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Jordan τ-derivations of Prime GPI-rings

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Abstract. Let R be a noncommutative prime ring, with maximal symmetric ring of quotients $Q_{ms}(R)$ and extended centriod C, and let τ be an anti-automorphism of R. An additive map $\delta\colon R \to Q_{ms}(R)$ is called a Jordan τ-derivation if $\delta(x^2) = \delta(x)x^\tau + x\delta(x)$ for all $x \in R$. In 2015 Lee and the author proved that any Jordan τ-derivation of R is X-inner if either R is not a GPI-ring or R is a PI-ring except when $\operatorname{char} R = 2$ and $\dim_C RC = 4$. In the paper we prove that, when R is a prime GPI-ring but is not a PI-ring, any Jordan τ-derivation is X-inner if either τ is of the second kind or both $\operatorname{char} R \neq 2$ and τ is of the first kind with $\deg \tau^2 \neq 2$.

1. Introduction

Throughout the paper, R is a prime ring with center $Z(R)$. Let $Q_{ml}(R)$ (resp. $Q_{ms}(R)$) be the maximal left (resp. symmetric) ring of quotients of R, and let $Q_s(R)$ be the Martindale symmetric ring of quotients of R. The center C of $Q_{ml}(R)$ is called the extended centroid of R. Note that $Q_s(R) \subseteq Q_{ms}(R) \subseteq Q_{ml}(R)$ and $Z(Q_{ms}(R)) = Z(Q_s(R)) = C$. We refer the readers to see [2] for more details.

By a derivation (resp. Jordan derivation) of R we mean an additive map $\delta\colon R \to Q_{ml}(R)$ satisfying $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in R$ (resp. $\delta(x^2) = \delta(x)x + x\delta(x)$ for all $x \in R$). Obviously, a derivation is a Jordan derivation, but, in general, the converse is not true. Herstein [9], Lee, and the author [15] characterized Jordan derivations of prime rings. They showed that an additive map $\delta\colon R \to Q_{ml}(R)$ is a Jordan derivation if and only if it is of the form $d + \mu$, where $d\colon R \to Q_{ml}(R)$ is a derivation and $\mu\colon R \to C$ is an additive map satisfying $2\mu = 0$ and $\mu(x^2) = 0$ for all $x \in R$. Moreover, Cusack [6] and Brešar [3] independently generalized Herstein's result, i.e., they proved that every Jordan derivation of a 2-torsion free semi-prime ring is a derivation.

Let $\tau\colon R \to R$ be an anti-automorphism of R. An additive map $\delta\colon R \to Q_{ms}(R)$ is called a Jordan τ-derivation of R if $\delta(x^2) = x\delta(x) + \delta(x)x^\tau$ for all $x \in R$. A Jordan τ-derivation δ of R is said to be inner (resp. X-inner) if there exists $a \in R$ (resp. $a \in Q_{ms}(R)$) such that $\delta(x) = ax^\tau - xa$ for $x \in R$.

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Let A be a unital ring. Suppose that $*\colon A \to A$ is an involution of A, i.e., $*$ is an anti-automorphism of A such that $(x^*)^* = x$ for all $x \in A$. The problem of the representability of quadratic forms by bilinear forms is connected with the structure of Jordan $*$-derivations (see [18, 19]). Brešar and Vukman showed that if a unital $*$-ring A contains $1/2$ and a central invertible skew-hermitian element μ (i.e., $\mu^* = -\mu$), then every Jordan $*$-derivation $\delta\colon A \to A$ is inner (see [4, Theorem 1]). In particular, every Jordan $*$-derivation $\delta\colon A \to A$ of a unital complex $*$-algebra A is inner because any complex $*$-algebra A always assumes $(\beta x)^* = \overline{\beta} x^*$ for all $x \in A$ and $\beta \in \mathbb{C}$.

Let H be a real Hilbert space with $\dim_{\mathbb{R}} H > 1$. Let $\mathcal{B}(H)$ stand for the algebra of all bounded linear operators on the Hilbert space H and let $\mathcal{A}$ be a standard operator algebra on H. Then $\mathcal{B}(H)$ can be endowed with a canonical involution, say $*$. It is known that $\mathcal{A}$ is a prime algebra over $\mathbb{R}$ with nonzero socle. Moreover, $Q_{ms}(\mathcal{A}) = Q_{ms}(\mathcal{B}(H)) = \mathcal{B}(H)$ (see [5, Theorem 1.3]). Šemrl proved that every Jordan $*$-derivation $\delta\colon \mathcal{B}(H) \to \mathcal{B}(H)$ is inner (see [17]), and that all Jordan $*$-derivations $\delta\colon \mathcal{A} \to \mathcal{B}(H)$ are X-inner (see [20]). Chuang et al. extended Šemrl's theorems above by proving the theorem: Let R be a prime ring, which is not a division ring. Let τ be an anti-automorphism of R and let $\delta\colon R \to Q_s(R)$ be a Jordan τ-derivation. If $\operatorname{char} R \neq 2$ and the socle of R is nonzero, then δ is X-inner (see [5, Theorem 1.2]). Moreover, the structure of Jordan $*$-derivations of prime rings is completely determined. To be precise, it was proved that every Jordan $*$-derivation of R is X-inner except when $\operatorname{char} R = 2$ and $\dim_C RC = 4$ (see [7, 13, 14]). Note that there exist non X-inner Jordan $*$-derivations when $\operatorname{char} R = 2$ and $\dim_C RC = 4$ (see [13, Theorem 3.1]).

The reader is referred to [2] for the definitions of PI-rings and GPI-rings. In [16], Lee and the author showed that any Jordan τ-derivation of R is X-inner if either R is not a GPI-ring or R is a PI-ring except when $\operatorname{char} R = 2$ and $\dim_C RC = 4$. In order to completely characterize Jordan τ-derivations of R, they raised the following question.

