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# A Mean Field Type Flow on a Closed Riemannian Surface with the Action of an Isometric Group

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Abstract. Let $(\Sigma, g)$ be a closed Riemannian surface, $\mathbf{G} = \{\sigma_1, \ldots, \sigma_N\}$ be an isometric group acting on it. Denote a positive integer $\ell = \min_{x \in \Sigma} I(x)$, where $I(x)$ is the number of all distinct points of the set $\{\sigma_1(x), \ldots, \sigma_N(x)\}$. By a method of flow due to Castéras (Pacific J. Math. 2015), we prove that the solution to the mean field equation

$$-\Delta_g u = 8\pi\ell \left( \frac{he^u}{\int_\Sigma he^u \, dv_g} - \frac{1}{\mathrm{Vol}_g(\Sigma)} \right)$$

exists under given conditions. This gives a new proof of Yang and Zhu's result in (Internat. J. Math. 2020). The case $\ell = 1$ was studied by Li and Zhu (Calc. Var. Partial Differential Equations 2019).

## 1. Introduction

Let $(\Sigma, g)$ be a closed Riemannian surface and $\Delta$ be the Laplace-Beltrami operator with respect to the metric $g$. The famous mean field equation is stated as follows:

$$-\Delta u = \rho \left( \frac{he^u}{\int_\Sigma he^u \, dv_g} - \frac{1}{\mathrm{Vol}_g(\Sigma)} \right), \tag{1.1}$$

where $\rho$ is some real number, $h \in C^\infty(\Sigma)$, and $\mathrm{Vol}_g(\Sigma)$ stands for the volume of $\Sigma$. For $\rho < 8\pi$, Ding, Jost, Li and Wang [12] proved that (1.1) has a solution when $h$ is a smooth positive function; for $\rho = 8\pi$, a sufficient condition for existence of solutions to (1.1) is given by Yang and Zhu [23] when $h \geq 0$ and $h \not\equiv 0$. When $\Sigma$ is a flat torus, it was independently proved by Nolasco and Tarantello [20] that (1.1) has a solution for $\rho = 8\pi$. While the problem on $\mathbb{S}^2$ is much more complicated and known as the Nirenberg problem. For works in this direction, we refer the reader to [4,5,9–11,15,18,19]. When $\rho \in (8\pi, 4\pi^2)$ and $h \equiv 1$, Struwe and Tarantello [22] pointed out that the solutions of (1.1) are nontrivial under the assumption that $\Sigma$ is flat torus with a fundamental domain. For $\rho \in (8\pi, 16\pi)$, it was proved by Ding, Jost, Li and Wang [13] that (1.1) exists a non-minimal solution. In the case $\rho \neq 8N\pi$, $\forall\, N \in \mathbb{N}$, Chen and Lin [6,7] obtained a degree-counting formula

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for (1.1) provided that the genus of $\Sigma$ is positive. Later, the result was generalized by Malchiodi [17] to $\rho \in (8m\pi, 16m\pi)$ ($m \in \mathbb{Z}^+$) when $\Sigma$ is a general Riemannian surface. For the recent work, Li and Zhu [16] showed that under certain assumptions, (1.1) has a smooth solution with $\rho = 8\pi$ on a closed Riemannian surface.

Let $\mathbf{G} = \{\sigma_1, \ldots, \sigma_N\}$ be a finite isometric group acting on a closed Riemannian surface $(\Sigma, g)$, and $u\colon \Sigma \to \mathbb{R}$ be a measurable function, we say that $u \in \mathscr{I}_{\mathbf{G}}$ if $u$ is $\mathbf{G}$-invariant, namely $u(\sigma_i(x)) = u(x)$ for any $1 \leq i \leq N$ and almost every $x \in \Sigma$. Define a Hilbert space

$$\mathscr{H}_{\mathbf{G}} = \left\{ u \in W^{1,2}(\Sigma, g) \cap \mathscr{I}_{\mathbf{G}} : \int_\Sigma u \, dv_g = 0 \right\} \tag{1.2}$$

with an inner product $\langle u, v\rangle_{\mathscr{H}_{\mathbf{G}}} = \int_\Sigma \langle \nabla u, \nabla v\rangle \, dv_g$, where $\langle \nabla u, \nabla v\rangle$ stands for the Riemannian inner product of $\nabla u$ and $\nabla v$. Denote

$$\ell = \min_{x \in \Sigma} I(x) \tag{1.3}$$

with $I(x) = \sharp \mathbf{G}(x)$, where $\sharp A$ stands for the number of all distinct points in the set $A$, and $\mathbf{G}(x) = \{\sigma_1(x), \ldots, \sigma_N(x)\}$ for any $x \in \Sigma$. Recently, Yang and Zhu [25] extended Ding, Jost, Li and Wang's result [12] to $(\Sigma, g)$ with an isometric group action $\mathbf{G}$. Precisely, for $\rho = 8\pi\ell$ and $u \in \mathscr{H}_{\mathbf{G}}$, they considered the functionals

$$\widetilde{J}_{8\pi\ell(1-\epsilon)}(u) = \frac{1}{2} \int_\Sigma |\nabla_g u|^2 \, dv_g - 8\pi\ell(1-\epsilon) \log \int_\Sigma h e^u \, dv_g,$$

where $h$ is a smooth positive function and $h(\sigma(x)) = h(x)$ for all $\sigma \in \mathbf{G}$ and all $x \in \Sigma$. For any $0 < \epsilon < 1$, it follows from Chen [8] and a direct method of variation that $\widetilde{J}_{8\pi\ell(1-\epsilon)}$ attains its minimum at some minimizer $u_\epsilon$. While if $\widetilde{J}_{8\pi\ell}$ has no minimizer on $\mathscr{H}_{\mathbf{G}}$, using a method of blow-up analysis, they obtain

$$\inf_{u \in \mathscr{H}_{\mathbf{G}}} \widetilde{J}_{8\pi\ell}(u) \geq -4\pi\ell \max_{x \in \Sigma} (2 \log(\pi\ell h(x)) + \widetilde{A}_x) - 8\pi\ell, \tag{1.4}$$

where $\widetilde{A}_x = \lim_{r \to 0} (\widetilde{G}_x(y) + 4 \log r)$ is a constant, $r$ denotes the geodesic distance between $x$ and $y$, $\widetilde{G}_x$ is a Green function satisfying

$$\Delta_g \widetilde{G}_x = \frac{8\pi\ell}{\mathrm{Vol}_g(\Sigma)} - 8\pi \sum_{i=1}^{\ell} \delta_{\sigma_i(x)} \quad \text{and} \quad \int_\Sigma \widetilde{G}_x \, dv_g = 0.$$

Clearly, the minimizer is a solution of (1.1). Moreover, for works of related issues, we refer the reader to Fang and Yang [14] and Yang and Zhu [24].

Castéras [2] investigated a gradient flow related to the mean field equation (1.1). Continuing [2], Castéras [3] obtained the global existence of the flow. The mean field type flow in [2, 3] is presented as follows:

$$
\begin{cases} \frac{\partial}{\partial t} e^v = \Delta v - Q + \rho \frac{e^v}{\int_\Sigma e^v \, dv_g}, \\ v(x,0) = v_0(x), \end{cases} \tag{1.5}
$$

where $v_0 \in C^{2+\alpha}(\Sigma)$, $\alpha \in (0,1)$ is the initial data and $Q \in C^\infty(\Sigma)$ is a given function such that $\int_\Sigma Q \, dv_g = \rho$. It is a gradient flow involving the functional

$$
J_\rho(v(t)) = \frac{1}{2} \int_\Sigma |\nabla v(t)|^2 \, dv_g + \int_\Sigma Q v(t) \, dv_g - \rho \log \int_\Sigma e^{v(t)} \, dv_g. \tag{1.6}
$$

Suppose $h \in C^\infty(\Sigma)$ is a positive function, and $h$ satisfies

$$
\Delta \log h = Q - \rho. \tag{1.7}
$$

Using the flow due to [2, 3], Li and Zhu [16] gave a new proof to the results of [12]. Motivated by [16, 25], it is natural for us to consider the same question as in [25] by the method of flow. Our aim is to prove the convergence of the mean field type flow (1.5) on $(\Sigma, g)$ with an isometric group action. Different from Yang and Zhu [25], it is not required to assume $\int_\Sigma v \, dv_g = 0$ in our paper. Here we define a Hilbert space

$$
\mathscr{H}^n_{\mathbf{G}} = \left\{ v \in W^{n,2}(\Sigma, g) \cap \mathscr{I}_{\mathbf{G}} \right\}, \quad n = 1, 2, \tag{1.8}
$$

where $\mathscr{I}_{\mathbf{G}}$ is defined as in (1.2).

