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Frame-based Average Sampling in Multiply Generated Shift-invariant Subspaces of Mixed Lebesgue Spaces

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Abstract. In this paper, we mainly discuss the nonuniform average sampling and reconstruction in multiply generated shift-invariant subspaces

$$V_{p,q}(\Phi_r) = \Big\{ \sum_{k_1\in\mathbf{Z}} \sum_{k_2\in\mathbf{Z}^d} c^T(k_1,k_2)\Phi_r(\cdot - k_1, \cdot - k_2) : (c(k_1,k_2))_{(k_1,k_2)\in\mathbf{Z}\times\mathbf{Z}^d} \in \big(\ell^{p,q}(\mathbf{Z}\times\mathbf{Z}^d)\big)^r \Big\}$$

of mixed Lebesgue spaces $L^{p,q}(\mathbf{R}\times\mathbf{R}^d)$, $1 \le p, q \le \infty$, where $\Phi_r = (\varphi_1, \varphi_2, \ldots, \varphi_r)^T$ with $\varphi_i \in L^{p,q}(\mathbf{R}\times\mathbf{R}^d)$ and $c = (c_1, c_2, \ldots, c_r)^T$ with $c_i \in \ell^{p,q}(\mathbf{Z}\times\mathbf{Z}^d)$, $i = 1, 2, \ldots, r$, under the assumption that the family $\{\varphi_i(x - k_1, y - k_2) : (k_1, k_2) \in \mathbf{Z}\times\mathbf{Z}^d, 1 \le i \le r\}$ constitutes a (p, q)-frame of $V_{p,q}(\Phi_r)$. First, iterative approximation projection algorithms for two kinds of average sampling functionals are established. Then, we estimate the convergence rates of the corresponding algorithms.

1. Introduction

Mixed Lebesgue spaces were firstly introduced in [4], although the initial realization of their necessity goes back to [12]. In the last years, there are many groups of people working on mixed Lebesgue spaces from the viewpoint of harmonic analysis and operator theory, refer to [5–8, 10, 11, 13–16, 21, 22, 26] and references therein.

For $1 \le p, q \le \infty$, the mixed Lebesgue space $L^{p,q}(\mathbf{R}\times\mathbf{R}^d)$ denotes the Banach space of all functions f such that

$$\|f\|_{L^{p,q}(\mathbf{R}\times\mathbf{R}^d)} = \big\| \|f(x_1,x_2)\|_{L^q_{x_2}(\mathbf{R}^d)} \big\|_{L^p_{x_1}(\mathbf{R})} < \infty.$$

Similarly, $\ell^{p,q} = \ell^{p,q}(\mathbf{Z}\times\mathbf{Z}^d)$ is the Banach space of all complex valued sequences $c = (c(k_1,k_2))_{(k_1,k_2)\in\mathbf{Z}\times\mathbf{Z}^d}$ such that

$$\|c\|_{\ell^{p,q}} = \big\| \|c(k_1,k_2)\|_{\ell^q_{k_2}(\mathbf{Z}^d)} \big\|_{\ell^p_{k_1}(\mathbf{Z})} < \infty.$$

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Sampling plays an important role in signal and information processing, and sampling theory in shift-invariant subspaces of classical Lebesgue spaces $L^p(\mathbf{R}^d)$ had been generally studied in the past years, such as [1–3, 9, 28, 29]. Since mixed Lebesgue space is a suitable tool for modeling and measuring time variant signals due to its separate integrability for different variables, some sampling results in bandlimited subspaces, shift-invariant subspaces and reproducing kernel subspaces of mixed Lebesgue spaces $L^{p,q}(\mathbf{R}\times\mathbf{R}^d)$ are given recently [17, 18, 20, 23, 24, 27, 30, 31]. However, the existed works in shift-invariant spaces always assume that the family $\{\varphi_i(x-k_1, y-k_2) : (k_1,k_2)\in\mathbf{Z}\times\mathbf{Z}^d, 1\le i\le r\}$ has (p,q)-stability [25], instead of a possibly redundant frame.

In this paper, we will study the average sampling and reconstruction problem in the multiply generated shift-invariant subspaces

$$V_{p,q}(\Phi_r) = \Big\{ \sum_{k_1\in\mathbf{Z}}\sum_{k_2\in\mathbf{Z}^d} c^T(k_1,k_2)\Phi_r(\,\cdot-k_1,\,\cdot-k_2) : (c(k_1,k_2))_{(k_1,k_2)\in\mathbf{Z}\times\mathbf{Z}^d}\in\big(\ell^{p,q}(\mathbf{Z}\times\mathbf{Z}^d)\big)^r \Big\}$$

of mixed Lebesgue spaces $L^{p,q}(\mathbf{R}\times\mathbf{R}^d)$, $1\le p,q\le\infty$, where T denotes the transpose, $\Phi_r = (\varphi_1,\varphi_2,\dots,\varphi_r)^T$ with $\varphi_i\in L^{p,q}(\mathbf{R}\times\mathbf{R}^d)$ and $c=(c_1,c_2,\dots,c_r)^T$ with $c_i\in\ell^{p,q}(\mathbf{Z}\times\mathbf{Z}^d)$, $i=1,2,\dots,r$, based on the following assumptions:

(A1) The family $\{\varphi_i(\,\cdot-k_1,\,\cdot-k_2) : (k_1,k_2)\in\mathbf{Z}\times\mathbf{Z}^d, 1\le i\le r\}$ is a (p,q)-frame for $V_{p,q}(\Phi_r)$, which is defined and characterized in [19].

Definition 1.1. The family $\{\varphi_i(x_1-k_1,x_2-k_2) : i=1,\dots,r,(k_1,k_2)\in\mathbf{Z}\times\mathbf{Z}^d\}$ is called a (p,q)-frame of $V_{p,q}(\Phi_r)$ if $\varphi_i\in L^{p',q'}(\mathbf{R}\times\mathbf{R}^d)$ and there exists a positive constant $A\ge 1$ (depending on p, q and Φ_r) such that for any $f\in V_{p,q}(\Phi_r)$,

$$A^{-1}\|f\|_{L^{p,q}} \le \sum_{i=1}^{r}\Big\|\big\{\langle f,\varphi_i(\,\cdot-k_1,\,\cdot-k_2)\rangle\big\}_{(k_1,k_2)\in\mathbf{Z}\times\mathbf{Z}^d}\Big\|_{\ell^{p,q}} \le A\|f\|_{L^{p,q}},$$

where $\langle\,\cdot\,\rangle$ denotes the dual product between $L^{p,q}(\mathbf{R}\times\mathbf{R}^d)$ and $L^{p',q'}(\mathbf{R}\times\mathbf{R}^d)$, p' and q' are the conjugate numbers of p and q, respectively.

(A2) The generators φ_i, $1\le i\le r$ are continuous and belong to the mixed Wiener amalgam space $W(L^{1,1}) = W(L^{1,1})(\mathbf{R}\times\mathbf{R}^d)$ whose general forms are defined as all measurable functions satisfying

$$\|f\|_{W(L^{p,q})} = \bigg(\sum_{k_1\in\mathbf{Z}}\sup_{x_1\in[0.1]}\bigg(\sum_{k_2\in\mathbf{Z}^d}\sup_{x_2\in[0.1]^d}|f(x_1+k_1,x_2+k_2)|^q\bigg)^{p/q}\bigg)^{1/p}<\infty$$

for $1\le p,q<\infty$. The cases $p=\infty$ or $q=\infty$ obey the usual adjustment.