Question 1.1. Let R be a prime GPI-ring, which is not commutative, with an anti-automorphism τ. Suppose that neither R is a PI-ring nor R is a division ring. Is any Jordan τ-derivation of R X-inner?

We remark that, by Martindale's theorem [10, Theorem 3], if R is both a prime GPI-ring and a division ring, then it is a PI-ring and Question 1.1 is solved by [16, Theorem 2.9] in this case. Hence Question 1.1 is reduced to the case that R is a prime GPI-ring but is not a PI-ring. Given an automorphism (resp. anti-automorphism) g of R, g can be uniquely extended to an automorphism (resp. anti-automorphism) of $Q_s(R)$ (see [2, Proposition 2.5.3] for the automorphism case and [2, Proposition 2.5.4] for the anti-automorphism case). An automorphism (resp. anti-automorphism) g is said to be of the first kind if $\beta^g = \beta$ for all $\beta \in C$. Otherwise, g is said to be of the second kind. We first give an affirmative answer to Question 1.1 when τ is of the second kind.

Theorem 1.2. *Let R be a noncommutative prime ring with an anti-automorphism τ. If τ is of the second kind, then any Jordan τ-derivation of R is X-inner except when* $\operatorname{char} R = 2$ *and* $\dim_C RC = 4$.

We next consider the case that τ is of the first kind. By an X-inner automorphism we mean an automorphism of the form $x \mapsto uxu^{-1}$ for all $x \in R$, where $u \in Q_s(R)$. Kharchenko proved that, given an automorphism σ of a prime GPI-ring, if σ is of the first kind, then it is X-inner (see [8, Proof of Proposition 2]). By Kharchenko's theorem, τ^2 is X-inner when R is a prime GPI-ring and τ is of the first kind. The complexity of the question depends on that of τ^2.

Definition 1.3. Let R be a prime GPI-ring with an automorphism σ of the first kind. Then there exists $u \in Q_s(R)$ such that $x^\sigma = uxu^{-1}$ for all $x \in R$. We say that $\deg \sigma = m$ if u is algebraic of minimal degree m over C. Moreover, $\deg \sigma = \infty$ if u is not algebraic over C.

Clearly, $\deg \sigma$ is independent of the element u we choose. Also, if $\deg \tau^2 = 1$, then τ is an involution and Question 1.1 has been solved in [13, Theorem 1.2]. The following is the second main theorem of the paper.

Theorem 1.4. *Let R be a prime GPI-ring with* $\operatorname{char} R \neq 2$ *and an anti-automorphism τ of the first kind. If* $\deg \tau^2 \neq 2$*, then any Jordan τ-derivation of R is X-inner.*

We remark that the case of $\deg \tau^2 = 2$ keeps unknown.

2. Proof of Theorem 1.2

Throughout the section, R is a prime ring with an anti-automorphism τ of the second kind. To characterize Jordan τ-derivations of R, we need some results concerning functional identities. In [12], Lee dealt with functional identities on prime rings with an automorphism. For our purpose, we will follow his viewpoint to get useful results about functional identities.

We first introduce some notations. For maps $f\colon R^{r-1} \to Q_{ml}(R)$ and $g\colon R^{r-2} \to Q_{ml}(R)$, we write

$$f^i(\overline{x}_r) = f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r)$$

and

$$g^{ij}(\overline{x}_r) = g^{ji}(\overline{x}_r) = g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_r)$$

where $\overline{x}_r = (x_1, \ldots, x_r) \in R^r$ and $1 \leq i < j \leq r$. Our purpose is to prove the following theorem.

Theorem 2.1. *Let R be a prime ring with an anti-automorphism τ of the second kind. Suppose that $E_{it}, F_{\ell 1}\colon R^{r-1} \to Q_{ml}(R)$ are $(r-1)$-additive maps such that*

$$\sum_{i=1}^{r} E_{i1}^{i}(\overline{x}_r)x_i + \sum_{i=1}^{r} E_{i2}^{i}(\overline{x}_r)x_i^{\tau} + \sum_{\ell=1}^{r} x_\ell F_{\ell 1}^{\ell}(\overline{x}_r) \in C \tag{2.1}$$

for $\overline{x}_r \in R^r$, where $1 \leq i, \ell \leq r$ and $t = 1, 2$. If R is not a PI-ring, then there exist a nonzero ideal I of R, $(r-2)$-additive maps $p_{it\ell 1}\colon I^{r-2} \to Q_{ml}(R)$, and $(r-1)$-additive maps $\lambda_{i1}\colon I^{r-1} \to C$ such that

$$E_{i1}^{i}(\overline{x}_r) = \sum_{\substack{1\leq \ell \leq r \\ \ell \neq i}} x_\ell p_{i1\ell 1}^{i\ell}(\overline{x}_r) + \lambda_{i1}^{i}(\overline{x}_r), \quad E_{i2}^{i}(\overline{x}_r) = \sum_{\substack{1\leq \ell \leq r \\ \ell \neq i}} x_\ell p_{i2\ell 1}^{i\ell}(\overline{x}_r)$$

and

$$F_{\ell 1}^{\ell}(\overline{x}_r) = -\sum_{\substack{1\leq i \leq r \\ i \neq \ell}} p_{i1\ell 1}^{i\ell}(\overline{x}_r)x_i - \sum_{\substack{1\leq i \leq r \\ i \neq \ell}} p_{i2\ell 1}^{i\ell}(\overline{x}_r)x_i^{\tau} - \lambda_{\ell 1}^{\ell}(\overline{x}_r)$$

for all $\overline{x}_r \in I^r$, where $1 \leq i, \ell \leq r$ and $t = 1, 2$.