Then our main result reads

**Theorem 1.1.** *Let $(\Sigma, g)$ be a closed Riemannian surface, $\mathbf{G} = \{\sigma_1, \dots, \sigma_\ell\}$ be an isometric group acting on it. Define a function space $\mathscr{H}^1_{\mathbf{G}}$ as in (1.8) and a function $I(x)$ as in (1.3). Let $v(t) \in \mathscr{H}^1_{\mathbf{G}}$ be the solution of (1.5), and $Q$ be a smooth function in (1.6), satisfying $Q(\sigma(x)) = Q(x)$ for all $\sigma \in \mathbf{G}$ and all $x \in \Sigma$. Suppose that $I(x) \equiv \ell$ for all $x \in \Sigma$, and that $2 \log h(x) + \widetilde{A}_x$ achieves its maximum at some point $p \in \Sigma$, where $h(x)$ and $\widetilde{A}_x$ are defined in (1.7) and (1.4) respectively. If in addition*

$$
Q(p) > 2K(p), \tag{1.9}
$$

*where $K(p)$ denotes the Gaussian curvature of $(\Sigma, g)$ at $p$, then for $\rho = 8\pi\ell$, there exists an initial data $v_0 \in C^{2+\alpha}(\Sigma)$ such that $v(t)$ converges in $H^2(\Sigma)$ to a solution $v_\infty \in C^\infty(\Sigma)$ of*

$$
-\Delta v_\infty + Q = 8\pi\ell \frac{e^{v_\infty}}{\int_M e^{v_\infty} \, dv_g}. \tag{1.10}
$$

The proof of Theorem 1.1 is based on the works of [2,3,16] related with a gradient flow. Let us describe its outline. To prove the convergence of the flow in (1.5) with $\rho = 8\pi\ell$, we first study some properties of the flow and then we get the compactness theorem. It is shown that we have the following alternative: either $v(t_k)$ is compact or $v(t_k)$ blows up, where $v(t_k)$ is a subsequence of $v(t)$ as $t_k \to \infty$. Next, we suppose blow-up happens. By blow-up analysis, we derive

$$\begin{aligned}\lim_{t\to+\infty} J_{8\pi\ell}(v(t)) \geq -4\pi\ell \max_{x\in\Sigma}(2\log(\pi\ell h(x)) + \widetilde{A}_x) - 8\pi\ell \\ + 8\pi\ell \int_\Sigma \log h \, dv_g - \frac{1}{2}\int_\Sigma |\nabla \log h|^2 \, dv_g,\end{aligned}$$

where $h(x)$ and $\widetilde{A}_x$ are defined in (1.7) and (1.4) respectively. However, under the hypothesis (1.9), we construct a sequence of initial data $v_{0,\varepsilon}$ such that

$$\begin{aligned}J_{8\pi\ell}(v_{0,\varepsilon}) < -4\pi\ell \max_{x\in\Sigma}(2\log(\pi\ell h(x)) + \widetilde{A}_x) - 8\pi\ell \\ + 8\pi\ell \int_\Sigma \log h \, dv_g - \frac{1}{2}\int_\Sigma |\nabla \log h|^2 \, dv_g,\end{aligned}$$

which makes a contradiction, since $J_{8\pi\ell}(v(t))$ decreases in $t$. Thus, we exclude the blow-up phenomenon. According to the monotonicity of $J_{8\pi\ell}(v(t))$, under some appropriate initial data $v_{0,\varepsilon}$, we finally prove the solution of (1.5) converges to a solution $v_\infty \in C^\infty(\Sigma)$ of (1.10). Since the equation (1.10) is equivalent to the mean field equation (1.1), we conclude that (1.1) has a smooth solution for $\rho = 8\pi\ell$. This ends the proof of Theorem 1.1. For the special case $\mathbf{G} = \{\mathrm{Id}\}$, where $\mathrm{Id}\colon \Sigma \to \Sigma$ is the identity map, our results are reduced to that of Li and Zhu [16]. Though the method we employ is similar to [16], there are many technical difficulties to be smoothed. Furthermore, by the symmetric properties of $(\Sigma, g)$, we deal with the singular points in constructing Green functions to derive the lower bound of $J_{8\pi\ell}(v(t))$.

According to Yang and Zhu [24], one can raise the same question for the functional

$$J_{\alpha,\beta}(u) = \frac{1}{2}\int_\Sigma \left(|\nabla_g u|^2 - \alpha u^2\right) dv_g - \beta \log \int_\Sigma h e^u \, dv_g$$

on a function space $\mathscr{H} = \left\{u \in W^{1,2}(\Sigma, g) : \int_\Sigma u \, dv_g = 0\right\}$. It is also interesting to consider the existence of solutions to (1.1) through the method of flow.

Note that $\frac{\partial}{\partial t}\int_\Sigma e^{v(t)} \, dv_g = 0$ by (1.5). This leads to $\int_\Sigma e^{v(t)} \, dv_g = C$. Hereafter, we can assume without loss of generality that $\int_\Sigma e^{v(t)} \, dv_g = 1$. The remaining part of this paper is to prove Theorem 1.1. Throughout this paper, we assume the volume of $\Sigma$ equals to 1, and we write $v_k = v(t_k)$ for simplicity. Moreover, sequence and subsequence are not distinguished, and various constants are often denoted by the same $C$ from line to line.

## 2. Proof of Theorem 1.1

In this section, we begin by studying some properties of the flow. Following the same arguments of [3, Theorem 0.1], we can obtain the global solution of the flow (1.5) on a closed Riemann surface with an isometric group action. As an obvious analogue of Proposition 2.1 in [16], we prove

**Proposition 2.1.** *Let $v(t) \in \mathscr{H}^1_{\mathbf{G}}$ be the solution of* (1.5) *with $\rho = 8\pi\ell$. For all $t \geq 0$, there holds*

$$J_{8\pi\ell}(v(t)) \geq -C, \tag{2.1}$$

*where $C > 0$ is a constant not depending on $t$ and $\mathscr{H}^1_{\mathbf{G}}$ is defined in* (1.8).

*Proof.* Denote $\overline{v} = \int_\Sigma v\, dv_g$. Since $\int_\Sigma Q\, dv_g = 8\pi\ell$, we have

$$J_{8\pi\ell}(v(t)) = \frac{1}{2}\int_\Sigma |\nabla v(t)|^2\, dv_g + \int_\Sigma Q(v(t) - \overline{v})\, dv_g - 8\pi\ell \log \int_\Sigma e^{v(t)-\overline{v}}\, dv_g. \tag{2.2}$$

According to Chen [8], one gets by Young's inequality

$$\log \int_\Sigma e^{v-\overline{v}}\, dv_g \leq \log \int_\Sigma e^{\frac{1}{16\pi\ell}\|\nabla v\|_2^2 + 4\pi\ell \frac{v^2}{\|\nabla v\|_2^2}}\, dv_g \leq \frac{1}{16\pi\ell}\int_\Sigma |\nabla v|^2\, dv_g + C. \tag{2.3}$$

Inserting (2.3) into (2.2), we obtain

$$J_{8\pi\ell}(v(t)) \geq \int_\Sigma Q(v(t) - \overline{v})\, dv_g - C. \tag{2.4}$$

In view of (1.5) and (1.7), applying the integration by parts, one has

$$\begin{aligned} \int_\Sigma Q(v(t) - \overline{v})\, dv_g &= \int_\Sigma \Delta v \cdot \log h\, dv_g \\ &= \int_\Sigma \frac{\partial e^v}{\partial t} \log h\, dv_g + \int_\Sigma (Q - 8\pi\ell e^v) \log h\, dv_g. \end{aligned} \tag{2.5}$$