(A3) We assume that there exists some $\delta_0 > 0$ such that $\mathrm{osc}_{(\delta_0,x)}(\varphi_i) \in W(L^{1,1})$ for all $1 \le i \le r$. Here, $\mathrm{osc}_{(\delta_0,x)}(\varphi)$ is the partial oscillation (or modulus of continuity) about variable x defined by

$$\mathrm{osc}_{(\delta,x)}(\varphi)(x,y) = \sup_{|x'|\le\delta} |\varphi(x+x',y) - \varphi(x,y)|.$$

Moreover, we define the partial oscillation about the variable y as

$$\mathrm{osc}_{(\delta,y)}(f)(x,y) = \sup_{|y'|\le\delta} |f(x,y+y') - f(x,y)|$$

and the oscillation

$$\mathrm{osc}_{\delta}(f)(x,y) = \sup_{|x'|\le\delta,|y'|\le\delta} |f(x+x',y+y') - f(x,y)|.$$

In fact, if φ belongs to the classical Wiener amalgam space $W(L^1)(\mathbf{R}^{d+1})$ defined by

$$\|\varphi\|_{W(L^1)} = \sum_{k_1\in\mathbf{Z}} \sum_{k_2\in\mathbf{Z}^d} \sup_{(x,y)\in[0,1]^{d+1}} |\varphi(x+k_1,y+k_2)| < \infty,$$

then we can obtain

$$\begin{aligned}
&\|\operatorname{osc}_{(\delta_0,x)}(\varphi)\|_{W(L^{1,1})} \\
&= \sum_{k_1\in\mathbf{Z}} \sup_{x\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{y\in[0,1]^d} \sup_{|x'|\le\delta_0} |\varphi(x+x'+k_1,y+k_2) - \varphi(x+k_1,y+k_2)| \\
&\le \|\varphi\|_{W(L^{1,1})} + \sum_{k_1\in\mathbf{Z}} \sum_{k_2\in\mathbf{Z}^d} \sup_{(x,y)\in[0,1]^{d+1}} \sup_{|x'|\le\delta_0} |\varphi(x+x'+k_1,y+k_2)| \\
&\le (4+2[\delta_0])\|\varphi\|_{W(L^1)} < \infty.
\end{aligned}$$

With the requirements (A1) and (A2), we know from [19] that

(i) The space $V_{p,q}(\Phi_r)$ is a closed linear subspace of $L^{p,q}(\mathbf{R}\times\mathbf{R}^d)$ and there exists a positive constant $B \ge 1$ (depending on p, q and Φ_r) such that

$$B^{-1}\|f\|_{L^{p,q}} \le \inf_{f=\sum_{i=1}^r \varphi_i *' c_i} \sum_{i=1}^r \|c_i\|_{\ell^{p,q}} \le B\|f\|_{L^{p,q}}, \quad \forall\, f \in V_{p,q}(\Phi_r), \tag{1.1}$$

where $\varphi_i *' c_i = \sum_{k_1\in\mathbf{Z}} \sum_{k_2\in\mathbf{Z}^d} c_i(k_1,k_2)\varphi_i(\cdot - k_1, \cdot - k_2)$.

(ii) There exists $\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_r \in W(L^{1,1}) \cap V_{p,q}(\Phi_r)$ such that for every $f \in V_{p,q}(\Phi_r)$,

$$\begin{aligned}
f &= \sum_{i=1}^r \sum_{k_1\in\mathbf{Z}} \sum_{k_2\in\mathbf{Z}^d} \big\langle f, \widetilde{\varphi}_i(\cdot - k_1, \cdot - k_2)\big\rangle \varphi_i(\cdot - k_1, \cdot - k_2) \\
&= \sum_{i=1}^r \sum_{k_1\in\mathbf{Z}} \sum_{k_2\in\mathbf{Z}^d} \big\langle f, \varphi_i(\cdot - k_1, \cdot - k_2)\big\rangle \widetilde{\varphi}_i(\cdot - k_1, \cdot - k_2).
\end{aligned}$$

The paper is organized as follows. In Section 2, we give some lemmata which will provide the theoretical basis for the subsequent sections. In Section 3, iterative approximation projection algorithms for two kinds of average sampling schemes are given. In Section 4, we estimate the corresponding convergence rates.

2. Some lemmata

In this section, we will give some lemmata which will be used in the subsequent sections.

Lemma 2.1. *If $f \in W(L^{1,1})$ and $g \in L^1(\mathbf{R}^{d+1})$ then $f * g \in W(L^{1,1})$ and*

$$\|f * g\|_{W(L^{1,1})} \le 2^{d+1}\|f\|_{W(L^{1,1})}\|g\|_{L^1}.$$

Proof. By the definition of convolution, we can obtain

$$\begin{aligned}
&\|f * g\|_{W(L^{1,1})}\\
&\le \sum_{k_1\in\mathbf{Z}} \sup_{x\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{y\in[0,1]^d} \int_{\mathbf{R}}\int_{\mathbf{R}^d} |f(x+k_1-s,y+k_2-t)||g(s,t)|\,dtds\\
&\le \sum_{\ell_1\in\mathbf{Z}}\sum_{\ell_2\in\mathbf{Z}^d} \int_0^1\int_{[0,1]^d} \sum_{k_1\in\mathbf{Z}} \sup_{x\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{y\in[0,1]^d} |f(x+k_1-s-\ell_1,y+k_2-t-\ell_2)|\\
&\quad\times |g(s+\ell_1,t+\ell_2)|\,dtds\\
&\le \sum_{\ell_1\in\mathbf{Z}}\sum_{\ell_2\in\mathbf{Z}^d} \sup_{s\in[0,1]}\sup_{t\in[0,1]^d} \left(\sum_{k_1\in\mathbf{Z}} \sup_{x\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{y\in[0,1]^d} |f(x+k_1-s-\ell_1,y+k_2-t-\ell_2)|\right)\\
&\quad\times \int_0^1\int_{[0,1]^d} |g(s+\ell_1,t+\ell_2)|\,dtds\\
&\le \sum_{\ell_1\in\mathbf{Z}}\sum_{\ell_2\in\mathbf{Z}^d} \left(\sum_{k_1\in\mathbf{Z}} \sup_{x\in[0,1]}\sup_{s\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{y\in[0,1]^d}\sup_{t\in[0,1]^d} |f(x+k_1-s,y+k_2-t)|\right)\\
&\quad\times \int_0^1\int_{[0,1]^d} |g(s+\ell_1,t+\ell_2)|\,dtds\\
&\le 2^{d+1}\|f\|_{W(L^{1,1})}\|g\|_{L^1}.
\end{aligned}$$

This completes the proof. □

Lemma 2.2. [24, Theorem 3.4] *Suppose that $1 \le p,q \le \infty$ and $g \in W(L^{1,1})(\mathbf{R}^{d+1})$. Then for any $f \in L^{p,q}(\mathbf{R}^{d+1})$, the sequence*

$$d(k_1,k_2) = \int_{\mathbf{R}}\int_{\mathbf{R}^d} f(x_1,x_2)\overline{g(x_1-k_1,x_2-k_2)}\,dx_1dx_2$$

belongs to $\ell^{p,q}(\mathbf{Z}\times\mathbf{Z}^d)$ and $\|d\|_{\ell^{p,q}} \le \|f\|_{L^{p,q}}\|g\|_{W(L^{1,1})}$.

Hence, the operator P defined by

$$Pf = \sum_{i=1}^{r} \sum_{k_1 \in \mathbf{Z}} \sum_{k_2 \in \mathbf{Z}^d} \big\langle f, \widetilde{\varphi}_i(\cdot - k_1, \cdot - k_2) \big\rangle \varphi_i(\cdot - k_1, \cdot - k_2)$$

is a bounded projection from $L^{p,q}(\mathbf{R} \times \mathbf{R}^d)$ onto $V_{p,q}(\Phi_r)$.