Corollary 2.2. *Let R be a prime ring with an anti-automorphism τ of the second kind. Suppose that $E_i, F_\ell\colon R^{r-1} \to Q_{ml}(R)$ are $(r-1)$-additive maps such that*

$$\sum_{i=1}^{r} E_{i}^{i}(\overline{x}_r)x_i^{\tau} + \sum_{\ell=1}^{r} x_\ell F_{\ell}^{\ell}(\overline{x}_r) \in C$$

for $\overline{x}_r \in R^r$, where $1 \leq i, \ell \leq r$. If R is not a PI-ring, then there exist a nonzero ideal I of R and $(r-2)$-additive maps $p_{i\ell}\colon I^{r-2} \to Q_{ml}(R)$ such that

$$E_{i}^{i}(\overline{x}_r) = \sum_{\substack{1\leq \ell \leq r \\ \ell \neq i}} x_\ell p_{i\ell}^{i\ell}(\overline{x}_r) \quad \text{and} \quad F_{\ell}^{\ell}(\overline{x}_r) = -\sum_{\substack{1\leq i \leq r \\ i \neq \ell}} p_{i\ell}^{i\ell}(\overline{x}_r)x_i^{\tau}$$

for all $\overline{x}_r \in I^r$, where $1 \leq i, \ell \leq r$.

Proof. By Theorem 2.1, there exist a nonzero ideal I of R, $(r-2)$-additive maps $p_{it\ell 1}\colon I^{r-2} \to Q_{ml}(R)$, and $(r-1)$-additive maps $\lambda_{i1}\colon I^{r-1} \to C$ such that

$$0 = \sum_{\substack{1\leq \ell \leq r \\ \ell \neq i}} x_\ell p_{i1\ell 1}^{i\ell}(\overline{x}_r) + \lambda_{i1}^{i}(\overline{x}_r), \quad E_{i}^{i}(\overline{x}_r) = \sum_{\substack{1\leq \ell \leq r \\ \ell \neq i}} x_\ell p_{i2\ell 1}^{i\ell}(\overline{x}_r),$$

and

$$F_{\ell}^{\ell}(\overline{x}_r) = -\sum_{\substack{1\leq i \leq r \\ i \neq \ell}} p_{i1\ell 1}^{i\ell}(\overline{x}_r)x_i - \sum_{\substack{1\leq i \leq r \\ i \neq \ell}} p_{i2\ell 1}^{i\ell}(\overline{x}_r)x_i^{\tau} - \lambda_{\ell 1}^{\ell}(\overline{x}_r)$$

for all $\overline{x}_r \in I^r$, where $1 \leq i, \ell \leq r$ and $t = 1, 2$. In view of [1, Theorem 2.4], $p_{i1\ell 1} = 0$ and $\lambda_{i1} = 0$ for $1 \leq i, \ell \leq r$. The proof is complete by putting $p_{i\ell} = p_{i2\ell 1}$. $\square$

To begin the proof of Theorem 2.1, we first give the following lemma.

Lemma 2.3. *Suppose that $E_i, F_\ell\colon R^{r-1} \to Q_{ml}(R)$ are $(r-1)$-additive maps such that*

$$\sum_{i=1}^{r} E_i^i(\overline{x}_r)x_i + \sum_{\ell=1}^{r} F_\ell^\ell(\overline{x}_r)x_\ell^\tau \in C \tag{2.2}$$

for $\overline{x}_r \in R^r$, where $1 \le i, \ell \le r$. If R is not a PI-*ring, then there exists a nonzero ideal I of R such that $E_i^i = 0 = F_\ell^\ell$ on I^r for $1 \le i, \ell \le r$.*

Before proving it, we define the following notation (see [12]). For a map $f\colon R^{r-1} \to Q_{ml}(R)$ and $t \ne i$, we write

$$f^i(\overline{x}_r; \{y\}_t) = f(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_r)$$

where $z_t = y$ and $z_j = x_j$ for $j \ne t$, i.e., we replace x_t by y in $f^i(\overline{x}_r)$.

Proof of Lemma 2.3. Let $A := \{1, 2, \ldots, r\}$ and

$$L := \{\ell \in A \mid \text{there exists a nonzero ideal } J \text{ of } R \text{ such that } F_\ell^\ell = 0 \text{ on } J^r\}.$$

We proceed the proof by induction on $r - |L|$.

Suppose first that $r - |L| = 0$, i.e., $L = A$. Then there exists a nonzero ideal J such that $F_\ell^\ell = 0$ on J^r for all $\ell = 1, \ldots, r$. Thus $\sum_{i=1}^{r} E_i^i(\overline{x}_r)x_i \in C$ for all $\overline{x}_r \in J^r$. By [1, Theorem 2.4], $E_i^i = 0$ on J^r for all $i = 1, \ldots, r$, as asserted.

Suppose next that $r - |L| \ge 1$. Without loss of generality, we may assume that $r \notin L$. Then, for any nonzero ideal U of R, $F_r^r \ne 0$ on U^r. Fix $\beta \in C$ with $\beta^\tau \ne \beta$ and choose a nonzero ideal K of R such that $\beta K \subseteq R$. Then, by (2.2), we have

$$\begin{aligned} &\sum_{i=1}^{r-1} \big(E_i^i(\overline{x}_r; \{\beta x_r\}_r) - \beta E_i^i(\overline{x}_r)\big)x_i \\ &+ \sum_{\ell=1}^{r-1} \big(F_\ell^\ell(\overline{x}_r; \{\beta x_r\}_r) - \beta F_\ell^\ell(\overline{x}_r)\big)x_\ell^\tau + (\beta^\tau - \beta)F_r^r(\overline{x}_r)x_r^\tau \in C \end{aligned} \tag{2.3}$$

for all $\overline{x}_r \in K^r$. Let $K_1 = K \cap K^\tau$. Then K_1 is an ideal of R such that $K_1^{\tau^{-1}} \subseteq K$ and, by (2.3), we have

$$\sum_{i=1}^{r-1} \widetilde{E}_i^i(\overline{x}_r)x_i + F_r^r(\overline{x}_r)x_r + \sum_{\ell=1}^{r-1} \widetilde{F}_\ell^\ell(\overline{x}_r)x_\ell^\tau \in C \tag{2.4}$$

