ for each $t \in \mathbb{R}$.

Let $\widetilde{x}(t)$ satisfy the integral equation

$$
\begin{aligned}
\widetilde{x}(t) &= \int_{-\infty}^{t} T(t-s)\widetilde{g}_1(s, \widetilde{x}(s))\,ds + \int_{-\infty}^{t} T(t-s)\widetilde{f}_1(s, \widetilde{x}(s))\,dW(s)\\
&+ \int_{-\infty}^{t}\int_{|y|_V<1} T(t-s)\widetilde{G}_1(s, \widetilde{x}(s-), y)\widetilde{N}(ds,dy)\\
&+ \int_{-\infty}^{t}\int_{|y|_V>1} T(t-s)\widetilde{F}_1(s, \widetilde{x}(s-), y)N(ds,dy) + \sum_{t_i<t} T(t-t_i)\widetilde{I}_{i,1}(\widetilde{x}(t_i)).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&E\|x_n(t)-\widetilde{x}(t)\|^2\\
&\le 5E\bigg\| \int_{-\infty}^{t} T(t-s)[g_1(s+s_n, x_n(s)) - \widetilde{g}_1(s, \widetilde{x}(s))]\,ds \bigg\|^2\\
&\quad + 5E\bigg\| \int_{-\infty}^{t} T(t-s)[f_1(s+s_n, x_n(s)) - \widetilde{f}_1(s, \widetilde{x}(s))]\,dW(s) \bigg\|^2\\
&\quad + 5E\bigg\| \int_{-\infty}^{t}\int_{|y|_V<1} T(t-s)[G_1(s+s_n, x_n(s-), y) - \widetilde{G}_1(s, \widetilde{x}(s-), y)]\widetilde{N}(ds,dy) \bigg\|^2
\end{aligned}
$$

$$+ 5E\Big\| \int_{-\infty}^{t} \int_{|y|_V>1} T(t-s)[F_1(s+s_n, x_n(s-), y) - \widetilde{F}_1(s, \widetilde{x}(s-), y)] N(ds, dy)\Big\|^2$$
$$+ 5^{p-1}E\Big\| \sum_{t_i<t+s_n} T(t+s_n-t_i)I_{i,1}(x_n(t_i)) - \sum_{t_i<t} T(t-t_i)\widetilde{I}_{i,1}(\widetilde{x}(t_i))\Big\|^2$$
$$:= \sum_{j=1}^{5} \Lambda_j.$$

By (H6), for any $\varepsilon > 0$, there exist $\delta_1 > 0$ and a bounded subset $K \subset L^2(\mathbb{P}, \mathbb{H})$ such that $x, \overline{x} \subset K$ and $E\|x - \overline{x}\|^2 < \delta_1$ imply that

$$E\|g_1(t, x) - g_1(t, \overline{x})\|^2 < \varepsilon, \quad E\|f_1(t, x) - f_1(t, \overline{x})\|^2_{L_2^0} < \varepsilon,$$
$$\int_{|y|_V<1} E\|G_1(t, x, y) - G_1(t, \overline{x}, y)\|^2 \, v(dy) < \varepsilon,$$
$$\int_{|y|_V>1} E\|F_1(t, x, y) - F_1(t, \overline{x}, y)\|^2 \, v(dy) < \varepsilon$$

for each $t \in \mathbb{R}$. For the above $\delta_1 > 0$, there exists $\delta_2 > 0$ such that $\|x_n(t) - \widetilde{x}(t)\|^2 < \delta_1$ for all $n > \delta_2$ and all $t \in \mathbb{R}$. Therefore,

$$E\|g_1(s+s_n, x_n(s)) - g_1(s+s_n, \widetilde{x}(s))\|^2 < \varepsilon, \tag{3.9}$$
$$E\|f_1(s+s_n, x_n(s)) - f_1(s+s_n, \widetilde{x}(s))\|^2_{L_2^0} < \varepsilon, \tag{3.10}$$
$$\int_{|y|_V<1} E\|G_1(s+s_n, x_n(s), y) - G_1(s+s_n, \widetilde{x}(s), y)\|^2 \, v(dy) < \varepsilon, \tag{3.11}$$
$$\int_{|y|_V>1} E\|F_1(s+s_n, x_n(s), y) - F_1(s+s_n, \widetilde{x}(s), y)\|^2 \, v(dy) < \varepsilon \tag{3.12}$$

for all $n > \delta_2$ and all $s + s_n \in \mathbb{R}$. Using (3.9) and Hölder's inequality, we have

$$\begin{aligned}\Lambda_1 &\le 10M^2 \Big(\int_{-\infty}^{t} e^{-\delta(t-s)} \, ds \Big) \\ &\quad \times \Big[\int_{-\infty}^{t} e^{-\delta(t-s)} E\|g_1(s+s_n, x_n(s)) - g_1(s+s_n, \widetilde{x}(s))\|^2 \, ds \\ &\quad + \int_{-\infty}^{t} e^{-\delta(t-s)} E\|g_1(s+s_n, \widetilde{x}(s)) - \widetilde{g}_1(s, \widetilde{x}(s)) \, ds \Big] \\ &\le 10M^2 \frac{1}{\delta^{p-1}} \Big[\frac{1}{\delta}\varepsilon + \sup_{t\in\mathbb{R}} \varepsilon_n^{(1)}(t) \Big],\end{aligned}$$

where $\varepsilon_n^{(1)}(t) = \sum_j^4 \varepsilon_n^{(1j)}(t)$, and

$$\varepsilon_n^{11}(t) = \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)} E\|g_1(s+s_n, \widetilde{x}(s)) - \widetilde{g}_1(s, \widetilde{x}(s))\|^2 \, ds,$$

$$\varepsilon_n^{(12)}(t) = \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\delta(t-s)} E\|g_1(s+s_n, \widetilde{x}(s)) - \widetilde{g}_1(s, \widetilde{x}(s))\|^2 \, ds,$$

$$\varepsilon_n^{(13)}(t) = \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\delta(t-s)} E\|g_1(s+s_n, \widetilde{x}(s)) - \widetilde{g}_1(s, \widetilde{x}(s))\|^2 \, ds,$$

$$\varepsilon_n^{(14)}(t) = \int_{t_i}^{t} e^{-\delta(t-s)} E\|g_1(s+s_n, \widetilde{x}(s)) - \widetilde{g}_1(s, \widetilde{x}(s))\|^2 \, ds.$$

By (3.6), there exists $N_1 \in \mathbb{N}$ such that

$$E\|g_1(s+s_n, \widetilde{x}(s)) - \widetilde{g}_1(s, \widetilde{x}(s))\|^2 < \varepsilon$$

for all $s \in [t_j + \eta, t_{j+1} - \eta]$, $j \in \mathbb{Z}$, $j \leq i$, and $t - s \geq t - t_i + t_i - (t_{j+1} - \eta) \geq s - t_i + \gamma(i-1-j) + \eta$, whenever $n \geq N_1$. Then,

$$\begin{aligned}
\varepsilon_n^{(11)}(t) &\leq \varepsilon \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\delta(t-s)} \, ds \leq \frac{\varepsilon}{\delta} \sum_{j=-\infty}^{i-1} e^{-\delta(t-t_{j+1}+\eta)} \\
&\leq \frac{\varepsilon}{\delta} \sum_{j=-\infty}^{i-1} e^{-\delta\alpha(i-j-1)} \leq \frac{\varepsilon}{\delta(1-e^{-\delta\alpha})}, \\
\varepsilon_n^{(12)}(t) &\leq 2\big[\|g_1\|_\infty^2 + \|\widetilde{g}_1\|_\infty^2\big] \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-\delta(t-s)} \, ds \\
&\leq 2\big[\|g_1\|_\infty^2 + \|\widetilde{g}_1\|_\infty^2\big] \varepsilon e^{\delta\eta} \sum_{j=-\infty}^{i-1} e^{-\delta(t-t_j)} \\
&\leq 2\big[\|g_1\|_\infty^2 + \|\widetilde{g}_1\|_\infty^2\big] \varepsilon e^{\delta\eta} e^{-\delta(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\delta\alpha(i-j)} \\
&\leq \frac{2\big[\|g_1\|_\infty^2 + \|\widetilde{g}_1\|_\infty^2\big] e^{\delta\alpha/2}}{1-e^{-\delta\alpha}} \varepsilon.
\end{aligned}$$