Lemma 2.3. [24, Theorem 3.1] *Assume that $1 \le p, q \le \infty$ and $\varphi \in W(L^{1,1})$. Then for any $c \in \ell^{p,q}(\mathbf{Z} \times \mathbf{Z}^d)$, the function $f = \sum_{k_1 \in \mathbf{Z}} \sum_{k_2 \in \mathbf{Z}^d} c(k_1, k_2) \varphi(\cdot - k_1, \cdot - k_2)$ belongs to $L^{p,q}(\mathbf{R} \times \mathbf{R}^d)$ and*

$$\|f\|_{L^{p,q}} \le \|c\|_{\ell^{p,q}} \|\varphi\|_{W(L^{1,1})}.$$

Lemma 2.4. *Suppose that $\varphi_i \in W(L^{1,1})$ for all $1 \le i \le r$ and that $\{\varphi_i(\cdot - k_1, \cdot - k_2) : (k_1, k_2) \in \mathbf{Z} \times \mathbf{Z}^d, 1 \le i \le r\}$ is a (p, q)-frame for $V_{p,q}(\Phi_r)$, then for any $f \in V_{p,q}(\Phi_r)$, one has*

$$\|f\|_{L^{p,q}} \le \|f\|_{W(L^{p,q})} \le B \left(\max_{1 \le i \le r} \|\varphi_i\|_{W(L^{1,1})} \right) \|f\|_{L^{p,q}}. \tag{2.1}$$

Proof. Suppose that $f(x_1, x_2) = \sum_{i=1}^{r} f_i(x_1, x_2)$ with

$$f_i(x_1, x_2) = \sum_{j_1 \in \mathbf{Z}} \sum_{j_2 \in \mathbf{Z}^d} c_i(j_1, j_2) \varphi_i(x_1 - j_1, x_2 - j_2).$$

Write $b_{x_1+k_1-j_1}(\cdot) = \sup_{x_2 \in [0,1]^d} |\varphi_i(x_1 + k_1 - j_1, x_2 + \cdot)|$ and $c_{j_1}(\cdot) = c_i(j_1, \cdot)$. Then

$$\begin{aligned} &\left(\sum_{k_2 \in \mathbf{Z}^d} \sup_{x_2 \in [0,1]^d} |f_i(x_1 + k_1, x_2 + k_2)|^q \right)^{1/q} \\ &\le \left(\sum_{k_2 \in \mathbf{Z}^d} \left(\sum_{j_1 \in \mathbf{Z}} c_{j_1} * b_{x_1+k_1-j_1}(k_2) \right)^q \right)^{1/q} \\ &\le \sum_{j_1 \in \mathbf{Z}} \|c_{j_1} * b_{x_1+k_1-j_1}\|_{\ell^q} \le \sum_{j_1 \in \mathbf{Z}} \|c_{j_1}\|_{\ell^q_{j_2}} \|b_{x_1+k_1-j_1}\|_{\ell^1_{j_2}}. \end{aligned} \tag{2.2}$$

Let $a_{j_1} = \|c_{j_1}\|_{\ell^q_{j_2}}$ and $d_{k_1-j_1} = \sup_{x_1 \in [0,1]} \|b_{x_1+k_1-j_1}\|_{\ell^1_{j_2}}$. Then

$$\begin{aligned} &\left(\sum_{k_1 \in \mathbf{Z}} \sup_{x_1 \in [0,1]} \left(\sum_{j_1 \in \mathbf{Z}} \|c_{j_1}\|_{\ell^q_{j_2}} \|b_{x_1+k_1-j_1}\|_{\ell^1_{j_2}} \right)^p \right)^{1/p} \\ &\le \|a * d\|_{\ell^p} \le \|a\|_{\ell^p_{j_1}} \|d\|_{\ell^1_{j_1}} = \|c_i\|_{\ell^{p,q}} \|\varphi_i\|_{W(L^{1,1})}. \end{aligned}$$

This together with (2.2) obtains $\|f_i\|_{W(L^{p,q})} \le \|c_i\|_{\ell^{p,q}} \|\varphi_i\|_{W(L^{1,1})}$. Then it follows from (1.1) that

$$\begin{aligned}\|f\|_{W(L^{p,q})} \le \sum_{i=1}^{r} \|f_i\|_{W(L^{p,q})} &\le \left(\sum_{i=1}^{r} \|c_i\|_{\ell^{p,q}}\right)\left(\max_{1\le i\le r} \|\varphi_i\|_{W(L^{1,1})}\right) \\ &\le B\left(\max_{1\le i\le r} \|\varphi_i\|_{W(L^{1,1})}\right)\|f\|_{L^{p,q}},\end{aligned}$$

which is the right-hand side of (2.1). The left side can be easily verified. □

Note that $W(L^{p,q}) \subset L^{\infty,\infty}(\mathbf{R}\times\mathbf{R}^d)$. Then Lemma 2.4 shows that the convergence in $L^{p,q}$-sense also means the uniform convergence.

Lemma 2.5. *Suppose that $\varphi \in W(L^{1,1})$, then $\operatorname{osc}_{(\delta,y)}(\varphi) \in W(L^{1,1})$ for any $\delta > 0$, and*

$$\|\operatorname{osc}_{(\delta,y)}(\varphi)\|_{W(L^{1,1})} \le (1 + (2[\delta]+3)^d)\|\varphi\|_{W(L^{1,1})}.$$

Proof. For any $\delta > 0$, it follows from the definition of $W(L^{1,1})$-norm that

$$\begin{aligned}&\|\operatorname{osc}_{(\delta,y)}(\varphi)\|_{W(L^{1,1})} \\ &\le \|\varphi\|_{W(L^{1,1})} + \sum_{k_1\in\mathbf{Z}} \sup_{x_1\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{x_2\in[0,1]^d} \sup_{|y_2|\le\delta} |\varphi(x_1+k_1, x_2+k_2+y_2)| \\ &\le \|\varphi\|_{W(L^{1,1})} + \sum_{k_1\in\mathbf{Z}} \sup_{x_1\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{x_2\in\left[-[\delta]-1,[\delta]+2\right]^d} |\varphi(x_1+k_1, x_2+k_2)| \\ &\le (1 + (2[\delta]+3)^d)\|\varphi\|_{W(L^{1,1})}.\end{aligned}$$

Finally, the desired result follows from $\varphi \in W(L^{1,1})$. □

Lemma 2.6. *If $\varphi \in W(L^{1,1})$ is continuous and $\operatorname{osc}_{(\delta_0,x)}(\varphi) \in W(L^{1,1})$ for some $\delta_0 > 0$, then*

$$\|\operatorname{osc}_{\delta_0}(\varphi)\|_{W(L^{1,1})} \le (2 + (2[\delta_0]+3)^d)\|\varphi\|_{W(L^{1,1})} + (2[\delta_0]+3)^d\|\operatorname{osc}_{(\delta_0,x)}(\varphi)\|_{W(L^{1,1})} \tag{2.3}$$

and $\lim_{\delta\to 0}\|\operatorname{osc}_{\delta}(\varphi)\|_{W(L^{1,1})} = 0$.

Proof. Since $\varphi \in W(L^{1,1})$ and $\operatorname{osc}_{(\delta_0,x)}(\varphi) \in W(L^{1,1})$, this together with Lemma 2.5 obtains

$$\begin{aligned}&\|\operatorname{osc}_{\delta_0}(\varphi)\|_{W(L^{1,1})} \\ &= \sum_{k_1\in\mathbf{Z}} \sup_{x\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{y\in[0,1]^d} \sup_{|x'|\le\delta_0,|y'|\le\delta_0} |\varphi(x+x'+k_1, y+y'+k_2) - \varphi(x+k_1, y+k_2)| \\ &\le \|\varphi\|_{W(L^{1,1})} + \|\operatorname{osc}_{(\delta_0,y)}(\varphi)\|_{W(L^{1,1})} \\ &\quad + \sum_{k_1\in\mathbf{Z}} \sup_{x\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{y\in[0,1]^d} \sup_{|y'|\le\delta_0} \operatorname{osc}_{(\delta_0,x)}(\varphi)(x+k_1, y+y'+k_2) \\ &\le (2 + (2[\delta_0]+3)^d)\|\varphi\|_{W(L^{1,1})} + (2[\delta_0]+3)^d\|\operatorname{osc}_{(\delta_0,x)}(\varphi)\|_{W(L^{1,1})}.\end{aligned}$$

