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Boundedness of the Discrete Square Function on Nondoubling Parabolic Manifolds with Ends

Hong Chuong Doan

Abstract. Let M be a nondoubling parabolic manifold with ends. In this study, we will investigate the L^p boundedness of the discrete square function in terms of the Littlewood-Paley decomposition. It should be pointed out that, in our setting, the doubling condition of the underlying space and the regularity estimate of the kernel are missing.

1. Introduction

We first recall some basic facts about the parabolic manifolds with ends which have been studied in [11]. Let M be a complete noncompact Riemannian manifold and $K \subset M$ be a connected compact subset of M with nonempty interior and smooth boundary such that $M \setminus K$ has k noncompact connected components $E_1, E_2, \ldots, E_k$. We refer to each E_i $(i = 1, \ldots, k)$ as an end of M and K as its central part.

In many cases, each E_i is isometric to the exterior of a compact set in another manifold M_i; therefore, we refer to M as the connected sum of $M_1, M_2, \ldots, M_k$ and write

$$M = M_1 \sharp M_2 \sharp \cdots \sharp M_k.$$

For a fix integer $N > 0$, take an arbitrary integer $m \in [1, N]$. The manifold $\mathcal{R}^m$ is defined by

$$\mathcal{R}^1 = \mathbb{R}_+ \times S^{N-1} \quad \text{and} \quad \mathcal{R}^m = \mathbb{R}^m \times S^{N-m} \quad \text{for all } m \geq 2.$$

For these manifolds, their topological dimension are N but their "*dimension at infinity*" could vary from 1 to N, that is, dimension at infinity of $\mathcal{R}^m$ is m in the sense that $V(x, r) \approx r^m$ for all $r \geq 1$. Thus, we can construct a finite connected sum of the $\mathcal{R}^m$'s:

$$M = \mathcal{R}^{m_1} \sharp \mathcal{R}^{m_2} \sharp \cdots \sharp \mathcal{R}^{m_k},$$

where $m_1, m_2, \ldots, m_k \in [1, N]$.

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Let us recall that a manifold is called parabolic if any positive super-harmonic function on M is constant. This implies that $m_1, m_2, \ldots, m_k \le 2$. For other equivalent definitions of parabolic manifolds, we refer the readers to [10, p. 164] and [11, p. 5].

For the sake of simplicity, we narrow down our study to the case $M = \mathcal{R}^1 \sharp \mathcal{R}^2$. However, the result is still valid on other settings, for example $\mathcal{R}^1 \sharp \mathcal{R}^1 \sharp \mathcal{R}^2$, with a slight modification.

For $x \in M$, we set

$$|x| = \sup_{y \in K} d(x, y)$$

where d is the geodesic distance in M and K is the central part of M. It can be verified that x is separated from 0 on M and $|x| \approx 1 + d(x, K)$. The notation "$\approx$" means that there exist constants c and C such that $c(1 + d(x, K)) \le |x| \le C(1 + d(x, K))$.

We denote by $B(x, r)$ the geodesic ball centered x with radius r and $V(x, r)$ the measure of $B(x, r)$. It can be easily checked that the manifold M is a nondoubling space based on the following properties:

(a) $V(x, r) \approx r^2$ for all $x \in M$, when $r \le 1$;

(b) $V(x, r) \approx r$ for $B(x, r) \subset \mathcal{R}^1$, when $r > 1$; and

(c) $V(x, r) \approx r^2$ for either $x \in \mathcal{R}^1 \setminus K$, $r > 2|x|$ or $x \in \mathcal{R}^2$, $r > 1$.

Let Δ be the Laplace-Beltrami operator acting on M. Consider a dyadic partition of unit, i.e., $\varphi_0 \in C_0^\infty(\mathbb{R})$ and $\varphi \in C_0^\infty(0, \infty)$ such that $\operatorname{supp} \varphi \subset [1/2, 2]$ and $\operatorname{supp} \varphi_0 \subset [0, 2]$, $\varphi_0^{(2\nu+1)}(0) = 0$ for all $\nu \ge 0$ and

$$1 = \varphi_0(\lambda) + \sum_{k>0} \varphi(2^{-k}\lambda) = \sum_{k \ge 0} \varphi_k(\lambda)$$

for all $\lambda \ge 0$, where $\varphi_k(\lambda) = \varphi(2^{-k}\lambda)$ for all $k > 0$.

In this study, we are interested in studying the estimates of the square function defined by

$$S_\Delta f := \left(\left|\varphi_0(\sqrt{\Delta}) f\right|^2 + \sum_{k>0} \left|\varphi(2^{-k}\sqrt{\Delta}) f\right|^2 \right)^{1/2}. \tag{1.1}$$

This problem has been a topic of interest in harmonic analysis with a great deal of attention. The foundations of this subject can be found in [12]. Other versions (with different square functions) can be found in [15, Chapter 4]. For more examples, see [5, 13] and the references therein.

Although the theory of Littlewood-Paley inequalities is well developed and understood, most of previous results require either the doubling condition (see [5, 13]) or the regularity estimate for the heat kernels (see [1]). Recent progress has been made in [2] in which the two necessary conditions above are not satisfied. Inspired by that work, we would like to investigate the Littlewood-Paley inequality on nondoubling parabolic manifolds with ends. More specifically, we obtain the following result:

Theorem 1.1. *Let Δ be the Laplace-Beltrami operator on the manifold $\mathcal{R}^1\sharp\mathcal{R}^2$ and S_Δ the discrete square function defined by* (1.1)*. Then*

$$\|S_\Delta f\|_{L^p} \approx \|f\|_{L^p} \tag{1.2}$$

for all $1 < p < \infty$.

It should be noted that the standard Calderón-Zygmund decomposition cannot be applied to the case where variables lie in different ends due to the blowing up effect of nondoubling volumes of balls. Furthermore, the heat kernels upper bounds depend heavily on both the distance of variables from the central part K and the modulus of variables against the value of the time scaling t. For example, if $x \in \mathcal{R}^1$, $y \in \mathcal{R}^2$, $|x|, |y| \le \sqrt{t}$ and $t > 1$,

$$p_t(x,y) \approx \frac{1}{t}\left(1 + \frac{|x|}{\sqrt{t}} \log \frac{\sqrt{t}}{|y|}\right)$$

where $p_t(x,y)$ denotes the heat kernel of the heat semi-group $e^{-t\Delta}$ acting on $\mathcal{R}^1\sharp\mathcal{R}^2$.

Therefore, to achieve the desired estimate (1.2), we had to separately investigate the heat kernel upper bounds and carry out a subtle decomposition. The approach taken here is derived from the sharp heat kernel estimates in [11] and the ideas in [2, 3, 8].

Our result can be extended for the non-negative self-adjoint operators $L \in L^2(M)$ which satisfy the similar heat kernel upper bounds in [11].

The paper is organised as follows. In Section 2, we address some needed kernel estimates based on the heat kernel estimates in [11]. The proof of our main results will be shown in Section 3.

2. Kernel estimates

The following estimates are taken from [11].

Theorem 2.1. [11] *Let Δ be the Laplace-Beltrami operator acting on the manifold $\mathcal{R}^1\sharp\mathcal{R}^2$. The heat kernel $p_t(x,y)$ associated with the heat semigroup $e^{-t\Delta}$ satisfies the following estimates*

(1) *If $t \le 1$, then*

$$p_t(x,y) \approx \frac{C}{V(x,\sqrt{t})} e^{-b\frac{d^2(x,y)}{t}}$$

for all $x, y \in \mathcal{R}^1\sharp\mathcal{R}^2$.

(2) *If* $t > 1$, *then*

(i) *for* $x, y \in K$
$$p_t(x,y) \approx \frac{C}{t} e^{-b\frac{d^2(x,y)}{t}}.$$

(ii) *for* $x \in \mathcal{R}^2 \setminus K$ *and* $y \in K$

(ii$_1$) $|x| > \sqrt{t}$,
$$p_t(x,y) \approx \frac{C}{t} e^{-b\frac{d^2(x,y)}{t}}.$$

(ii$_2$) $|x| \leq \sqrt{t}$,
$$p_t(x,y) \approx \frac{C}{t}.$$

(iii) *for* $x \in \mathcal{R}^1 \setminus K$ *and* $y \in K$

(iii$_1$) $|x| > \sqrt{t}$,
$$p_t(x,y) \approx C\frac{\log t}{t} e^{-b\frac{d^2(x,y)}{t}}.$$

(iii$_2$) $|x| \leq \sqrt{t}$,
$$p_t(x,y) \approx \frac{C}{t}\left(1 + \frac{|x|}{\sqrt{t}}\log t\right) e^{-b\frac{d^2(x,y)}{t}}.$$

(iv) *for* $x \in \mathcal{R}^1 \setminus K$ *and* $y \in \mathcal{R}^2 \setminus K$

(iv$_1$) $|y| > \sqrt{t}$,
$$p_t(x,y) \approx \frac{C}{t} e^{-b\frac{d^2(x,y)}{t}}.$$

(iv$_2$) $|x|, |y| \leq \sqrt{t}$,
$$p_t(x,y) \approx \frac{C}{t}\left(1 + \frac{|x|}{\sqrt{t}}\log\frac{\sqrt{t}}{|y|}\right).$$

(iv$_3$) $|x| > \sqrt{t} \geq |y|$,
$$p_t(x,y) \approx \frac{C}{t}\log\frac{\sqrt{t}}{|y|} e^{-b\frac{d^2(x,y)}{t}}.$$

(v) *for* $x, y \in \mathcal{R}^1 \setminus K$

(v$_1$) $|x|, |y| \leq \sqrt{t}$,
$$p_t(x,y) \approx \frac{C}{t}\left(1 + \frac{|x||y|}{\sqrt{t}} + \frac{|x| + |y|}{\sqrt{t}}\log t\right) e^{-b\frac{d^2(x,y)}{t}}.$$

(v$_2$) $|x| > \sqrt{t} \geq |y|$,
$$p_t(x,y) \approx \frac{C}{t}(|y| + \log t) e^{-b\frac{d^2(x,y)}{t}}.$$

(v$_3$) $|y| > \sqrt{t} \geq |x|$,
$$p_t(x,y) \approx \frac{C}{t}(|x| + \log t) e^{-b\frac{d^2(x,y)}{t}}.$$

(v$_4$) $|x|, |y| > \sqrt{t}$,

$$p_t(x,y) \approx \frac{C}{\sqrt{t}} e^{-b\frac{d^2(x,y)}{t}}.$$

(vi) *for* $x, y \in \mathcal{R}^2 \setminus K$

$$p_t(x,y) \approx \frac{C}{t} e^{-b\frac{d^2(x,y)}{t}}.$$

We recall here a version of Phramén-Lindelöf Theorem (see [6, Proposition 2.2]).