We estimate the two integrals on the right-hand side of (2.5) respectively. Taking the derivative with respect to $t$ of $J_{8\pi\ell}(v(t))$ in (1.6), one can check that

$$\frac{\partial}{\partial t} J_{8\pi\ell}(v(t)) = \int_\Sigma (-\Delta v + Q - 8\pi\ell e^v)\frac{\partial v}{\partial t}\, dv_g = -\int_\Sigma \left(\frac{\partial v}{\partial t}\right)^2 e^v\, dv_g, \tag{2.6}$$

due to (1.5). Since $Q \in C^\infty(\Sigma)$ and $h \in C^\infty(\Sigma)$, it follows from the Hölder inequality and (2.6) that

$$\begin{aligned} \int_\Sigma \frac{\partial e^v}{\partial t} \log h\, dv_g &\geq -\max_\Sigma |\log h| \left(\int_\Sigma \left(\frac{\partial v}{\partial t}\right)^2 e^v\, dv_g\right)^{1/2} \\ &= -\max_\Sigma |\log h| \left(-\frac{\partial}{\partial t} J_{8\pi\ell}(v(t))\right)^{1/2}, \end{aligned} \tag{2.7}$$

and that

$$\int_\Sigma (Q - 8\pi\ell e^v)\log h\, dv_g \geq \int_\Sigma Q\log h\, dv_g - 8\pi\ell \max_\Sigma |\log h|. \tag{2.8}$$

Combing (2.4), (2.5), (2.7) and (2.8), we obtain

$$J_{8\pi\ell}(v(t)) \geq -8\pi\ell\max_\Sigma|\log h| + \int_\Sigma Q\log h\, dv_g - \max_\Sigma|\log h|\left(-\frac{\partial}{\partial t}J_{8\pi\ell}(v(t))\right)^{1/2} - C. \tag{2.9}$$

If $\max_\Sigma |\log h| = 0$, we can get the desired result directly. In the following, suppose $\max_\Sigma |\log h| > 0$. Then (2.9) can be rewritten as

$$\frac{J_{8\pi\ell}(v(t))}{\max_\Sigma|\log h|} \geq -8\pi\ell + \frac{\int_\Sigma Q\log h\, dv_g - C}{\max_\Sigma|\log h|} - \left(-\frac{\partial}{\partial t}J_{8\pi\ell}(v(t))\right)^{1/2}. \tag{2.10}$$

Denote

$$\xi = \frac{1}{\max_\Sigma|\log h|}, \quad \zeta = -8\pi\ell + \frac{\int_\Sigma Q\log h\, dv_g - C}{\max_\Sigma|\log h|}. \tag{2.11}$$

We claim that for any $t \geq 0$, there holds

$$\xi J_{8\pi\ell}(v(t)) - \zeta \geq 0. \tag{2.12}$$

For otherwise, there exists some $t_0 > 0$ such that for all $t \geq t_0$,

$$\xi J_{8\pi\ell}(v(t)) - \zeta < 0. \tag{2.13}$$

By (2.10), we have for any $t_1 \geq t_0$ that

$$\int_{t_0}^{t_1} -\frac{dJ_{8\pi\ell}(v(t))}{(-\xi J_{8\pi\ell}(v(t)) + \zeta)^2} \geq t_1 - t_0,$$

namely,

$$\left(-\xi(t_1 - t_0) + \frac{1}{-\xi J_{8\pi\ell}(v(t_0)) + \zeta}\right)(-\xi J_{8\pi\ell}(v(t_1)) + \zeta) \geq 1.$$

Letting $t_1 \to +\infty$, we find

$$\lim_{t\to+\infty} J_{8\pi\ell}(v(t)) \geq \frac{\xi}{\zeta},$$

which contradicts (2.13). Then, inserting (2.11) into (2.12), one gets by (1.7)

$$J_{8\pi\ell}(v(t)) \geq 8\pi\ell\int_\Sigma \log h\, dv_g - \int_\Sigma |\nabla \log h|^2\, dv_g - 8\pi\ell\max_\Sigma|\log h| - C.$$

Noting that $h \in C^\infty(\Sigma)$, we conclude (2.1). This completes the proof. $\square$

In view of (2.6), we can see that $J_{8\pi\ell}(v(t))$ decreases with respect to $t$. By integrating (2.6) from 0 to $t$, one finds

$$\int_0^t \int_\Sigma \left(\frac{\partial v}{\partial t}\right)^2 e^v \, dv_g dt = J_{8\pi\ell}(v(0)) - J_{8\pi\ell}(v(t)).$$

This together with Proposition 2.1 leads to

$$\int_0^\infty \int_\Sigma \left(\frac{\partial v}{\partial t}\right)^2 e^v \, dv_g dt < C < +\infty.$$

Thus, there exists a sequence $t_k \to +\infty$ such that

$$\lim_{t_k \to +\infty} \int_\Sigma \left(\frac{\partial v_k}{\partial t}\right)^2 e^{v_k} \, dv_g \to 0 \tag{2.14}$$

as $k \to +\infty$.

To proceed, we need the following estimate, which is similar to Proposition 2.1 in [3].

**Proposition 2.2.** *For $\rho = 8\pi\ell$, if $v(t) \in \mathscr{H}^1_{\mathbf{G}}$ is the solution of* (1.5), *then*

$$-\frac{\partial}{\partial t} e^{v(x,t)} + 8\pi\ell e^{v(x,t)} \geq -C, \quad \forall\, t \geq 0,\ \forall\, x \in \Sigma, \tag{2.15}$$

*where the constant $C > 0$ not depending on $t$, and $\mathscr{H}^1_{\mathbf{G}}$ is defined in* (1.8).

Since no new idea comes out in its proof, we omit the details here but refer the readers to [3]. Thanks to (2.14) and (2.15), we can see that the conditions in [2, Theorem 1.2] are satisfied by $v_k$. Following [2], we describe the compactness theorem as below.

**Theorem 2.3.** *Define a function space $\mathscr{H}^1_{\mathbf{G}}$ as in* (1.8). *Let $v(t) \in \mathscr{H}^1_{\mathbf{G}}$ be the solution of* (1.5) *with $\rho = 8\pi\ell$. Then for a sequence $t_k \to +\infty$, we have for $k \to +\infty$, either*
(i) *there exists a constant $C$ not depending on $k$ such that*

$$\|v_k\|_{H^2(\Sigma)} \leq C,$$

*or* (ii) *there exists a sequence of points $\{x_k\}$ and a sequence of real positive numbers $\{R_k\} \to 0$ such that*

$$\lim_{k \to +\infty} \int_{B_{2R_k(\sigma_i(x_k))}} e^{v_k} \, dv_g = \frac{1}{\ell}, \quad \forall\, i = 1, \ldots, \ell,$$

*where $B_{2R_k(\sigma_i(x_k))} \subset \Sigma$ denotes a geodesic ball centered at $\sigma_i(x_k)$ with radius $2R_k$, and*

$$\lim_{k \to +\infty} \int_{\Sigma \setminus \bigcup_{i=1}^{\ell} B_{2R_k(\sigma_i(x_k))}} e^{v_k} \, dv_g = 0.$$

In what follows, the sequence $v_k \subseteq H^2(\Sigma)$ is said to be compact if it is uniformly bounded in $H^2(\Sigma)$. Theorem 2.3 shows that we have the following alternative: either $v_k$ is compact or $v_k$ blows up. Subsequently, we will exclude the blow-up phenomenon to occur.

## 3. Blow-up analysis

Recall that $\mathscr{H}^2_{\mathbf{G}}$ is defined in (1.8). In this section, we study the asymptotic behavior of non-compact solutions $v_k \subseteq \mathscr{H}^2_{\mathbf{G}}$ in Theorem 2.3. Set

$$\Delta v_k = Q + \frac{\partial e^{v_k}}{\partial t} - 8\pi\ell e^{v_k} := F_k \tag{3.1}$$

where $\int_\Sigma e^{v_k}\, dv_g = 1$. Similar to [16], we discuss the convergence of $v_k$ near and away from the blow-up point $x_0$. Based on the blow-up analysis, we finally calculate

$$\begin{aligned}\lim_{t\to+\infty} J_{8\pi\ell}(v(t)) \geq -4\pi\ell \max_{x\in\Sigma}(2\log(\pi\ell h(x)) + \widetilde{A}_x) - 8\pi\ell \\ + 8\pi\ell \int_\Sigma \log h\, dv_g - \frac{1}{2}\int_\Sigma |\nabla \log h|^2\, dv_g,\end{aligned}$$

where $h(x)$ and $\widetilde{A}_x$ are defined in (1.7) and (1.4) respectively.