Thus, $\operatorname{osc}_{\delta_0}(\varphi) \in W(L^{1,1})$. Then for any $\varepsilon > 0$, there exists an integer $L_0 > 0$ such that

$$\sum_{k_1\in\mathbf{Z}} \sup_{x_1\in[0,1]} \sum_{|k_2|\geq L_0} \sup_{x_2\in[0,1]^d} \operatorname{osc}_{\delta_0}(\varphi)(x_1+k_1, x_2+k_2) < \frac{\varepsilon}{3}, \tag{2.4}$$

$$\sum_{|k_1|\geq L_0} \sup_{x_1\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{x_2\in[0,1]^d} \operatorname{osc}_{\delta_0}(\varphi)(x_1+k_1, x_2+k_2) < \frac{\varepsilon}{3}. \tag{2.5}$$

Moreover, since φ is continuous, there exists $\delta_1 > 0$ such that

$$\operatorname{osc}_{\delta}(\varphi)(x_1+k_1, x_2+k_2) \leq \frac{\varepsilon}{3(2L_0+1)^{d+1}}$$

for all $(x_1, x_2) \in [0,1]\times[0,1]^d$, $|k_1| \leq L_0$, $|k_2| \leq L_0$ and $0 < \delta < \delta_1$. Then

$$\sum_{|k_1|\leq L_0} \sup_{x_1\in[0,1]} \sum_{|k_2|\leq L_0} \sup_{x_2\in[0,1]^d} \operatorname{osc}_{\delta}(\varphi)(x_1+k_1, x_2+k_2) \leq \frac{\varepsilon}{3}. \tag{2.6}$$

Let $\delta' = \min\{\delta_0, \delta_1\}$. It follows from (2.4), (2.5) and (2.6) that for any $0 < \delta \leq \delta'$, one has

$$\begin{aligned}\|\operatorname{osc}_{\delta}(\varphi)\|_{W(L^{1,1})} &= \sum_{k_1\in\mathbf{Z}} \sup_{x_1\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{x_2\in[0,1]^d} \operatorname{osc}_{\delta}(\varphi)(x_1+k_1, x_2+k_2)\\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Thus, $\lim_{\delta\to 0}\|\operatorname{osc}_{\delta}(\varphi)\|_{W(L^{1,1})} = 0$. □

Lemma 2.7. *Suppose that $\varphi \in W(L^{1,1})$ is continuous and $\operatorname{osc}_{(\delta_0,x)}(\varphi) \in W(L^{1,1})$ for some $\delta_0 > 0$. If $\psi \in L^1(\mathbf{R}^{d+1})$ satisfies $\int_{\mathbf{R}}\int_{\mathbf{R}^d}\psi(x,y)\,dxdy = 1$, then $\varphi^a(x,y) = \varphi(x,y) - \varphi * \psi_a^*(x,y)$ satisfies*

$$\lim_{a\to 0}\|\varphi^a\|_{W(L^{1,1})} = 0,$$

where $\psi_a(\cdot) = \frac{1}{a^{d+1}}\psi\big(\frac{\cdot}{a}\big)$ and $\psi_a^*(\cdot) = \overline{\psi_a(-\,\cdot\,)}$.

Proof. Let $\Omega_{\delta_0} = \{(s,t) \in \mathbf{R}\times\mathbf{R}^d : |s| \leq \delta_0, |t| \leq \delta_0\}$. Then it follows from the definition of $\varphi^a(x,y)$ that

$$\begin{aligned}|\varphi^a(x,y)| &\leq \left(\int_{\sqrt{s^2+|t|^2}\geq\delta_0} + \int_{\Omega_{\delta_0}}\right)|\varphi(x,y)-\varphi(x+s,y+t)||\psi_a(s,t)|\,dsdt\\ &=: I_1(x,y) + I_2(x,y).\end{aligned}$$

Now, we estimate $\|I_1\|_{W(L^{1,1})}$ and $\|I_2\|_{W(L^{1,1})}$, respectively. In fact,

$$\begin{aligned}&\|I_1\|_{W(L^{1,1})}\\ &\leq \sum_{k_1\in\mathbf{Z}} \sup_{x\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{y\in[0,1]^d} \int_{\sqrt{s^2+|t|^2}\geq\delta_0} |\varphi(x+k_1, y+k_2)||\psi_a(s,t)|\,dsdt\\ &\quad + \sum_{k_1\in\mathbf{Z}} \sup_{x\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{y\in[0,1]^d} \int_{\sqrt{s^2+|t|^2}\geq\delta_0} |\varphi(x+k_1+s, y+k_2+t)||\psi_a(s,t)|\,dsdt\\ &=: I_{1,1} + I_{1,2}.\end{aligned} \tag{2.7}$$

Since $\psi \in L^1(\mathbf{R}^{d+1})$, then changing the variables obtains

$$I_{1,1} \le \|\varphi\|_{W(L^{1,1})} \int_{\sqrt{s^2+|t|^2}\ge \frac{\delta_0}{a}} |\psi(s,t)|\, dsdt \to 0 \quad \text{as } a \to 0. \tag{2.8}$$

Let $\mathbf{Z}_0^{d+1} = \big\{(\ell_1,\ell_2) \in \mathbf{Z}\times\mathbf{Z}^d : \sqrt{(\ell_1+s)^2+|\ell_2+t|^2} \ge \delta_0 \text{ for any } (s,t)\in[0,1]\times[0,1]^d\big\}$. Then we have

$$\begin{aligned}
&I_{1,2}\\
&= \sum_{k_1\in\mathbf{Z}} \sup_{x\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{y\in[0,1]^d} \int_{\sqrt{s^2+|t|^2}\ge\delta_0} |\varphi(x+k_1+s,y+k_2+t)||\psi_a(s,t)|\, dsdt\\
&\le \sum_{k_1\in\mathbf{Z}} \sup_{x\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{y\in[0,1]^d} \sum_{(\ell_1,\ell_2)\in\mathbf{Z}_0^{d+1}} \int_0^1 \int_{[0,1]^d} |\varphi(x+k_1+\ell_1+s,y+k_2+\ell_2+t)|\\
&\quad \times |\psi_a(\ell_1+s,\ell_2+t)|\, dtds\\
&\le \sum_{(\ell_1,\ell_2)\in\mathbf{Z}_0^{d+1}} \int_0^1 \int_{[0,1]^d} \left(\sum_{k_1\in\mathbf{Z}} \sup_{x\in[0,1]} \sup_{s\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{y\in[0,1]^d} \sup_{t\in[0,1]^d} |\varphi(x+k_1+s,y+k_2+t)| \right)\\
&\quad \times |\psi_a(\ell_1+s,\ell_2+t)|\, dtds\\
&\le 2^{d+1}\|\varphi\|_{W(L^{1,1})} \int_{\sqrt{s^2+|t|^2}\ge\frac{\delta_0}{a}} |\psi(s,t)|\, dsdt \to 0 \quad \text{as } a\to 0.
\end{aligned}$$

This together with (2.7) and (2.8) proves that $\lim_{a\to0}\|I_1\|_{W(L^{1,1})} = 0$.