Lemma 2.2. [6] *Suppose that function F is analytic in $\{z \in \mathbb{C} : \Re z > 0\}$ and satisfies the following estimates*

$$|F(z)| \leq \alpha \ \text{ for all } z \in \mathbb{C}_+ \quad \text{and} \quad |F(\Re z)| \leq \beta e^{\nu \Re z} e^{-\frac{\delta}{\Re z}}$$

for some $\alpha, \beta, \delta > 0$ and $\nu \geq 0$. Then

$$|F(z)| \leq \alpha e^{-\Re \frac{\delta}{z}}$$

for all $z \in \mathbb{C}_+$.

Next, we will investigate the estimate of the heat kernel of the operator $e^{-z\Delta}$ with $z \in \mathbb{C}$ and $\Re z > 0$.

Proposition 2.3. *Let $z := t + is$ where $t > 0$ and $s \in \mathbb{R}$. Let $\theta = \arg z$. The kernel of the operator $e^{-z\Delta}$ satisfies the following estimate:*

$$|p_z(x,y)| \lesssim \left(\frac{1}{\sqrt{t}} + \frac{1}{t}\right) e^{-b\frac{d^2(x,y)}{|z|}\cos\theta}$$

for all $x, y \in \mathcal{R}^1 \sharp \mathcal{R}^2$.

This proof is an adaption of the ideas in [2, 9]. However, in practically, we had to employed particular treatments for the cases where the log terms occurred.

Proof of Proposition 2.3. We have

$$e^{-z\Delta} = e^{-\left(\frac{t}{2}+is\right)\Delta} e^{-\frac{t}{2}\Delta}.$$

This implies that

$$p_z(x,y) = e^{-\left(\frac{t}{2}+is\right)\Delta}\left[p_{\frac{t}{2}}(\,\cdot\,,y)\right](x) = \int_{\mathcal{R}^1\sharp\mathcal{R}^2} p_{\frac{t}{2}+is}(x,w) p_{\frac{t}{2}}(w,y)\, d\mu(w).$$

Applying Hölder's inequality,

$$|p_z(x,y)| \leq \left\|p_{\frac{t}{2}+is}(x,\cdot\,)\right\|_2 \left\|p_{\frac{t}{2}}(\,\cdot\,,y)\right\|_2. \tag{2.1}$$

We now consider the following cases.

Case 1*:* $t \le 2$. Applying Theorem 2.1(1) for the kernel $p_{\frac{t}{2}}$,

$$p_{\frac{t}{2}}(w,u) \lesssim \frac{1}{t} e^{-b\frac{d^2(w,u)}{t}}$$

for all $w, u \in \mathcal{R}^1 \sharp \mathcal{R}^2$. Thus

$$\left\| p_{\frac{t}{2}}(\,\cdot\,,u) \right\|_2^2 \lesssim \int_{\mathcal{R}^1\sharp\mathcal{R}^2} \left| \frac{1}{t} e^{-b\frac{d^2(w,u)}{t}} \right|^2 d\mu(w) \lesssim \frac{1}{t} \int_{\mathcal{R}^1\sharp\mathcal{R}^2} \frac{1}{t} e^{-2b\frac{d^2(x,y)}{t}}\, d\mu(w).$$

Hence

$$\left\| p_{\frac{t}{2}}(\,\cdot\,,u) \right\|_2 \lesssim \frac{1}{\sqrt{t}} \quad \text{for all } u \in \mathcal{R}^1\sharp\mathcal{R}^2. \tag{2.2}$$

Moreover,

$$p_{\frac{t}{2}+is}(x,w) = e^{is\Delta}\left[p_{\frac{t}{2}}(\,\cdot\,,x)\right](w),$$

and then it follows

$$\left\| p_{\frac{t}{2}+is}(x,\cdot) \right\|_2 \lesssim \|e^{is\Delta}\|_{2\to 2} \left\| p_{\frac{t}{2}}(\,\cdot\,,x) \right\|_2 \lesssim \left\| p_{\frac{t}{2}}(\,\cdot\,,x) \right\|_2 \lesssim \frac{1}{\sqrt{t}}$$

where the last inequality was implied by (2.2).

Substituting both estimates of $\left\| p_{\frac{t}{2}}(\,\cdot\,,u) \right\|_2$ and $\left\| p_{\frac{t}{2}+is}(x,\cdot) \right\|_2$ into (2.1), we deduce that

$$|p_z(x,y)| \lesssim \frac{1}{t} \quad \text{for all } x, y \in \mathcal{R}^1\sharp\mathcal{R}^2.$$

Case 2*:* $t > 2$. It should be noted that

$$\begin{aligned} \left\| p_{\frac{t}{2}}(\,\cdot\,,u) \right\|_2^2 &= \int_{\mathcal{R}^1\sharp\mathcal{R}^2} \left| p_{\frac{t}{2}}(w,u) \right|^2 d\mu(w) = \left(\int_{\mathcal{R}^2\setminus K} + \int_{\mathcal{R}^1\setminus K} + \int_K \right) \left| p_{\frac{t}{2}}(w,u) \right|^2 d\mu(w) \\ &=: I + II + III. \end{aligned}$$

To estimate

$$I = \int_{\mathcal{R}^2\setminus K} \left| p_{\frac{t}{2}}(w,u) \right|^2 d\mu(w),$$

let us consider the following cases.

(i_1) If $u \in \mathcal{R}^2 \setminus K$, by Theorem 2.1(2)(vi),

$$I \lesssim \int_{\mathcal{R}^2\setminus K} \left| \frac{1}{t} e^{-b\frac{d^2(w,u)}{t}} \right|^2 d\mu(w) \lesssim \frac{1}{t}.$$

(i_2) If $u \in K$, by Theorem 2.1(2)(ii_1) and (2)(ii_2),

$$\begin{aligned} I &= \left(\int_{|w| \le \sqrt{t}} + \int_{|w| > \sqrt{t}} \right) \left| p_{\frac{t}{2}}(w,u) \right|^2 d\mu(w) \\ &\lesssim \int_{|w| \le \sqrt{t}} \left| \frac{1}{t} \right|^2 d\mu(w) + \int_{|w| > \sqrt{t}} \left| \frac{1}{t} e^{-b\frac{d^2(w,u)}{t}} \right|^2 d\mu(w) \\ &=: I_1 + I_2. \end{aligned}$$

Similar to (i_1), we get that

$$I_2 \lesssim \frac{1}{t}.$$

Since $|w| \le \sqrt{t}$ and $w \in \mathcal{R}^2 \setminus K$, it can be verified that

$$I_1 = \frac{1}{t^2} \int_{|w| \le \sqrt{t}} d\mu(w) \lesssim \frac{1}{t}.$$

(i_3) If $u \in \mathcal{R}^1 \setminus K$ and $|u| \le |w|$,

$$I = \left(\int_{|w| \le \sqrt{t}} + \int_{|w| > \sqrt{t}} \right) \left| p_{\frac{t}{2}}(w,u) \right|^2 d\mu(w) =: I_3 + I_4.$$

Applying Theorem 2.1(2)(iv_2) for I_3,

$$\begin{aligned} I_3 &\lesssim \int_{|w| \le \sqrt{t}} \left| \frac{1}{t} \left(1 + \frac{|u|}{\sqrt{t}} \log \frac{\sqrt{t}}{|w|} \right) \right|^2 d\mu(w) \\ &\lesssim \int_{|w| \le \sqrt{t}} \frac{1}{t^2} d\mu(w) + \int_{|w| \le \sqrt{t}} \frac{1}{t^2} \log \frac{\sqrt{t}}{|w|} d\mu(w) \\ &\quad + \int_{|w| \le \sqrt{t}} \frac{1}{t^2} \log^2 \frac{\sqrt{t}}{|w|} d\mu(w) \quad (\text{since } |u| \le |w| \le \sqrt{t}) \\ &=: I_{31} + I_{32} + I_{33}. \end{aligned}$$

Similar to I_1, we have that

$$I_{31} \lesssim \frac{1}{t}.$$

As for I_{32} and I_{33},

$$I_{32} \lesssim \sum_j \int_{2^{-(j+1)}\sqrt{t} < |w| \le 2^{-j}\sqrt{t}} \frac{j+1}{t^2} d\mu(w) \lesssim \frac{1}{t} \sum_j \frac{j+1}{2^{2j}}$$

and

$$I_{33} \lesssim \sum_j \int_{2^{-(j+1)}\sqrt{t} < |w| \le 2^{-j}\sqrt{t}} \frac{(j+1)^2}{t^2} d\mu(w) \lesssim \frac{1}{t} \sum_j \frac{(j+1)^2}{2^{2j}},$$

since $\frac{\sqrt{t}}{|w|} \le 2^{j+1}$.

Since both series $\sum_j \frac{j+1}{2^{2j}}$ and $\sum_j \frac{(j+1)^2}{2^{2j}}$ converge, $I_{32} \lesssim \frac{1}{t}$ and $I_{33} \lesssim \frac{1}{t}$. Combining estimates of I_{31}, I_{32}, I_{33},

$$I_3 \lesssim \frac{1}{t}.$$

As for I_4, by Theorem 2.1(2)(iv$_1$),

$$I_4 \lesssim \int_{|w|>\sqrt{t}} \left| \frac{1}{t} e^{-b\frac{d^2(w,u)}{t}} \right|^2 d\mu(w) \lesssim \frac{1}{t} \int_{|w|>\sqrt{t}} \frac{1}{t} e^{-2b\frac{d^2(w,u)}{t}} d\mu(w) \lesssim \frac{1}{t}.$$

(i$_4$) If $u \in \mathcal{R}^1 \setminus K$ and $|u| > |w|$,

$$I = \left(\int_{|w|>\sqrt{t}} + \int_{|w|\le\sqrt{t}} \right) \left| p_{\frac{t}{2}}(w,u) \right|^2 d\mu(w) =: I_5 + I_6.$$

As for I_5,

$$I_5 \lesssim \int_{|w|>\sqrt{t}} \left| \frac{1}{t} e^{-b\frac{d^2(w,u)}{t}} \right|^2 d\mu(w) \lesssim \frac{1}{t}.$$

As for I_6, we consider the following cases:

(a) If $|u| > \sqrt{t}$,

$$\begin{aligned} I_6 &\lesssim \int_{|w|\le\sqrt{t}} \left| \frac{1}{t} \log \frac{\sqrt{t}}{|w|} e^{-b\frac{d^2(w,u)}{t}} \right|^2 d\mu(w) \\ &\lesssim \int_{|w|\le\sqrt{t}} \frac{1}{t^2} \log^2 \frac{\sqrt{t}}{|w|} e^{-2b\frac{d^2(w,u)}{t}} d\mu(w) \lesssim \frac{1}{t}. \end{aligned}$$

The last inequality is argued similarly to I_{33}.