Similarly, one has

$$\varepsilon_n^{(13)}(t) \leq \widetilde{M_1}\varepsilon, \quad \varepsilon_n^{(14)}(t) \leq \widetilde{M_2}\varepsilon,$$

where $\widetilde{M_2}$, $\widetilde{M_2}$ are some positive constants. Therefore, we get that $\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \varepsilon_n^{(1)}(t) = 0$. Using (3.10) and the Itô integral, we have

$$\begin{aligned}
\Lambda_2 &\leq 5M^2 \int_{-\infty}^{t} e^{-2\delta(t-s)} E\|f_1(s+s_n, x_n(s)) - \widetilde{f}_1(s, \widetilde{x}(s))\|_{L_2^0}^2 \, ds \\
&\leq 10M^2 \Big[\int_{-\infty}^{t} e^{-2\delta(t-s)} E\|f_1(s+s_n, x_n(s)) - f_1(s+s_n, \widetilde{x}(s))\|_{L_2^0}^2 \, ds \\
&\qquad + \int_{-\infty}^{t} e^{-2\delta(t-s)} E\|f_1(s+s_n, \widetilde{x}(s)) - \widetilde{f}_1(s, \widetilde{x}(s))\|_{L_2^0}^2 \, ds \Big] \\
&\leq 10M^2 \Big[\frac{1}{2\delta}\varepsilon + \sup_{t\in\mathbb{R}} \varepsilon_n^{(2)}(t) \Big],
\end{aligned}$$

where $\varepsilon_n^{(2)}(t) = \sum_{j=1}^{4} \varepsilon_n^{2j}(t)$, and

$$\varepsilon_n^{(21)}(t) = \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-2\delta(t-s)} E\|f_1(s+s_n, \widetilde{x}(s)) - \widetilde{f}_1(s, \widetilde{x}(s))\|_{L_2^0}^2 \, ds,$$

$$\varepsilon_n^{(22)}(t) = \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} e^{-2\delta(t-s)} E\|f_1(s+s_n, \widetilde{x}(s)) - \widetilde{f}_1(s, \widetilde{x}(s))\|_{L_2^0}^2 \, ds,$$

$$\varepsilon_n^{(23)}(t) = \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-2\delta(t-s)} E\|f_1(s+s_n, \widetilde{x}(s)) - \widetilde{f}_1(s, \widetilde{x}(s))\|_{L_2^0}^2 \, ds,$$

$$\varepsilon_n^{(24)}(t) = \int_{t_i}^{t} e^{-2\delta(t-s)} E\|f_1(s+s_n, \widetilde{x}(s)) - \widetilde{f}_1(s, \widetilde{x}(s))\|_{L_2^0}^2 \, ds.$$

By (3.7), there exists $N_2 \in \mathbb{N}$ such that

$$E\|f_1(s+s_n, \widetilde{x}_s) - \widetilde{f}_1(s, \widetilde{x}_s)\|_{L_2^0}^2 < \varepsilon$$

for all $s \in [t_j + \eta, t_{j+1} - \eta]$, $j \in \mathbb{Z}$, $j \le i$, and $t - s \ge t - t_i + t_i - (t_{j+1} - \eta) \ge s - t_i + \gamma(i - 1 - j) + \eta$, whenever $n \ge N_2$. Then,

$$\begin{aligned}
\varepsilon_n^{(21)}(t) &\le \varepsilon \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-2\delta(t-s)} \, ds \le \frac{1}{2\delta} \varepsilon \sum_{j=-\infty}^{i-1} e^{-2\delta(t - t_{j+1} + \eta)} \\
&\le \frac{1}{2\delta} \varepsilon \sum_{j=-\infty}^{i-1} e^{-2\delta\alpha(i-j-1)} \le \frac{1}{2\delta(1 - e^{-2\delta\alpha})} \varepsilon, \\
\varepsilon_n^{(22)}(t) &\le 2\big[\|f_1\|_\infty^2 + \|\widetilde{f}_1\|_\infty^2\big] \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_{j+1}+\eta} e^{-2\delta(t-s)} \, ds \\
&\le 2\big[\|f_1\|_\infty^2 + \|\widetilde{f}_1\|_\infty^2\big] \varepsilon e^{2\delta\eta} \sum_{j=-\infty}^{i-1} e^{-2\delta(t-t_j)} \\
&\le 2\big[\|f_1\|_\infty^2 + \|\widetilde{f}_1\|_\infty^2\big] \varepsilon e^{2\delta\eta} e^{-2\delta(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-2\delta\alpha(i-j)} \\
&\le \frac{2^{p-1}\big[\|f_1\|_\infty^2 + \|\widetilde{f}_1\|_\infty^2\big] e^{\delta\alpha} \varepsilon}{1 - e^{-2\delta\alpha}}.
\end{aligned}$$

Similarly, one has

$$\varepsilon_n^{(23)}(t) \le \widetilde{M}_3 \varepsilon, \quad \varepsilon_n^{(24)}(t) \le \widetilde{M}_4 \varepsilon,$$

where $\widetilde{M}_3$, $\widetilde{M}_4$ are some positive constants. Therefore, we get that $\lim_{n\to\infty} \varepsilon_n^{(2)}(t) = 0$.

Using (3.11) and the properties of the integral for the Poisson random measure, we have

$$\begin{aligned}
\Lambda_3 &\le 5M^2 \int_{-\infty}^{t} \int_{|y|_V<1} e^{-2\delta(t-s)} E\|G_1(s+s_n, x_n(s-), y) - \widetilde{G}_1(s, \widetilde{x}(s-), y)\|^2 \, v(dy)\, ds \\
&\le 10M^2 \Big[\int_{-\infty}^{t} \int_{|y|_V<1} e^{-2\delta(t-s)} E\|G_1(s+s_n, x_n(s-), y) - G_1(s+s_n, \widetilde{x}(s-), y)\|^2 \, v(dy)\, ds \\
&\qquad + \int_{-\infty}^{t} \int_{|y|_V<1} e^{-2\delta(t-s)} E\|G_1(s+s_n, \widetilde{x}(s-), y) - \widetilde{G}_1(s, \widetilde{x}(s-), y)\|^2 \, v(dy)\, ds \Big] \\
&\le 10M^2 \Big[\frac{1}{2\delta}\varepsilon + \sup_{t\in\mathbb{R}} \varepsilon_n^{(3)}(t) \Big],
\end{aligned}$$

where $\varepsilon_n^{(3)}(t) = \sum_{j=1}^{4} \varepsilon_n^{3j}(t)$, and

$$\begin{aligned}
\varepsilon_n^{(31)}(t) &= \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} \int_{|y|_V<1} e^{-2\delta(t-s)} E\|G_1(s+s_n, \widetilde{x}(s-), y) - \widetilde{G}_1(s, \widetilde{x}(s-), y)\|^2 \, v(dy)\, ds, \\
\varepsilon_n^{(32)}(t) &= \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} \int_{|y|_V<1} e^{-2\delta(t-s)} E\|G_1(s+s_n, \widetilde{x}(s-), y) - \widetilde{G}_1(s, \widetilde{x}(s-), y)\|^2 \, v(dy)\, ds, \\
\varepsilon_n^{(33)}(t) &= \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} \int_{|y|_V<1} e^{-2\delta(t-s)} E\|G_1(s+s_n, \widetilde{x}(s-), y) - \widetilde{G}_1(s, \widetilde{x}(s-), y)\|^2 \, v(dy)\, ds, \\
\varepsilon_n^{(34)}(t) &= \int_{t_i}^{t} \int_{|y|_V<1} e^{-2\delta(t-s)} E\|G_1(s+s_n, \widetilde{x}(s-), y) - \widetilde{G}_1(s, \widetilde{x}(s-), y)\|^2 \, v(dy)\, ds.
\end{aligned}$$