### 3.1. Asymptotic behavior near the blow-up point

According to [2, Proposition 3.1], the convergence of $v_k$ is described as follows:

**Proposition 3.1.** *Let $v_k \subseteq \mathscr{H}^2_{\mathbf{G}}$ be a sequence of non-compact solutions of* (3.1)*, satisfying* (2.14) *and* (2.15)*. Denote*

$$\widetilde{v}_k = v_k(\exp_{x_k}(r_k\cdot)) + 2\log r_k,$$

*where $\exp_{x_k}$ represents the exponential map centered in $x_k$ and $\mathscr{H}^2_{\mathbf{G}}$ is in* (1.8)*. Then, there exist a sequence of points $x_k$ and a sequence of real numbers $r_k$ such that as $k \to +\infty$, $\widetilde{v}_k \to \widetilde{v}_\infty$ in $C^\alpha_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\alpha \in (0,1)$, and weakly in $H^2_{\mathrm{loc}}(\mathbb{R}^2)$, where $\widetilde{v}_\infty$ is the solution of*

$$-\Delta \widetilde{v}_\infty = 8\pi\ell e^{\widetilde{v}_\infty}.$$

*Moreover, there exist $\lambda > 0$ and $\widetilde{x}_0 \in \mathbb{R}^2$ such that*

$$\widetilde{v}_\infty(x) = 2\log\frac{2\lambda}{1+(\lambda|x-\widetilde{x}_0|)^2} + \log\frac{1}{4\pi\ell}.$$

Since the proof of Proposition 3.1 is an obvious analog of that of [2, Proposition 3.1], we omit it, but refer the reader to [2] for details.

### 3.2. Convergence away from the blow-up point

Similar to [16], we have the following two observations of $v_k$, which are essential in our analysis. The difference is that $v_k$ is $\mathbf{G}$-invariant in our case, namely $v_k(\sigma_i(x)) = v_k(x)$ for any $1 \leq i \leq N$ and almost every $x \in \Sigma$. The first key observation is the following:

**Proposition 3.2.** *Let* $v_k \subseteq \mathscr{H}^2_{\mathbf{G}}$ *be a sequence of non-compact solutions of* (3.1), *satisfying* (2.14) *and* (2.15). *Then for any* $1 < p < 2$, *there holds*

$$\|v_k - \overline{v}_k\|_{W^{1,p}(\Sigma)} \leq C,$$

*where the constant* $C > 0$ *is independent of* $k$, *and* $\mathscr{H}^2_{\mathbf{G}}$ *is defined in* (1.8).

*Proof.* By the result of [25, Proposition 5], there exists a unique Green function $\widetilde{G}_x(y)$ on $(\Sigma, g)$, which is a distributional solution to

$$\Delta_g \widetilde{G}_x = \sum_{i=1}^{\ell} \delta_{\sigma_i(x)} - \ell. \tag{3.2}$$

Note that $v_k(\sigma(x)) = v_k(x)$ for all $\sigma \in \mathbf{G}$ and all $x \in \Sigma$. Then it follows from [1, Theorem 4.13] that

$$v_k(x) - \overline{v}_k = \frac{1}{\ell}\int_\Sigma \widetilde{G}_x(y) F_k(y)\, dv_g(y) \quad \text{for a.e. } x \in \Sigma, \tag{3.3}$$

and that

$$|\nabla v_k(x)| \leq \frac{1}{\ell}\int_\Sigma |\nabla \widetilde{G}_x(y)||F_k(y)|\, dv_g(y) \leq C \int_\Sigma \frac{1}{|x-y|}|F_k(y)|\, dv_g(y). \tag{3.4}$$

Combing (2.15) and (3.1), we deduce that

$$\|F_k\|_{L^1(\Sigma)} \leq 16\pi\ell + \left\| \frac{\partial e^{v_k}}{\partial t} \right\|_{L^1(\Sigma)} \leq C. \tag{3.5}$$

This together with (3.4), Jensen's inequality and Fubini's Theorem gives

$$\begin{aligned} \int_{B_r(x^*)} |\nabla v_k(x)|^p\, dv_g(x) &\leq \int_{B_r(x^*)} \int_\Sigma \|F_k\|^{p-1}_{L^1(\Sigma)} \frac{|F_k(y)|}{|x-y|^p}\, dv_g(y) dv_g(x) \\ &\leq C \sup_{y\in\Sigma} \int_{B_r(x^*)} \frac{1}{|x-y|^p}\, dv_g(x) \\ &\leq C r^{2-p}, \end{aligned} \tag{3.6}$$

where $B_r(x^*) \subset \Sigma$ denotes a ball centered at $x^*$ with radius $r > 0$. Noticing $\Sigma$ is compact, for any $1 < p < 2$, we can see $\|\nabla v_k(x)\|_{L^p(\Sigma)} \leq C$ by (3.6). Then by Poincaré's inequality, we get the desired result. This achieves the proof of the proposition. $\square$

To get the convergence of $v_k$ away from the blow-up point, we also need the proposition as below.

**Proposition 3.3.** *Let* $v_k \subseteq \mathscr{H}^2_{\mathbf{G}}$ *be a sequence of non-compact solutions of* (3.1), *satisfying* (2.14) *and* (2.15). *Then for each* $V \subset\subset \Sigma \setminus \left\{ \bigcup_{i=1}^{\ell} \sigma_i(x_0) \right\}$, *there exist constants* $C > 0$ *and* $\alpha > 1$ *such that*

$$\int_V e^{\alpha(v_k - \overline{v}_k)}\, dv_g \leq C,$$

*where* $\mathscr{H}^2_{\mathbf{G}}$ *is defined in* (1.8).

*Proof.* Let $V$ be any subset satisfying $V \subset\subset \Sigma \setminus \{\bigcup_{i=1}^{\ell} \sigma_i(x_0)\}$. Note that $v_k$ is non-compact. By the results of [2, Proposition 2.1] and Theorem 2.3, we have for any $x \in V$, there exists $r_x > 0$ such that for some $\delta_x > 0$

$$\int_{B_{r_x}(x)} |F_k| < 4\pi - \delta_x \tag{3.7}$$

in $B_{r_x}(x) \subset \Sigma \setminus \{\bigcup_{i=1}^{\ell} \sigma_i(x_0)\}$. Then we can find an integer $m$ satisfying $\overline{V} \subset \bigcup_{j=1}^{m} B_{r_{x_j}/2}(x_j)$, where $x_j \in V$. In view of (3.3), for $x \in B_{r_{x_j}/2}(x_j)$, one has by (3.5) that

$$\begin{aligned} e^{\alpha(v_k(x)-\overline{v}_k)} &= e^{\frac{\alpha}{\ell}\left(\int_{B_{r_{x_j}}(x_j)} \widetilde{G}_x(y)F_k(y)\,dv_g(y) + \int_{\Sigma\setminus B_{r_{x_j}}(x_j)} \widetilde{G}_x(y)F_k(y)\,dv_g(y)\right)} \\ &\le Ce^{\frac{\alpha}{\ell}\int_{B_{r_{x_j}}(x_j)} \widetilde{G}_x(y)F_k(y)\,dv_g(y)}, \end{aligned} \tag{3.8}$$

where $\alpha > 0$ is a constant and $\widetilde{G}_x(y)$ is in (3.2).