Since $\lim_{\delta\to0}\|\operatorname{osc}_\delta(\varphi)\|_{W(L^{1,1})} = 0$, then for any $\epsilon>0$, there exists a $0<\delta_2<\delta_0$ such that

$$\|\operatorname{osc}_\delta(\varphi)\|_{W(L^{1,1})} < \epsilon, \quad \forall\, \delta < \delta_2.$$

Now, we begin to estimate $\|I_2\|_{W(L^{1,1})}$:

$$\begin{aligned}
\|I_2\|_{W(L^{1,1})} &\le \sum_{k_1\in\mathbf{Z}} \sup_{x\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{y\in[0,1]^d} \int_{\Omega_{\delta_0}} \operatorname{osc}_{\max\{|s|,|t|\}}(\varphi)(x+k_1,y+k_2)|\psi_a(s,t)|\, dsdt\\
&\le \sum_{k_1\in\mathbf{Z}} \sup_{x\in[0,1]} \sum_{k_2\in\mathbf{Z}^d} \sup_{y\in[0,1]^d} \Big[\int_{\Omega_{\delta_2/2}} \operatorname{osc}_{\delta_2/2}(\varphi)(x+k_1,y+k_2)|\psi_a(s,t)|\, dsdt\\
&\qquad + \int_{\sqrt{s^2+|t|^2}\ge\delta_2/2} \operatorname{osc}_{\delta_0}(\varphi)(x+k_1,y+k_2)|\psi_a(s,t)|\, dsdt \Big]\\
&\le \|\operatorname{osc}_{\delta_2/2}(\varphi)\|_{W(L^{1,1})} \int_{\mathbf{R}}\int_{\mathbf{R}^d} |\psi_a(s,t)|\, dtds\\
&\quad + \|\operatorname{osc}_{\delta_0}(\varphi)\|_{W(L^{1,1})} \int_{\sqrt{s^2+|t|^2}\ge\delta_2/2} |\psi_a(s,t)|\, dsdt\\
&\le \epsilon\|\psi\|_{L^1} + \|\operatorname{osc}_{\delta_0}(\varphi)\|_{W(L^{1,1})} \int_{\sqrt{s^2+|t|^2}\ge\frac{\delta_2}{2a}} |\psi(s,t)|\, dsdt.
\end{aligned}$$

This together with (2.3) obtains $\lim_{a\to0}\|I_2\|_{W(L^{1,1})} = 0$ and the final result holds. □

Lemma 2.8. *Let φ, ψ and φ^a be as in Lemma 2.7. Then*

$$\|\operatorname{osc}_\gamma(\varphi^a)\|_{W(L^{1,1})} \le \big(1+2^{d+1}\|\psi\|_{L^1}\big)\|\operatorname{osc}_\gamma(\varphi)\|_{W(L^{1,1})}$$

and $\lim_{\gamma\to 0}\|\operatorname{osc}_\gamma(\varphi^a)\|_{W(L^{1,1})}=0$.

Proof. Note that

$$\begin{aligned}\operatorname{osc}_\gamma(\varphi^a)(x,y) &\le \operatorname{osc}_\gamma(\varphi)(x,y)+\int_{\mathbf{R}}\int_{\mathbf{R}^d}\operatorname{osc}_\gamma(\varphi)(x+s,y+t)|\psi_a(s,t)|\,dtds\\ &= \operatorname{osc}_\gamma(\varphi)(x,y)+\operatorname{osc}_\gamma(\varphi)*|\psi_a^*|(x,y).\end{aligned}$$

Then it follows from Lemma 2.1 that

$$\|\operatorname{osc}_\gamma(\varphi^a)\|_{W(L^{1,1})} \le \big(1+2^{d+1}\|\psi\|_{L^1}\big)\|\operatorname{osc}_\gamma(\varphi)\|_{W(L^{1,1})}.$$

Since $\lim_{\gamma\to 0}\|\operatorname{osc}_\gamma(\varphi)\|_{W(L^{1,1})}=0$, the desired result is proved. □

3. Average sampling and reconstruction

In this section, we will discuss the nonuniform average sampling in the multiply generated shift-invariant subspaces $V_{p,q}(\Phi_r)$.

Definition 3.1. A set $\Gamma=\{\gamma_{j,k}=(x_j,y_k): x_j\in\mathbf{R}, y_k\in\mathbf{R}^d, j\in\mathbb{J}_1, k\in\mathbb{J}_2\}$ is called a γ_0-dense set if

$$\mathbf{R}^{d+1}=\bigcup_{j,k}B_\gamma(x_j,y_k),\quad \forall\,\gamma>\gamma_0.$$

Here, $B_\gamma(x_j,y_k)$ is an open ball in $\mathbf{R}^{d+1}$ with center (x_j,y_k) and radius γ, $\mathbb{J}_1$ and $\mathbb{J}_2$ are countable index sets.

Definition 3.2. The family $U=\{u_{j,k}(x,y)\}_{j\in\mathbb{J}_1,k\in\mathbb{J}_2}$ is a bounded uniform partition of unity (BUPU) associated with Γ if

(i) $0\le u_{j,k}(x,y)\le 1$ for all $(x,y)\in\mathbf{R}\times\mathbf{R}^d$, $j\in\mathbb{J}_1$, $k\in\mathbb{J}_2$;

(ii) $u_{j,k}(x,y)$ is supported in $B_\gamma(x_j,y_k)$ for every $j\in\mathbb{J}_1$, $k\in\mathbb{J}_2$;

(iii) $\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}u_{j,k}(x,y)=1$ for all $(x,y)\in\mathbf{R}\times\mathbf{R}^d$.

Obviously, $u_{j,k}(x,y)=\frac{\chi_{B(x_j,\gamma/\sqrt{2})}(x)}{\sum_{j'\in\mathbb{J}_1}\chi_{B(x_{j'},\gamma/\sqrt{2})}(x)}\cdot\frac{\chi_{B(y_k,\gamma/\sqrt{2})}(y)}{\sum_{k'\in\mathbb{J}_2}\chi_{B(y_{k'},\gamma/\sqrt{2})}(y)}$, $j\in\mathbb{J}_1$, $k\in\mathbb{J}_2$ is an example of bounded uniform partition of unity. Here, $B(y_k,\gamma/\sqrt{2})$ denotes a ball in $\mathbf{R}^d$ with radius $\gamma/\sqrt{2}$ centered at y_k.

If $f\in V_{p,q}(\Phi_r)$, define the interpolation operator

$$Q_\Gamma f=\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}f(x_j,y_k)u_{j,k}.$$

Then we have the following conclusion.

Lemma 3.3. *Suppose that $\{\varphi_i\}_{1\le i\le r}$ satisfies the assumptions* (A2) *and* (A3)*, then for any $\gamma>0$ and $f=\sum_{i=1}^r\sum_{k_1\in\mathbf{Z}}\sum_{k_2\in\mathbf{Z}^d}c_i(k_1,k_2)\varphi_i(\cdot-k_1,\cdot-k_2)\in V_{p,q}(\Phi_r)$, one has*

$$\|Q_\Gamma f\|_{L^{p,q}}\le\|c\|_{\ell^{p,q}}\max_{1\le i\le r}\big\{\|\varphi_i\|_{W(L^{1,1})}+\|\operatorname{osc}_\gamma(\varphi_i)\|_{W(L^{1,1})}\big\}.$$

Here, $\|c\|_{\ell^{p,q}}=\sum_{i=1}^r\|c_i\|_{\ell^{p,q}}$ for $c=(c_1,c_2,\dots,c_r)^T\in\big(\ell^{p,q}(\mathbf{Z}\times\mathbf{Z}^d)\big)^r$.