(b) If $|u| \le \sqrt{t}$, using the same arguments as those of I_3, we have

$$I_6 \lesssim \int_{|w|\le\sqrt{t}} \left| \frac{1}{t} \left(1 + \frac{|u|}{\sqrt{t}} \log \frac{\sqrt{t}}{|w|} \right) \right|^2 d\mu(w) \lesssim \frac{1}{t}.$$

Combining all the above cases,

$$I \lesssim \frac{1}{t}.$$

Next, we investigate the integral

$$II = \int_{\mathcal{R}^1\setminus K} \left| p_{\frac{t}{2}}(w,u) \right|^2 d\mu(w).$$

(ii$_1$) If $u \in K$,

$$II = \left(\int_{|w|>\sqrt{t}} + \int_{|w|\le\sqrt{t}} \right) \left| p_{\frac{t}{2}}(w,u) \right|^2 d\mu(w) =: II_1 + II_2.$$

Applying Theorem 2.1(2)(iii$_1$),

$$\begin{aligned} II_1 &\lesssim \int_{|w|>\sqrt{t}} \left|\frac{\log t}{t} e^{-b\frac{d^2(w,u)}{t}}\right|^2 d\mu(w) \lesssim \int_{|w|>\sqrt{t}} \frac{1}{t}\frac{\log^2 t}{t} e^{-2b\frac{d^2(w,u)}{t}}\, d\mu(w) \\ &\lesssim \frac{1}{t}\int_{|w|>\sqrt{t}} \frac{1}{\sqrt{t}} e^{-2b\frac{d^2(w,u)}{t}}\, d\mu(w) \quad (\text{since } \log t \lesssim t^{1/4}) \\ &\lesssim \frac{1}{t}. \end{aligned}$$

Applying Theorem 2.1(2)(iii$_2$),

$$\begin{aligned} II_2 &\lesssim \int_{|w|\le\sqrt{t}} \left|\frac{1}{t}\left(1+\frac{|w|}{\sqrt{t}}\log t\right) e^{-b\frac{d^2(w,u)}{t}}\right|^2 d\mu(w) \\ &\lesssim \int_{|w|\le\sqrt{t}} \frac{1}{t^2}\left(1+\frac{|w|}{\sqrt{t}}\log t\right)^2 e^{-2b\frac{d^2(w,u)}{t}}\, d\mu(w). \end{aligned}$$

Using the same technique as those for II_1,

$$II_2 \lesssim \frac{1}{t}.$$

(ii$_2$) If $u \in \mathcal{R}^2 \setminus K$ and $|w| \le |u|$,

$$II = \left(\int_{|w|>\sqrt{t}} + \int_{|w|\le\sqrt{t}}\right) \left|p_{\frac{t}{2}}(w,u)\right|^2 d\mu(w) =: II_3 + II_4.$$

As for II_3, applying Theorem 2.1(2)(iv$_1$),

$$II_3 \lesssim \int_{|w|>\sqrt{t}} \left|\frac{1}{t} e^{-b\frac{d^2(w,u)}{t}}\right|^2 d\mu(w) \lesssim \frac{1}{t}.$$

As for II_4, we consider the following cases:

(a) If $|u| > \sqrt{t}$,

$$II_4 \lesssim \int_{|w|>\sqrt{t}} \left|\frac{1}{t} e^{-b\frac{d^2(w,u)}{t}}\right|^2 d\mu(w) \lesssim \frac{1}{t}.$$

The last inequality is argued similarly to II_3.

(b) If $|u| \le \sqrt{t}$,

$$\begin{aligned} II_4 &\lesssim \int_{|w|\le\sqrt{t}} \left|\frac{1}{t}\left(1+\frac{|w|}{\sqrt{t}}\log\frac{\sqrt{t}}{|u|}\right)\right|^2 d\mu(w) \\ &\lesssim \int_{|w|\le\sqrt{t}} \frac{1}{t^2}\left(1+\frac{w}{\sqrt{t}}\log\frac{\sqrt{t}}{|u|}+\frac{|w|^2}{t}\log^2\frac{\sqrt{t}}{|u|}\right) d\mu(w) \\ &\lesssim \int_{|w|\le\sqrt{t}} \frac{1}{t^2}\left(1+\log\frac{\sqrt{t}}{|u|}+\log^2\frac{\sqrt{t}}{|u|}\right) d\mu(w) \quad (\text{since } |w| \le \sqrt{t}). \end{aligned}$$

Since $\log t \lesssim t^{\alpha}$, $\alpha > 0$ and $|u| \le \sqrt{t}$, we get

$$II_4 \lesssim \frac{1}{t}.$$

(ii$_3$) If $u \in \mathcal{R}^2 \setminus K$ and $|w| > |u|$,

$$II = \left(\int_{|w|>\sqrt{t}} + \int_{|w|\le\sqrt{t}} \right) \left| p_{\frac{t}{2}}(w,u) \right|^2 d\mu(w) =: II_5 + II_6.$$

As for II_6, by Theorem 2.1(2)(iv$_2$),

$$II_6 \lesssim \int_{|w|\le\sqrt{t}} \left| \frac{1}{t} \left(1 + \frac{|w|}{\sqrt{t}} \log \frac{\sqrt{t}}{|u|} \right) \right|^2 d\mu(w).$$

Using the same arguments as in the estimate of II_4,

$$II_6 \lesssim \frac{1}{t}.$$

As for II_5, we consider the following cases:

(a) If $|u| > \sqrt{t}$, by Theorem 2.1(2)(iv$_1$),

$$II_5 \lesssim \int_{|w|>\sqrt{t}} \frac{1}{t^2} e^{-b\frac{d^2(w,u)}{t}} \, d\mu(w) \le \frac{1}{t}.$$

(b) If $|u| \le \sqrt{t} < |w|$, by Theorem 2.1(2)(iv$_3$),

$$II_5 \lesssim \int_{|w|>\sqrt{t}} \frac{1}{t^2} \log^2 \frac{\sqrt{t}}{|u|} e^{-2b\frac{d^2(w,u)}{t}} \, d\mu(w).$$

Again, using the inequality $\log t \lesssim t^{\alpha}$ and the assumption $|u| \le \sqrt{t}$,

$$II_5 \lesssim \int_{|w|>\sqrt{t}} \frac{1}{t\sqrt{t}} e^{-2b\frac{d^2(w.u)}{t}} \, d\mu(w) \lesssim \frac{1}{t}.$$

(ii$_4$) If $u \in \mathcal{R}^1 \setminus K$ and $|u| \le |w|$,

$$II \lesssim \left(\int_{|w|\le\sqrt{t}} + \int_{|w|>\sqrt{t}} \right) \left| p_{\frac{t}{2}}(w,u) \right|^2 d\mu(w) =: II_7 + II_8.$$

By Theorem 2.1(2)(v$_1$),

$$II_7 \lesssim \int_{|w|\le\sqrt{t}} \left(\frac{1}{t} + \frac{|w||u|}{t\sqrt{t}} + \frac{|w|+|u|}{t\sqrt{t}} \log t \right)^2 e^{-2b\frac{d^2(w,u)}{t}} \, d\mu(w).$$

By a straightforward calculation we get that

$$\frac{1}{t} + \frac{|w||u|}{t\sqrt{t}} + \frac{|w|+|u|}{t\sqrt{t}} \log t \lesssim \frac{1}{\sqrt{t}}.$$

Thus

$$II_7 \lesssim \int_{|w|\le\sqrt{t}} \frac{1}{t} e^{-2b\frac{d^2(w,u)}{t}}\, d\mu(w) \lesssim \frac{1}{\sqrt{t}}.$$

As for II_8, we consider the following cases:

(a) If $|u| > \sqrt{t}$, by Theorem 2.1(2)(v_4),

$$II_8 \lesssim \int_{|w|>\sqrt{t}} \frac{1}{t} e^{-2b\frac{d^2(w,u)}{t}}\, d\mu(w) \lesssim \frac{1}{\sqrt{t}}.$$

(b) If $|u| \le \sqrt{t}$, by Theorem 2.1(2)(v3),

$$II_8 \lesssim \int_{|w|>\sqrt{t}} \left(\frac{|u|}{t} + \frac{\log t}{t}\right)^2 e^{-2b\frac{d^2(w,u)}{t}}\, d\mu(w) \lesssim \int_{|w|>\sqrt{t}} \frac{1}{t} e^{-2b\frac{d^2(w,u)}{t}}\, d\mu(w).$$

The last inequality was obtained by using the same arguments as in II_7. Hence,

$$II_8 \lesssim \frac{1}{\sqrt{t}}.$$

(ii_5) If $u \in \mathcal{R}^1 \setminus K$ and $|u| > |w|$,

$$II \lesssim \left(\int_{|w|\le\sqrt{t}} + \int_{|w|>\sqrt{t}}\right) \left|p_{\frac{t}{2}(w,u)}\right|^2 d\mu(w) =: II_9 + II_{10}.$$

Due to the symmetry of $p_{\frac{t}{2}}(w,u)$, the estimates for II_9 and II_{10} can be done similarly to those in case (ii_4); therefore, we omit details.

We note that the estimate of the last integral

$$III = \int_K \left|p_{\frac{t}{2}}(w,u)\right|^2 d\mu(w)$$

can be done similarly to cases I, I_1 and II_2; hence,

$$III \lesssim \frac{1}{t}.$$

Combining all the above estimates, we obtain that

$$|p_z(x,y)| \lesssim \frac{1}{\sqrt{t}} + \frac{1}{t}$$

for all $x, y \in \mathcal{R}^1 \sharp \mathcal{R}^2$ and $z = t + is$ with $t > 0$.

We now set

$$F(z) := p_z(x,y) \left[(\Re z)^{-1/2} + (\Re z)^{-1}\right]^{-1}. \tag{2.3}$$

Then from (2.3),

$$|F(z)| \lesssim 1 \quad \text{for all } z \in \mathbb{C}_+. \tag{2.4}$$

By a simple calculation we can obtain the upper bound of the heat kernel $p_t(x,y)$:

$$|p_t(x,y)| \lesssim \left(\frac{1}{\sqrt{t}} + \frac{1}{t}\right) e^{-b\frac{d^2(x,y)}{t}}$$

for all $x, y \in \mathcal{R}^1\sharp\mathcal{R}^2$ and $t > 0$.

This implies that

$$|F(t)| \lesssim e^{-b\frac{d^2(x,y)}{t}} \quad \text{for all } t > 0. \tag{2.5}$$

Combining (2.4), (2.5) and then applying Phragmén-Lindelöf Theorem (see Lemma 2.2) we get that

$$|p_z(x,y)| \lesssim \left(\frac{1}{\sqrt{t}} + \frac{1}{t}\right) e^{-b\frac{d^2(x,y)}{|z|}\cos\theta}$$

for all $x, y \in \mathcal{R}^1\sharp\mathcal{R}^2$. □

We denote by $K_{F(\sqrt{\Delta})}(x,y)$ the kernel of $F(\sqrt{\Delta})$. By the same approach as those in [2], we obtain the following estimates (Lemmas 2.4 and 2.5). Details, therefore, are omitted.

Lemma 2.4. *Let $R \geq 1$. There exists a constant $C > 0$ so that for any Borel function F supported in $[0, R]$, we have*

$$\int_{\mathcal{R}^1\sharp\mathcal{R}^2} \left|K_{F(\sqrt{\Delta})}(x,y)\right|^2 d\mu(x) \leq CR^2\|F\|^2_{L^\infty}$$

for all $y \in \mathcal{R}^1\sharp\mathcal{R}^2$.