By (3.6), there exists $N_3 \in \mathbb{N}$ such that

$$\int_{|y|_V<1} E\|G_1(s+s_n, \widetilde{x}(s-), y) - \widetilde{G}_1(s, \widetilde{x}(s-), y)\|^2 \, v(dy) < \varepsilon$$

for all $s \in [t_j+\eta, t_{j+1}-\eta]$, $j \in \mathbb{Z}$, $j \le i$, and $t-s \ge t-t_i+t_i-(t_{j+1}-\eta) \ge s-t_i+\gamma(i-1-j)+\eta$, whenever $n \ge N_3$. Then,

$$\begin{aligned}
\varepsilon_n^{(31)}(t) &\le \varepsilon \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-2\delta(t-s)}\, ds \le \frac{1}{2\delta}\varepsilon \sum_{j=-\infty}^{i-1} e^{-2\delta(t-t_{j+1}+\eta)} \\
&\le \frac{1}{2\delta}\varepsilon \sum_{j=-\infty}^{i-1} e^{-2\delta\alpha(i-j-1)} \le \frac{1}{2\delta(1-e^{-2\delta\alpha})}\varepsilon, \\
\varepsilon_n^{(32)}(t) &\le 2\big[\|G_1\|_\infty^2 + \|\widetilde{G}_1\|_\infty^2\big] \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_{j+1}+\eta} e^{-2\delta(t-s)}\, ds \\
&\le 2\big[\|G_1\|_\infty^2 + \|\widetilde{G}_1\|_\infty^2\big]\varepsilon e^{2\delta\eta} \sum_{j=-\infty}^{i-1} e^{-2\delta(t-t_j)} \\
&\le 2\big[\|G_1\|_\infty^2 + \|\widetilde{G}_1\|_\infty^2\big]\varepsilon e^{2\delta\eta} e^{-2\delta(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-2\delta\alpha(i-j)} \\
&\le \frac{2\big[\|G_1\|_\infty^2 + \|\widetilde{G}_1\|_\infty^2\big] e^{2\delta\alpha}}{1-e^{-2\delta\alpha}}\varepsilon,
\end{aligned}$$

where $\|G_1\|_\infty^2 = \sup_{s\in\mathbb{R}} \int_{|y|_V<1} E\|G_1(s+s_n, \widetilde{x}(s-), y)\|^2\, v(dy)$ and $\|\widetilde{G}_1\|_\infty^2 = \int_{|y|_V<1} E\|\widetilde{G}_1(s, \widetilde{x}(s-), y)\|^2\, v(dy)$. Similarly, one has

$$\varepsilon_n^{(33)}(t) \le \widetilde{M}_5\varepsilon, \quad \varepsilon_n^{(34)}(t) \le \widetilde{M}_6\varepsilon,$$

where $\widetilde{M}_5$, $\widetilde{M}_6$ are some positive constants. Therefore, we get that $\lim_{n\to\infty} \varepsilon_n^{(3)}(t) = 0$.

Using (3.12) and the properties of the integral for the Poisson random measure, we have

$$\begin{aligned}
\Lambda_4 &\le 10M^2 \int_{-\infty}^t \int_{|y|_V>1} e^{-2\delta(t-s)} E\|F_1(s+s_n, x_n(s-), y) - \widetilde{F}_1(s, \widetilde{x}(s-), y)\|^2\, v(dy)\, ds \\
&\quad + 10M^2 \frac{b}{\delta} \int_{-\infty}^t \int_{|y|_V>1} e^{-\delta(t-s)} E\|F_1(s+s_n, x_n(s-), y) - \widetilde{F}_1(s, \widetilde{x}(s-), y)\|^2\, v(dy)\, ds \\
&\le 20M^2 \bigg[\int_{-\infty}^t \int_{|y|_V>1} e^{-2\delta(t-s)} E\|F_1(s+s_n, x_n(s-), y) - F_1(s+s_n, \widetilde{x}(s-), y)\|^2\, v(dy)\, ds \\
&\quad + \int_{-\infty}^t \int_{|y|_V>1} e^{-2\delta(t-s)} E\|F_1(s+s_n, \widetilde{x}(s-), y) - \widetilde{F}_1(s, \widetilde{x}(s-), y)\|^2\, v(dy)\, ds \bigg] \\
&\quad + 20M^2 \frac{b}{\delta} \bigg[\int_{-\infty}^t \int_{|y|_V>1} e^{-\delta(t-s)} E\|F_1(s+s_n, x_n(s-), y) - F_1(s+s_n, \widetilde{x}(s-), y)\|^2\, v(dy)\, ds \\
&\qquad + \int_{-\infty}^t \int_{|y|_V>1} e^{-\delta(t-s)} E\|F_1(s+s_n, \widetilde{x}(s-), y) - \widetilde{F}_1(s, \widetilde{x}(s-), y)\|^2\, v(dy)\, ds \bigg] \\
&\le 20M^2 \bigg[\left(\frac{1}{2\delta} + \frac{b}{\delta^2} \right) \varepsilon + \sup_{t\in\mathbb{R}} \varepsilon_n^{(3)}(t) \bigg],
\end{aligned}$$

where $\varepsilon_n^{(4)}(t) = \sum_{j=1}^4 \varepsilon_n^{4j}(t)$, and

$$\begin{aligned}
\varepsilon_n^{(41)}(t) &= \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} \int_{|y|_V>1} \left[e^{-2\delta(t-s)} + \frac{b}{\delta} e^{-\delta(t-s)} \right] \\
&\qquad \times E\|F_1(s+s_n, \widetilde{x}(s-), y) - \widetilde{F}_1(s, \widetilde{x}(s-), y)\|^2\, v(dy)\, ds, \\
\varepsilon_n^{(42)}(t) &= \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_j+\eta} \int_{|y|_V>1} \left[e^{-2\delta(t-s)} + \frac{b}{\delta} e^{-\delta(t-s)} \right] \\
&\qquad \times E\|F_1(s+s_n, \widetilde{x}(s-), y) - \widetilde{F}_1(s, \widetilde{x}(s-), y)\|^2\, v(dy)\, ds, \\
\varepsilon_n^{(43)}(t) &= \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} \int_{|y|_V>1} \left[e^{-2\delta(t-s)} + \frac{b}{\delta} e^{-\delta(t-s)} \right] \\
&\qquad \times E\|F_1(s+s_n, \widetilde{x}(s-), y) - \widetilde{F}_1(s, \widetilde{x}(s-), y)\|^2\, v(dy)\, ds, \\
\varepsilon_n^{(44)}(t) &= \int_{t_i}^{t} \int_{|y|_V>1} \left[e^{-2\delta(t-s)} + \frac{b}{\delta} e^{-\delta(t-s)} \right] \\
&\qquad \times E\|F_1(s+s_n, \widetilde{x}(s-), y) - \widetilde{F}_1(s, \widetilde{x}(s-), y)\|^2\, v(dy)\, ds.
\end{aligned}$$

By (3.8), there exists $N_4 \in \mathbb{N}$ such that

$$\int_{|y|_V>1} E\|F_1(s+s_n,\widetilde{x}(s-),y)-\widetilde{F}_1(s,\widetilde{x}(s-),y)\|^2\,v(dy)<\varepsilon$$

for all $s \in [t_j+\eta, t_{j+1}-\eta]$, $j \in \mathbb{Z}$, $j \le i$, and $t-s \ge t-t_i+t_i-(t_{j+1}-\eta) \ge s-t_i+\gamma(i-1-j)+\eta$, whenever $n \ge N_4$. Then,

$$\begin{aligned}
\varepsilon_n^{(41)}(t) &\le \varepsilon \sum_{j=-\infty}^{i-1}\int_{t_j+\eta}^{t_{j+1}-\eta}\left[e^{-2\delta(t-s)}+\frac{b}{\delta}e^{-\delta(t-s)}\right]ds\\
&\le \frac{1}{2\delta}\varepsilon\sum_{j=-\infty}^{i-1}e^{-2\delta(t-t_{j+1}+\eta)}+\frac{b}{\delta^2}\varepsilon\sum_{j=-\infty}^{i-1}e^{-\delta(t-t_{j+1}+\eta)}\\
&\le \frac{1}{2\delta}\varepsilon\sum_{j=-\infty}^{i-1}e^{-2\delta\alpha(i-j-1)}+\frac{b}{\delta^2}\varepsilon\sum_{j=-\infty}^{i-1}e^{-\delta\alpha(i-j-1)}\\
&\le \left[\frac{1}{2\delta(1-e^{-2\delta\alpha})}+\frac{b}{\delta^2(1-e^{-\delta\alpha})}\right]\varepsilon,
\end{aligned}$$