Set $\beta(y) = |F_k(y)\chi_{B_{r_{x_j}}(x_j)}| / \|F_k(y)\chi_{B_{r_{x_j}}(x_j)}\|_{L^1(\Sigma)}$. This together with (3.8) yields

$$\begin{aligned} &\int_{B_{r_{x_j}/2}(x_j)} e^{\alpha(v_k(x)-\overline{v}_k)}\,dv_g(x) \\ &\le C\int_{B_{r_{x_j}/2}(x_j)} \int_{\Sigma} \beta(y) e^{\frac{\alpha}{\ell}\|F_k(y)\chi_{B_{r_{x_j}}(x_j)}\|_{L^1(\Sigma)}|\widetilde{G}_x(y)|}\,dv_g(y)dv_g(x) \\ &\le C\sup_{y\in\Sigma}\int_{\Sigma}\left(\frac{1}{|x-y|}\right)^{\frac{\alpha}{2\pi\ell}\|F_k(y)\chi_{B_{r_{x_j}}(x_j)}\|_{L^1(\Sigma)}}\,dv_g(x). \end{aligned}$$

The first inequality is a direct consequence of Jensen's inequality. The second one follows from [1, Theorem 4.13]. Due to (3.7) and $\ell \ge 1$, there exists the constant $\alpha > 1$ such that

$$\frac{\alpha}{2\pi\ell}\|F_k(y)\chi_{B_{r_{x_j}}(x_j)}\|_{L^1(\Sigma)} < 2$$

for each $j \in \{1, \ldots, m\}$. As a consequence,

$$\int_V e^{\alpha(v_k-\overline{v}_k)}\,dv_g \le \sum_{j=1}^{m}\int_{B_{r_{x_j}/2}(x_j)} e^{\alpha(v_k(x)-\overline{v}_k)}\,dv_g(x) \le C.$$

Therefore, Proposition 3.3 is established. □

Recall that $h$ is defined as in (1.7). Denote $\mu_k = v_k - \log h$. It is clear that

$$\Delta(\mu_k - \overline{\mu}_k) = 8\pi\ell + \frac{\partial e^{v_k}}{\partial t} - 8\pi\ell e^{v_k}, \tag{3.9}$$

and that $\mu_k \subseteq \mathscr{H}^2_{\mathbf{G}}$. Then we obtain the proposition as follows.

**Proposition 3.4.** *Let $\mu_k$ be defined as above. For $1 < p < 2$ and some $0 < \gamma < 1$, there holds*

$$\begin{cases} \mu_k - \overline{\mu}_k \rightharpoonup \widetilde{G}_{x_0} & \textit{weakly in } W^{1,p}(\Sigma), \\ \mu_k - \overline{\mu}_k \to \widetilde{G}_{x_0} & \textit{strongly in } W^{2,2}_{\mathrm{loc}}\big(\Sigma \setminus \big\{ \bigcup_{i=1}^{\ell} \sigma_i(x_0) \big\}\big), \\ \mu_k - \overline{\mu}_k \to \widetilde{G}_{x_0} & \textit{in } C^{\gamma}_{\mathrm{loc}}\big(\Sigma \setminus \big\{ \bigcup_{i=1}^{\ell} \sigma_i(x_0) \big\}\big) \end{cases}$$

*as $k \to +\infty$, where the Green function $\widetilde{G}_{x_0}$ satisfies*

$$\Delta \widetilde{G}_{x_0} = 8\pi\ell - 8\pi \sum_{i=1}^{\ell} \delta_{\sigma_i(x_0)} \quad \textit{and} \quad \int_{\Sigma} \widetilde{G}_{x_0} \, dv_g = 0. \tag{3.10}$$

*Moreover, $\widetilde{G}_{x_0}$ takes the form*

$$\widetilde{G}_{x_0}(x) = -4 \log r + \widetilde{A}_{x_0} + O(r) \tag{3.11}$$

*near $\sigma_i(x_0)$, where $\widetilde{A}_{x_0}$ is a constant, $r$ denotes the geodesic distance between $x$ and $\sigma_i(x_0)$, $i = 1, \ldots, \ell$.*

*Proof.* Observe that $\log h \in C^{\infty}(\Sigma)$. By employing Proposition 3.2, we see that $\|\mu_k - \overline{\mu}_k\|_{W^{1,p}(\Sigma)} \leq C$ for any $1 < p < 2$. Since $\widetilde{G}_{x_0}(x)$ is the unique solution of (3.10), up to a subsequence, we have $\mu_k - \overline{\mu}_k \rightharpoonup \widetilde{G}_{x_0}$ weakly in $W^{1,p}(\Sigma)$ as $k \to +\infty$. Recall that $V \subset\subset \Sigma \setminus \big\{ \bigcup_{i=1}^{\ell} \sigma_i(x_0) \big\}$. Due to Proposition 3.3, we obtain by using the Jensen's inequality that

$$\int_V e^{\alpha v_k(x)} \, dv_g = e^{\alpha \overline{v}_k} \int_V e^{\alpha (v_k(x) - \overline{v}_k)} \, dv_g \leq C \left( \int_{\Sigma} e^{v_k(x)} \, dv_g \right)^{\alpha} \leq C, \tag{3.12}$$

where $\alpha > 1$. Together with Hölder's inequality and (2.14), it leads to

$$\int_V \left| \frac{\partial e^{v_k}}{\partial t} \right|^r dv_g \leq \left( \int_V \left( \frac{\partial v_k}{\partial t} \right)^2 e^{v_k} \, dv_g \right)^{r/2} \left( \int_V e^{\alpha v_k(x)} \, dv_g \right)^{1 - r/2} \to 0 \tag{3.13}$$

as $k \to +\infty$, where $r = 2\alpha/(\alpha + 1) > 1$. Choose $\omega = \min\{r, \alpha\}$. Combining (3.12) and (3.13), we employ the elliptic estimate to (3.9), which yields $\|\mu_k - \overline{\mu}_k\|_{W^{2,\omega}_{\mathrm{loc}}(V)} \leq C$. And then Sobolev's embedding theorem implies that $\mu_k - \overline{\mu}_k \to \widetilde{G}_{x_0}$ in $C^{\gamma}_{\mathrm{loc}}\big(\Sigma \setminus \big\{ \bigcup_{i=1}^{\ell} \sigma_i(x_0) \big\}\big)$ for $0 < \gamma < 1$. Following the same arguments as in [16, Proposition 3.5], one can show that $\|\mu_k - \overline{\mu}_k - \widetilde{G}_{x_0}\|_{H^2(V)} \to 0$. By elliptic estimates, we obtain (3.11). This concludes the proof of Proposition 3.4. □

### 3.3. A lower bound of $J_{8\pi\ell}(v(t))$

In this subsection, we shall derive a lower bound of $J_{8\pi\ell}(v(t))$. Precisely, we have the following proposition.

**Proposition 3.5.** *Define a function space $\mathscr{H}^1_{\mathbf{G}}$ as in* (1.8). *Let $v(t) \in \mathscr{H}^1_{\mathbf{G}}$ be the solution of* (1.5) *with $\rho = 8\pi\ell$. Suppose $v_k$ is a noncompact sequence of $v(t)$. Then we have*

$$\begin{aligned}\lim_{t\to+\infty} J_{8\pi\ell}(v(t)) \geq & -4\pi\ell \max_{x\in\Sigma}(2\log(\pi\ell h(x)) + \widetilde{A}_x) - 8\pi\ell \\ & + 8\pi\ell \int_\Sigma \log h \, dv_g - \frac{1}{2}\int_\Sigma |\nabla \log h|^2 \, dv_g,\end{aligned} \tag{3.14}$$

*where where $h(x)$ and $\widetilde{A}_x$ are defined in* (1.7) *and* (3.11) *respectively.*

*Proof.* We prove the statement on the contrary. For otherwise, there exists a constant $\varepsilon > 0$ such that

$$\begin{aligned}\lim_{t\to+\infty} J_{8\pi\ell}(v(t)) < & -4\pi\ell \max_{x\in\Sigma}(2\log(\pi\ell h(x)) + \widetilde{A}_x) - 8\pi\ell \\ & + 8\pi\ell \int_\Sigma \log h \, dv_g - \frac{1}{2}\int_\Sigma |\nabla \log h|^2 \, dv_g - \varepsilon.\end{aligned} \tag{3.15}$$

Let $v_k \subseteq \mathscr{H}^2_{\mathbf{G}}$ be a sequence of non-compact solutions of (3.1), satisfying (2.14) and (2.15). Notice that $\int_\Sigma e^{v_k} = 1$. This leads to $\log \int_\Sigma e^{v_k} \, dv_g = 0$. Then (1.6) can be rewritten as

$$J_{8\pi\ell}(v_k) = \frac{1}{2}\int_\Sigma |\nabla v_k|^2 \, dv_g + \int_\Sigma Q v_k \, dv_g, \tag{3.16}$$

where $\int_\Sigma Q \, dv_g = 8\pi\ell$. Now, we estimate the two integrals on the right-hand side of (3.16) respectively. By Proposition 3.4,

$$v_k - \overline{v}_k \to \widetilde{G}_{x_0} + \log h - \int_\Sigma \log h \, dv_g \tag{3.17}$$

in $C^\gamma_{\mathrm{loc}}\big(\Sigma \setminus \big\{ \bigcup_{i=1}^\ell \sigma_i(x_0) \big\}\big)$, for some $0 < \gamma < 1$. This together with (1.7) gives that

$$\begin{aligned}\int_\Sigma Q v_k \, dv_g &= -\int_\Sigma \nabla \log h \cdot \nabla v_k \, dv_g + 8\pi\ell\overline{v}_k \\ &= -\int_\Sigma |\nabla \log h|^2 \, dv_g - \int_\Sigma \nabla \log h \cdot \nabla \widetilde{G}_{x_0} \, dv_g + 8\pi\ell\overline{v}_k + o_k(1).\end{aligned} \tag{3.18}$$