for all $\overline{x}_r \in K_1^r$, where

$$\widetilde{E}_i^i(\overline{x}_r) = (\beta^\tau - \beta)^{-1}\big(E_i^i(\overline{x}_r; \{\beta x_r^{\tau^{-1}}\}_r) - \beta E_i^i(\overline{x}_r; \{x_r^{\tau^{-1}}\}_r)\big)$$

and

$$\widetilde{F}^{\ell}_{\ell}(\overline{x}_r) = (\beta^{\tau} - \beta)^{-1}\big(F^{\ell}_{\ell}(\overline{x}_r; \{\beta x_r^{\tau^{-1}}\}_r) - \beta F^{\ell}_{\ell}(\overline{x}_r; \{x_r^{\tau^{-1}}\}_r)\big).$$

Set

$$L_1 := \{\ell \mid 1 \leq \ell \leq r-1, \text{there exists a nonzero ideal } J \text{ of } R \text{ such that } \widetilde{F}^{\ell}_{\ell} = 0 \text{ on } J^r\}.$$

Let $\ell \in L$. Then $1 \leq \ell \leq r-1$ and there is a nonzero ideal N of R such that $F^{\ell}_{\ell} = 0$ on N^r. By the definition of $\widetilde{F}^{\ell}_{\ell}$, there exists a nonzero ideal M of R contained in N such that $\widetilde{F}^{\ell}_{\ell} = 0$ on M^r and so $\ell \in L_1$. Thus $|L| \leq |L_1|$ and $r - |L| \geq r - |L_1| > (r-1) - |L_1|$. By applying the induction hypothesis on (2.4), we have $F^{\ell}_{\ell} = 0$ on W^r for some nonzero ideal W of R, a contradiction. $\square$

Proof of Theorem 2.1. Let $A := \{1, 2, \ldots, r\}$ and

$$L := \{i \in A \mid \text{there exists a nonzero ideal } J \text{ of } R \text{ such that } E^{i}_{i2} = 0 \text{ on } J^r\}.$$

We proceed the proof by induction on $r - |L|$.

Assume first that $r - |L| = 0$, i.e., $L = A$. Then $E^{i}_{i2} = 0$ on U^r for some nonzero ideal U of R and so (2.1) becomes

$$\sum_{i=1}^{r} E^{i}_{i1}(\overline{x}_r)x_i + \sum_{\ell=1}^{r} x_{\ell}F^{\ell}_{\ell 1}(\overline{x}_r) \in C$$

for $\overline{x}_r \in U^r$. Hence the result follows from [1, Corollary 2.11].

Assume next that $r - |L| \geq 1$. Without loss of generality, assume that $r \notin L$. Then $E^{r}_{r2} \neq 0$ on any nonzero ideal of R. Let $\beta \in C$ with $\beta^{\tau} \neq \beta$ and choose a nonzero ideal J of R such that $\beta J \subseteq R$. Then, by (2.1), we have

$$\begin{aligned}
&\sum_{i=1}^{r-1} \big(E^{i}_{i1}(\overline{x}_r; \{\beta x_r\}_r) - \beta E^{i}_{i1}(\overline{x}_r)\big)x_i \\
&+ \sum_{i=1}^{r-1} \big(E^{i}_{i2}(\overline{x}_r; \{\beta x_r\}_r) - \beta E^{i}_{i2}(\overline{x}_r)\big)x_i^{\tau} + E^{r}_{r2}(\overline{x}_r)(\beta^{\tau} - \beta)x_r^{\tau} \\
&+ \sum_{\ell=1}^{r-1} x_{\ell}\big(F^{\ell}_{\ell 1}(\overline{x}_r; \{\beta x_r\}_r) - \beta F^{\ell}_{\ell 1}(\overline{x}_r)\big) \in C
\end{aligned}$$

for all $\overline{x}_r \in J^r$. Let

$$\begin{aligned}
\widetilde{E}^{i}_{i1}(\overline{x}_r) &= (\beta^{\tau} - \beta)^{-1}\big(E^{i}_{i1}(\overline{x}_r; \{\beta x_r\}_r) - \beta E^{i}_{i1}(\overline{x}_r)\big), \\
\widetilde{E}^{i}_{i2}(\overline{x}_r) &= (\beta^{\tau} - \beta)^{-1}\big(E^{i}_{i2}(\overline{x}_r; \{\beta x_r\}_r) - \beta E^{i}_{i2}(\overline{x}_r)\big),
\end{aligned}$$

and

$$\widetilde{F}^{\ell}_{\ell 1}(\overline{x}_r) = (\beta^{\tau} - \beta)^{-1}\big(F^{\ell}_{\ell 1}(\overline{x}_r; \{\beta x_r\}_r) - \beta F^{\ell}_{\ell 1}(\overline{x}_r)\big).$$

Then

$$\sum_{i=1}^{r-1}\widetilde{E}^i_{i1}(\overline{x}_r)x_i+\sum_{i=1}^{r-1}\widetilde{E}^i_{i2}(\overline{x}_r)x_i^\tau+E^r_{r2}(\overline{x}_r)x_r^\tau+\sum_{\ell=1}^{r-1}x_\ell\widetilde{F}^\ell_{\ell 1}(\overline{x}_r)\in C \tag{2.5}$$

for all $\overline{x}_r\in J^r$. Choose a nonzero ideal J_1 of R contained in J so that $J_1^{\tau^{-1}}\subseteq J$. By (2.5), we have

$$\sum_{i=1}^{r-1}\widetilde{E}^i_{i1}(\overline{x}_r;\{x_r^{\tau^{-1}}\}_r)x_i+\sum_{i=1}^{r-1}\widetilde{E}^i_{i2}(\overline{x}_r;\{x_r^{\tau^{-1}}\}_r)x_i^\tau+E^r_{r2}(\overline{x}_r)x_r$$
$$+\sum_{\ell=1}^{r-1}x_\ell\widetilde{F}^\ell_{\ell 1}(\overline{x}_r;\{x_r^{\tau^{-1}}\}_r)\in C$$

for all $\overline{x}_r\in J_1^r$. Set $G^i_{i2}(\overline{x}_r):=\widetilde{E}^i_{i2}(\overline{x}_r;\{x_r^{\tau^{-1}}\}_r)$ and

$$L_1:=\{i\mid 1\le i\le r-1,\text{there exists a nonzero ideal } J \text{ of } R \text{ such that } G^i_{i2}=0 \text{ on } J^r\}.$$