Proof. By Lemma 2.3, one has

$$\begin{aligned}\|f-Q_\Gamma f\|_{L^{p,q}}&=\Big\|f(x,y)\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}u_{j,k}(x,y)-\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}f(x_j,y_k)u_{j,k}(x,y)\Big\|_{L^{p,q}}\\&\le\Big\|\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}|f(x,y)-f(x_j,y_k)|u_{j,k}(x,y)\Big\|_{L^{p,q}}\\&\le\Big\|\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}\operatorname{osc}_\gamma(f)(x,y)u_{j,k}(x,y)\Big\|_{L^{p,q}}\\&=\|\operatorname{osc}_\gamma(f)\|_{L^{p,q}}\le\sum_{i=1}^r\|c_i\|_{\ell^{p,q}}\|\operatorname{osc}_\gamma(\varphi_i)\|_{W(L^{1,1})}.\end{aligned}\tag{3.1}$$

Moreover, we can obtain

$$\begin{aligned}\|Q_\Gamma f\|_{L^{p,q}}&\le\|f-Q_\Gamma f\|_{L^{p,q}}+\|f\|_{L^{p,q}}\\&\le\sum_{i=1}^r\|c_i\|_{\ell^{p,q}}\|\operatorname{osc}_\gamma(\varphi_i)\|_{W(L^{1,1})}+\sum_{i=1}^r\|c_i\|_{\ell^{p,q}}\|\varphi_i\|_{W(L^{1,1})}.\end{aligned}$$

This completes the proof. □

3.1. The first average sampling

The first average sampling scheme is convolution sampling which is defined as

$$\langle f(\cdot),\psi_a(\cdot-(x_j,y_k))\rangle=f*\psi_a^*(x_j,y_k),\quad j\in\mathbb{J}_1,\ k\in\mathbb{J}_2,\tag{3.2}$$

where $\psi\in L^1(\mathbf{R}^{d+1})$ satisfies $\int_{\mathbf{R}}\int_{\mathbf{R}^d}\psi(x,y)\,dxdy=1$.

Based on the average samples obtained in (3.2), define the prereconstruction operator

$$A_{\Gamma,a}f=\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}f*\psi_a^*(x_j,y_k)u_{j,k}.$$

Lemma 3.4. *Let P be a bounded projection from $L^{p,q}(\mathbf{R}\times\mathbf{R}^d)$ onto $V_{p,q}(\Phi_r)$. Suppose that $\{\varphi_i\}_{1\le i\le r}$ satisfies the assumptions* (A1)–(A3)*, then there exist $\gamma_0>0$ and $a_0>0$ such that for every γ-dense set Γ with $\gamma\le\gamma_0$ and for every positive $a\le a_0$, the operator $I-PA_{\Gamma,a}$ is a contraction on $V_{p,q}(\Phi_r)$.*

Proof. Suppose that $f=\sum_{i=1}^{r}\sum_{k_1\in\mathbf{Z}}\sum_{k_2\in\mathbf{Z}^d}c_i(k_1,k_2)\varphi_i(\cdot-k_1,\,\cdot-k_2)\in V_{p,q}(\Phi_r)$. Since $Pf=f$, then

$$\begin{aligned}\|f-PA_{\Gamma,a}f\|_{L^{p,q}}&=\|f-PQ_\Gamma f+PQ_\Gamma f-PA_{\Gamma,a}f\|_{L^{p,q}}\\&\le\|f-PQ_\Gamma f\|_{L^{p,q}}+\|PQ_\Gamma f-PA_{\Gamma,a}f\|_{L^{p,q}}\\&\le\|P\|_{\mathrm{op}}\big(\|f-Q_\Gamma f\|_{L^{p,q}}+\|Q_\Gamma f-A_{\Gamma,a}f\|_{L^{p,q}}\big),\end{aligned}\tag{3.3}$$

where $\|P\|_{\mathrm{op}}$ denotes the operator norm of P. Moreover, it follows from (1.1) and (3.1) that

$$\begin{aligned}\|f-Q_\Gamma f\|_{L^{p,q}}&\le\sum_{i=1}^{r}\|c_i\|_{\ell^{p,q}}\|\operatorname{osc}_\gamma(\varphi_i)\|_{W(L^{1,1})}\\&\le B\|f\|_{L^{p,q}}\max_{1\le i\le r}\|\operatorname{osc}_\gamma(\varphi_i)\|_{W(L^{1,1})}.\end{aligned}\tag{3.4}$$

Let $\varphi_i^a(x,y)=\varphi_i(x,y)-\varphi_i*\psi_a^*(x,y)$, $1\le i\le r$. Note that

$$Q_\Gamma f-A_{\Gamma,a}f=Q_\Gamma\left(\sum_{i=1}^{r}\sum_{k_1\in\mathbf{Z}}\sum_{k_2\in\mathbf{Z}^d}c_i(k_1,k_2)\varphi_i^a(\cdot-k_1,\,\cdot-k_2)\right).$$

This together with Lemma 3.3 and (1.1) obtains

$$\begin{aligned}\|Q_\Gamma f-A_{\Gamma,a}f\|_{L^{p,q}}&=\left\|\sum_{i=1}^{r}Q_\Gamma\Big(\sum_{k_1\in\mathbf{Z}}\sum_{k_2\in\mathbf{Z}^d}c_i(k_1,k_2)\varphi_i^a(\cdot-k_1,\,\cdot-k_2)\Big)\right\|_{L^{p,q}}\\&\le\sum_{i=1}^{r}\|c_i\|_{\ell^{p,q}}\max_{1\le i\le r}\big\{\|\varphi_i^a\|_{W(L^{1,1})}+\|\operatorname{osc}_\gamma(\varphi_i^a)\|_{W(L^{1,1})}\big\}\\&\le B\|f\|_{L^{p,q}}\max_{1\le i\le r}\big\{\|\varphi_i^a\|_{W(L^{1,1})}+\|\operatorname{osc}_\gamma(\varphi_i^a)\|_{W(L^{1,1})}\big\}.\end{aligned}$$

This together with (3.3) and (3.4) obtains

$$\begin{aligned}&\|f-PA_{\Gamma,a}f\|_{L^{p,q}}\\&\le B\|P\|_{\mathrm{op}}\left(\max_{1\le i\le r}\|\operatorname{osc}_\gamma(\varphi_i)\|_{W(L^{1,1})}+\max_{1\le i\le r}\big\{\|\varphi_i^a\|_{W(L^{1,1})}+\|\operatorname{osc}_\gamma(\varphi_i^a)\|_{W(L^{1,1})}\big\}\right)\|f\|_{L^{p,q}}.\end{aligned}$$

Therefore, it follows from Lemmas 2.6, 2.7 and 2.8 that we can choose positive numbers γ_0 and a_0 such that for any $\gamma\le\gamma_0$ and $a\le a_0$, one has

$$B\|P\|_{\mathrm{op}}\left(\max_{1\le i\le r}\|\operatorname{osc}_\gamma(\varphi_i)\|_{W(L^{1,1})}+\max_{1\le i\le r}\big\{\|\varphi_i^a\|_{W(L^{1,1})}+\|\operatorname{osc}_\gamma(\varphi_i^a)\|_{W(L^{1,1})}\big\}\right)<1,$$

which means that $I-PA_{\Gamma,a}$ is a contraction on $V_{p,q}(\Phi_r)$. □

Theorem 3.5. *Let P, $\{\varphi_i\}_{1\le i\le r}$, Γ and $A_{\Gamma,a}$ be as in Lemma 3.4. Then any signal $f\in V_{p,q}(\Phi_r)$ can be recovered from its average samples $\{\langle f(\cdot),\psi_a(\cdot-(x_j,y_k))\rangle\}_{j\in\mathbb{J}_1,k\in\mathbb{J}_2}$ by the iterative approximation projection algorithm*

$$f_1=PA_{\Gamma,a}f\quad\text{and}\quad f_{n+1}=PA_{\Gamma,a}(f-f_n)+f_n.$$

Here, f_n uniformly converges to f, and also in $W(L^{p,q})$ and $L^{p,q}$-norms just as

$$\|f-f_n\|_{L^{p,q}}\le\|f-f_n\|_{W(L^{p,q})}\le M\alpha_1^n\|f\|_{L^{p,q}} \tag{3.5}$$

for some $\alpha_1=\alpha_1(\gamma,a,\Phi,\psi)<1$ and $M<\infty$.