To calculate the integral $\int_\Sigma |\nabla v_k|^2 \, dv_g$, we divide it into three parts, namely

$$\begin{aligned}&\int_{\Sigma\setminus\bigcup_{i=1}^\ell B_\delta(\sigma_i(x_k))} |\nabla v_k|^2 \, dv_g + \int_{\bigcup_{i=1}^\ell B_\delta(\sigma_i(x_k))\setminus B_{Rr_k}(\sigma_i(x_k))} |\nabla v_k|^2 \, dv_g \\ &+ \int_{\bigcup_{i=1}^\ell B_{Rr_k}(\sigma_i(x_k))} |\nabla v_k|^2 \, dv_g,\end{aligned}$$

where $\delta > 0$. Then the above parts shall be estimated respectively. We begin with the first part. It follows from Proposition 3.4 that

$$\begin{aligned}&\int_{\Sigma\setminus\bigcup_{i=1}^\ell B_\delta(\sigma_i(x_k))} |\nabla v_k|^2 \, dv_g \\ &= \int_{\Sigma\setminus\bigcup_{i=1}^\ell B_\delta(\sigma_i(x_k))} |\nabla(\mu_k + \log h)|^2 \, dv_g \\ &= \int_{\Sigma\setminus\bigcup_{i=1}^\ell B_\delta(\sigma_i(x_0))} \Big(|\nabla \widetilde{G}_{x_0}|^2 + 2\nabla\widetilde{G}_{x_0} \cdot \nabla \log h + |\nabla \log h|^2\Big) \, dv_g + o_k(1).\end{aligned} \tag{3.19}$$

In a normal coordinate system $\{x_1, x_2\}$ near $x_0$, by elliptic estimates, $\widetilde{G}_{x_0}$ can be represented by

$$\widetilde{G}_{x_0}(x) = -4\log r + \widetilde{A}_{x_0} + b_1x_1 + b_2x_2 + c_1x_1^2 + 2c_2x_1x_2 + c_3x_2^2 + O(r^3), \tag{3.20}$$

where $\widetilde{A}_{x_0}$, $b_1$, $b_2$, $c_1$, $c_2$, $c_3$ are constants, $r(x)$ denotes the geodesic distance between $x$ and $\sigma_i(x_0)$, $i = 1, \ldots, \ell$. Using the divergence theorem, we calculate by (3.10) and (3.20) that

$$\begin{aligned}\int_{\Sigma\setminus\bigcup_{i=1}^{\ell} B_\delta(\sigma_i(x_0))} |\nabla\widetilde{G}_{x_0}|^2\, dv_g &= -\int_{\bigcup_{i=1}^{\ell}\partial B_\delta(\sigma_i(x_0))} \widetilde{G}_{x_0}\cdot\frac{\partial\widetilde{G}_{x_0}}{\partial n}\, ds_g \\ &\quad - \int_{\Sigma\setminus\bigcup_{i=1}^{\ell} B_\delta(\sigma_i(x_0))} \widetilde{G}_{x_0}\left(8\pi\ell - 8\pi\sum_{i=1}^{\ell}\delta_{\sigma_i(x_0)}\right) dv_g \\ &= -32\pi\ell\log\delta + 8\pi\ell\widetilde{A}_{x_0} + o_\delta(1).\end{aligned} \tag{3.21}$$

Inserting (3.21) into (3.19), one has

$$\begin{aligned}&\int_{\Sigma\setminus\bigcup_{i=1}^{\ell} B_\delta(\sigma_i(x_k))} |\nabla v_k|^2\, dv_g \\ &= \int_{\Sigma\setminus\bigcup_{i=1}^{\ell} B_\delta(\sigma_i(x_0))} (2\nabla\widetilde{G}_{x_0}\cdot\nabla\log h\, dv_g + |\nabla\log h|^2)\, dv_g \\ &\quad - 32\pi\ell\log\delta + 8\pi\ell\widetilde{A}_{x_0} + o_\delta(1) + o_k(1).\end{aligned} \tag{3.22}$$

Next we estimate the integral of $v_k$ on the annulus. Since $v_k \subseteq \mathscr{H}^2_{\mathbf{G}}$, it yields to

$$\int_{\bigcup_{i=1}^{\ell} B_\delta(\sigma_i(x_k))\setminus B_{Rr_k}(\sigma_i(x_k))} |\nabla v_k|^2\, dv_g = \ell\int_{B_\delta(x_k)\setminus B_{Rr_k}(x_k)} |\nabla v_k|^2\, dv_g. \tag{3.23}$$

We use a technique of [12] to get the estimate on the annulus. Set

$$\phi_k = \inf_{\partial B_{Rr_k}(x_k)} v_k, \quad \psi_k = \sup_{\partial B_\delta(x_k)} v_k, \quad \varphi_k = \phi_k - \psi_k + 2\log r_k + \overline{v}_k. \tag{3.24}$$

In view of Proposition 3.1 and (3.17), we see that as $k \to +\infty$

$$\phi_k + 2\log r_k \to \inf_{|x|=R}\widetilde{v}_\infty(x) \quad \text{in } C^\alpha_{\mathrm{loc}}(\mathbb{R}^2),$$

and

$$\psi_k - \overline{v}_k \to \sup_{\partial B_\delta(x_0)}\left(\widetilde{G}_{x_0} + \log h - \int_\Sigma v\log h\, dv_g\right) \quad \text{in } C^\gamma_{\mathrm{loc}}\left(\Sigma\setminus\left\{\bigcup_{i=1}^{\ell}\sigma_i(x_0)\right\}\right),$$

where $0 < \alpha < 1$ and $0 < \gamma < 1$. Then,

$$\varphi_k \to \inf_{|x|=R}\widetilde{v}_\infty(x) - \sup_{\partial B_\delta(x_0)}\left(\widetilde{G}_{x_0} + \log h - \int_\Sigma v\log h\, dv_g\right) \tag{3.25}$$

as $k \to +\infty$. Sequently, we proceed in a normal coordinate system near $x_k$. Let $T(\psi_k, \phi_k)$ be a set of all smooth functions $u \in \mathbb{R}^2$ with $u|_{\partial \mathbb{B}_\delta(0)} = \psi_k$ and $u|_{\partial \mathbb{B}_{Rr_k}(0)} = \phi_k$. It is not difficult to see that $\inf_{u \in T(\psi_k,\phi_k)} \int_{\mathbb{B}_\delta(0)\setminus \mathbb{B}_{Rr_k}(0)} |\nabla u|^2\, dx$ is attained by some function $h$ satisfying $\Delta h = 0$ in $\mathbb{B}_\delta(0) \setminus \mathbb{B}_{Rr_k}(0)$ with $h|_{\partial \mathbb{B}_\delta(0)} = \psi_k$, $h|_{\partial \mathbb{B}_{Rr_k}(0)} = \phi_k$. Then it follows that

$$h(x) = \frac{\phi_k(\log \delta - \log r) + \psi_k(\log r - \log Rr_k)}{\log \delta - \log Rr_k},$$

and that

$$\int_{\mathbb{B}_\delta(0)\setminus \mathbb{B}_{Rr_k}(0)} |\nabla h|^2\, dx = \frac{2\pi(\phi_k - \psi_k)^2}{\log \delta - \log Rr_k}. \tag{3.26}$$

