EFFICIENT CRITERIA FOR THE STABILIZATION OF PLANAR LINEAR SYSTEMS BY HYBRID FEEDBACK CONTROLS

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We suggest some criteria for the stabilization of planar linear systems via linear hybrid feedback controls. The results are formulated in terms of the input matrices. For instance, this enables us to work out an algorithm which is directly suitable for a computer realization. At the same time, this algorithm helps to check easily if a given linear 2×2 system can be stabilized (a) by a linear ordinary feedback control or (b) by a linear hybrid feedback control.

1. Introduction

Consider a linear control 2×2 system

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{1.1}$$

on $[0,\infty)$, where $x \in \mathbb{R}^2$ is the state variable of the system, $y \in \mathbb{R}^m$ is the output variable, $u \in \mathbb{R}^\ell$ is the control variable, and B and C are given real matrices of the sizes $2\times\ell$ and $m\times 2$, respectively.

If the pair (A,B) is controllable, or more generally, stabilizable, and $\operatorname{rank} C = 2$ (which describes the case of complete observability of the solutions), then it is always possible (see, e.g., [5, 6]) to achieve exponential stability of the zero solution to the control system (1.1) with an arbitrary matrix A. In such a case, there exists a linear ordinary feedback control of the form $u = Gy$ with an $\ell\times m$ matrix G, which yields exponential stability.

Similarly, if $\operatorname{rank} B = 2$ and the pair (A,C) is observable, or at least detectable, then again a suitable linear feedback control of the form $u = Gy$ solves the stabilization problem for system (1.1).

However, it is known that in practice, neither the condition $\operatorname{rank} B = 2$ nor the complete observability of the solutions (i.e., $\operatorname{rank} C = 2$) can be unavailable. The most interesting situation for applications is, therefore, the case when $\operatorname{rank} B = \operatorname{rank} C = 1$.

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A simple example of such a system is *the harmonic oscillator* with the external force as the control, where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{1.2}$$

Here, the displacement variable x_1 is available for measurements, while the controller can only change the velocity variable x_2 (we assume that $x = \left(\begin{smallmatrix} x_1 & x_2 \end{smallmatrix}\right)^\top \in \mathbb{R}^2$). This control system is both controllable and observable, but it cannot be stabilized by ordinary (even nonlinear and discontinuous) output feedback controls of the form $u = f(y)$ (see, e.g., [1]).

However, as it was shown by Artstein [1], there exists a *hybrid feedback control* which provides asymptotical stability of the zero solution to (1.1) with the matrices from (1.2).

A hybrid feedback control includes essentially two features (see Section 3 for the formal definitions): a discrete time controller (an automaton) attached to the given dynamical system (i.e., to (1.1) in our case) via the matrices B and C, and a switching algorithm describing when and how a control u should be changed. Artstein's example shows that such a hybrid feedback control may help even when the ordinary feedback fails to stabilize the system.

In [2, 3], the following result is obtained for B and C being nonzero matrices of rank 1: system (1.1) is stabilizable by a linear hybrid feedback control (LHFC) if and only if for at least one $\alpha \in \mathbb{R}$, the matrix $A + \alpha BC$ does not have nonnegative real eigenvalues. This result gives a necessary and sufficient stabilization condition, and it is straightforward that making use of hybrid feedback controls provides a better stabilization criterion compared to any one we can obtain exploiting ordinary feedback controls.

However, the shortcoming of this criterion is that it does not give any explicit description of how its assumptions can be verified in practice. In other words, it does not suggest any efficient, finite-step algorithm in terms of the given matrices (A,B,C), which would answer the question when system (1.1) admits a stabilizing feedback control.

In contrast to [2, 3], the present paper aims at

(1) finding verifiable criteria for LHFC stabilization of system (1.1),

(2) constructing efficient algorithms (which should also be "computer-friendly"), which can easily test a specific system (1.1) in terms of the input matrices (A,B,C) to find out whether the zero solution to (1.1) can be stabilized by an ordinary feedback linear control or by an LHFC.