Proof. Let $e_n=f-f_n$. Then

$$e_{n+1}=f-f_{n+1}=(I-PA_{\Gamma,a})e_n.$$

Set

$$\alpha_1=B\|P\|_{\mathrm{op}}\left(\max_{1\le i\le r}\|\operatorname{osc}_\gamma(\varphi_i)\|_{W(L^{1,1})}+\max_{1\le i\le r}\left\{\|\varphi_i^a\|_{W(L^{1,1})}+\|\operatorname{osc}_\gamma(\varphi_i^a)\|_{W(L^{1,1})}\right\}\right). \tag{3.6}$$

By Lemma 3.4, we may choose $\gamma>0$ and $a>0$ small enough such that $\alpha_1<1$ and

$$\|e_{n+1}\|_{L^{p,q}}\le\alpha_1\|e_n\|_{L^{p,q}}\le\alpha_1^{n+1}\|f\|_{L^{p,q}}.$$

Thus, it follows from Lemma 2.4 that (3.5) holds with $M=B\big(\max_{1\le i\le r}\|\varphi_i\|_{W(L^{1,1})}\big)$. □

3.2. The second average sampling

The second average sampling scheme is

$$\langle f,\psi_{j,k}\rangle=\int_{\mathbf{R}}\int_{\mathbf{R}^d}f(x,y)\psi_{j,k}(x,y)\,dxdy,\quad j\in\mathbb{J}_1,\ k\in\mathbb{J}_2,$$

where the average sampling functionals $\{\psi_{j,k}:j\in\mathbb{J}_1,k\in\mathbb{J}_2\}$ satisfy

(i) $\int_{\mathbf{R}}\int_{\mathbf{R}^d}\psi_{j,k}(x,y)\,dxdy=1$ for all $j\in\mathbb{J}_1$, $k\in\mathbb{J}_2$;

(ii) There exists an $R>0$ such that $\int_{\mathbf{R}}\int_{\mathbf{R}^d}|\psi_{j,k}(x,y)|\,dxdy\le R$ for all $j\in\mathbb{J}_1$, $k\in\mathbb{J}_2$;

(iii) $\operatorname{supp}\psi_{j,k}\subset B_a(x_j,y_k)$ for some $a>0$.

Define the quasi-reconstruction operator

$$A_\Gamma f=\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}\langle f,\psi_{j,k}\rangle u_{j,k}.$$

Then we have the following conclusion.

Theorem 3.6. *Let P be a bounded projection from $L^{p,q}(\mathbf{R}\times\mathbf{R}^d)$ onto $V_{p,q}(\Phi_r)$. Suppose that $\{\varphi_i\}_{1\le i\le r}$ satisfies the assumptions* (A1)–(A3)*, then there exist $\gamma>0$ and $a>0$ such that any $f\in V_{p,q}(\Phi_r)$ can be recovered from its average samples $\{\langle f,\psi_{j,k}\rangle\}_{j\in\mathbb{J}_1,k\in\mathbb{J}_2}$ by the iterative algorithm*

$$f_1=PA_\Gamma f\quad\text{and}\quad f_{n+1}=PA_\Gamma(f-f_n)+f_n,$$

which converges uniformly and also satisfies

$$\|f-f_n\|_{L^{p,q}}\le\|f-f_n\|_{W(L^{p,q})}\le M\alpha_2^n\|f\|_{L^{p,q}}$$

for some $\alpha_2=\alpha_2(\gamma,a,\Phi,R)<1$.

Proof. Let $f=\sum_{i=1}^r\sum_{k_1\in\mathbf{Z}}\sum_{k_2\in\mathbf{Z}^d}c_i(k_1,k_2)\varphi_i(\cdot-k_1,\cdot-k_2)\in V_{p,q}(\Phi_r)$. Then

$$\begin{aligned}\|f-PA_\Gamma f\|_{L^{p,q}}&=\|f-PQ_\Gamma f+PQ_\Gamma f-PA_\Gamma f\|_{L^{p,q}}\\&\le\|P\|_{\mathrm{op}}\big(\|f-Q_\Gamma f\|_{L^{p,q}}+\|Q_\Gamma f-A_\Gamma f\|_{L^{p,q}}\big).\end{aligned}\tag{3.7}$$

Note that

$$\mathrm{osc}_a(f)\le\sum_{i=1}^r\sum_{k_1\in\mathbf{Z}}\sum_{k_2\in\mathbf{Z}^d}|c_i(k_1,k_2)|\,\mathrm{osc}_a(\varphi_i)(\cdot-k_1,\cdot-k_2).$$

Then, it follows from Lemma 3.3 that

$$\begin{aligned}&\|Q_\Gamma f-A_\Gamma f\|_{L^{p,q}}\\&\le\Big\|\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}\int_{\mathbf{R}}\int_{\mathbf{R}^d}|f(x_j,y_k)-f(s,t)||\psi_{j,k}(s,t)|\,dsdtu_{j,k}(x,y)\Big\|_{L^{p,q}}\\&\le R\Big\|\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{J}_2}\mathrm{osc}_a(f)(x_j,y_k)u_{j,k}(x,y)\Big\|_{L^{p,q}}\\&\le R\|c\|_{\ell^{p,q}}\max_{1\le i\le r}\big\{\|\mathrm{osc}_a(\varphi_i)\|_{W(L^{1,1})}+\|\mathrm{osc}_\gamma(\mathrm{osc}_a(\varphi_i))\|_{W(L^{1,1})}\big\}\\&\le RB\|f\|_{L^{p,q}}\max_{1\le i\le r}\big\{\|\mathrm{osc}_a(\varphi_i)\|_{W(L^{1,1})}+\|\mathrm{osc}_\gamma(\mathrm{osc}_a(\varphi_i))\|_{W(L^{1,1})}\big\}\\&\le RB\|f\|_{L^{p,q}}\max_{1\le i\le r}\big\{2\|\mathrm{osc}_a(\varphi_i)\|_{W(L^{1,1})}+\|\mathrm{osc}_\gamma(\varphi_i)\|_{W(L^{1,1})}+\|\mathrm{osc}_{a+\gamma}(\varphi_i)\|_{W(L^{1,1})}\big\}.\end{aligned}$$

This together with (3.7) obtains

$$\begin{aligned}&\|f-PA_\Gamma f\|_{L^{p,q}}\\&\le B\|P\|_{\mathrm{op}}\Big(\max_{1\le i\le r}\|\mathrm{osc}_\gamma(\varphi_i)\|_{W(L^{1,1})}\\&\qquad+R\max_{1\le i\le r}\big\{2\|\mathrm{osc}_a(\varphi_i)\|_{W(L^{1,1})}+\|\mathrm{osc}_\gamma(\varphi_i)\|_{W(L^{1,1})}+\|\mathrm{osc}_{a+\gamma}(\varphi_i)\|_{W(L^{1,1})}\big\}\Big)\|f\|_{L^{p,q}}.\end{aligned}$$

Take $\gamma > 0$ and $a > 0$ small enough such that

(3.8)
$$\begin{aligned}\alpha_2 = B\|P\|_{\mathrm{op}}\Big(&\max_{1\le i\le r} \|\operatorname{osc}_\gamma(\varphi_i)\|_{W(L^{1,1})} \\ &+ R\max_{1\le i\le r}\big\{2\|\operatorname{osc}_a(\varphi_i)\|_{W(L^{1,1})} + \|\operatorname{osc}_\gamma(\varphi_i)\|_{W(L^{1,1})} + \|\operatorname{osc}_{a+\gamma}(\varphi_i)\|_{W(L^{1,1})}\big\}\Big) \\ &< 1.\end{aligned}$$

Then the operator $I - PA_\Gamma$ is a contraction on $V_{p,q}(\Phi_r)$ and the remained proof is similar to that of Theorem 3.5. □

4. Convergence rate

In this section, we estimate the convergence rate α_1 in Theorem 3.5 and α_2 in Theorem 3.6 when Φ_r and the average sampling functionals satisfy some additional regularity conditions.