Let $i\in L$ and $i\neq r$. Then there exists a nonzero ideal N of R such that $E^i_{i2}=0$ on N^r. From the definition of G^i_{i2}, there is a nonzero ideal M of R contained in N such that $G^i_{i2}=0$ on M^r, and so $i\in L_1$. Thus

$$r-|L|\ge r-|L_1|>(r-1)-|L_1|.$$

By the induction hypothesis, there exist a nonzero ideal J_2 of R contained in J_1 and $(r-2)$-additive maps $p_{r2\ell 1}\colon J_2^{r-2}\to Q_{ml}(R)$ such that

$$E^r_{r2}(\overline{x}_r)=\sum_{\ell=1}^{r-1}x_\ell p^{r\ell}_{r2\ell 1}(\overline{x}_r)$$

for all $\overline{x}_r\in J_2^r$. Substituting it into (2.1), we have

$$\sum_{i=1}^{r}E^i_{i1}(\overline{x}_r)x_i+\sum_{i=1}^{r-1}E^i_{i2}(\overline{x}_r)x_i^\tau+\sum_{\ell=1}^{r-1}x_\ell\big(F^\ell_{\ell 1}(\overline{x}_r)+p^{r\ell}_{r2\ell 1}(\overline{x}_r)x_r^\tau\big)+x_rF^r_{r1}(\overline{x}_r)\in C \tag{2.6}$$

for all $\overline{x}_r\in J_2^r$. By the induction hypothesis, there are a nonzero ideal I of R contained in J_2 and $(r-2)$-additive maps $p_{i2\ell 1}\colon I^{r-2}\to Q_{ml}(R)$ such that

$$E^i_{i2}(\overline{x}_r)=\sum_{\substack{\ell=1\\ \ell\neq i}}^{r}x_\ell p^{i\ell}_{i2\ell 1}(\overline{x}_r)$$

for all $\overline{x}_r\in I^r$ and $1\le i\le r-1$. Thus (2.6) becomes

$$\sum_{i=1}^{r}E^i_{i1}(\overline{x}_r)x_i+\sum_{\ell=1}^{r}x_\ell\left(F^\ell_{\ell 1}(\overline{x}_r)+\sum_{\substack{i=1\\ i\neq\ell}}^{r}p^{i\ell}_{i2\ell 1}(\overline{x}_r)x_i^\tau\right)\in C$$

for all $\overline{x}_r \in I^r$. According to [1, Corollary 2.11], there exist $(r-2)$-additive maps $p_{i1\ell 1}\colon I^{r-2} \to Q_{ml}(R)$ and $(r-1)$-additive maps $\lambda_{i1}\colon I^{r-1} \to C$ such that

$$E^i_{i1}(\overline{x}_r) = \sum_{\substack{1\leq \ell\leq r\\ \ell\neq i}} x_\ell p^{i\ell}_{i1\ell 1}(\overline{x}_r) + \lambda^i_{i1}(\overline{x}_r)$$

and

$$F^\ell_{\ell 1}(\overline{x}_r) + \sum_{\substack{i=1\\ i\neq \ell}}^{r} p^{i\ell}_{i2\ell 1}(\overline{x}_r)x_i^\tau = -\sum_{\substack{1\leq i\leq r\\ i\neq \ell}} p^{i\ell}_{i1\ell 1}(\overline{x}_r)x_i - \lambda^\ell_{\ell 1}(\overline{x}_r)$$

for all $\overline{x}_r \in I^r$, where $1 \leq i, \ell \leq r$, as asserted. □

Note that if $\delta\colon R \to Q_{ms}(R)$ is a Jordan τ-derivation, then

$$\delta(xy + yx) = \delta(x)y^\tau + y\delta(x) + \delta(y)x^\tau + x\delta(y) \tag{2.7}$$

for all $x, y \in R$.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. According to [16, Theorem 2.9], we can assume that R is not a PI-ring. Let $\delta\colon R \to Q_{ms}(R)$ be a Jordan τ-derivation. Define the bi-additive map $B\colon R\times R \to Q_{ms}(R)$ by $B(x,y) = \delta(xy+yx)$ for $x, y \in R$. It follows from [14, Lemma 2.3] that

$$B(xw, yz) - B(x, wyz) = B(zxw, y) - B(zx, wy)$$

for all $x, y, z, w \in R$. So, by (2.7), we have

$$\begin{aligned}
&\big(\delta(yz)w^\tau - \delta(wyz)\big)x^\tau + \big(\delta(xw)z^\tau - \delta(zxw)\big)y^\tau \\
&+ \big(\delta(wy)x^\tau - \delta(y)w^\tau x^\tau\big)z^\tau + \big(\delta(zx)y^\tau - \delta(x)z^\tau y^\tau\big)w^\tau \\
&+ x\big(w\delta(yz) - \delta(wyz)\big) + y\big(z\delta(xw) - \delta(zxw)\big) \\
&+ z\big(x\delta(wy) - xw\delta(y)\big) + w\big(y\delta(zx) - yz\delta(x)\big) = 0
\end{aligned}$$