2. Notations and relevant facts of control theory

We define by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ the sets of all natural, real, and complex numbers, respectively. The set $\mathbb{R}$ will in the sequel be naturally identified with $\{z \mid \operatorname{Im} z = 0\} \subset \mathbb{C}$. By $\langle \cdot, \cdot \rangle$ and $|\cdot|$ we mean the scalar product and the Euclidean norm in $\mathbb{R}^2$, respectively. We write $\operatorname{Span}\{b\}$ for the one-dimensional vector space containing a given vector $b \in \mathbb{R}^2$. We also put $\mathbb{C}_- := \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$, $\mathbb{C}_+ := \mathbb{C} \setminus \mathbb{C}_-$, and $\mathbb{C}_s^2 := \{(\lambda_1,\lambda_2) \in \mathbb{C}^2 \mid \lambda_1 = \bar{\lambda}_2\}$.

Let $M(\ell,m)$ denote the set of all real $\ell \times m$ matrices. Matrices will often be addressed as linear operators in the appropriate vector spaces. In the sequel, I and Θ will stand for the identity 2×2 matrix and the zero 2×2 matrix, respectively. Given a matrix $D \in M(2,2)$, we will denote its spectrum by $\sigma(D)$.

In what follows, we will consider system (1.1) for arbitrary but fixed matrices $A \in M(2,2)$, $B \in M(2,\ell)$, and $C \in M(m,2)$ ($\ell, m \in \mathbb{N}$). We also suppose that $\sigma(A) = \{\lambda_1, \lambda_2\}$. Moreover, if $\sigma(A) \subset \mathbb{R}$, then we suppose, without loss of generality, that $\lambda_1 \le \lambda_2$ (the case $\lambda_1 = \lambda_2$ is not excluded either).

The characteristic and the minimal polynomials of the matrix A will be denoted by $\pi_A(\lambda)$ and $p_A(\lambda)$, respectively. Clearly, $\pi_A(\lambda) = \lambda^2 - \operatorname{tr} A \cdot \lambda + \det A$. The decomposition $\mathbb{C} = \mathbb{C}_- \sqcup \mathbb{C}_+$ implies also a special factorization of the minimal polynomial $p_A = p_A^- p_A^+$, where the zeros of $p_A^-(\lambda)$ and $p_A^+(\lambda)$ belong to $\mathbb{C}_-$ and $\mathbb{C}_+$, respectively. The notation $\langle A|B\rangle$ is used for the controllability space of the pair (A,B), that is, $\langle A|B\rangle := B(\mathbb{R}^\ell) + AB(\mathbb{R}^\ell)$.

We recall some well-known facts (see, e.g., [5, 6]) from the theory of control linear systems, which are summarized in Definitions 2.1, 2.3, and 2.6 and Lemmas 2.2, 2.4, 2.5, 2.7, 2.8, and 2.9. Although some of the results are quite general, we will formulate them for the case of 2×2 systems, as it is the case of interest in this paper.

Definition 2.1. A matrix A is called stable if $\sigma(A) \subset \mathbb{C}_-$.

Lemma 2.2. *The following conditions are equivalent:*

(1) *A is stable;*
(2) $\operatorname{tr} A < 0$, $\det A > 0$*;*
(3) *the trivial solution to $\dot{x} = Ax$ is asymptotically stable.*

Definition 2.3. The pair (A,B) is controllable if $\langle A|B\rangle = \mathbb{R}^2$. The pair (A,C) is observable if the pair $(A^\top, C^\top)$ is controllable.