$$\begin{aligned}
\varepsilon_n^{(42)}(t) &\le 2\big[\|F_1\|_\infty^2+\|\widetilde{F}_1\|_\infty^2\big]\sum_{j=-\infty}^{i-1}\int_{t_j}^{t_{j+1}+\eta}\big[e^{-2\delta(t-s)}+e^{-\delta(t-s)}\big]\,ds\\
&\le 2\big[\|F_1\|_\infty^2+\|\widetilde{F}_1\|_\infty^2\big]\varepsilon\left[e^{2\delta\eta}\sum_{j=-\infty}^{i-1}e^{-2\delta(t-t_j)}+e^{\delta\eta}\sum_{j=-\infty}^{i-1}e^{-\delta(t-t_j)}\right]\\
&\le 2\big[\|F_1\|_\infty^2+\|\widetilde{F}_1\|_\infty^2\big]\varepsilon\left[e^{2\delta\eta}e^{-2\delta(t-t_i)}\sum_{j=-\infty}^{i-1}e^{-2\delta\alpha(i-j)}+e^{\delta\eta}e^{-\delta(t-t_i)}\sum_{j=-\infty}^{i-1}e^{-\delta\alpha(i-j)}\right]\\
&\le 2\big[\|F_1\|_\infty^2+\|\widetilde{F}_1\|_\infty^2\big]\left[\frac{e^{2\delta\alpha}}{1-e^{-2\delta\alpha}}+\frac{e^{\delta\alpha}}{1-e^{-\delta\alpha}}\right]\varepsilon,
\end{aligned}$$

where $\|F_1\|_\infty^2=\sup_{s\in\mathbb{R}}\int_{|y|_V>1}E\|F_1(s+s_n,\widetilde{x}(s-),y)\|^2\,v(dy)$ and $\|\widetilde{F}_1\|_\infty^2=\int_{|y|_V>1} E\|\widetilde{F}_1(s,\widetilde{x}(s-),y)\|^2\,v(dy)$. Similarly, one has

$$\varepsilon_n^{(43)}(t)\le \widetilde{M}_7\varepsilon,\quad \varepsilon_n^{(44)}(t)\le \widetilde{M}_8\varepsilon,$$

where $\widetilde{M}_7$, $\widetilde{M}_8$ are some positive constants. Therefore, we get that $\lim_{n\to\infty}\varepsilon_n^{(4)}(t)=0$.

Since $\{t_i^j\}$, $i\in Z$, $j=0,1,\ldots$, are equipotentially almost periodic, then for $\varepsilon>0$, there exist the sequence of real numbers $\{s_n\}$ and sequence of integer numbers $\{\alpha_n\}$, such that $t_i<t\le t_{i+1}$, $|t-t_i|>\varepsilon$, $|t-t_{i+1}|>\varepsilon$, $i\in\mathbb{Z}$, one has $t+s_n>t+s_n+\varepsilon>t_{i+\alpha_n}$ and $t_{i+\alpha_n+1}>t_{i+1}+s_n-\varepsilon>t+s_n$, that is $t_{i+\alpha_n}<t+s_n<t_{i+\alpha_n+1}$. By (H6), for any $\varepsilon>0$, there exist $\delta_3>0$ and a bounded subset $B\subset L^2(\mathbb{P},\mathbb{H})$ such that $x,\overline{x}\subset B$ and $E\|x-\overline{x}\|^2<\delta_3$ imply that

$$E\|I_{i,1}(x(t_i))-I_{i,1}(\overline{x}(t_i))\|^2<\varepsilon$$

for $i \in \mathbb{Z}$. For the above $\delta_3 > 0$, there exists $N_5 > 0$ such that $E\|x_n(t_{i+\alpha_i}) - \widetilde{x}(t_{i+\alpha_i})\|^2 < \delta_3$, $E\|\widetilde{x}(t_{i+\alpha_n}) - \widetilde{x}(t_i)\|^2 < \delta_3$ for all $n > N_5$ and all $i \in \mathbb{Z}$. Therefore,

$$E\|I_{i+\alpha_n,1}(x_n(t_{i+\alpha_i})) - I_{i+\alpha_n,1}(\widetilde{x}(t_{i+\alpha_i}))\|^2 < \varepsilon, \tag{3.13}$$

$$E\|I_{i+\alpha_n,1}(\widetilde{x}(t_{i+\alpha_n})) - I_{i+\alpha_n,1}(\widetilde{x}(t_i))\|^2 < \varepsilon \tag{3.14}$$

for all $n > N_5$ and all $s + s_n \in \mathbb{R}$. Then by (3.13), (3.14) and Hölder's inequality again, we have

$$\begin{aligned}\Lambda_5 \le\ & 15\bigg(\sum_{t_i<t} e^{-\delta(t-t_i)}\bigg) \\ & \times \bigg(\sum_{t_i<t} e^{-\delta(t-t_i)}\big[E\|I_{i+\alpha_n,1}(x_n(t_i+\alpha_n)) - I_{i+\alpha_n,1}(\widetilde{x}(t_i+\alpha_n))\|^2 \\ & \quad + E\|I_{i+\alpha_n,1}(\widetilde{x}(t_{i+\alpha_n})) - I_{i+\alpha_n,1}(\widetilde{x}(t_i))\|^2 + E\|I_{i+\alpha_n,1}(\widetilde{x}(t_i)) - \widetilde{I}_{i,1}(\widetilde{x}(t_i))\|^2\big]\bigg) \\ \le\ & \frac{15}{(1-e^{-\delta\alpha})^2}[2\varepsilon + \varepsilon_n^{(5)}],\end{aligned}$$

where $\varepsilon_n^{(5)} = E\|I_{i+\alpha_n,1}(\widetilde{x}(t_i+\alpha_n)) - \widetilde{I}_{i,1}(\widetilde{x}(t_i))\|^2$. By (3.8), we have $\lim_{n\to\infty} \varepsilon_n^{(5)} = 0$.

By above estimations, we have for all $t \in \mathbb{R}$,

$$E\|x_n(t) - \widetilde{x}(t)\|^2 \le \vartheta_1 \varepsilon_n(t) + \vartheta_2 \varepsilon,$$

where $\varepsilon_n(t) = \sum_{j=1}^5 \varepsilon_n^{(j)}(t)$, and ϑ_1, ϑ_2 are given constants. Therefore,

$$\sup_{t\in\mathbb{R}} E\|x_n(t) - \widetilde{x}(t)\|^2 \le \vartheta_1 \sup_{t\in\mathbb{R}} \varepsilon_n(t) + \vartheta_2 \varepsilon.$$

By $\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \varepsilon_n(t) = 0$ with ε sufficiently small, it follows that

$$\sup_{t\in\mathbb{R}} \|x_n(t) - \widetilde{x}(t)\|^2 \to \infty \quad \text{as } n \to \infty$$

for all $t \in \mathbb{R}$. Since $(\Phi x)(t + s_n)$ has the the same distribution as $x_n(t)$, it follows that $(\Phi x)(t + s_n) \to \widetilde{x}(t)$ in distribution as $n \to \infty$. Hence Φx has almost periodic in one-dimensional distributions. Note that the sequence $(E\|x_n(t)\|^2)$ is uniformly integrable, thus $(E\|\Phi x(t + s_n)\|^2)$ is also uniformly integrable, so the family $(E\|x_n(t)\|^2)_{t\in\mathbb{R}}$ is uniformly integrable. Next, we prove that Φx is almost periodic in distribution. For fixed $\tau \in \mathbb{R}$, let $\xi_n = \Phi x(\tau + s_n)$, $g_1^n = g_1(s + s_n, x)$, $f_1^n = f_1(s + s_n, x)$, $G_1^n = \int_{|y|_V<1} G_1(s + s_n, x, y)\widetilde{N}(ds, dy)$, $F_1^n = \int_{|y|_V>1} F_1(s + s_n, x, y)N(ds, dy)$ and $I_{i,1}^n = I_{i+\alpha_n,1}(x)$, $i \in \mathbb{Z}$. By the foregoing, (ξ_n) converges in distribution to some variable $\Phi x(\tau)$. We deduce that (ξ_n) is tight, so (ξ_n, W) is tight also. We can thus choose a subsequence (still noted s_n for simplicity) such that (ξ_n, W) converges in distribution to $(\Phi x(\tau), W)$. Similarly as the proof of [8, Property 3.1], for every $T \geq \tau$, $\Phi x(\cdot + s_n)$ converges in distribution on $PC_T([\tau, T], L^2(\mathbb{P}, \mathbb{H}))$ to the solution to