Lemma 2.5. *Let $R \geq 1$. For any $s > 0$ there exists a constant $C > 0$ such that*

$$\int_{\mathcal{R}^1\sharp\mathcal{R}^2} \left|p_{\frac{1+i\tau}{R^2}}(x,y)\right|^2 (1 + Rd(x,y))^{2s}\, d\mu(x) \lesssim R^2(1 + |\tau|)^{2s+2}$$

for all $y \in \mathcal{R}^1\sharp\mathcal{R}^2$ and $\tau > 0$.

The result below is a technical tool in the proof of our main result. For references on other analogous versions, see [2, Lemma 2.5] and [9, Lemma 4.3].

Lemma 2.6. *Let $s > 0$ and $R \geq 1$. Then we have*

(i) *For any $\epsilon > 0$ there exists a constant $C = C(\epsilon)$ such that*

$$\int_{\mathcal{R}^1\sharp\mathcal{R}^2} \left|K_{F(\sqrt{\Delta})}(x,y)\right|^2 (1 + Rd(x,y))^{2s}\, d\mu(x) \lesssim R^2\|\delta_R F\|^2_{W^\infty_{s+\epsilon}}$$

for all $y \in \mathcal{R}^1\sharp\mathcal{R}^2$ and all Borel functions F supported in $[R/4, R]$.

(ii) *For every $\epsilon > 0$ there exists a constant $C = C(\epsilon)$ such that*

$$\int_{\mathcal{R}^1\sharp\mathcal{R}^2} \left|K_{F(\sqrt{\Delta})}(x,y)\right|^2 (1+d(x,y))^{2s}\, d\mu(x) \lesssim \|F\|^2_{W^\infty_{2s+\epsilon}}$$

for all $y \in \mathcal{R}^1\sharp\mathcal{R}^2$ and all smooth functions F supported in $[0,2]$ with $F^{(2\nu+1)}(0) = 0$ for all $\nu \geq 0$.

The proof of this lemma can be done similarly to that in [2]; therefore, details are not provided here. We then use this lemma to get the following estimates.

Lemma 2.7. *Let $s > 2$ and $0 < \lambda \leq 1$. Then we have*

(i) *For any $\epsilon > 0$ there exists a constant $C = C(s,\epsilon)$ such that*

$$\left|K_{F(\lambda\sqrt{\Delta})}(x,y)\right| \leq \frac{C}{\lambda^2}\left(1+\frac{d(x,y)}{\lambda}\right)^{-s} \|F\|_{W^\infty_{s+\epsilon}} \tag{2.6}$$

for all $x, y \in \mathcal{R}^1\sharp\mathcal{R}^2$ and all Borel functions F supported in $[1/2, 2]$.

(ii) *For any $\epsilon > 0$ there exists a constant $C = C(s,\epsilon)$ such that*

$$\left|K_{F(\sqrt{\Delta})}(x,y)\right| \leq C(1+d(x,y))^{-s}\|F\|_{W^\infty_{2s+\epsilon}} \tag{2.7}$$

for all $x, y \in \mathcal{R}^1\sharp\mathcal{R}^2$ and all Borel functions F supported in $[0,2]$ with $F^{(2\nu+1)}(0) = 0$ for all $\nu \geq 0$.

Proof. (i) Let $\phi \in C^\infty(\mathbb{R})$ such that $\operatorname{supp}\phi \subset [1/4, 4]$, $\phi(x) \geq 0$ and $\phi(x) = 1$ for all $x \in [1/2, 2]$. Then we have

$$F(\lambda\sqrt{\Delta}) = F(\lambda\sqrt{\Delta})\phi(\lambda\sqrt{\Delta})$$

which implies that

$$K_{F(\lambda\sqrt{\Delta})}(x,y) = \int_{\mathcal{R}^1\sharp\mathcal{R}^2} K_{F(\lambda\sqrt{\Delta})}(x,z) K_{\phi(\lambda\sqrt{\Delta})}(z,y)\, d\mu(z).$$

Applying Hölder's inequality, we get that

$$\begin{aligned}
\left|K_{F(\lambda\sqrt{\Delta})}(x,y)\right| &\leq \left[\int_{\mathcal{R}^1\sharp\mathcal{R}^2} \left|K_{F(\lambda\sqrt{\Delta})}(x,z)\right|^2 d\mu(z)\right]^{1/2} \left[\int_{\mathcal{R}^1\sharp\mathcal{R}^2} \left|K_{\phi(\lambda\sqrt{\Delta})}(z,y)\right|^2 d\mu(z)\right]^{1/2} \\
&\leq \left(1+\frac{d(x,y)}{\lambda}\right)^{-s} \left[\int_{\mathcal{R}^1\sharp\mathcal{R}^2} \left|K_{F(\lambda\sqrt{\Delta})}(x,z)\right|^2 \left(1+\frac{d(x,z)}{\lambda}\right)^{2s} d\mu(z)\right]^{1/2} \\
&\quad \times \left[\int_{\mathcal{R}^1\sharp\mathcal{R}^2} \left|K_{\phi(\lambda\sqrt{\Delta})}(z,y)\right|^2 \left(1+\frac{d(z,y)}{\lambda}\right)^{2s} d\mu(z)\right]^{1/2}.
\end{aligned}$$

Using Lemma 2.6(i), we obtain

$$\left|K_{F(\lambda\sqrt{\Delta})}(x,y)\right| \leq \frac{C}{\lambda^2}\left(1+\frac{d(x,y)}{\lambda}\right)^{-s} \|F\|_{W^\infty_{s+\epsilon}}\|\phi\|_{W^\infty_{s+\epsilon}}.$$

This gives (2.6).

(ii) Let $\phi \in C^\infty(\mathbb{R})$ be an even function such that $\operatorname{supp}\phi \in [-3,3]$, $\phi \geq 0$ and $\phi = 1$ on $[-2,2]$.

Let us consider

$$F(\sqrt{\Delta}) = F(\sqrt{\Delta})\phi(\sqrt{\Delta}).$$

We then have

$$K_{F(\sqrt{\Delta})}(x,y) = \int_{\mathcal{R}^1\sharp\mathcal{R}^2} K_{F(\sqrt{\Delta})}(x,z)K_{\phi(\sqrt{\Delta})}(z,y)\,d\mu(z).$$

Applying Hölder's inequality, we get that

$$\begin{aligned}
\left|K_{F(\sqrt{\Delta})}(x,y)\right| &\leq \left[\int_{\mathcal{R}^1\sharp\mathcal{R}^2}\left|K_{F(\sqrt{\Delta})}(x,z)\right|^2 d\mu(z)\right]^{1/2}\left[\int_{\mathcal{R}^1\sharp\mathcal{R}^2}\left|K_{\phi(\sqrt{\Delta})}(z,y)\right|^2 d\mu(z)\right]^{1/2}\\
&\leq (1+d(x,y))^{-s}\left[\int_{\mathcal{R}^1\sharp\mathcal{R}^2}\left|K_{F(\sqrt{\Delta})}(x,z)\right|^2(1+d(x,z))^{2s}\,d\mu(z)\right]^{1/2}\\
&\quad\times\left[\int_{\mathcal{R}^1\sharp\mathcal{R}^2}\left|K_{\phi(\sqrt{\Delta})}(z,y)\right|^2(1+d(z,y))^{2s}\,d\mu(z)\right]^{1/2}.
\end{aligned}$$

Applying Lemma 2.6(ii),

$$\left|K_{F(\sqrt{\Delta})}(x,y)\right| \leq (1+d(x,y))^{-s}\|F\|_{W^\infty_{2s+\epsilon}}\|\phi\|_{W^\infty_{2s+\epsilon}}.$$

Thus, (2.7) holds. □

3. The Littlewood-Paley inequality

Proposition 3.1. *Let $s > 1$. If F is a smooth function such that*

$$\|F\|_{W^\infty_{2s}([0,2])} + \sup_{j\geq 1}\|\eta\delta_{2^j}F\|_{W^\infty_s} < \infty \tag{3.1}$$

where $\delta_t F = F(t\,\cdot)$ for all $t > 0$ and $\eta \in C^\infty_c(0,\infty)$ is a fixed function, not identically zero. Then the spectral multiplier $F(\sqrt{\Delta})$ is of weak type $(1,1)$, and hence $F(\sqrt{\Delta})$ is of weak type $(1,1)$ and is bounded on $L^p(\mathcal{R}^1\sharp\mathcal{R}^2)$ for all $1 < p < \infty$.

Proof. Let $\varphi_0 \in C^\infty_0(\mathbb{R})$ and $\varphi \in C^\infty_0(0,\infty)$ such that $\operatorname{supp}\varphi_0 \subset [-2,2]$, $\operatorname{supp}\varphi \subset [1/2,2]$ and

$$1 = \varphi_0(\lambda) + \sum_{j\geq 1}\varphi(2^{-j}\lambda) = \sum_{j\geq 0}\varphi_j(\lambda)$$

where $\varphi_j = \varphi(2^{-j}\lambda)$ for all $j \geq 1$.

Hence,

$$F(\lambda) = \sum_{j\geq 0} \varphi_j(\lambda) F(\lambda) =: \sum_{j\geq 0} F_j(\lambda).$$

So

$$F(\sqrt{\Delta}) = \sum_{j\geq 0} F_j(\sqrt{\Delta}).$$

Moreover, from (3.1),

$$\|F_0\|_{W_s^\infty} < \infty \quad \text{and} \quad \|\delta_{2^j} F_j\|_{W_s^\infty} < \infty, \quad \forall\, j > 0. \tag{3.2}$$

Let $f \in L^1(X)$ and $\lambda > 0$. We have

$$\begin{aligned}
\mu\left(\left\{x \in \mathcal{R}^1 \sharp \mathcal{R}^2 : \left|F(\sqrt{\Delta}) f(x)\right| > \lambda\right\}\right) &\leq \mu\left(\left\{x \in \mathcal{R}^1 \sharp \mathcal{R}^2 : \left|F(\sqrt{\Delta}) f_1(x)\right| > \frac{\lambda}{3}\right\}\right) \\
&\quad + \mu\left(\left\{x \in \mathcal{R}^1 \sharp \mathcal{R}^2 : \left|F(\sqrt{\Delta}) f_2(x)\right| > \frac{\lambda}{3}\right\}\right) \\
&\quad + \mu\left(\left\{x \in \mathcal{R}^1 \sharp \mathcal{R}^2 : \left|F(\sqrt{\Delta}) f_3(x)\right| > \frac{\lambda}{3}\right\}\right) \\
&=: I + II + III
\end{aligned}$$

where $f_1(x) = f(x)\chi_{\mathcal{R}^2 \setminus K}$, $f_2(x) = f(x)\chi_{\mathcal{R}^1 \setminus K}$ and $f_3(x) = f(x)\chi_K$.

It can be observed that if each of terms above is bounded by $\frac{C}{\lambda}\|f\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)}$, the weak type $(1,1)$ of the operator $F(\sqrt{\Delta})$ is fully satisfied.