Lemma 2.4. (I) *The following conditions are equivalent:*

(1) *the pair (A,B) is controllable;*
(2) $\operatorname{rank}(B \ AB) = 2$*;*
(3) *for all $\Lambda \in \mathbb{C}_s^2$, there exists $F \in M(\ell,2)$ such that $\sigma(A+BF) = \Lambda$.*

(II) *The following conditions are equivalent:*

(1) *the pair (A,C) is observable;*
(2) $\operatorname{rank}\begin{pmatrix} C \\ CA \end{pmatrix} = 2$*;*
(3) *for all $\Lambda \in \mathbb{C}_s^2$, there exists $F \in M(2,m)$ such that $\sigma(A+FC) = \Lambda$.*

Lemma 2.5. *If* $\operatorname{rank} B \ge 2$, *then* (A,B) *is controllable, and if* $\operatorname{rank} C \ge 2$, *then* (A,C) *is observable.*

Definition 2.6. The pair (A,B) is called stabilizable if there exists $F \in M(\ell,2)$ such that the matrix $A + BF$ is stable.

The pair (A,C) is called detectable if there exists $F \in M(2,m)$ such that the matrix $A + FC$ is stable.

Lemma 2.7. *The pair (A,B) is stabilizable if and only if* $\ker p_A^+(A) \subset \langle A|B\rangle$.

Lemma 2.8. *The pair (A,C) is detectable if and only if $(A^\top, C^\top)$ is stabilizable.*

Lemma 2.9. *If the pair (A,B) is controllable, then (A,B) is stabilizable, and if the pair (A,C) is observable, then (A,C) is detectable.*

Remark 2.10. We point out that the converse to Lemma 2.9 is not true in general. Indeed, for the matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = (1\ 0)^\top$, the pair (A,B) is stabilizable but not controllable and the pair $(A,B^\top)$ is detectable but not observable.

3. Definitions of linear hybrid feedback controls and hybrid feedback stabilization

Definition 3.1. By a discrete automaton we mean in the sequal a 6-tuple $\Delta = (Q, I, \mathcal{M}, T, j, q_0)$, where

(i) Q is a finite set of all possible automaton states (locations);
(ii) the finite set I contains the input alphabet;
(iii) the transition map $\mathcal{M} : Q \times I \to Q$ indicates the location after a transition time, based on the previous location q and input $i \in I$ at the time of transition;
(iv) $T : Q \to (0,\infty)$ is a mapping which sets a period $T(q)$ between transitions times;
(v) $j : \mathbb{R}^m \to I$ is a function with property $j(\lambda y) = j(y)$, $y \in \mathbb{R}^m$, $\lambda > 0$;
(vi) $q_0 = q(0)$ is the state of the automaton at the initial time.

In [1, 4], a similar definition (without condition (v)) is considered. We add (v) to the standard requirements as we are going to use LHFCs in this particular paper (see Definition 3.2).

Intuitively, the automaton follows the output y and uses this information to determine switching times and the values of the new continuous piece of the control function.

For any automaton Δ satisfying (i)–(vi), we can iteratively define a special feedback operator F_Δ. Given $y : [0,\infty) \to \mathbb{R}^m$, the function $F_\Delta y : [0,\infty) \to Q$ is defined by the following:

(1) $(F_\Delta y)(0) = q_0$, $t_1 = T(q_0)$, $(F_\Delta y)(t) \equiv q_0$, $t \in [0,t_1)$;
(2) $(F_\Delta y)(t_1) = \mathcal{M}(q_0, j(y(t_1))) := q(t_1)$, $t_2 = t_1 + T(q(t_1))$, $(F_\Delta y)(t) \equiv q(t_1)$, $t \in [t_1,t_2)$;
(3) if $t_1,\dots,t_k$ and the values $(F_\Delta y)(t)$ for $t \in [0,t_k)$ are already known, then t_{k+1} and $(F_\Delta y)(t)$ are defined for $t \in [t_k,t_{k+1})$ by the equalities

$$\begin{gathered} (F_\Delta y)(t_k) = \mathcal{M}(q(t_{k-1}), j(y(t_k))) := q(t_k), \qquad t_{k+1} = t_k + T(q(t_k)), \\ (F_\Delta y)(t) \equiv q(t_k), \quad t \in [t_k, t_{k+1}). \end{gathered} \tag{3.1}$$

The sequence $\{t_k\}_{k=0}^\infty$ ($t_0 = 0$), constructed in the definition of $F_\Delta y$, determines when the automaton should switch between locations. Note that the sequence $\{t_k\}$ is allowed to depend on the output function $y(\cdot)$.