for all $x, y, z, w \in R$. According to Corollary 2.2, there exist a nonzero ideal I_1 of R and bi-additive maps $r_{13}, r_{23}, r_{43}\colon I_1^2 \to Q_{ml}(R)$ such that

$$x\big(\delta(wy) - w\delta(y)\big) = -r_{13}(y,w)x^\tau - r_{23}(x,w)y^\tau - r_{43}(x,y)w^\tau$$

for all $x, y, w \in I_1$. Again, there are additive maps $p, q\colon I_2 \to Q_{ml}(R)$ so that

$$\delta(wy) - w\delta(y) = -p(y)w^\tau - q(w)y^\tau$$

for all $y, w \in I_2$, where I_2 is a nonzero ideal of R contained in I_1. Let $x, y, t \in I_2$. We have

$$\delta(xy) = x\delta(y) - p(y)x^\tau - q(x)y^\tau. \tag{2.8}$$

Replacing x by tx in (2.8), we obtain

$$\delta(txy) = tx\delta(y) - p(y)x^\tau t^\tau - q(tx)y^\tau.$$

Left-multiplying (2.8) by t, we get $t\delta(xy) = tx\delta(y) - tp(y)x^\tau - tq(x)y^\tau$. Thus,

$$\delta(txy) - t\delta(xy) = \big(tq(x) - q(tx)\big)y^\tau - p(y)x^\tau t^\tau + tp(y)x^\tau.$$

Replacing x, y by t, xy respectively in (2.8), we have

$$\delta(txy) - t\delta(xy) = -p(xy)t^\tau - q(t)y^\tau x^\tau.$$

Comparing the two equalities above, we see that

$$\big(tq(x) - q(tx)\big)y^\tau + \big(p(xy) - p(y)x^\tau\big)t^\tau + \big(tp(y) + q(t)y^\tau\big)x^\tau = 0$$

for all $x, y, t \in I_2$. By Lemma 2.3,

$$tq(x) = q(tx), \quad p(xy) = p(y)x^\tau \quad \text{and} \quad tp(y) = -q(t)y^\tau$$

for all $x, y, t \in I_3$, where I_3 is a nonzero ideal of R contained in I_2. According to [11, Lemma 2.1], there is $a \in Q_{ml}(R)$ such that $q(x) = xa$ for $x \in I_3$. So

$$tp(y) = -q(t)y^\tau = -tay^\tau$$

for $t, y \in I_3$, i.e., $I_3\big(p(y) + ay^\tau\big) = 0$ for all $y \in I_3$. Thus, $p(y) = -ay^\tau$ and it follows from (2.8) that

$$\delta(xy) - x\delta(y) = ay^\tau x^\tau - xay^\tau$$

for $x, y \in I_3$. Let $\widetilde{\delta}\colon I_3 \to Q_{ml}(R)$ be defined by $\widetilde{\delta}(x) = ax^\tau - xa$ for all $x \in I_3$. Then $\widetilde{\delta}(xy) = ay^\tau x^\tau - xya$ and

$$(\widetilde{\delta} - \delta)(xy) = xay^\tau - xya - x\delta(y) = x(\widetilde{\delta} - \delta)(y)$$

for $x, y \in I_3$. So there exists $c \in Q_{ml}(R)$ such that $(\widetilde{\delta} - \delta)(x) = xc$ for all $x \in I_3$ (see [11, Lemma 2.1]). Define $J := \widetilde{\delta} - \delta$, a Jordan τ-derivation of I_3. Thus, $x^2c = J(x^2) = xJ(x) + J(x)x^\tau = x^2c + xcx^\tau$ for all $x \in I_3$; that is, $xcx^\tau = 0$ for all $x \in I_3$. By [5, Lemma 2.2], $c = 0$ follows, i.e., $\delta = \widetilde{\delta}$ on I_3. Therefore, $\delta(x) = ax^\tau - xa$ for $x \in I_3$. By [16, Lemma 2.6], $a \in Q_{ms}(R)$. Finally, we will show that δ is X-inner. Let $x \in I_3$ and $y \in R$. Then

$$\delta(xy + yx) = ay^\tau x^\tau + ax^\tau y^\tau - xya - yxa$$

and

$$\begin{aligned}\delta(xy + yx) &= \delta(x)y^\tau + y\delta(x) + \delta(y)x^\tau + x\delta(y)\\ &= ax^\tau y^\tau - xay^\tau + yax^\tau - yxa + \delta(y)x^\tau + x\delta(y).\end{aligned}$$

Comparing these equations, we have

$$\big(\delta(y) - (ay^\tau - ya)\big)x^\tau + x\big(\delta(y) - (ay^\tau - ya)\big) = 0$$

for all $x \in I_3$ and $y \in R$. Fix $y \in R$ and set $q := \delta(y) - (ay^\tau - ya)$. Then, for $x, z \in I_3$,

$$xzq = -q(xz)^\tau = -qz^\tau x^\tau = zqx^\tau = -zxq$$

and so $(xz + zx)q = 0$ for all $x, z \in I_3$. This implies $(xz + zx)q = 0$ for all $x, z \in Q_{ml}(R)$. If $\operatorname{char} R \neq 2$, let $z = 1$ and so $2xq = 0$ for all $x \in Q_{ml}(R)$ implying $q = 0$. If $\operatorname{char} R = 2$, then $[Q_{ml}(R), Q_{ml}(R)]q = 0$ forcing $q = 0$. Hence $\delta(y) = ay^\tau - ya$ for all $y \in R$, as desired. $\square$

3. Proof of Theorem 1.4

Throughout the section, R is a prime GPI-ring with an anti-automorphism τ of the first kind. Let $u \in Q_s(R)$ be fixed such that $x^{\tau^2} = uxu^{-1}$ for all $x \in R$.