3.1. Estimate of I

We will adapt some ideas in [2, 3, 8] to our present situation. Since f_1 is supported in $\mathcal{R}^2 \setminus K$ which is a homogeneous space (in the sense of Coifman and Weiss, [4]), we can employ the Calderón-Zygmund decomposition to decompose the integrable function

$$f_1 = g_1 + b_1 = g_1 + \sum_i b_{1,i}$$

such that

(a) $\operatorname{supp} g_1 \subset \mathcal{R}^2 \setminus K$ and $|g_1(x)| \lesssim \lambda$ for almost all $x \in \mathcal{R}^2 \setminus K$;

(b) there exists a sequence of balls $Q_{1,i}$ so that $\operatorname{supp} b_{1,i} \subset Q_{1,i} \subset \mathcal{R}^2 \setminus K$ and

$$\int_{\mathcal{R}^2 \setminus K} |b_{1,i}(y)|\, d\mu(y) \lesssim \lambda \mu(Q_{1,i});$$

(c) $\sum_i \mu(Q_{1,i}) \lesssim \frac{1}{\lambda} \int_{\mathcal{R}^2 \setminus K} |f_1(y)|\, d\mu(y) \lesssim \frac{1}{\lambda}\|f_1\|_{L^1(\mathcal{R}^2 \setminus K)}$;

(d) $\sum_i \chi_{Q_{1,i}} \lesssim 1$.

We then have

$$I \leq \mu\left(\left\{x \in \mathcal{R}^1 \sharp \mathcal{R}^2 : \left|F(\sqrt{\Delta})g_1(x)\right| > \frac{\lambda}{6}\right\}\right) + \mu\left(\left\{x \in \mathcal{R}^1 \sharp \mathcal{R}^2 : \left|F(\sqrt{\Delta})b_1(x)\right| > \frac{\lambda}{6}\right\}\right)$$
$$=: I_1 + I_2.$$

By using the L^2 boundedness of $F(\sqrt{\Delta})$ and the fact that $|g_1(x)| \lesssim \lambda$,

$$I_1 \lesssim \frac{1}{\lambda^2}\|g_1\|^2_{L^2(\mathcal{R}^2\backslash K)} \lesssim \frac{1}{\lambda}\|g_1\|_{L^1(\mathcal{R}^2\backslash K)}.$$

Conditions (b) and (c) imply $\|g_1\|_{L^1(\mathcal{R}^2\backslash K)} \lesssim \|f_1\|_{L^1(\mathcal{R}^2\backslash K)}$. Hence,

$$I_1 \lesssim \frac{1}{\lambda}\|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)}.$$

In order to estimate the term I_2, we decompose

$$F(\sqrt{\Delta})b_1 = \sum_i F(\sqrt{\Delta})\big(I - e^{-r_i^2\Delta}\big)b_{1,i} + \sum_i F(\sqrt{\Delta})e^{-r_i^2\Delta}b_{1,i}$$

where r_i is the radius of the ball $Q_{1,i}$.

Hence,

$$\begin{aligned} I_2 &\leq \mu\left(\left\{x \in \mathcal{R}^1 \sharp \mathcal{R}^2 : \left|\sum_i F(\sqrt{\Delta})\big(I - e^{-r_i^2\Delta}\big)b_{1,i}(x)\right| > \frac{\lambda}{12}\right\}\right) \\ &\quad + \mu\left(\left\{x \in \mathcal{R}^1 \sharp \mathcal{R}^2 : \left|\sum_i F(\sqrt{\Delta})e^{-r_i^2\Delta}b_{1,i}(x)\right| > \frac{\lambda}{12}\right\}\right) \\ &=: I_{21} + I_{22}. \end{aligned}$$

For the term I_{21}, by condition (c),

$$\begin{aligned} I_{21} &\lesssim \sum_i \mu(Q_{1,i}) + \frac{1}{\lambda}\sum_i \int_{\mathcal{R}^1\sharp\mathcal{R}^2\backslash Q_{1,i}} \left|F(\sqrt{\Delta})\big(I - e^{-r_i^2\Delta}\big)b_{1,i}\right| d\mu(x) \\ &\lesssim \frac{1}{\lambda}\|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)} + \frac{1}{\lambda}\sum_i\sum_{j\geq 0}\int_{\mathcal{R}^1\sharp\mathcal{R}^2\backslash Q_{1,i}} \left|F_{j,r_i}(\sqrt{\Delta})b_{1,i}\right| d\mu(x) \end{aligned} \tag{3.3}$$

where $F_{j,r_i}(\lambda) = F_j(\lambda)(1 - e^{-r_i^2\lambda^2})$ for each $j \geq 0$.

Taking s' with $s > s' > 1$, for each i and $j \geq 0$, by Hölder's inequality,

$$\begin{aligned} &\int_{\mathcal{R}^1\sharp\mathcal{R}^2\backslash Q_{1,i}} \left|K_{F_{j,r_i}(\sqrt{\Delta})}(x,y)\right| d\mu(x) \\ &\leq \left(\int_{\mathcal{R}^1\sharp\mathcal{R}^2\backslash Q_{1,i}} \left|K_{F_{j,r_i}(\sqrt{\Delta})}(x,y)\right|^2 (1 + 2^j d(x,y))^{2s'} d\mu(x)\right)^{1/2} \\ &\quad\times \left(\int_{\mathcal{R}^1\sharp\mathcal{R}^2\backslash Q_{1,i}} (1 + 2^j d(x,y))^{-2s'} d\mu(x)\right)^{1/2}. \end{aligned}$$

A straightforward calculation gives

$$\left(\int_{\mathcal{R}^1\sharp\mathcal{R}^2\setminus Q_{1,i}} (1+2^j d(x,y))^{-2s'}\, d\mu(x)\right)^{1/2} \lesssim 2^j(1+2^j r_i)^{1-s'}.$$

On the other hand, since $\operatorname{supp} F_{0,r_i} \subset [0,2]$ and $\operatorname{supp} F_{j,r_i} \subset [2^{j-1},2^{j+1}]$ with $j>0$, applying Lemma 2.6,

$$\left(\int_{\mathcal{R}^1\sharp\mathcal{R}^2\setminus Q_{1,i}} \left|K_{F_{j,r_i}(\sqrt{\Delta})}(x,y)\right|^2 (1+2^j d(x,y))^{2s'}\, d\mu(x)\right)^{1/2} \lesssim 2^j\|\delta_{2^j}F_{j,r_i}\|_{W_s^\infty}.$$

Similar to the previous arguments,

$$\left(\int_{\mathcal{R}^1\sharp\mathcal{R}^2\setminus Q_{1,i}} \left|K_{F_{0,r_i}(\sqrt{\Delta})}(x,y)\right|^2 (1+d(x,y))^{2s'}\, d\mu(x)\right)^{1/2} \lesssim \|F_{0,r_i}\|_{W_s^\infty}.$$

Fix $k>s$. Observe that, by (3.2),

$$\left\|\delta_{2^j}F_{j,r_i}\right\|_{W_s^\infty} \lesssim \left\|\delta_{2^j}F_j\right\|_{W_s^\infty}\left\|\delta_{2^j}(1-e^{r_i^2})\right\|_{C^k([1/4,1])} \lesssim \left(\frac{2^j r_i}{1+2^j r_i}\right)^2, \quad j>0$$

and

$$\|F_{0,r_i}\|_{W_s^\infty} \lesssim \left(\frac{r_i}{1+r_i}\right)^2.$$

As a consequence,

$$\int_{\mathcal{R}^1\sharp\mathcal{R}^2\setminus Q_{1,i}} \left|K_{j,r_i}(\sqrt{\Delta})(x,y)\right| d\mu(x) \le \left(\frac{2^j r_i}{1+2^j r_i}\right)^2 (1+2^j r_i)^{1-s'}, \quad j\ge 0.$$

This, along with condition (b), implies that for each $j\ge 0$,

$$\begin{aligned}\int_{\mathcal{R}^1\sharp\mathcal{R}^2\setminus Q_{1,i}} \left|F_{j,r_i}(\sqrt{\Delta})b_{1,i(x)}\right| d\mu(x) &\lesssim \left(\frac{2^j r_i}{1+2^j r_i}\right)^2 (1+2^j r_i)^{1-s'}\|b_{1,i}\|_{L^1(\mathcal{R}^2\setminus K)}\\ &\lesssim \left(\frac{2^j r_i}{1+2^j r_i}\right)^2 (1+2^j r_i)^{1-s'}\lambda\mu(Q_{1,i}).\end{aligned}$$

Substituting this into (3.3),

$$\begin{aligned} I_{21} &\lesssim \frac{1}{\lambda}\|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)} + \sum_i\sum_{j\ge 0}\left(\frac{2^j r_i}{1+2^j r_i}\right)^2 (1+2^j r_i)^{1-s'}\lambda\mu(Q_{1,i})\\ &\lesssim \frac{1}{\lambda}\|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)} + \sum_i \lambda\mu(Q_{1,i}) \lesssim \frac{1}{\lambda}\|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)}\end{aligned}$$

where condition (c) was applied for the last inequality.

We now turn to estimate I_{22}. For $x\in\mathcal{R}^1\sharp\mathcal{R}^2$ and i we consider the following cases.

Case 1*:* $0 \le r_i \le 1$. By Theorem 2.1(1),

$$\left|e^{-r_i^2\Delta} b_{1,i}(x)\right| \lesssim \int_{Q_{1,i}} \frac{1}{r_i^2} e^{-b\frac{d^2(x,y)}{r_i^2}} |b_{1,i}(y)|\, d\mu(y).$$

Using triangle inequality,

$$\sup_{z\in Q_{1,i}} e^{-b\frac{d^2(x,z)}{r_i^2}} \lesssim \inf_{z\in Q_{1,i}} e^{-b\frac{d^2(x,z)}{r_i^2}}. \tag{3.4}$$

Combining with condition (b) and the fact that $\mu(Q_{1,i}) \approx r_i^2$,

$$\begin{aligned}\left|e^{-r_i^2\Delta} b_{1,i}(x)\right| &\lesssim \int_{Q_{1,i}} \frac{1}{r_i^2} e^{-b\frac{d^2(x,y)}{r_i^2}} |b_{1,i}(y)|\, d\mu(y)\\ &\lesssim \lambda \inf_{z\in Q_{1,i}} e^{-b\frac{d^2(x,z)}{r_i^2}} \lesssim \lambda \int_{Q_{1,i}} \frac{1}{r_i^2} e^{-b\frac{d^2(x,y)}{r_i^2}} \chi_{Q_{1,i}}(y)\, d\mu(y).\end{aligned}$$

This implies, for $u \in L^2(\mathcal{R}^1\sharp\mathcal{R}^2)$,

$$\left|\langle |u|, e^{-r_i^2\Delta} b_{1,i}\rangle\right| \lesssim \lambda \int_{Q_{1,i}} \chi_{Q_{1,i}}(y) \int_{\mathcal{R}^1\sharp\mathcal{R}^2} \frac{1}{r_i^2} e^{-b\frac{d^2(x,y)}{r_i^2}} |u(x)|\, d\mu(x) d\mu(y) \lesssim \lambda \langle \mathcal{M}u, \chi_{Q_{1,i}}\rangle.$$

Since the Hardy-Littlewood maximal function $\mathcal{M}$ is bounded on $L^2(\mathcal{R}^1\sharp\mathcal{R}^2)$,

$$\left\|\sum_i e^{-r_i^2\Delta} b_{1,i}\right\|^2_{L^2(\mathcal{R}^1\sharp\mathcal{R}^2)} \lesssim \lambda^2 \left\|\sum_i \chi_{Q_{1,i}}\right\|^2_{L^2(\mathcal{R}^2\setminus K)} \lesssim \lambda \left(\sum_i \mu(Q_{1,i})\right) \lesssim \lambda \|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)}$$

where the third inequality was achieved by condition (c).