Definition 3.2. The pair $(\Delta, \{G_q\})$, where Δ is a discrete automaton and $\{G_q \mid q \in Q\} \subset M(\ell,m)$, will be addressed as an LHFC; dependence between the control function $u(\cdot)$ and the output function $y(\cdot)$ is defined by $u(t) = G_{q(t_k)} y(t)$, $t \in [t_k,t_{k+1})$ and $k = 0,1,\dots$, where $\{t_k\}_{k=0}^\infty$ is the corresponding sequence of the switching times.

The set of all LHFCs will in the sequel be denoted by $\mathcal{LH}$, *while* $u = (\Delta, \{G_q\}) \in \mathcal{LH}$ will stand for a specific control. According to Definition 3.2, system (1.1), governed by a control $u = (\Delta, \{G_q\}) \in \mathcal{LH}$ (in short, the u-governed system (1.1)), is equivalent to the nonlinear functional differential equation

$$\dot{x}(t) = \left(A + BG_{(F_\Delta Cx)(t)}C\right)x(t), \quad t \in [0, \infty). \tag{3.2}$$

The dynamics of system (1.1), governed by an LHFC u, is a triple $H(t) = (x(t), q(t), \tau(t))$, where $x(\cdot)$ is a solution to (1.1), $q(t)$ is the automaton's location at instance t, and $\tau(t)$ is the time remaining till the next transition instance (see [1]). The function $H(\cdot) : [0, \infty) \to \mathbb{R}^2 \times Q \times (0, \infty)$ is also called a *hybrid trajectory* of system (1.1).

Typical switching procedures (with examples) for systems with LHFC are described in [1, 4] in detail. In [4], some general properties of hybrid trajectories for linear and nonlinear finite-dimensional systems are discussed. In the same paper, one can find a review of the authors' results on some properties of the hybrid dynamics.

We mention here the main existence result from [4], which has a direct relevance to system (1.1) governed by a hybrid feedback control.

Lemma 3.3. *For any $u \in \mathcal{LH}$ and for any $\alpha \in \mathbb{R}^2$, there exists the unique hybrid trajectory $(x(\cdot), q(\cdot), \tau(\cdot))$ of the u-governed system (1.1) with the property $x(0) = \alpha$ (evidently, $x \equiv 0$ if $\alpha = 0$).*

In the sequel, we define by $\mathcal{LH}_1 \subset \mathcal{LH}$ the class of those LHFCs, for which Q contains only one point. Clearly, the class $\mathcal{LH}_1$ can naturally be identified with the class of ordinary linear feedback controls of the form $u = Gy$ with G being an appropriate matrix.

Definition 3.4 [1]. System (1.1) is said to be stabilizable by a control $u \in \mathcal{LH}$ (u-stab.) if the trivial solution to (1.1) is uniformly asymptotically stable. In other words,

(a) for any $\varepsilon > 0$, there is $\delta > 0$ such that every solution $x(\cdot)$ with the property $|x(0)| < \delta$ satisfies the estimate $|x(t)| < \varepsilon$ for $t \geq 0$;
(b) for every solution $x(\cdot)$, $|x(t)| \to 0$ as $t \to \infty$, the convergence being uniform with respect to initial points $x(0) \in K$ for any bounded $K \subset \mathbb{R}^2$.

Definition 3.5. Let $\mathcal{U} \subset \mathcal{LH}$. System (1.1) is called $\mathcal{U}$-stabilizable ($\mathcal{U}$-stab.) if there exists $u \in \mathcal{U}$ such that (1.1) is u-stab. A matrix triple (A_1, B_1, C_1) is called u-stab. or $\mathcal{U}$-stab. if the corresponding system (1.1) with $A = A_1$, $B = B_1$, and $C = C_1$ is u-stab. or $\mathcal{U}$-stab.