Lemma 3.1. *$u^\tau u = uu^\tau \in C$.*

Proof. Let $x \in R$. Then

$$u^\tau ux = u^\tau x^{\tau^2} u = (x^\tau u)^\tau u = (u(x^\tau)^{\tau^{-2}})^\tau u = xu^\tau u.$$

Hence $u^\tau u = uu^\tau \in C$. $\square$

Now, by Lemma 3.1, we fix $\beta := u^\tau u = uu^\tau \in C$ and so $u^\tau = \beta u^{-1}$. Since R is a prime GPI-ring, RC is a primitive ring with nonzero socle and so $Q_{ms}(RC) = Q_s(RC)$. In view of [5, Theorem 1.2], the aim of this section is to extend δ to a Jordan τ-derivation of RC when $\operatorname{char} R \neq 2$. The following result plays a key role.

Lemma 3.2. *Let $f\colon R \to Q_{ml}(R)$ be an additive map such that*

$$xf(y) + f(y)x^\tau = yf(x) + f(x)y^\tau \tag{3.1}$$

for all $x, y \in R$. If $\deg \tau^2 > 2$, then $f = 0$ on some nonzero ideal I of R.

Proof. Choose a nonzero ideal such that $uI_1 \subseteq R$. Then, by replacing x with ux in (3.1),

$$uxf(y)u + \beta f(y)x^\tau = yf(ux)u + f(ux)y^\tau u$$

for $x \in I_1$. Also, by (3.1),

$$uxf(y)u + uf(y)x^\tau u = uyf(x)u + uf(x)y^\tau u.$$

Comparing the two equations, we have

$$\beta f(y)x^\tau - uf(y)x^\tau u = yf(ux)u - uyf(x)u + \big(f(ux) - uf(x)\big)y^\tau u \tag{3.2}$$

for $x \in I_1$. Replacing y by uy in (3.2),

$$\beta f(uy)x^\tau - uf(uy)x^\tau u = uyf(ux)u - u^2yf(x)u + \beta\big(f(ux) - uf(x)\big)y^\tau.$$

Also, by (3.2),

$$u\beta f(y)x^\tau - u^2f(y)x^\tau u = uyf(ux)u - u^2yf(x)u + u\big(f(ux) - uf(x)\big)y^\tau u.$$

Comparing the two equations, we have

$$\begin{aligned} &\beta\big(f(uy) - uf(y)\big)x^\tau - u\big(f(uy) - uf(y)\big)x^\tau u \\ &= \beta\big(f(ux) - uf(x)\big)y^\tau - u\big(f(ux) - uf(x)\big)y^\tau u \end{aligned}$$

for all $x, y \in I_1$. By [1, Theorem 2.5], $f(uy) = uf(y)$ for all $y \in I_1$. So (3.2) becomes

$$\beta f(y)x^\tau - uf(y)x^\tau u = yuf(x)u - uyf(x)u \tag{3.3}$$

for all $x, y \in I_1$. Choose a nonzero ideal I_2 of R contained in I_1 such that $uI_2 \subseteq I_1$. Replacing x by ux in (3.3), we have

$$\beta^2f(y)x^\tau - \beta uf(y)x^\tau u = yu^2f(x)u^2 - uyuf(x)u^2$$

for all $x, y \in I_2$. On the other hand, (3.3) implies

$$\beta^2f(y)x^\tau - \beta uf(y)x^\tau u = \beta\big(yuf(x)u - uyf(x)u\big)$$

for all $x, y \in I_2$. Comparing the two equations, we have

$$y\big(\beta uf(x) - u^2f(x)u\big) + uy\big(uf(x)u - \beta f(x)\big) = 0$$

for all $x, y \in I_2$. Similarly, $uf(x)u = \beta f(x)$ for all $x \in I_2$. Thus (3.3) becomes

$$\beta f(y)x^\tau - uf(y)x^\tau u = \beta yf(x) - uyf(x)u \tag{3.4}$$

for all $x, y \in I_2$. Choose a nonzero ideal I_3 of R contained in I_2 such that $I_3u \subseteq I_2$. Replacing y by yu in (3.4),

$$\begin{aligned} \beta f(yu)x^\tau - uf(yu)x^\tau u &= \beta yuf(x) - uyuf(x)u = \beta^2yf(x)u^{-1} - \beta uyf(x) \\ &= \beta\big(\beta yf(x) - uyf(x)u\big)u^{-1} = \beta^2f(y)x^\tau u^{-1} - \beta uf(y)x^\tau \end{aligned}$$

for all $x, y \in I_3$. So we get

$$\beta^2f(y)x^\tau - \big(\beta uf(y) + \beta f(yu)\big)x^\tau u + uf(yu)x^\tau u^2 = 0$$

for all $x, y \in I_3$. By [1, Theorem 2.5], $f(y) = 0$ for all $y \in I_3$, as desired. □

Remark 3.3. In Lemma 3.2, the case for $\deg \tau^2 = 1$ had been solved by Beidar and Martindale III (see [1]). However, the solution of (3.1) is still unknown when $\deg \tau^2 = 2$. If we can solve it, then the same result holds for $\deg \tau^2 = 2$ in Theorem 1.4.

Lemma 3.4. *Suppose* $\deg \tau^2 > 2$. *Then, for each* $\alpha \in C$, *there exists an nonzero ideal* I *of* R *such that* $\alpha I \subseteq R$ *and* $\delta(\alpha x) = \alpha\delta(x)$ *for all* $x \in I$.