$$\begin{aligned} x(t) &= T(t-\tau)\varphi(\tau) + \int_\tau^t T(t-s) g_1(s, x(s))\, ds + \int_\tau^t T(t-s) f_1(s, x(s))\, dW(s) \\ &\quad + \int_\tau^t \int_{|y|_V<1} T(t-s) G_1(s, x(t), y) \widetilde{N}(ds, dy) \\ &\quad + \int_\tau^t \int_{|y|_V>1} T(t-s) F_1(s, x(t), y) N(ds, dy) + \sum_{\tau<t_i<t} T(t-t_i) I_{i,1}(x(t_i)). \end{aligned}$$

Note that Φx does not depend on the chosen interval $[\tau, T]$, thus the convergence takes place on $PC_T([\tau, T], L^2(\mathbb{P}, \mathbb{H}))$. Hence Φx is almost periodic in distribution.

(2) $\Upsilon x \in WPAP_T^0(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}), \rho)$.

In fact, for $r > 0$, one has

$$\begin{aligned} &\frac{1}{\mu(r,\rho)} \int_{-r}^r E\|\Upsilon(t)\|^2\, dt \\ &\leq 5 \frac{1}{\mu(r,\rho)} \int_{-r}^r E\left\| \int_{-\infty}^t T(t-s) g_2(s, x(s))\, ds \right\|^2 \rho(t)\, dt \\ &\quad + 5 \frac{1}{\mu(r,\rho)} \int_{-r}^r E\left\| \int_{-\infty}^t T(t-s) f_2(s, x(s))\, dW(s) \right\|^2 \rho(t)\, dt \\ &\quad + 5 \frac{1}{\mu(r,\rho)} \int_{-r}^r E\left\| \int_{-\infty}^t \int_{|y|_V<1} T(t-s) G_2(s, x(s-), y) \widetilde{N}(ds, dy) \right\|^2 \rho(t)\, dt \\ &\quad + 5 \frac{1}{\mu(r,\rho)} \int_{-r}^r E\left\| \int_{-\infty}^t \int_{|y|_V>1} T(t-s) F_2(s, x(s-), y) N(ds, dy) \right\|^2 \rho(t)\, dt \\ &\quad + 5 \frac{1}{\mu(r,\rho)} \int_{-r}^r E\left\| \sum_{t_i<t} T(t-t_i) I_{i,2}(x(t_i)) \right\|^2 \rho(t)\, dt \\ &:= \sum_{j=1}^5 \Pi_j. \end{aligned}$$

By (H2) and Hölder's inequality, we have

$$\begin{aligned} \Pi_1 &= 5 \frac{1}{\mu(r,\rho)} \int_{-r}^r E\left\| \int_0^\infty T(s) g_2(t-s, x(t-s))\, ds \right\|^2 \rho(t)\, dt \\ &\leq 5M^2 \frac{1}{\mu(r,\rho)} \int_{-r}^r \left(\int_0^\infty e^{-\delta s}\, ds \right) \int_0^\infty e^{-\delta s} E\|g_2(t-s, x_{t-s})\|^2\, ds\, \rho(t)\, dt \\ &= 5M^2 \left(\int_0^\infty e^{-\delta s}\, ds \right) \int_0^\infty e^{-\delta s}\, ds\, \frac{1}{\mu(r,\rho)} \int_{-r}^r E\|g_2(t-s, x(t-s))\|^2 \rho(t)\, dt. \end{aligned}$$

Since $g_2 \in WPAP_T^0(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}), \rho)$, it follows that

$$\frac{1}{\mu(r,\rho)} \int_{-r}^r E\left\| \int_{-\infty}^t T(t-s) g_2(s, x(s))\, ds \right\|^2 dt \to 0 \quad \text{as } r \to \infty$$

for all $s \in \mathbb{R}$. Using the Lebesgue's dominated convergence theorem, we have $\Pi_1 \to 0$ as $r \to \infty$.

By (H2) and the Itô integral, we have

$$\begin{aligned}
\Pi_2 &= 5\frac{1}{\mu(r,\rho)}\int_{-r}^{r} E\left\|\int_0^\infty T(s)f_2(t-s,x(t-s))\,dW(s)\right\|^2\rho(t)\,dt \\
&\le 5M^2\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\int_0^\infty e^{-2s}E\|f_2(t-s,x(t-s))\|_{L_2^0}^2\,ds\,\rho(t)\,dt \\
&= 5M^2\left(\int_0^\infty e^{-2\delta s}\,ds\right)\frac{1}{\mu(r,\rho)}\int_{-r}^{r} E\|f_2(t-s,x(t-s))\|_{L_2^0}^2\rho(t)\,dt.
\end{aligned}$$

Since $f_2 \in WPAP_T^0(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}), L^2(\mathbb{P},L_2^0),\rho)$, it follows that

$$\frac{1}{\mu(r,\rho)}\int_{-r}^{r} E\left\|\int_{-\infty}^{t} T(t-s)f_2(t-s,x(t-s))\,dW(s)\right\|^2\rho(t)\,dt \to 0 \quad \text{as } r\to\infty$$

for all $s \in \mathbb{R}$. Using the Lebesgue's dominated convergence theorem, we have $\Pi_2 \to 0$ as $r \to \infty$.

By (H2) and the properties of the integral for the Poisson random measure, we have

$$\begin{aligned}
\Pi_3 &\le 5M^2\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\int_0^\infty\int_{|y|_V<1} e^{-2s}E\|G_2(t-s,x(t-s),y)\|^2\,v(dy)\,ds\,\rho(t)\,dt \\
&= 5M^2\left(\int_0^\infty e^{-2\delta s}\,ds\right)\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\int_{|y|_V<1} E\|G_2(t-s,x(t-s),y)\|^2\,v(dy)\rho(t)\,dt.
\end{aligned}$$

Since $G_2 \in PWPAP_T^0(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\times V, L^2(\mathbb{P},\mathbb{H}),\rho)$, it follows that

$$\frac{1}{\mu(r,\rho)}\int_{-r}^{r} E\left\|\int_{-\infty}^{t}\int_{|y|_V<1} T(t-s)G_2(t-s,x(t-s),y)\widetilde{N}(ds,dy)\right\|^2\rho(t)\,dt \to 0 \text{ as } r\to\infty$$

for all $s \in \mathbb{R}$. Using the Lebesgue's dominated convergence theorem, we have $\Pi_3 \to 0$ as $r \to \infty$.

Similarly, we have

$$\begin{aligned}
\Pi_4 &\le 10M^2\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\int_0^\infty\int_{|y|_V>1} e^{-2s}E\|F_2(t-s,x(t-s),y)\|^2\,v(dy)\,ds\,\rho(t)\,dt \\
&\quad + 10M^2\frac{b}{\delta}\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\int_0^\infty\int_{|y|_V>1} e^{-s}E\|F_2(t-s,x(t-s),y)\|^2\,v(dy)\,ds\,\rho(t)\,dt \\
&= 10M^2\left(\int_0^\infty e^{-2\delta s}\,ds\right)\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\int_{|y|_V>1} E\|F_2(t-s,x(t-s),y)\|^2\,v(dy)\rho(t)\,dt \\
&\quad + 10M^2\frac{b}{\delta}\left(\int_0^\infty e^{-\delta s}\,ds\right)\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\int_{|y|_V>1} E\|F_2(t-s,x(t-s),y)\|^2\,v(dy)\rho(t)\,dt.
\end{aligned}$$

Since $F_2 \in PWPAP_T^0(\mathbb{R}\times L^2(\mathbb{P}\times V,\mathbb{K}), L^2(\mathbb{P},\mathbb{H}),\rho)$, it follows that

$$\frac{1}{\mu(r,\rho)}\int_{-r}^{r} E\left\|\int_{-\infty}^{t}\int_{|y|_V<1} T(t-s)F_2(t-s,x(t-s),y)\widetilde{N}(ds,dy)\right\|^2\rho(t)\,dt \to 0 \text{ as } r\to\infty$$

for all $s \in \mathbb{R}$. Using the Lebesgue's dominated convergence theorem, we have $\Pi_4 \to 0$ as $r \to \infty$.