4. Some elementary and well-known facts about hybrid stabilization

Lemma 4.1. *Putting* $\operatorname{rank} B = \ell_1$ *and* $\operatorname{rank} C = m_1$, *let $B_1 \in M(2, \ell_1)$ be a matrix consisting of ℓ_1 linearly independent columns of the matrix B, and $C_1 \in M(m_1, 2)$ a matrix consisting of m_1 linearly independent rows of the matrix C. Then the following statements are valid.*

(1) *For all $G \in M(\ell, m)$, there exists the unique matrix $G_1 \in M(\ell_1, m_1)$ such that*

$$BGC = B_1 G_1 C_1. \tag{4.1}$$

Conversely, for all $G_1 \in M(\ell_1, m_1)$, there exists $G \in M(\ell, m)$ such that (4.1) is valid.

(2) *The triple* (A,B,C) *is* $\mathcal{LH}$*-stab. (resp.,* $\mathcal{LH}_1$*-stab.) if and only if* (A,B_1,C_1) *is* $\mathcal{LH}$*-stab. (resp.,* $\mathcal{LH}_1$*-stab.).*

The first statement of the lemma is just a simple exercise from the matrix algebra, while the second statement is a straightforward corollary from the first if one takes into account the definition of the classes $\mathcal{LH}$ and $\mathcal{LH}_1$ in Section 3.

Lemma 4.2. *The triple* (A,B,C) *is* $\mathcal{LH}_1$*-stab. if and only if there exists* $G \in M(\ell,m)$ *such that the matrix* $A+BGC$ *is stable.*

Corollary 4.3. *Let* $B = \begin{pmatrix} b_1 & b_2 \end{pmatrix}^\top \neq 0$ *and* $C = \begin{pmatrix} c_1 & c_2 \end{pmatrix} \neq 0$. *Then the triple* (A,B,C) *is* $\mathcal{LH}_1$*-stab. if and only if there exists* $\alpha \in \mathbb{R} : \sigma(A+\alpha BC) \subset \mathbb{C}_-$.

Corollary 4.4. *Assume that one of the following statements is valid:*

(1) *the pair* (A,B) *is stabilizable and* $\operatorname{rank} C = 2$,
(2) *the pair* (A,C) *is detectable and* $\operatorname{rank} B = 2$.

Then (A,B,C) *is* $\mathcal{LH}_1$*-stab.*

Proof. Suppose that the first statement is valid. By Lemma 4.1, one can then assume that $C \in M(2,2)$, $\det C \neq 0$. By Definition 2.6, there exists $F \in M(\ell,2)$ such that the matrix $A+BF$ is stable. Then $A+BGC$ is stable, where $G = FC^{-1}$. According to Lemma 4.2, (A,B,C) is $\mathcal{LH}_1$-stab. Case (2) can be treated similarly. □

Corollary 4.5. *If* $\operatorname{rank} B \geq 2$ *and* $\operatorname{rank} C \geq 2$, *then* (A,B,C) *is* $\mathcal{LH}_1$*-stab.*

In [2, 3], the following result is proved.

Theorem 4.6. *Let* $b = \begin{pmatrix} b_1 & b_2 \end{pmatrix}^\top \neq 0$ *and* $c = \begin{pmatrix} c_1 & c_2 \end{pmatrix} \neq 0$. *The triple* (A,B,C) *is* $\mathcal{LH}$*-stab. if and only if there exists* $\alpha \in \mathbb{R} : \sigma(A+\alpha BC) \cap [0,\infty) = \varnothing$ *(in other words,* $A+\alpha BC$ *does not have nonnegative real eigenvalues).*

The results of this section show that if we wish to construct an algorithm which would test whether a given triple (A,B,C) provides $\mathcal{LH}_1$-stabilizability or $\mathcal{LH}$-stabilizability of system (1.1), then we need to do the following:

(1) study the cases when the pair (A,B) is not stabilizable or the pair (A,C) is not detectable,
(2) find efficient algorithms for verifying the assumptions of Corollary 4.3 and Theorem 4.6.