Define a function space

$$\mathscr{W}_k(\psi_k, \phi_k) = \left\{ v_k \in \mathscr{H}^2_{\mathbf{G}}(B_\delta(x_k) \setminus B_{Rr_k}(x_k)) : v_k|_{\partial B_\delta(x_k)} = \psi_k, v_k|_{\partial B_{Rr_k}(x_k)} = \phi_k \right\},$$

where $\mathscr{H}^2_{\mathbf{G}}$ is in (1.8). Let $\widetilde{v}_k = \max\{\psi_k, \min\{v_k, \phi_k\}\}$. Then $\widetilde{v}_k \in \mathscr{W}_k(\psi_k, \phi_k)$ and in a normal coordinate system near $x_k$, there holds by

$$\int_{B_\delta(x_k)\setminus B_{Rr_k}(x_k)} |\nabla v_k|^2\, dv_g \geq \int_{B_\delta(x_k)\setminus B_{Rr_k}(x_k)} |\nabla \widetilde{v}_k|^2\, dv_g \geq \int_{\mathbb{B}_\delta(0)\setminus \mathbb{B}_{Rr_k}(0)} |\nabla h|^2\, dx.$$

This together with (3.23), (3.24) and (3.26), one can easily check that

$$\begin{aligned} &\int_{\bigcup_{i=1}^{\ell} B_\delta(\sigma_i(x_k))\setminus B_{Rr_k}(\sigma_i(x_k))} |\nabla v_k|^2\, dv_g \\ &\geq \frac{2\pi\ell(\phi_k - \psi_k)^2}{\log \delta - \log Rr_k} \\ &\geq 2\pi\ell \left(2 + \frac{\overline{v}_k}{\log r_k}\right)^2 \left(-\log r_k + \log R - \log \delta - \frac{C_{R,\delta}}{\log r_k}\right) \\ &\quad + 4\pi\ell \left(2 + \frac{\overline{v}_k}{\log r_k}\right)\left(\varphi_k + \frac{\varphi_k C_{R,\delta}}{(\log r_k)^2}\right) + \frac{\ell C'_{R,\delta}\overline{v}_k}{2(\log r_k)^2} + o_k(1), \end{aligned} \tag{3.27}$$

where $C_{R,\delta}$ and $C'_{R,\delta}$ are constants relying only on $\delta$ and $R$.

Finally, we compute the integral $\int_{\bigcup_{i=1}^{\ell} B_{Rr_k}(\sigma_i(x_k))} |\nabla v_k|^2\, dv_g$. Thanks to Proposition 3.1, we obtain

$$\begin{aligned} &\int_{\bigcup_{i=1}^{\ell} B_{Rr_k}(\sigma_i(x_k))} |\nabla v_k|^2\, dv_g \\ &= \ell(1 + o_k(1)) \int_{B_R(0)} |\widetilde{v}_\infty(x)|^2\, dx \\ &\geq 16\pi\ell(1 + o_k(1)) \left(\log(1 + \lambda^2(R - |\widetilde{x}_0|)^2) - \frac{\lambda^2(R - |\widetilde{x}_0|)^2}{1 + \lambda^2(R - |\widetilde{x}_0|)^2}\right). \end{aligned} \tag{3.28}$$

Inserting (3.18), (3.22), (3.27) and (3.28) into (3.16), we conclude that

(3.29)
$$\begin{aligned}
J_{8\pi\ell}(v_k) \geq & -\pi\ell\log r_k\left(2-\frac{\overline{v}_k}{\log r_k}\right)^2+\frac{\ell C'_{R,\delta}\overline{v}_k}{2(\log r_k)^2}+2\pi\ell\left(2+\frac{\overline{v}_k}{\log r_k}\right)\left(\varphi_k+\frac{C_{R,\delta}\varphi_k}{(\log r_k)^2}\right)\\
& -16\pi\ell\log\delta+4\pi\ell\widetilde{A}_{x_0}+\pi\ell\left(2+\frac{\overline{v}_k}{\log r_k}\right)^2\left(\log R-\log\delta-\frac{C_{R,\delta}}{\log r_k}\right)\\
& +8\pi\ell(1+o_k(1))\left(\log(1+\lambda^2(R-|\widetilde{x}_0|)^2)-\frac{\lambda^2(R-|\widetilde{x}_0|)^2}{1+\lambda^2(R-|\widetilde{x}_0|)^2}\right)+o_\delta(1)\\
& -\int_{\bigcup_{i=1}^{\ell}B_\delta(\sigma_i(x_0))}\nabla\widetilde{G}_{x_0}\cdot\nabla\log h\,dv_g\\
& -\frac{1}{2}\int_{\Sigma\setminus\bigcup_{i=1}^{\ell}B_\delta(\sigma_i(x_0))}|\nabla\log h|^2\,dv_g+o_k(1).
\end{aligned}$$

Using the divergence theorem, one has

$$\begin{aligned}
&\int_{\bigcup_{i=1}^{\ell}B_\delta(\sigma_i(x_0))}\nabla\widetilde{G}_{x_0}\cdot\nabla\log h\,dv_g\\
&=\ell\left(\int_{\partial B_\delta(x_0)}\frac{\partial\widetilde{G}_{x_0}}{\partial n}\log h\,ds_g-\int_{B_\delta(x_0)}\Delta\widetilde{G}_{x_0}\log h\,dv_g\right)\\
&=o_\delta(1).
\end{aligned}\tag{3.30}$$

Moreover, (3.29) implies that

$$J_{8\pi\ell}(v_k)\geq(C-\pi\ell\log r_k)\left(2-\frac{\overline{v}_k}{\log r_k}+O\left(-\frac{1}{\log r_k}\right)\right)^2+C.$$

Note that $J_{8\pi\ell}(v_k)\leq J_{8\pi\ell}(v_0)$. Then it follows that

$$\left|2-\frac{\overline{v}_k}{\log r_k}\right|\leq\frac{C}{(-\ell\log r_k)^{1/2}}.$$

Letting $k\to+\infty$ leads to $\overline{v}_k/\log r_k\to 2$. Together with (3.25), (3.29) and (3.30), we finally arrive at

$$\begin{aligned}
\lim_{k\to\infty}J_{8\pi\ell}(v_k)\geq & -4\pi\ell\max_{x\in\Sigma}(2\log(\pi\ell h(x))+\widetilde{A}_x)-8\pi\ell\\
& +8\pi\ell\int_\Sigma\log h\,dv_g-\frac{1}{2}\int_\Sigma|\nabla\log h|^2\,dv_g,
\end{aligned}\tag{3.31}$$

by passing to the limit $k\to+\infty$ first and then $\delta\to 0$, $R\to+\infty$. Notice that $J_{8\pi\ell}(v(t))$ decreases in $t$. According to (3.15), we can find some $t_0>0$ such that

$$\begin{aligned}
J_{8\pi\ell}(v(t_0))< & -4\pi\ell\max_{x\in\Sigma}(2\log(\pi\ell h(x))+\widetilde{A}_x)-8\pi\ell\\
& +8\pi\ell\int_\Sigma\log h\,dv_g-\frac{1}{2}\int_\Sigma|\nabla\log h|^2\,dv_g-\frac{\varepsilon}{2}.
\end{aligned}$$

Then when $t_k > t_0$, we see that $J_{8\pi\ell}(v(t_k)) \leq J_{8\pi\ell}(v(t_0))$, namely,

$$\begin{aligned} J_{8\pi\ell}(v(t_k)) < -4\pi\ell \max_{x\in\Sigma}(2\log(\pi\ell h(x)) + \widetilde{A}_x) - 8\pi\ell \\ + 8\pi\ell \int_\Sigma \log h \, dv_g - \frac{1}{2}\int_\Sigma |\nabla \log h|^2 \, dv_g, \end{aligned}$$

which contradicts with (3.31). Thus the proposition is proved. □

## 4. Completion of the proof of Theorem 1.1

In this section, we will complete the proof of Theorem 1.1. Under the assumptions of Theorem 1.1, we shall construct a sequence of initial data $v_{0,\varepsilon}$ to show

$$\begin{aligned} J_{8\pi\ell}(v_{0,\varepsilon}) < -4\pi\ell \max_{x\in\Sigma}(2\log(\pi\ell h(x)) + \widetilde{A}_x) - 8\pi\ell \\ + 8\pi\ell \int_\Sigma \log h \, dv_g - \frac{1}{2}\int_\Sigma |\nabla \log h|^2 \, dv_g, \end{aligned}$$

where $\widetilde{A}_x$ is defined as in (3.11). Observe from (2.6) that $J_{8\pi\ell}(v(t)) \leq J_{8\pi\ell}(v_{0,\varepsilon})$ as $t \to +\infty$. This yields to a contradiction with Proposition 3.5. Therefore we conclude that $v_k$ is compact. Then we follow the idea of [3] to get the convergence of the flow. This finishes the proof of the theorem.