For a given $i \in \mathbb{Z}$, define the function $(\mathcal{V}x)(t)$ by $(\mathcal{V}x)(t) = T(t - t_i)I_{i,2}(x(t_i))$, $t_i < t \le t_{i+1}$, then

$$\begin{aligned}\lim_{t\to\infty} E\|(\mathcal{V}x)(t)\|^2 &= \lim_{t\to\infty} E\|T(t-t_i)I_{i,2}(x(t_i))\|^2 \\ &\le \lim_{t\to\infty} M^p e^{-p\delta(t-t_i)} \sup_{i\in\mathbb{Z}} E\|I_{i,2}\|_\infty^2 = 0.\end{aligned}$$

Thus $\mathcal{V}x \in PC_T^0(\mathbb{R}, L^2(\mathbb{P},\mathbb{H})) \subset WPAP_T^0(\mathbb{R}, L^2(\mathbb{P},\mathbb{H}),\rho)$. Define $\mathcal{V}_j x\colon \mathbb{R} \to L^2(\mathbb{P},\mathbb{H})$ by

$$(\mathcal{V}_j x)(t) = T(t - t_{i-j})I_{i-j,2}(x(t_i)), \quad t_i < t \le t_{i+1},\ j \in \mathbb{N}.$$

So $\mathcal{V}_j x \in WPAP_T^0(\mathbb{R}, L^2(\mathbb{P},\mathbb{H}),\rho)$. Moreover,

$$\begin{aligned}E\|(\mathcal{V}_j x)(t)\|^2 &= E\|T(t-t_{i-j})I_{i-j,2}(x(t_i))\|^2 \\ &\le M^2 e^{-2\delta(t-t_{i-j})} \sup_{i\in\mathbb{Z}} E\|I_{i-j,2}\|_\infty^2 \\ &\le M^2 e^{-2\delta(t-t_i)} e^{-2\delta\alpha j} \sup_{i\in\mathbb{Z}} E\|I_{i-j,2}\|_\infty^2.\end{aligned}$$

Therefore, the series $\sum_{j=0}^\infty \mathcal{V}_j x$ is uniformly convergent on $\mathbb{R}$. By Lemma 2.12, one has

$$\sum_{t_i<t} T(t-t_i)I_{i,2}(x(t_i)) = \sum_{j=0}^\infty (\mathcal{V}_j x)(t) \in WPAP_T^0(\mathbb{R}, L^2(\mathbb{P},\mathbb{H}),\rho),$$

that is

$$\frac{1}{\mu(r,\rho)} \int_{-r}^r E\left\| \sum_{t_i<t} T(t-t_i)I_{i,2}(x(t_i)) \right\|^2 \rho(t)\,dt \to 0 \quad \text{as } r \to \infty.$$

Using the Lebesgue's dominated convergence theorem, we have $\Pi_5 \to 0$ as $r \to \infty$.

Hence

$$\lim_{n\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^r E\|\Upsilon(t)\|^2 \rho(t)\,dt = 0,$$

which is mean that $\Upsilon x \in WPAP_T^0(\mathbb{R}, L^p(\mathbb{P},\mathbb{H}),\rho)$. Therefore, x is piecewise weighted pseudo almost periodic in distribution mild solution to system (1.1)–(1.2). This completes the proof. □

4. Exponential stable of mild solution

In this section, we present the exponential stable behavior of a piecewise weighted pseudo almost periodic in distribution mild solution of (1.1)–(1.2). To do this, we also need the following assumptions:

(B1) For any $\varsigma > 0$, there exist constant $\varrho, l_1, l_2, c_i > 0$, $i \in \mathbb{Z}$, such that, for all $t \in \mathbb{R}$ and $x \in L^2(\mathbb{P}, \mathbb{H})$ with $E\|x\|^2 > \varrho$,

$$\sup_{t\in\mathbb{R}} \big[E\|g(t,x)\|^2 + E\|f(t,x)\|^2_{L_2^0}\big] \le \varsigma l_1 \|x\|^2, \quad (t,x) \in \mathbb{R} \times L^2(\mathbb{P},\mathbb{K}),$$

$$\sup_{t\in\mathbb{R}} \bigg[\int_{|y|_V<1} E\|G(t,x,y)\|^2\, v(dy) + \int_{|y|_V>1} E\|F(t,x,y)\|^2\, v(dy)\bigg]$$
$$\le \varsigma l_2 \|x\|^p, \quad (t,x,y) \in \mathbb{R} \times L^2(\mathbb{P},\mathbb{K}) \times V,$$
$$E\|I_i(x)\|^2 \le \varsigma c_i E\|x\|^2, \quad i \in \mathbb{Z},\ x \in L^2(\mathbb{P},\mathbb{K}).$$

(B2) There exist constants $0 < \beta < \delta$, such that

$$\sup_{t\in\mathbb{R}, E\|x\|^2\le\varrho} \big\{e^{\beta t}\big[E\|g(t,x)\|^2 + E\|f(t,x)\|^2_{L_2^0}\big]\big\} < \infty, \quad (t,x) \in \mathbb{R} \times L^2(\mathbb{P},\mathbb{H}),$$

$$\sup_{t\in\mathbb{R}, E\|x\|^2\le\varrho} \bigg\{e^{\beta t}\bigg[\int_{|y|_V<1} E\|G(t,x,y)\|^2\, v(dy) + \int_{|y|_V>1} E\|F(t,x,y)\|^2\, v(dy)\bigg]\bigg\}$$
$$< \infty, \quad (t,x,y) \in \mathbb{R} \times L^2(\mathbb{P},\mathbb{H}) \times V,$$
$$\sup_{E\|x\|^2\le\varrho} \big\{e^{\beta t_i} E\|I_i(x)\|^2\big\} < \infty, \quad i \in \mathbb{Z},\ x \in L^2(\mathbb{P},\mathbb{H}).$$

Theorem 4.1. *Assume that assumptions of Theorem* 3.2 *hold and, in addition, hypotheses* (B1) *and* (B2) *are satisfied. Then the piecewise weighted pseudo almost periodic in distribution mild solution of* (1.1)–(1.2) *is exponentially stable in mean square.*

Proof. Let $x(\cdot)$ be a fixed point of Ψ in $\mathbb{Y}$. By Theorem 3.2, any fixed point of Ψ is a mild solution of the system (1.1)–(1.2). We now can choose a positive constant β such that $0 < \beta < \delta$, and

$$e^{\beta t} E\|x(t)\|^2 \le \Gamma_1,$$

where

$$\begin{aligned}
\Gamma_1 &= 10e^{\beta t} E\bigg\|\int_{-\infty}^t T(t-s)g(s,x(s))\,ds\bigg\|^2 \\
&\quad + 10e^{\beta t} E\bigg\|\int_{-\infty}^t T(t-s)f(s,x(s))\,dW(s)\bigg\|^2 \\
&\quad + 10e^{\beta t} E\bigg\|\int_{-\infty}^t \int_{|y|_V<1} T(t-s)G(s,x(s-),y)\widetilde{N}(ds,dy)\bigg\|^2 \\
&\quad + 10e^{\beta t} E\bigg\|\int_{-\infty}^t \int_{|y|_V>1} T(t-s)F(s,x(s-),y)N(ds,dy)\bigg\|^2 \\
&\quad + 10e^{\beta t} E\bigg\|\sum_{t_i<t} T(t-t_i)I_i(x(t_i))\bigg\|^2 := \sum_{j=1}^5 \widetilde{\mu}_j
\end{aligned}$$

and on the other hand, by using (B1), (B2) and combine Step 2 in Theorem 3.2, we have for all $t \in \mathbb{R}$ and $x \in L^2(\mathbb{P}, \mathbb{H})$, $y \in V$,

$$E\|g(t,x)\|^2 + E\|f(t,x)\|^2_{L_2^0} \le \varsigma l_1 E\|x\|^2 + \nu_1 e^{-\beta t}, \tag{4.1}$$

$$\int_{|y|_V<1} E\|G(t,x,y)\|^2 \, v(dy) + \int_{|y|_V>1} E\|F(t,x,y)\|^2 \, v(dy) \le \varsigma l_2 E\|x\|^2 + \nu_2 e^{-\beta t}, \tag{4.2}$$

$$E\|I_i(x)\|^2 \le \varsigma c_i E\|x\|^2 + \nu_3, \quad i \in \mathbb{Z}. \tag{4.3}$$