5. The cases where (A,B) is not stabilizable and (A,C) is not detectable

Everywhere in Sections 5, 6, and 7, excluding Theorem 5.8, we assume that $B = \begin{pmatrix} b_1 & b_2 \end{pmatrix}^\top \neq 0$ and $C = \begin{pmatrix} c_1 & c_2 \end{pmatrix} \neq 0$.

Lemma 5.1. *The pair* (A,B) *is controllable if and only if for all* $\lambda \in \sigma(A) \cap \mathbb{R}$, $B \notin \ker(A - \lambda I)$.

Proof. By Lemma 2.4, the pair (A,B) is not controllable if and only if $\det\begin{pmatrix} B & AB \end{pmatrix} = 0$, which implies that $\lambda \in \sigma(A) \cap \mathbb{R}$, $B \in \ker(A - \lambda I)$. □

Corollary 5.2. *If* $\operatorname{Im}\lambda_i \neq 0$, *then* (A,B) *is controllable.*

Lemma 5.3. *For* $\lambda_1 \geq 0$, (A,B) *is controllable if and only if* (A,B) *is stabilizable.*

Proof. For $\lambda_1 \geq 0, [p_A^+(A) = p_A(A) = \Theta] \Rightarrow [\ker p_A^+(A) = \mathbb{R}^2]$. If (A,B) is stabilizable, then the latter implication and Lemma 2.7 give the relation $\langle A|B\rangle = \mathbb{R}^2$. According to Definition 2.3, the pair (A,B) is therefore controllable. The converse statement follows easily from Lemma 2.9. □

Lemma 5.4. *Let* $\lambda_1 < 0 \leq \lambda_2$. *Then the following statements are true:*

(1) $[(A,B)$ *is controllable*$] \Leftrightarrow [B \notin \ker(A-\lambda_i I)$, $i = 1,2]$;
(2) $[(A,B)$ *is not controllable, but stabilizable*$] \Leftrightarrow [B \in \ker(A-\lambda_2 I) \setminus \ker(A-\lambda_1 I)]$;
(3) $[(A,B)$ *is not stabilizable*$] \Leftrightarrow [B \in \ker(A-\lambda_1 I) \setminus \ker(A-\lambda_2 I)]$.

Proof. The first statement follows from Lemma 5.1. The case $B \in \ker(A-\lambda_i I)$, $i = 1,2$, is irrelevant. Indeed, under this assumption, we get $(\lambda_1 - \lambda_2)B = 0$, which contradicts the conditions $\lambda_1 \neq \lambda_2$ and $B \neq 0$.

Let $B \in \ker(A-\lambda_1 I) \triangle \ker(A-\lambda_2 I)$. Then $\det(B\ AB) = 0$, which implies that $\langle A|B\rangle = \operatorname{Span}\{B\}$. Taking into account that $p_A^+(\lambda) = \lambda - \lambda_2$ and using Lemma 2.7, we obtain that $[(A,B)$ is stabilizable $] \Leftrightarrow [\ker(A-\lambda_2 I) = \operatorname{Span}\{B\}] \Leftrightarrow [B \in \ker(A-\lambda_2 I)]$. □

Lemmas 2.4, 2.8, 5.1, 5.3, and 5.4 yield the following theorem.

Theorem 5.5. *For the matrices A, B, C from (1.1),* $[(A,B)$ *is not stabilizable* $] \Leftrightarrow [(\lambda_1 \geq 0, \det(B\ AB) = 0) \vee (\lambda_1 < 0 \leq \lambda_2,\ AB = \lambda_1 B)]$; $[(A,C)$ *is not detectable*$] \Leftrightarrow [(\lambda_1 \geq 0, \det\begin{pmatrix} C \\ CA\end{pmatrix} = 0) \vee (\lambda_1 < 0 \leq \lambda_2, CA = \lambda_1 C)]$.