### 4.1. Exclusion of blow-up phenomenon

We first exclude the blow-up phenomenon. Pick up some point $p \in \Sigma$ such that

$$2\log(\pi\ell h(p)) + \widetilde{A}_p = \max_{x\in\Sigma}(2\log(\pi\ell h(x)) + \widetilde{A}_x). \tag{4.1}$$

Notice that $I(x) = \sharp\mathbf{G}(x) \equiv \ell$ for all $x \in \Sigma$ and $\mathbf{G} = \{\sigma_1, \ldots, \sigma_\ell\}$. Then $I(p) = \ell$ and $\sigma_1(p), \ldots, \sigma_\ell(p)$ are different points on $\Sigma$. For some $\delta > 0$, choose a normal coordinate system $(B_\delta(x_0), \exp_p^{-1}; \{y^1, y^2\})$ near $p$. By [25], $\widetilde{G}_p$ can be written as

$$\widetilde{G}_p(\exp_p(y)) = -4\log r + \widetilde{A}_p + b_1 y^1 + b_2 y^2 + c_1(y^1)^2 + 2c_2 y^1 y^2 + c_3 (y^2)^2 + O(r^3), \tag{4.2}$$

where $r = |y| = d_g(p, \exp_p(y))$, $\widetilde{A}_p$ is a constant. Following the arguments of [25, Section 5], we define

$$\phi_\epsilon(x) = \begin{cases} c - 2\log\left(1 + \frac{r^2}{8\epsilon^2}\right) + \widetilde{A}_p + \alpha(\exp_p^{-1}(\sigma_i^{-1}(x))), & x \in B_{R\epsilon}(\sigma_i(p)),\ i = 1, \ldots, \ell, \\ \widetilde{G}_p(x) - \eta(\sigma_i^{-1}(x))\beta(\exp_p^{-1}(\sigma_i^{-1}(x))), & x \in B_{2R\epsilon}(\sigma_i(p)) \setminus B_{R\epsilon}(\sigma_i(p)), \\ \widetilde{G}_p(x), & x \in \Sigma \setminus \bigcup_{i=1}^{\ell} B_{2R\epsilon}(\sigma_i(p)), \end{cases}$$

where $\widetilde{A}_p$ is defined in (4.2), $R$ and $c$ are constants depending only on $\epsilon$ and will be determined later, $r = r(x)$ denotes the geodesic distance between $x$ and $\sigma_i(p)$ for $x \in$

$B_{R\epsilon}(\sigma_i(p))$, $\eta \in C_0^\infty(B_{2R\epsilon}(p))$ is a cut-off function, satisfying $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{R\epsilon}(p)$ and $|\nabla_g \eta| \le 4/(R\epsilon)$, $\alpha(y) = b_1 y^1 + b_2 y^2$ and $\beta(y) = c_1(y^1)^2 + 2c_2 y^1 y^2 + c_3(y^2)^2 + O(r^3)$.

Set $\widetilde{v}_{0,\varepsilon} = (\phi_\epsilon - \overline{\phi}_\epsilon) + \log h$. In view of (1.6), we obtain

$$J_{8\pi\ell}(\widetilde{v}_{0,\varepsilon}) = \frac{1}{2}\int_\Sigma |\nabla_g \phi_\epsilon|^2 \, dv_g - 8\pi\ell \log \int_\Sigma h e^{\phi_\epsilon} \, dv_g + 8\pi\ell \overline{\phi}_\epsilon + 8\pi\ell \int_\Sigma \log h \, dv_g - \frac{1}{2}\int_\Sigma |\nabla \log h|^2 \, dv_g. \tag{4.3}$$

By the result of [25], it then follows from (4.3) that

$$\begin{aligned} J_{8\pi\ell}(\widetilde{v}_{0,\varepsilon}) = & -8\pi\ell - 4\pi\ell \widetilde{A}_p - 8\pi\ell \log(\pi\ell h(p)) \\ & - 32\pi\ell \left( 8\pi\ell - 2K(p) + b_1^2 + b_2^2 + \frac{\Delta h(p)}{h(p)} + \frac{2(k_1 b_1 + k_2 b_2)}{h(p)} + o_\epsilon(1) \right) \epsilon^2 \log \frac{1}{\epsilon} \\ & + 8\pi\ell \int_\Sigma \log h \, dv_g - \frac{1}{2}\int_\Sigma |\nabla \log h|^2 \, dv_g, \end{aligned} \tag{4.4}$$

where $b_1$ and $b_2$ are defined in (4.2), $(k_1, k_2) = \nabla h(p)$. Since $\Delta \log h = Q - 8\pi\ell$, there holds

$$\frac{\Delta h(p)}{h(p)} = Q - 8\pi\ell + \frac{k_1^2 + k_2^2}{h(p)^2}. \tag{4.5}$$

Under the hypothesis $Q(p) > 2K(p)$, we have by (4.5) that

$$\begin{aligned} & 8\pi\ell - 2K(p) + b_1^2 + b_2^2 + \frac{\Delta h(p)}{h(p)} + \frac{2(k_1 b_1 + k_2 b_2)}{h(p)} \\ & = Q(p) - 2K(p) + \left(\frac{k_1 + b_1 h(p)}{h(p)}\right)^2 + \left(\frac{k_2 + b_2 h(p)}{h(p)}\right)^2 > 0. \end{aligned} \tag{4.6}$$

Inserting (4.6) into (4.4), by (4.1), we find

$$\begin{aligned} J_{8\pi\ell}(\widetilde{v}_{0,\varepsilon}) < & -4\pi\ell \max_{x\in\Sigma} (2\log(\pi\ell h(x)) + \widetilde{A}_x) - 8\pi\ell \\ & + 8\pi\ell \int_\Sigma \log h \, dv_g - \frac{1}{2}\int_\Sigma |\nabla \log h|^2 \, dv_g. \end{aligned} \tag{4.7}$$

Observe that $\widetilde{v}_{0,\varepsilon}$ is the function of Lipschitz. Clearly, $\widetilde{v}_{0,\varepsilon}$ can be modified into a smooth function $\widehat{v}_{0,\varepsilon}$, and $J_{8\pi\ell}(\widehat{v}_{0,\varepsilon})$ satisfies (4.7). Then, choose some constant $c_0$ such that $\int_\Sigma e^{\widehat{v}_{0,\varepsilon} + c_0} \, dv_g = 1$. Denote $v_{0,\varepsilon} = \widehat{v}_{0,\varepsilon} + c_0$. As a consequence,

$$\begin{aligned} J_{8\pi\ell}(v_{0,\varepsilon}) < & -4\pi\ell \max_{x\in\Sigma} (2\log(\pi\ell h(x)) + \widetilde{A}_x) - 8\pi\ell \\ & + 8\pi\ell \int_\Sigma \log h \, dv_g - \frac{1}{2}\int_\Sigma |\nabla \log h|^2 \, dv_g, \end{aligned}$$

which contradicts to (3.14). Thus, we conclude that blow-up can't happen and the sequence $v_k$ is compact.

### 4.2. The convergence

We follow the ideas of Catéras in [3] for the study of convergence. As $k \to +\infty$, note that

$$\begin{aligned}\int_\Sigma (\Delta v_k - \Delta v_\infty)^2\, dv_g &= \int_\Sigma \left(8\pi\ell\big(e^{v_\infty} - e^{v_k}\big) + \frac{\partial e^{v_k}}{\partial t}\right)^2 dv_g \\ &\le C \int_\Sigma \big(e^{v_\infty} - e^{v_k}\big)^2\, dv_g + C \int_\Sigma \left|\frac{\partial v_k}{\partial t}\right|^2 e^{v_k}\, dv_g \to 0,\end{aligned}$$

where $v_\infty$ is a solution of (1.10). By the result of Simon [21], we finally obtain that

$$\|v(t) - v_\infty\|_{H^2(\Sigma)} \to 0 \quad \text{as } k \to +\infty.$$

Therefore, Theorem 1.1 is established.

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