Theorem 4.1. *Suppose that $\{\varphi_i\}_{1\le i\le r}$ and ψ satisfy the conditions of Theorem* 3.5*, and that $|\nabla\varphi_i| \in W(L^1)(\mathbf{R}^{d+1})$ for every $i = 1,\dots,r$ and $\|\psi\|_{1,\eta} := \int_{\mathbf{R}}\int_{\mathbf{R}^d} |\psi(s,t)||(s,t)|^\eta\, dsdt < \infty$ for some $0 < \eta \le 1$. Then the convergence rate α_1 in Theorem* 3.5 *satisfies*

$$\begin{aligned}\alpha_1 \le B\|P\|_{\mathrm{op}}\Big(&\sqrt{2}\gamma\big(2 + 2^{d+1}\|\psi\|_{L^1}\big)(3 + 2[\gamma])^{d+1}\max_{1\le i\le r}\big\||\nabla\varphi_i|\big\|_{W(L^1)} + a^\eta\|\psi\|_{1,\eta} \\ &\times \max_{1\le i\le r}\big((1 + 2^{d+1})\delta_0^{-\eta}\|\varphi_i\|_{W(L^{1,1})} + \sqrt{2}\delta_0^{1-\eta}(3 + 2[\delta_0])^{d+1}\big\||\nabla\varphi_i|\big\|_{W(L^1)}\big)\Big).\end{aligned} \tag{4.1}$$

Proof. Note that

$$\begin{aligned}|\varphi(x_1 + y_1, x_2 + y_2) - \varphi(x_1, x_2)| &= \left|\int_0^1 (y_1, y_2)\cdot\nabla\varphi(x_1 + sy_1, x_2 + sy_2)\, ds\right| \\ &\le \int_0^1 |(y_1, y_2)||\nabla\varphi|(x_1 + sy_1, x_2 + sy_2)\, ds \\ &\le |(y_1, y_2)| \sup_{|t_1|\le|y_1|,|t_2|\le|y_2|} |\nabla\varphi|(x_1 + t_1, x_2 + t_2),\end{aligned}$$

where $\cdot$ in the first equality denotes the vector inner product. Therefore, we can obtain

$$\begin{aligned}\operatorname{osc}_\gamma(\varphi)(x_1, x_2) &= \sup_{|y_1|\le\gamma,|y_2|\le\gamma} |\varphi(x_1 + y_1, x_2 + y_2) - \varphi(x_1, x_2)| \\ &\le \sqrt{2}\gamma \sup_{|t_1|\le\gamma,|t_2|\le\gamma} |\nabla\varphi|(x_1 + t_1, x_2 + t_2).\end{aligned}$$

Moreover, one has

$$
\begin{aligned}
&\|\operatorname{osc}_\gamma(\varphi)\|_{W(L^{1,1})}\\
&=\sum_{k_1\in\mathbf{Z}}\sup_{x_1\in[0,1]}\sum_{k_2\in\mathbf{Z}^d}\sup_{x_2\in[0,1]^d}|\operatorname{osc}_\gamma(\varphi)(x_1+k_1,x_2+k_2)|\\
&\le\sqrt{2}\gamma\sum_{k_1\in\mathbf{Z}}\sup_{x_1\in[0,1]}\sum_{k_2\in\mathbf{Z}^d}\sup_{x_2\in[0,1]^d}\sup_{|t_1|\le\gamma,|t_2|\le\gamma}|\nabla\varphi|(x_1+t_1+k_1,x_2+t_2+k_2)\\
&\le\sqrt{2}\gamma\sum_{k_1\in\mathbf{Z}}\sum_{k_2\in\mathbf{Z}^d}\sup_{y_1\in[-\gamma,1+\gamma]}\sup_{y_2\in[-\gamma,1+\gamma]^d}|\nabla\varphi|(y_1+k_1,y_2+k_2)\\
&\le\sqrt{2}\gamma(3+2[\gamma])^{d+1}\big\||\nabla\varphi|\big\|_{W(L^1)}.
\end{aligned}
\tag{4.2}
$$

Next, we estimate the $W(L^{1,1})$-norm of $\varphi^a=\varphi-\varphi*\psi_a^*$. Following the proof of Lemma 2.7, we have

$$
\|I_1\|_{W(L^{1,1})}\le(1+2^{d+1})a^\eta\delta_0^{-\eta}\|\varphi\|_{W(L^{1,1})}\|\psi\|_{1,\eta}.
\tag{4.3}
$$

Moreover, one has

$$
\begin{aligned}
I_2(x,y)&\le\int_{\Omega_{\delta_0}}|(s,t)|\sup_{|s_1|\le\delta_0,|t_1|\le\delta_0}|\nabla\varphi|(x+s_1,y+t_1)|\psi_a(s,t)|\,dsdt\\
&\le\sqrt{2}a^\eta\delta_0^{1-\eta}\|\psi\|_{1,\eta}\sup_{|s_1|\le\delta_0,|t_1|\le\delta_0}|\nabla\varphi|(x+s_1,y+t_1).
\end{aligned}
$$

This together with (4.3) obtains

$$
\begin{aligned}
\|\varphi^a\|_{W(L^{1,1})}&\le(1+2^{d+1})a^\eta\delta_0^{-\eta}\|\varphi\|_{W(L^{1,1})}\|\psi\|_{1,\eta}\\
&\quad+\sqrt{2}a^\eta\delta_0^{1-\eta}(3+2[\delta_0])^{d+1}\big\||\nabla\varphi|\big\|_{W(L^1)}\|\psi\|_{1,\eta}.
\end{aligned}
$$

Finally, it follows from Lemma 2.8 and (3.6) that the desired result (4.1) holds. □

Theorem 4.2. *Suppose that* $\{\varphi_i\}_{1\le i\le r}$ *and* $\{\psi_{j,k}:j\in\mathbb{J}_1,k\in\mathbb{J}_2\}$ *satisfy the conditions of Theorem* 3.6*, and that* $|\nabla\varphi_i|\in W(L^1)(\mathbf{R}^{d+1})$ *for every* $i=1,\dots,r$*. Then the convergence rate* α_2 *in Theorem* 3.6 *satisfies*

$$
\begin{aligned}
\alpha_2&\le\sqrt{2}B\|P\|_{\mathrm{op}}\big(\gamma(1+R)(3+2[\gamma])^{d+1}+2aR(3+2[a])^{d+1}+(a+\gamma)R(3+2[a+\gamma])^{d+1}\big)\\
&\quad\times\max_{1\le i\le r}\big\||\nabla\varphi_i|\big\|_{W(L^1)}.
\end{aligned}
$$

Proof. The proof of Theorem 4.2 is a direct consequence of (3.8) and (4.2). □

Remark 4.3. Note that the convergence rates α_1 in Theorem 3.5 and α_2 in Theorem 3.6 are given as

$$
\alpha_1=B\|P\|_{\mathrm{op}}\left(\max_{1\le i\le r}\|\operatorname{osc}_\gamma(\varphi_i)\|_{W(L^{1,1})}+\max_{1\le i\le r}\big\{\|\varphi_i^a\|_{W(L^{1,1})}+\|\operatorname{osc}_\gamma(\varphi_i^a)\|_{W(L^{1,1})}\big\}\right)
$$

and

$$\alpha_2 = B\|P\|_{\mathrm{op}}\Big(\max_{1\le i\le r} \|\operatorname{osc}_\gamma(\varphi_i)\|_{W(L^{1,1})} + R \max_{1\le i\le r} \big\{2\|\operatorname{osc}_a(\varphi_i)\|_{W(L^{1,1})} + \|\operatorname{osc}_\gamma(\varphi_i)\|_{W(L^{1,1})} + \|\operatorname{osc}_{a+\gamma}(\varphi_i)\|_{W(L^{1,1})}\big\}\Big).$$

Although we know from Lemmas 2.6, 2.7 and 2.8 that both $\alpha_1 < 1$ and $\alpha_2 < 1$ can be achieved by taking $\gamma > 0$ and $a > 0$ small enough, we do not have the expression of the dependence of α_1 and α_2 on γ and a. Under suitable conditions, Theorems 4.1 and 4.2 obtain explicit bounds for α_1 and α_2, respectively, which allow us to find sufficient conditions for density γ and size a such that α_1 and α_2 can be arbitrary small.

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