Remark 5.6. The condition $\lambda_1 \geq 0$ is equivalent to $\operatorname{tr} A \geq 0$ and $\operatorname{tr}^2 A \geq 4\det A \geq 0$, and the condition $\lambda_1 < 0 \leq \lambda_2$ is equivalent to $[\det A < 0] \vee [\det A = 0$ and $\operatorname{tr} A < 0]$.

Lemma 5.7. *If the pair* (A,B) *is not controllable, then for all* $F = (f_1\ f_2)$, $\sigma(A+BF) = \{\lambda^*, \lambda_j\}$ *for some* $j \in \{1,2\}$, *where* $\lambda^* = \lambda_i + FB$, $i \neq j$; *in this case* $B \in \ker(A+BF-\lambda^* I)$.

Proof. By virtue of Lemma 5.4, $B \in \ker(A-\lambda_i I)$ for at least one $i \in \{1,2\}$. Let $\lambda^* = \lambda_i + FB$. Then $(A+BF-\lambda^* I)B = (A-\lambda_i I)B = 0$, that is, $\lambda^* \in \sigma(A+BF)$ and $B \in \ker(A+BF-\lambda^* I)$.

Clearly, the zeros λ_1 and λ_2 of the polynomial $\pi_A(\lambda)$ and the zeros λ^* and λ^{**} of the polynomial $\pi_{A+BF}(\lambda)$ are related to each other in the following way: $\lambda_1 + \lambda_2 = \operatorname{tr} A$ and $\lambda^* + \lambda^{**} = \operatorname{tr}(A+BF)$. These equalities imply that $\lambda^* + \lambda^{**} = \operatorname{tr} A + FB = \lambda_1 + \lambda_2 + FB = \lambda^* + \lambda_j$, where $j \neq i$. Thus, $\lambda^{**} = \lambda_j$. □

Theorem 5.8. *Let* $A \in M(2,2)$, $B \in M(2,\ell)$, $C \in M(m,2)$, *and either the pair* (A,B) *is not stabilizable or the pair* (A,C) *is not detectable. Then* (A,B,C) *is not* $\mathcal{LH}$*-stab.*

Proof. Assume that (A,B) is not stabilizable. By virtue of Lemmas 2.5 and 2.9, one has that $\operatorname{rank} B \leq 1$, and according to Lemma 4.1, one can assume, without loss of generality, that $B = (b_1\ b_2)^\top$. If $B = 0$, then the statement of the theorem is evident. Let $B \neq 0$. By Theorem 5.5, only one of the following cases can occur:

(1) $0 \leq \lambda_1 < \lambda_2$;
(2) $\lambda_1 < 0 \leq \lambda_2$;

(3) $\lambda := \lambda_1 = \lambda_2 \geq 0$, $\ker(A - \lambda I) = \mathbb{R}^2$;
(4) $\lambda := \lambda_1 = \lambda_2 \geq 0$, $\dim\ker(A - \lambda I) = 1$.

Using Lemma 5.4, one can easily show that in cases (2) and (3),

$$B \in \ker(A - \lambda_1 I), \qquad \ker(A - \lambda_2 I) \setminus \operatorname{Span}\{B\} \neq \varnothing. \tag{5.1}$$

In case (1), either (5.1) or its counterpart, where λ_1 and λ_2 are interchanged, is true. We assume, with no loss of generality, that (5.1) holds true in case (1).

We prove the theorem for cases (1), (2), and (3) simultaneously by choosing $D \in \mathbb{R}^2$ so that

$$B \in \ker(A - \lambda_1 I), \qquad D \in \ker(A - \lambda_2 I) \setminus \operatorname{Span}\{B\}. \tag{5.2}$$

Let $P : \mathbb{R}^2 \to \mathbb{R}$ be the projector onto the subspace $\operatorname{Span}\{D\}$ along the subspace $\operatorname{Span}\{B\}$ so that $PB = 0$, $PD = D$, $(I - P)B = B$, and $(I - P)D = 0$.