Proof. Choose a nonzero ideal I_1 of R such that $\alpha I_1 \subseteq R$. Let $x, y \in I_1$. Then

$$\delta((\alpha x)y + y(\alpha x)) = \delta(\alpha x)y^\tau + y\delta(\alpha x) + \alpha\delta(y)x^\tau + \alpha x\delta(y)$$

and

$$\delta(x(\alpha y) + (\alpha y)x) = \alpha\delta(x)y^\tau + \alpha y\delta(x) + \delta(\alpha y)x^\tau + x\delta(\alpha y).$$

Comparing the two equations, we have

$$\big(\delta(\alpha y) - \alpha\delta(y)\big)x^\tau + x\big(\delta(\alpha y) - \alpha\delta(y)\big) = \big(\delta(\alpha x) - \alpha\delta(x)\big)y^\tau + y\big(\delta(\alpha x) - \alpha\delta(x)\big).$$

By Lemma 3.2, there is a nonzero ideal I of R contained in I_1 such that $\delta(\alpha x) = \alpha\delta(x)$ for all $x \in I$. $\square$

Lemma 3.5. *Suppose* $\deg \tau^2 > 2$. *Then every Jordan* τ*-derivation* δ *of* R *can be extended to a Jordan* τ*-derivation of* RC.

Applying Lemma 3.5, we are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. In view of [16, Theorem 2.9], we can assume that R is not a PI-ring. By [13, Theorem 1.2], we also assume that $\deg \tau^2 > 2$. Recall that if R is a prime GPI-ring and RC is a division ring, then R is a PI-ring. So RC is not a division ring. Let $\delta\colon R \to Q_{ms}(R)$ be a Jordan τ-derivation of R. Since the socle of RC is nonzero, $Q_{ms}(RC) = Q_s(RC)$. By Lemma 3.5, δ can be extended to a Jordan τ-derivation $\widetilde{\delta}\colon RC \to Q_s(RC)$ of RC. According to [5, Theorem 1.2], there exists $a \in Q_s(RC)$ such that $\widetilde{\delta}(x) = ax^\tau - xa$ for $x \in RC$. In particular, $\delta(x) = ax^\tau - xa$ for all $x \in R$. As a consequence of [2, Proposition 2.1.10], $Q_{ms}(R) = Q_{ms}(RC) = Q_s(RC)$ and so $a \in Q_{ms}(R)$. Hence δ is X-inner, as desired. $\square$

Proof of Lemma 3.5. Choose a subset $\{w_i\}_{i\in\Phi}$ of R which is a basis of RC over C, where Φ is a nonempty well-ordered set. Then any element of RC can be written as the form $\sum_{i\in\Phi} \alpha_i w_i$, where $\alpha_i = 0$ for all but finitely many $i \in \Phi$. Recall that $Q_{ms}(R) = Q_{ms}(RC)$. Define $\widetilde{\delta}\colon RC \to Q_{ms}(RC)$ by

$$\widetilde{\delta}\left(\sum_{i\in\Phi} \alpha_i w_i\right) = \sum_{i\in\Phi} \alpha_i\delta(w_i).$$

Then it is clearly a well-defined additive map since $\{w_i\}_{i\in\Phi}$ forms a basis of RC over C. We claim that $\widetilde{\delta}|_R = \delta$. Let $x \in R$. Write $x = \sum_{i\in\Phi} \alpha_i w_i$. By Lemma 3.4, there is a nonzero ideal I of R such that $\alpha_i I \subseteq R$ and $\delta(\alpha_i y) = \alpha_i \delta(y)$ for all $i \in \Phi$ and $y \in I$. Let $y \in I$. Then

$$\begin{aligned}
\delta(xy + yx) &= \sum_{i\in\Phi} \delta\big((\alpha_i y)w_i + w_i(\alpha_i y)\big) \\
&= \sum_{i\in\Phi} \delta(\alpha_i y)w_i^\tau + \delta(w_i)\alpha_i y^\tau + w_i\delta(\alpha_i y) + \alpha_i y\delta(w_i) \\
&= \delta(y)x^\tau + x\delta(y) + \sum_{i\in\Phi} \big(\alpha_i\delta(w_i)y^\tau + \alpha_i y\delta(w_i)\big).
\end{aligned}$$

Comparing this with (2.7), we have

$$\left(\delta(x) - \sum_{i\in\Phi} \alpha_i\delta(w_i)\right) y^\tau + y\left(\delta(x) - \sum_{i\in\Phi} \alpha_i\delta(w_i)\right) = 0$$

for all $y \in I$. By applying the same argument as the last part of the proof of Theorem 1.2, we have $\delta(x) = \sum_{i\in\Phi} \alpha_i\delta(w_i) = \widetilde{\delta}(x)$ and so the claim holds.

Finally we show that $\widetilde{\delta}$ is a Jordan τ-derivation. Let $x = \sum_{i\in\Phi} \alpha_i w_i \in RC$. For each $i, j \in \Phi$, write $w_i w_j = \sum_{k\in\Phi} \gamma_k^{ij} w_k \in R$, where $\gamma_k^{ij} \in C$. Then

$$\begin{aligned}
\widetilde{\delta}(x^2) &= \widetilde{\delta}\left(\sum_{i,j} \alpha_i\alpha_j w_i w_j\right) = \widetilde{\delta}\left(\sum_{i,j} \alpha_i\alpha_j \sum_{k\in\Phi} \gamma_k^{ij} w_k\right) = \sum_{i,j} \alpha_i\alpha_j \sum_{k\in\Phi} \gamma_k^{ij}\delta(w_k) \\
&= \sum_{i,j} \alpha_i\alpha_j\widetilde{\delta}(w_i w_j) = \sum_{i,j} \alpha_i\alpha_j\delta(w_i w_j) = \sum_{i} \alpha_i^2\delta(w_i^2) + \sum_{i<j} \alpha_i\alpha_j\delta(w_i w_j + w_j w_i) \\
&= \sum_{i} \alpha_i^2\big(w_i\delta(w_i) + \delta(w_i)w_i^\tau\big) + \sum_{i<j} \alpha_i\alpha_j\big(\delta(w_i)w_j^\tau + w_j\delta(w_i) + \delta(w_j)w_i^\tau + w_i\delta(w_j)\big) \\
&= x\widetilde{\delta}(x) + \widetilde{\delta}(x)x^\tau.
\end{aligned}$$

Hence the proof of Lemma 3.5 is complete. □

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