Case 2*:* $r_i > 1$. To get the estimate of $\left\|\sum_i e^{-r_i^2\Delta} b_{1,i}\right\|_{L^2(\mathcal{R}^1\sharp\mathcal{R}^2)}$, it suffices to show that the following 3 cases are held.

$$\left\|\sum_i e^{-r_i^2\Delta} b_{1,i}\right\|^2_{L^2(\mathcal{R}^2\setminus K)} \lesssim \lambda \|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)}, \tag{3.5}$$

$$\left\|\sum_i e^{-r_i^2\Delta} b_{1,i}\right\|^2_{L^2(\mathcal{R}^1\setminus K)} \lesssim \lambda \|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)} \tag{3.6}$$

and

$$\left\|\sum_i e^{-r_i^2\Delta} b_{1,i}\right\|^2_{L^2(K)} \lesssim \lambda \|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)}. \tag{3.7}$$

(ii_1) Estimate (3.5): It is noted that in this case $x \in \mathcal{R}^2 \setminus K$ and $y \in Q_{1,i} \subset \mathcal{R}^2 \setminus K$. Applying Theorem 2.1(2)(vi),

$$\left|e^{-r_i^2\Delta} b_{1,i}(x)\right| \lesssim \int_{Q_{1,i}} \frac{1}{r_i^2} e^{-b\frac{d^2(x,y)}{r_i^2}} |b_{1,i}(y)|\, d\mu(y).$$

Since the right-hand side of the above estimate has the same form as that in Case 1, we get that

$$\left|e^{-r_i^2\Delta}b_{1,i}(x)\right| \lesssim \lambda \int_{Q_{1,i}} \frac{1}{r_i^2} e^{-b\frac{d^2(x,y)}{r_i^2}} \chi_{Q_{1,i}}(y)\, d\mu(y).$$

This implies that for $u \in L^2(\mathcal{R}^2 \setminus K)$,

$$\left|\langle |u|, e^{-r_i^2\Delta}b_{1,i}\rangle\right| \lesssim \lambda \int_{Q_{1,i}} \chi_{Q_{1,i}}(y) \int_{\mathcal{R}^2\setminus K} \frac{1}{r_i^2} e^{-b\frac{d^2(x,y)}{r_i^2}} |u(x)|\, d\mu(x) d\mu(y) \lesssim \lambda \langle \mathcal{M}u, \chi_{Q_{1,i}}\rangle.$$

Hence,

$$\left|\left\langle |u|, \sum_i e^{-r_i^2\Delta}b_{1,i}\right\rangle\right| \lesssim \lambda \left\langle \mathcal{M}u, \sum_i \chi_{Q_{1,i}}\right\rangle.$$

Since the Hardy-Littlewood maximal function $\mathcal{M}$ is bounded on $L^2(\mathcal{R}^2 \setminus K)$,

$$\left\|\sum_i e^{-r_i^2\Delta}b_{1,i}\right\|^2_{L^2(\mathcal{R}^2\setminus K)} \lesssim \lambda^2 \left\|\sum_i \chi_{Q_{1,i}}\right\|^2_{L^2(\mathcal{R}^2\setminus K)} \lesssim \lambda \left(\sum_i \mu(Q_{1,i})\right) \lesssim \lambda \|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)}$$

which shows (3.5).

(ii$_2$) Estimate (3.7): We first consider the case $x \in K$ and $|y| > r_i$. Applying Theorem 2.1(2)(ii$_1$),

$$\left|e^{-r_i^2\Delta}b_{1,i}(x)\right| \lesssim \int_{Q_{1,i}} \frac{1}{r_i^2} e^{-b\frac{d^2(x,y)}{r_i^2}} |b_{1,i}(y)|\, d\mu(y).$$

Using the similar approach used in Case 1 together with the Gaussian upper bound of $\left|e^{-r_i^2\Delta}b_{1,i}(x)\right|$, we then get

$$\left|\left\langle |u|, \sum_i e^{-r_i^2\Delta}b_{1,i}\right\rangle\right| \lesssim \lambda \left\langle \mathcal{M}u, \sum_i \chi_{Q_{1,i}}\right\rangle \tag{3.8}$$

for any $u \in L^2(K)$.

For the case $x \in K$ and $|y| \le r_i$, by Theorem 2.1(2)(ii$_2$),

$$\left|e^{-r_i^2\Delta}b_{1,i}(x)\right| \lesssim \int_{Q_{1,i}} \frac{1}{r_i^2} |b_{1,i}(y)|\, d\mu(y).$$

So for any $u \in L^2(K)$,

$$\left|\langle |u|, e^{-r_i^2\Delta}b_{1,i}\rangle\right| \lesssim \lambda \int_{Q_{1,i}} \chi_{Q_{1,i}}(y) \int_K \frac{1}{r_i^2} |u(x)|\, d\mu(x) d\mu(y) \lesssim \lambda \langle \mathcal{M}u, \chi_{Q_{1,i}}\rangle.$$

Then,

$$\left|\left\langle |u|, \sum_i e^{-r_i^2\Delta}b_{1,i}\right\rangle\right| \lesssim \lambda \left\langle \mathcal{M}u, \sum_i \chi_{Q_{1,i}}\right\rangle. \tag{3.9}$$

Combining (3.8) and (3.9), we get (3.7).

(ii$_3$) Estimate (3.6): We note that

$$\begin{aligned}
\left|\langle |u|, e^{-r_i^2\Delta} b_{1,i}\rangle\right| &= \int_{\mathcal{R}^1\setminus K} e^{-r_i^2\Delta} b_{1,i}(x)|u(x)|\, d\mu(x) \\
&= \left(\int_{|y|>r_i} + \int_{|x|,|y|\le r_i} + \int_{|y|\le r_i<|x|}\right) e^{-r_i^2\Delta} b_{1,i}(x)|u(x)|\, d\mu(x) \\
&=: E_1 + E_2 + E_3.
\end{aligned}$$

As for E_1, applying Theorem 2.1(2)(iv$_1$),

$$E_1 \lesssim \int_{|y|>r_i} \left(\int_{Q_{1,i}} \frac{1}{r_i^2} e^{-b\frac{d^2(x,y)}{r_i^2}} |b_{1,i}(y)|\, d\mu(y)\right) |u(x)|\, d\mu(x).$$

It should be noted that the estimate (3.4) still holds for this setting; therefore,

$$\begin{aligned}
E_1 &\lesssim \lambda \int_{|y|>r_i} \int_{Q_{1,i}} \frac{1}{r_i^2} e^{-b\frac{d^2(x,y)}{r_i^2}} \chi_{Q_{1,i}}(y)\, d\mu(y)|u(x)|\, d\mu(x) \\
&\lesssim \lambda \int_{Q_{1,i}} \chi_{Q_{1,i}}(y) \int_{|y|>r_i} \frac{1}{r_i^2} e^{-b\frac{d^2(x,y)}{r_i^2}} |u(x)|\, d\mu(x)\, d\mu(y) \\
&\lesssim \lambda \langle \mathcal{M}u, \chi_{Q_{1,i}}\rangle.
\end{aligned}$$

As for E_2, by Theorem 2.1(2)(iv$_2$),

$$E_2 \lesssim \int_{|x|,|y|\le r_i} \left(\int_{Q_{1,i}} \frac{1}{r_i^2} \left[1 + \frac{|x|}{r_i} \log\frac{r_i}{|y|}\right] |b_{1,i}(y)|\, d\mu(y)\right) |u(x)|\, d\mu(x).$$

By observing that $|x|, |y| \le r_i$ and $\log r_i \lesssim r_i^\alpha$ for all $\alpha > 0$,

$$\begin{aligned}
E_2 &\lesssim \int_{|x|,|y|\le r_i} \left(\int_{Q_{1,i}} \frac{1}{r_i} |b_{1,i}(y)|\, d\mu(y)\right) |u(x)|\, d\mu(x) \\
&\lesssim \lambda \int_{Q_{1,i}} \chi_{Q_{1,i}}(y) \int_{|x|,|y|\le r_i} \frac{1}{r_i} |u(x)|\, d\mu(x)\, d\mu(y) \\
&\lesssim \lambda \langle \mathcal{M}u, \chi_{Q_{1,i}}\rangle.
\end{aligned}$$

As for E_3, by Theorem 2.1(2)(iv$_3$),

$$E_3 \lesssim \int_{|y|\le r_i<|x|} \left(\int_{Q_{1,i}} \frac{1}{r_i^2} \log\frac{r_i}{|y|} e^{-b\frac{d^2(x,y)}{r_i^2}} |b_{1,i}(y)|\, d\mu(y)\right) |u(x)|\, d\mu(x).$$

Again by $|y| \le r_i$ and $\log r_i \lesssim r_i$, we have

$$\begin{aligned}
E_3 &\lesssim \int_{|y|\le r_i<|x|} \left(\int_{Q_{1,i}} \frac{1}{r_i} e^{-b\frac{d^2(x,y)}{r_i^2}} |b_{1,i}(y)|\, d\mu(y) \right) |u(x)|\, d\mu(x) \\
&\lesssim \lambda \int_{Q_{1,i}} \chi_{Q_{1,i}}(y) \int_{|x|,|y|\le r_i} \frac{1}{r_i} e^{-b\frac{d^2(x,y)}{r_i^2}} |u(x)|\, d\mu(x)\, d\mu(y) \\
&\lesssim \lambda \langle \mathcal{M}u, \chi_{Q_{1,i}} \rangle.
\end{aligned}$$

Combining all estimates of E_1, E_2 and E_3, we obtain

$$\left| \left\langle |u|, \sum_i e^{-r_i^2 \Delta} b_{1,i} \right\rangle \right| \lesssim \lambda \left\langle \mathcal{M}u, \sum_i \chi_{Q_{1,i}} \right\rangle$$

for all $u \in L^2(\mathcal{R}^1 \setminus K)$. This implies (3.6).