Note that, for ς sufficiently small, we have

$$\begin{aligned} 20M^2\varsigma\bigg\{ \bigg[\frac{1}{\delta(\delta-\beta)}\frac{1}{2\delta-\beta}\bigg] l_1 + \bigg[\frac{1}{2\delta-\beta} + 2\frac{1}{2\delta-\beta} + \frac{b}{\delta(\delta-\beta)}\bigg] l_2 \\ + \frac{1}{1-e^{-\delta\alpha}(1-e^{-(\delta-\beta)\alpha})} \sup_{i\in\mathbb{Z}} c_i \bigg\} < 1. \end{aligned} \tag{4.4}$$

Then by (4.1) and Hölder's inequality, we have

$$\begin{aligned} \widetilde{\mu}_1 &\le 10M^2 e^{\beta t}\bigg(\int_{-\infty}^t e^{-\delta(t-s)} \, ds\bigg)\bigg(\int_{-\infty}^t e^{-\delta(t-s)} E\|g(s,x(s))\|^2 \, ds\bigg) \\ &\le 10M^2 \frac{1}{\delta} e^{\beta t}\bigg(\int_{-\infty}^t e^{-\delta(t-s)} [\varsigma l_1 E\|x(s)\|^2 + \nu_1 e^{-\beta s}] \, ds\bigg) \\ &\le 10M^2 \frac{1}{\delta}\bigg(\int_{-\infty}^t e^{-(\delta-\beta)(t-s)} [\varsigma l_1 e^{\beta s} E\|x(s)\|^2 + \nu_1] \, ds\bigg) \\ &\le 10M \frac{1}{\delta(\delta-\beta)}\Big[\varsigma l_1 \sup_{s\in\mathbb{R}} e^{\beta s} E\|x(s)\|^2 + \nu_1\Big]. \end{aligned}$$

By (4.2) and the Itô integral, we have

$$\begin{aligned} \widetilde{\mu}_2 &\le 10M^2 e^{\beta t} \int_{-\infty}^t e^{-2\delta(t-s)} E\|f(s,x(s))\|^2_{L_2^0} \, ds \\ &\le 10M^2 e^{\beta t} \int_{-\infty}^t e^{-\delta\delta(t-s)} [\varsigma l_1 E\|x(s)\|^2 + \nu_1 e^{-\beta s}] \, ds \\ &\le 10M^2 \int_{-\infty}^t e^{-(2\delta-\beta)(t-s)} [l_1 e^{\beta s} E\|x(s)\|^2 + \nu_1] \, ds \\ &\le 10M^2 \frac{1}{2\delta-\beta}\Big[\varsigma l_1 \sup_{s\in\mathbb{R}} e^{\beta s} E\|x(s)\|^2 + \nu_1\Big]. \end{aligned}$$

By (4.3) and the properties of the integral for the Poisson random measure, we have

$$\begin{aligned} \widetilde{\mu}_3 &\le 10M^2 e^{\beta t} \int_{-\infty}^t \int_{|y|_V<1} e^{-2\delta(t-s)} E\|G(s,x(s),y)\|^2 \, v(dy) \, ds \\ &\le 10M^2 e^{\beta t} \int_{-\infty}^t e^{-2\delta(t-s)} [\varsigma l_2 E\|x(s)\|^2 + \nu_2 e^{-\beta s}] \, ds \end{aligned}$$

$$
\begin{aligned}
&\le 10M^2 \int_{-\infty}^{t} e^{-(2\delta-\beta)(t-s)}[\varsigma l_2 e^{\beta s} E\|x(s)\|^2 + \nu_2]\, ds \\
&\le 10M^2 \frac{1}{2\delta-\beta}\Big[\varsigma l_2 \sup_{s\in\mathbb{R}} e^{\beta s} E\|x(s)\|^2 + \nu_2\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{\mu}_4 &\le 20M^2 e^{\beta t}\int_{-\infty}^{t}\int_{|y|_V>1} e^{-2\delta(t-s)} E\|F(s,x(s),y)\|^2\, v(dy)\, ds \\
&\quad + 20M^2 \frac{b}{\delta} e^{\beta t}\int_{-\infty}^{t}\int_{|y|_V>1} e^{-\delta(t-s)} E\|F(s,x(s),y)\|^2\, v(dy)\, ds \\
&\le 20M^2 e^{\beta t}\int_{-\infty}^{t} e^{-2\delta(t-s)}[\varsigma l_2 E\|x(s)\|^2 + \nu_2 e^{-\beta s}]\, ds \\
&\quad + 20M^2 \frac{b}{\delta} e^{\beta t}\int_{-\infty}^{t} e^{-\delta(t-s)}[\varsigma l_2 E\|x(s)\|^2 + \nu_2 e^{-\beta s}]\, ds \\
&\le 20M^2 \int_{-\infty}^{t} e^{-(2\delta-\beta)(t-s)}[\varsigma l_2 e^{\beta s} E\|x(s)\|^2 + \nu_2]\, ds \\
&\quad + 20M^2 \frac{b}{\delta}\int_{-\infty}^{t} e^{-(\delta-\beta)(t-s)}[\varsigma l_2 e^{\beta s} E\|x(s)\|^2 + \nu_2]\, ds \\
&\le 20M^2\bigg(\frac{1}{2\delta-\beta} + \frac{b}{\delta(\delta-\beta)}\bigg)\Big[\varsigma l_2 \sup_{s\in\mathbb{R}} e^{\beta s} E\|x(s)\|^2 + \nu_2\Big].
\end{aligned}
$$

By (4.3) and Hölder's inequality again, we have

$$
\begin{aligned}
\widetilde{\mu}_5 &\le 10M^2 e^{\beta t}\bigg(\sum_{t_i<t} e^{-\delta(t-t_i)}\bigg)\bigg(\sum_{t_i<t} e^{-\delta(t-t_i)} E\|I_i(x(t_i))\|^2\bigg) \\
&\le 10M^2 \frac{1}{1-e^{-\delta\alpha}} e^{\beta t}\bigg(\sum_{t_i<t} e^{-\delta(t-t_i)}[\varsigma c_i E\|x(t_i)\|^2 + \nu_3 e^{\beta t_i}]\bigg) \\
&\le 10M^2 \frac{1}{1-e^{-\delta\alpha}}\bigg(\sum_{t_i<t} e^{-(\delta-\beta)(t-t_i)}[\varsigma c_i e^{\beta t_i} E\|x(t_i)\|^2 + \nu_3]\bigg) \\
&\le 10M^2 \frac{1}{1-e^{-\delta\alpha}(1-e^{-(\delta-\beta)\alpha})}\Big[\sup_{i\in\mathbb{Z}} \varsigma c_i e^{\beta t_i} E\|x(t_i)\|^2 + \nu_3\Big].
\end{aligned}
$$