Hence, from the L^2 boundedness of $F(\sqrt{\Delta})$,

$$\begin{aligned}
I_{22} &\lesssim \frac{1}{\lambda^2} \left\| \sum_i F(\sqrt{\Delta}) e^{-r_i^2 \Delta} b_{1,i} \right\|^2_{L^2(\mathcal{R}^1 \sharp \mathcal{R}^2)} \\
&\lesssim \frac{1}{\lambda^2} \left\| \sum_i e^{-r_i^2 \Delta} b_{1,i} \right\|^2_{L^2(\mathcal{R}^2 \sharp \mathcal{R}^2)} \lesssim \frac{1}{\lambda} \|f\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)}.
\end{aligned}$$

Thus,

$$I_2 \lesssim \frac{1}{\lambda} \|f\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)}.$$

This combined with the estimate of I_1 induces that

$$I \lesssim \frac{1}{\lambda} \|f\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)}.$$

3.2. Estimate of II

We first note that

$$\begin{aligned}
II &\le \mu\left(\left\{ x \in \mathcal{R}^2 \setminus K : \left| F(\sqrt{\Delta}) f_2(x) \right| > \frac{\lambda}{9} \right\} \right) \\
&\quad + \mu\left(\left\{ x \in \mathcal{R}^1 : \left| F(\sqrt{\Delta}) f_2(x) \right| > \frac{\lambda}{9} \right\} \right) \\
&=: II_1 + II_2.
\end{aligned}$$

To estimate II_1, we note that the standard Calderón-Zygmund decomposition on nonhomogeneous spaces such as in [14, 16] cannot be applied since, in that decomposition, we only know the existence of a sequence of Calderón-Zygmund cubes instead of their exact positions.

Thus, to deal with this case, we use a Withney type decomposition of the level set Ω and then we make clever use of the upper bounds to handle the weak type estimates, without enlarging those cubes, which avoids the case of nondoubling measure. The genesis of this approach is an adaption of an idea from [14]. See [3] for a similar use of this idea.

We now split $\mathcal{R}^1 \setminus K$ into two parts according to function f_2 as follows:

$$F := \{y \in \mathcal{R}^1 \setminus K : \mathcal{M}_2 f_2(y) \leq \lambda\} \quad \text{and} \quad \Omega := \{y \in \mathcal{R}^1 \setminus K : \mathcal{M}_2 f_2(y) > \lambda\}$$

where $\mathcal{M}_2$ is the Hardy-Littlewood maximal function defined on $\mathcal{R}^1 \setminus K$.

Then we define

$$f_{2,\lambda}(y) := f_2(y)\chi_F(y) \quad \text{and} \quad f_2^\lambda(y) := f_2(y)\chi_\Omega(y)$$

for all $y \in \mathcal{R}^1 \setminus K$. Then we have

$$\begin{aligned} II_1 &\leq \left|\left\{x \in \mathcal{R}^2 \setminus K : \left|F(\sqrt{\Delta})f_{2,\lambda}(x)\right| > \frac{\lambda}{6}\right\}\right| \\ &\quad + \left|\left\{x \in \mathcal{R}^2 \setminus K : \left|F(\sqrt{\Delta})f_2^\lambda(x)\right| > \frac{\lambda}{6}\right\}\right| \\ &=: II_{11} + II_{12}. \end{aligned}$$

As for II_{11}, by using the L^2 boundedness of $F(\sqrt{\Delta})$, we obtain that

$$II_{11} \lesssim \frac{1}{\lambda^2}\|f_{2,\lambda}\|^2_{L^2(\mathcal{R}^1\setminus K)} \lesssim \frac{1}{\lambda}\|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)},$$

where we use the fact that $|f_{2,\lambda}(y)| = |f_2(y)|\chi_F(y) \leq |\mathcal{M}_2 f_2(y)|\chi_F(y) \leq \lambda$.

As for II_{12}, we consider the function f_2^λ. We now apply a covering lemma in [4] (see also [7, Lemma 5.5]) for the set Ω in the homogeneous space $\mathcal{R}^1$ to obtain a collection of balls $\{Q_{2,i} := B(y_i, r_i) : y_i \in \Omega, r_i = \frac{d(y_i,\Omega^c)}{2}, i = 1, 2\dots\}$ such that

(i) $\Omega = \bigcup_i Q_{2,i}$;

(ii) $\{B(y_i, \frac{r_i}{5})\}_{i=1}^\infty$ are disjoint;

(iii) there exists a universal constant C such that $\sum_k \chi_{Q_{2,k}}(y) \leq C$ for all $y \in \Omega$.

Hence, we can further decompose

$$f_2^\lambda(y) = \sum_i f_{2,i}^\lambda(y)$$

where $f_{2,i}^\lambda(y) = \frac{\chi_{Q_{2,i}}(y)}{\sum_k \chi_{Q_{2,k}}(y)} f_2^\lambda(y)$.

Applying Lemma 2.7, we have

$$\begin{aligned}\left|F(\sqrt{\Delta})f_{2,i}^{\lambda}(x)\right| &= \left|\sum_{j\geq 0} F_j(\sqrt{\Delta})f_{2,i}^{\lambda}(x)\right| \\ &= \sum_{j\geq 0}\int_{Q_{2,i}} \left|K_{F_j(\sqrt{\Delta})}(x,y)\right| |f_{2,i}^{\lambda}(y)|\, d\mu(y) \\ &\lesssim \sum_{j\geq 0}\int_{Q_{2,i}} \frac{1}{(2^{-j})^2}\left(1+\frac{d(x,y)}{2^{-j}}\right)^{-s} |f_{2,i}^{\lambda}(y)|\, d\mu(y)\end{aligned}$$

where $s > 2$.

Since $d(x,y) \approx |x| + |y|$ for all $x \in \mathcal{R}^2 \setminus K$ and $y \in \mathcal{R}^1 \setminus K$,

$$\begin{aligned}\left|F(\sqrt{\Delta})f_{2,i}^{\lambda}(x)\right| &\lesssim \sum_{j\geq 0}\int_{Q_{2,i}} \frac{1}{2^{-2j}}\frac{2^{-sj}}{(|x|+|y|)^s}|f_{2,i}^{\lambda}(y)|\, d\mu(y) \\ &\lesssim \sum_{j\geq 0}\int_{Q_{2,i}} \frac{1}{2^{(s-2)j}}\frac{1}{|x|^s}|f_{2,i}^{\lambda}(y)|\, d\mu(y),\end{aligned}$$

which implies that

$$\begin{aligned}\left|F(\sqrt{\Delta})f_{2,i}^{\lambda}(x)\right| &\lesssim \sum_{j\geq 0}\int_{Q_{2,i}} \frac{1}{2^{2j}}\frac{1}{|x|^2}|f_{2,i}^{\lambda}(y)|\, d\mu(y) \\ &\lesssim \frac{1}{|x|^2}\sum_{j\geq 0}\frac{1}{2^{2j}}\int_{Q_{2,i}} |f_{2,i}^{\lambda}(y)|\, d\mu(y) \\ &\lesssim \frac{1}{|x|^2}\int_{Q_{2,i}} |f_{2,i}^{\lambda}(y)|\, d\mu(y).\end{aligned}$$

Hence,

$$\left|F(\sqrt{\Delta})f_2^{\lambda}(x)\right| \lesssim \sum_i \left|F(\sqrt{\Delta})f_{2,i}^{\lambda}(x)\right| \lesssim \frac{1}{|x|^2}\sum_i\int_{Q_{2,i}} |f_{2,i}^{\lambda}(y)|\, d\mu(y) \lesssim \frac{1}{|x|^2}\|f_2\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)}.$$

This implies that

$$II_{12} \leq \mu\left(\left\{x \in \mathcal{R}^2 \setminus K : |x|^2 \lesssim \frac{1}{\lambda}\|f_2\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)}\right\}\right) \lesssim \frac{1}{\lambda}\|f_2\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)} \lesssim \frac{1}{\lambda}\|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)}.$$

As for II_2, we note that the underlying space $\mathcal{R}^1$ is a homogeneous space; therefore, we can employ the Calderón-Zygmund decomposition to decompose f_2 into two parts as follows:

$$f_2 = g_2 + b_2 = g_2 + \sum_i b_{2,i}$$

such that

(a′) $\operatorname{supp} g_2 \subset \mathcal{R}^1$ and $|g_2(x)| \lesssim \lambda$ for almost all $x \in \mathcal{R}^1$;

(b′) there exists a sequence of balls $B_{2,i}$ so that $\operatorname{supp} b_{2,i} \subset B_{2,i} \subset \mathcal{R}^1$ and

$$\int_{\mathcal{R}^1} |b_{2,i}(y)|\, d\mu(y) \lesssim \lambda \mu(B_{2,i});$$

(c′) $\sum_i \mu(B_{2,i}) \lesssim \frac{1}{\lambda} \int_{\mathcal{R}^1} |f_2(y)|\, d\mu(y) \lesssim \frac{1}{\lambda} \|f_2\|_{L^1(\mathcal{R}^1)}$;

(d′) $\sum_i \chi_{B_{2,i}} \lesssim 1$.

We then have

$$\begin{aligned} II_2 &\leq \mu\left(\left\{x \in \mathcal{R}^1 : \left|F(\sqrt{\Delta}) g_2(x)\right| > \frac{\lambda}{6}\right\}\right) + \mu\left(\left\{x \in \mathcal{R}^1 : \left|F(\sqrt{\Delta}) b_2(x)\right| > \frac{\lambda}{6}\right\}\right) \\ &=: II_{21} + II_{22}. \end{aligned}$$

By using the L^2 boundedness of $F(\sqrt{\Delta})$ and the fact that $|g_2(x)| \lesssim \lambda$,

$$II_{21} \lesssim \frac{1}{\lambda^2} \|g_2\|^2_{L^2(\mathcal{R}^1)} \lesssim \frac{1}{\lambda} \|g_2\|_{L^1(\mathcal{R}^1)}.$$

Conditions (b′) and (c′) imply $\|g_2\|_{L^1(\mathcal{R}^1)} \lesssim \|f_2\|_{L^1(\mathcal{R}^1}$. Hence,

$$II_{21} \lesssim \frac{1}{\lambda} \|f\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)}.$$

In order to estimate the term II_{22}, we decompose

$$F(\sqrt{\Delta}) b_2 = \sum_i F(\sqrt{\Delta}) \big(I - e^{-r_i^2 \Delta}\big) b_{2,i} + \sum_i F(\sqrt{\Delta}) e^{-r_i^2 \Delta} b_{2,i}$$

where r_i is the radius of the ball $B_{2,i}$.

Hence,

$$\begin{aligned} II_{22} &\leq \mu\left(\left\{x \in \mathcal{R}^1 : \left|\sum_i F(\sqrt{\Delta}) \big(I - e^{-r_i^2 \Delta}\big) b_{2,i}(x)\right| > \frac{\lambda}{12}\right\}\right) \\ &\quad + \mu\left(\left\{x \in \mathcal{R}^1 : \left|\sum_i F(\sqrt{\Delta}) e^{-r_i^2 \Delta} b_{2,i}(x)\right| > \frac{\lambda}{12}\right\}\right) \\ &=: II_{221} + II_{222}. \end{aligned}$$

Using the same arguments as those for I_{21} we can show that

$$II_{221} \lesssim \frac{1}{\lambda} \|f\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)}.$$

For the term II_{222}, we consider the following cases.

Case 1. If $0 \leq r_i \leq 1$, we can use the same arguments as those in Case 1 of I_{22} since the heat semigroup has the Gaussian upper bound. Hence

$$\left\|\sum_i e^{-r_i^2 \Delta} b_{2,i}\right\|^2_{L^2(\mathcal{R}^1 \sharp \mathcal{R}^2)} \lesssim \lambda \|f\|_{L^1(\mathcal{R}^1 \sharp \mathcal{R}^2)}.$$

Case 2. If $r_i > 1$, we will investigate the following inequalities

$$\left\|\sum_i e^{-r_i^2\Delta} b_{2,i}\right\|_{L^2(\mathcal{R}^1\setminus K)}^2 \lesssim \lambda \|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)} \tag{3.10}$$

and

$$\left\|\sum_i e^{-r_i^2\Delta} b_{2,i}\right\|_{L^2(K)}^2 \lesssim \lambda \|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)}. \tag{3.11}$$

We now estimate (3.10). We first split the following integral into 4 parts noting that $x, y \in \mathcal{R}^1 \setminus K$.

$$\begin{aligned}
\left|\langle |u|, e^{-r_i^2\Delta} b_{2,i}\rangle\right| &= \int_{\mathcal{R}^1\setminus K} e^{-r_i^2\Delta} b_{2,i}(x)|u(x)|\, d\mu(x) \\
&= \left(\int_{|x|,|y|>r_i} + \int_{|x|,|y|\le r_i} + \int_{|y|\le r_i<|x|} + \int_{|y|>r_i\ge|x|}\right) e^{-r_i^2\Delta} b_{2,i}(x)|u(x)|\, d\mu(x) \\
&=: H_1 + H_2 + H_3 + H_4.
\end{aligned}$$

As for H_1, by Theorem 2.1(2)(v$_4$), we have

$$H_1 \lesssim \int_{|x|,|y|>r_i} \left(\int_{B_{2,i}} \frac{1}{r_i} e^{-b\frac{d^2(x,y)}{r_i^2}} |b_{2,i}(y)|\, d\mu(y)\right) |u(x)|\, d\mu(x).$$

Using the same technique as that in the estimate E_1, we have

$$H_1 \lesssim \lambda \langle \mathcal{M}u, \chi_{B_{2,i}}\rangle.$$

As for H_2, by Theorem 2.1(2)(v$_1$), we have

$$H_2 \lesssim \int_{|x|,|y|\le r_i} \left(\int_{B_{2,i}} \frac{1}{r_i^2}\left(1 + \frac{|x||y|}{r_i} + \frac{|x|+|y|}{r_i}\log r_i^2\right) e^{-b\frac{d^2(x,y)}{r_i^2}} |b_{2,i}(y)|\, d\mu(y)\right) |u(x)|\, d\mu(x).$$

Since $|x|, |y| \le r_i$ and $\log r_i \lesssim r_i^\alpha$ for all $\alpha > 0$,

$$\begin{aligned}
H_2 &\lesssim \int_{|x|,|y|\le r_i} \left(\int_{B_{2,i}} \frac{1}{r_i} e^{-b\frac{d^2(x,y)}{r_i^2}} |b_{2,i}(y)|\, d\mu(y)\right) |u(x)|\, d\mu(x) \\
&\lesssim \lambda \int_{B_{2,i}} \chi_{B_{2,i}}(y) \int_{|x|,|y|\le r_i} \frac{1}{r_i} e^{-b\frac{d^2(x,y)}{r_i^2}} |u(x)|\, d\mu(x) d\mu(y) \\
&\lesssim \lambda \langle \mathcal{M}u, \chi_{B_{2,i}}\rangle.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
H_3 &\lesssim \int_{|y|\le r_i<|x|}\left(\int_{B_{2,i}}\frac{1}{r_i^2}(|y|+\log r_i^2)e^{-b\frac{d^2(x,y)}{r_i^2}}|b_{2,i}(y)|\,d\mu(y)\right)|u(x)|\,d\mu(x)\\
&\lesssim \int_{|y|\le r_i<|x|}\left(\int_{B_{2,i}}\frac{1}{r_i}e^{-b\frac{d^2(x,y)}{r_i^2}}|b_{2,i}(y)|\,d\mu(y)\right)|u(x)|\,d\mu(x)\\
&\lesssim \lambda\int_{B_{2,i}}\chi_{B_{2,i}}(y)\int_{|y|\le r_i<|x|}\frac{1}{r_i}e^{-b\frac{d^2(x,y)}{r_i^2}}|u(x)|\,d\mu(x)d\mu(y)\\
&\lesssim \lambda\langle \mathcal{M}u,\chi_{B_{2,i}}\rangle.
\end{aligned}
$$

For the last term H_4, we also get the same conclusion.

Hence, by combining the estimates of H_1, H_2, H_3 and H_4, we get that

$$
\left|\left\langle |u|,\sum_i e^{-r_i^2\Delta}b_{2,i}\right\rangle\right| \lesssim \lambda\left\langle \mathcal{M}u,\sum_i \chi_{B_{2,i}}\right\rangle
$$

for all $u\in L^2(\mathcal{R}^1\setminus K)$. This implies that

$$
\left\|\sum_i e^{-r_i^2\Delta}b_{2,i}\right\|^2_{L^2(\mathcal{R}^1\setminus K)} \lesssim \lambda\|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)}.
$$

The estimate (3.11) can be obtained by the same approach. So details are omitted.

Hence,

$$
\begin{aligned}
II_{222} &\lesssim \frac{1}{\lambda^2}\left\|\sum_i F(\sqrt{\Delta})e^{-r_i^2\Delta}b_{2,i}\right\|^2_{L^2(\mathcal{R}^1\sharp\mathcal{R}^2)}\\
&\lesssim \frac{1}{\lambda^2}\left\|\sum_i e^{-r_i^2\Delta}b_{2,i}\right\|^2_{L^2(\mathcal{R}^1\sharp\mathcal{R}^2)} \lesssim \frac{1}{\lambda}\|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)}.
\end{aligned}
$$

This implies

$$
II_{22}\lesssim \frac{1}{\lambda}\|f\|_{L^1(\mathcal{R}^1\sharp\mathcal{R}^2)}.
$$

The estimate of III can be done similarly to I; therefore, we omit details. □

This result will be used as an instrument in the proof of Theorem 1.1. We note that this proof is quite standard; therefore, for the sake of completeness, we provide it here (see [2, 9] for references).

Proof of Theorem 1.1. Let $\epsilon := \{\epsilon_j\}_{j\ge 0}$ be an arbitrary sequence with $\epsilon=\pm 1$. Let us define

$$
T_\epsilon f := \sum_{j\ge 0}\epsilon_j\varphi_j(\sqrt{L})f.
$$

Recall that $\varphi_0 \in C_0^\infty(\mathbb{R})$ and $\varphi \in C_0^\infty(0,\infty)$ are two functions satisfying $\operatorname{supp}\varphi_0 \subset [0,2]$ and $\varphi_0^{(2\nu+1)}(0) = 0$ for all $\nu \geq 0$, $\operatorname{supp}\varphi \subset [1/2,2]$. By a straightforward calculation we can show that the function $F_\epsilon(\lambda) = \sum_{j\geq 0}\epsilon_j\varphi_j(\lambda)$ satisfies condition (3.1) in Proposition 3.1. Thus,

$$\|T_\epsilon f\|_p \leq C\|f\|_p \quad \text{for } 1 < p < \infty,$$

where C is a dependent constant of ϵ.

Combining the previous fact with Khintchine's inequality and Fubini Theorem, we get that

$$\|S_\Delta f\|_p^p \lesssim \int_{\mathcal{R}^1\sharp\mathcal{R}^2} \mathbb{E}\{|T_\epsilon f(x)|^p\}\, dx = \mathbb{E}(\|T_\epsilon f\|_p^p) \lesssim \|f\|_p^p$$

which yields

$$\|S_\Delta f\|_p \lesssim \|f\|_p.$$

For the lower bound, it is well-known that we can find functions $\widetilde{\varphi}_0 \in C_0^\infty(\mathbb{R})$ and $\widetilde{\varphi} \in C_0^\infty(0,\infty)$ such that $\operatorname{supp}\widetilde{\varphi}_0 \subset [0,2]$ and $\widetilde{\varphi}_0^{(2\nu+1)}(0) = 0$ for all $\nu \geq 0$, $\operatorname{supp}\widetilde{\varphi} \subset [1/2,2]$ and

$$1 = \widetilde{\varphi}_0(\lambda)\varphi_0(\lambda) + \sum_{k>0}\widetilde{\varphi}(2^{-k}\lambda)\varphi(2^{-k}\lambda) = \sum_{k\geq 0}\widetilde{\varphi}_k(\lambda)\varphi_k(\lambda)$$

for all $\lambda \geq 0$.

This implies that

$$f = \sum_{k\geq 0}\widetilde{\varphi}_k(\sqrt{\Delta})\varphi_k(\sqrt{\Delta})f.$$

Hence, for $g \in L^{p'}(\mathcal{R}^1\sharp\mathcal{R}^2)$, by Hölder's inequality,

$$\begin{aligned}
\int_{\mathcal{R}^1\sharp\mathcal{R}^2} f(x)g(x)\, d\mu(x) &= \int_{\mathcal{R}^1\sharp\mathcal{R}^2} \sum_{k\geq 0}\varphi_k(\sqrt{\Delta})f(x)\widetilde{\varphi}_k(\sqrt{\Delta})g(x)\, d\mu(x) \\
&\leq \int_{\mathcal{R}^1\sharp\mathcal{R}^2} \left(\left|\varphi_0(\sqrt{\Delta})f\right|^2 + \sum_{k\geq 0}\left|\varphi(2^{-k}\sqrt{\Delta})f\right|^2\right)^{1/2} \\
&\qquad \times \left(\left|\widetilde{\varphi}_0(\sqrt{\Delta})g\right|^2 + \sum_{k\geq 0}\left|\widetilde{\varphi}(2^{-k}\sqrt{\Delta})g\right|^2\right)^{1/2} d\mu(x) \\
&=: \int_{\mathcal{R}^1\sharp\mathcal{R}^2} S_\Delta f\widetilde{S}_\Delta g\, d\mu \lesssim \|S_\Delta f\|_p\|\widetilde{S}_\Delta g\|_{p'}.
\end{aligned}$$

By the same approach as those in the proof of $\|S_\Delta f\|_p \lesssim \|f\|_p$, we obtain the following estimate:

$$\|\widetilde{S}_\Delta g\|_{p'} \lesssim \|g\|_{p'}.$$

Combining those estimates above,

$$\int_{\mathcal{R}^1\sharp\mathcal{R}^2} f(x)g(x)\, d\mu(x) \lesssim \|S_\Delta f\|_p\|g\|_{p'}$$

for all $g \in L^{p'}(\mathcal{R}^1 \sharp \mathcal{R}^2)$. This implies that

$$\|f\|_p \lesssim \|S_\Delta g\|_p.$$

The proof is complete. $\square$

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Hong Chuong Doan
Department of Economic Mathematics, University of Economics and Law, Vietnam
and
Vietnam National University, Hochiminh City, Vietnam
E-mail address: chuongdh@uel.edu.vn