Then we have for all $t \in \mathbb{R}$,

$$
\Gamma_1 \le \widetilde{L}_0 \sup_{s\in\mathbb{R}} e^{\beta s} E\|x(s)\|^2 + \widetilde{L}_1,
$$

where $\widetilde{L}_0 = 10M^2\varsigma\big\{\big[\frac{1}{\delta(\delta-\beta)}\frac{1}{2\delta-\beta}\big]l_1 + \big[\frac{1}{2\delta-\beta} + 2\frac{1}{2\delta-\beta} + \frac{b}{\delta(\delta-\beta)}\big]l_2 + \frac{1}{1-e^{-\delta\alpha}(1-e^{-(\delta-\beta)\alpha})}\sup_{i\in\mathbb{Z}} c_i\big\}$, $\widetilde{L}_1 = 10M^2\big\{\big[\frac{1}{\delta(\delta-\beta)}\frac{1}{2\delta-\beta}\big]\nu_1 + \big[\frac{1}{2\delta-\beta} + 2\frac{1}{2\delta-\beta} + \frac{b}{\delta(\delta-\beta)}\big]\nu_2 + \frac{1}{1-e^{-\delta\alpha}(1-e^{-(\delta-\beta)\alpha})}\nu_3\big\}$. Thus, from the above inequality, it follows that

$$
e^{\beta t} E\|x(t)\|^2 \le 2\widetilde{L}_0 \sup_{s\in\mathbb{R}} e^{\beta s} E\|x(s)\|^2 + 2\widetilde{L}_1.
$$

By (4.4), we get that $2\widetilde{L}_0 < 1$ and

$$\sup_{t\in\mathbb{R}} e^{\beta t} E\|x(t)\|^2 \le \frac{2\widetilde{L}_1}{1-2\widetilde{L}_0}.$$

Then we get that $E\|x(t)\|^2 \le \frac{2\widetilde{L}_1}{1-2\widetilde{L}_0} e^{-\beta t}$, which is implies that the piecewise weighted pseudo almost periodic in distribution mild solution of (1.1)–(1.2) is exponentially stable in mean square. The proof is completed. □

Remark 4.2. In particular, if the exponentially stable C_0-semigroup is replaced by an exponentially stable analytic semigroup in the system (1.1)–(1.2). Then, we are also concerned with the case which associated semigroup is analytic and the nonlinear term is the non-Lipschitz conditions, some sufficient conditions about the exponentially stable of mild solution for the new systems by using weaker conditions in the sense of the fractional power arguments are established. In this way, we have improved the stability results of this article.

5. An example

In this section, we shall illustrate the analytical results. Consider following partial stochastic differential equations of the form

$$\begin{aligned}\frac{\partial z(t,x)}{\partial t} &= \frac{\partial^2}{\partial x^2} z(t,x) + \kappa(t) b_1(x) z(t,x) + \kappa(t) b_2(x) z(t,x) \frac{\partial W}{\partial t}(t,x) \\ &\quad + \kappa(t) b_3(x) z(t,x) \frac{\partial Z}{\partial t}(t,x), \quad t\in\mathbb{R},\ x\in[0,\pi],\end{aligned} \tag{5.1}$$

$$z(t,0) = z(t,\pi) = 0, \quad t\in\mathbb{R}, \tag{5.2}$$

$$\triangle z(t_i,x) = \kappa(i) z(t_i,x), \quad i\in\mathbb{Z},\ x\in[0,\pi], \tag{5.3}$$

where W is a Q-Wiener process on $L^2([0,\pi])$ with $\operatorname{Tr} Q < \infty$ and Z is a Lévy pure jump process on $L^2([0,\pi])$ which is independent of W. The function $\kappa\colon \mathbb{R}\to\mathbb{R}$ is given by $\kappa(t) = \sin\frac{1}{1+\cos t+\cos\sqrt{2}t} + e^{2t}$ for $t\in\mathbb{R}$, and $b_j\colon [0,\pi]\to\mathbb{R}$, $j=1,2,3$, are continuous functions. In this system, $t_i = i + \frac{1}{4}|\sin i + \sin\sqrt{2}i|$, $\{t_i^j\}$, $i\in\mathbb{Z}$, $j\in\mathbb{Z}$ are equipotentially almost periodic and $\alpha = \inf_{i\in\mathbb{Z}}(t_{i+1}-t_i) > 0$, one can see [26] for more details.

Let $\mathbb{H} = V = L^2([0,\pi])$ with the norm $\|\cdot\|$ and define the operators $A\colon D(A)\subseteq H\to H$ by $A\omega = \omega''$ with the domain

$$D(A) := \{\omega\in\mathbb{H} : \omega, \omega' \text{ are absolutely continuous}, \omega''\in\mathbb{H}, \omega(0)=\omega(\pi)=0\}.$$

It is well known that A generates a strongly continuous semigroup $T(\cdot)$ which is compact, analytic and self-adjoint. Furthermore, A has a discrete spectrum; the eigenvalues are

$-n^2$, $n \in \mathbb{N}$, with the corresponding normalized eigenvectors $z_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, and the semigroup $(T(t))_{t\geq 0}$ is hyperbolic as $\sigma(A) \cap i\mathbb{R} = \emptyset$. Moreover, $\|T(t)\| \leq e^{-t}$ for $t \geq 0$ for all $t > 0$.

Set $\varphi(t)(x) = \varphi(t,x)$ for $x \in [0,\pi]$, $y \in V$. We define the the maps $g\colon \mathbb{R} \times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$, $f\colon \mathbb{R} \times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$, $G, F\colon \mathbb{R} \times L^2(\mathbb{P},\mathbb{H}) \times V \to L^2(\mathbb{P},\mathbb{H})$, $I_i\colon L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ by

$$\begin{aligned} g(t,\varphi)(x) &= \kappa(t) b_1(x) \varphi(t)(x), & f(t,\varphi)(x) &= \kappa(t) b_2(x) \varphi(t)(x), \\ h(t,\varphi)(x) &= \kappa(t) b_3(x) \varphi(t)(x), & I_i(\varphi)(x) &= \kappa(i) \varphi(x). \end{aligned}$$

Using these definitions, we can represent the system (5.1)–(5.3) in the abstract form (1.1)–(1.2), where

$$h(t,\varphi)\, dZ = \int_{|y|_V<1} G(t,\varphi(t-),y) \widetilde{N}(dt,dy) + \int_{|y|_V>1} F(t,\varphi(t-),y) N(dt,dy)$$

with

$$Z(t,x) = \int_{|y|_V<1} y \widetilde{N}(dt,dy) + \int_{|y|_V>1} y N(dt,dy)$$

and

$$G(t,\varphi(t-),y) = h(t,\varphi) y \cdot 1_{|y|_V<1}, \quad F(t,\varphi(t-),y) h(t,\varphi) y \cdot 1_{|y|_V>1}.$$

Obviously, $\kappa(\cdot)$ is the almost periodic component and the weighted function

$$\rho(t) = \begin{cases} 1 & \text{for } t < 0, \\ e^{-2t} & \text{for } t \geq 0. \end{cases}$$

Then, $\lim_{r\to\infty} \mu(r,\rho) = \infty$ and hence $\rho \in \mathbb{U}_\infty$. The function e^{2t} is the weighted component which satisfies

$$\lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} e^t \rho(t)\, dt = 0.$$

It follows that g, f are weighted pseudo almost periodic and G, F are weighted pseudo almost periodic functions. Then, it satisfies all the assumptions given in Theorem 4.1. Therefore, the piecewise weighted pseudo almost periodic mild solution of (5.1)–(5.3) exponentially stable in mean square.

6. Conclusions

This paper considers a class of impulsive partial stochastic differential equations driven by Lévy noise in Hilbert spaces under non-Lipschitz conditions. Firstly, by means of exponential stable property, the stochastic analysis techniques and a fixed-point theorem for condensing maps, we discuss the existence of mild solutions to systems (1.1)–(1.2) under the nonlinear terms and the jump operators satisfy the non-Lipschitz conditions. Secondly, we present exponential stability of mean square piecewise pseudo almost periodic mild solutions to these equations. Finally, an example is provided to show the effectiveness of the proposed results.

Some interesting questions deserve further investigation. One may propose more realistic but complex equations, for example, we will be devoted to studying a class of impulsive partial stochastic differential equations with Lévy noise and not instantaneous impulses. Moreover, it is of significance to study fractional impulsive partial stochastic differential equations driven by Lévy noise. Another problem of interest is to consider optimal mild solutions of systems governed by impulsive partial differential equations under both Rosenblatt process and Lévy process.

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Fangxia Lu
School of Mathematics and Statistics, Hexi University, Zhangye, Gansu 734000, China
E-mail address: `zhylfx@163.com`