We choose an arbitrary $F \in M(1,2)$ and consider a trajectory $x(\cdot)$ of the equation

$$\dot{x} = (A + BF)x \tag{5.3}$$

such that $Px(t_0) \neq 0$ at some instance $t_0 \geq 0$.

Evidently,

$$\delta(t_0) := \frac{d|Px|}{dt}\bigg|_{t=t_0} = \frac{1}{|Px|}\langle Px, P\dot{x}\rangle\bigg|_{t=t_0} = \frac{1}{|Px|}\langle Px, P(A+BF)x\rangle\bigg|_{t=t_0}. \tag{5.4}$$

Conditions (5.2) imply that for some $\mu \in \mathbb{R}$,

$$\begin{aligned} P(A+BF)x &= PAx = PA\big(Px + (I-P)x\big) \\ &= PAPx + \mu PAB = \lambda_2 PX + \mu\lambda_1 PB = \lambda_2 PX. \end{aligned} \tag{5.5}$$

Then (5.4) and $\lambda_2 \geq 0$ yield

$$\delta(t_0) = \frac{1}{|Px|}\cdot \lambda_2 \cdot |Px|^2\bigg|_{t=t_0} = \lambda_2 \left| Px(t_0) \right| \geq 0. \tag{5.6}$$

The last relation can be used to verify the following properties:

$$\left| Px(\cdot) \right| \text{ does not decrease on } [t_0, \infty), \quad \left| x(t) \right| \geq \beta \left| Px(t_0) \right|, \quad t \geq t_0, \tag{5.7}$$

where $\beta > 0$ does not depend on F.

We now fix some $u = (\Delta, \{G_q\}) \in \mathcal{LH}$ and consider an arbitrary trajectory $(x(\cdot), q(\cdot), \tau(\cdot))$ of the u-governed system (1.1) with $\alpha := |Px(0)| \neq 0$. Let $\{t_n\}_{n=0}^{\infty}$ be the corresponding sequence of the automaton's switching instances, $t_0 = 0$. Then for all $[t_n, t_{n+1})$, $n \in \mathbb{N} \cup \{0\}$, the first component $x(\cdot)$ of the hybrid trajectory satisfies (5.3), where $F = G_q C$

for some $q \in Q$. Due to (5.7),

$$|x(t)| \geq \beta |Px(t_n)| \geq \beta |Px(0)| = \alpha\beta > 0, \quad t \in [t_n, t_{n+1}). \tag{5.8}$$

As n is arbitrary, $x(t) \not\to 0$, $t \to \infty$. Thus, the theorem is proved for cases (1), (2), and (3).

The proof of case (4) will be divided into two parts.

(a) If $CB \neq 0$, then there exists $G \in M(1,m)$ such that $GBC \neq 0$. By Lemma 5.7, $\sigma(A_G) = \{\lambda, \lambda^*\}$, where $A_G = A + BGC$ and $\lambda^* = \lambda + GCB \neq \lambda$. Clearly, the pair (A_G, B) is not stabilizable. It has already been proved that in case (1) or (2) the triple (A_G, B, C) is not $\mathcal{LH}$-stab. According to Definitions 3.2 and 3.5, the triple (A, B, C) is not $\mathcal{LH}$-stab. either.

(b) If $CB = 0$, then (see Lemma 5.7) for all $G \in M(1,m)$, one has $B \in \ker(A_G - \lambda I)$. Hence, the solution to $\dot{x} = A_G x$, $x(0) = B$ is given by $x(t) = e^{A_G t}B = e^{\lambda t}B$, $t \geq 0$. The last equality does not depend on G, so that it constitutes a solution to the u-governed system (1.1) satisfying $x(0) = B$ for any $u \in \mathcal{LH}$. Since $\lambda \geq 0$, system (1.1) is not $\mathcal{LH}$-stab.

The theorem can be proved in a similar manner if (A, C) is not detectable. □

6. Efficient criteria for $\mathcal{LH}_1$-stabilizability of the triple (A, B, C)

Everywhere in Sections 6 and 7, it is assumed that $A \in M(2,2)$, $B \in M(2,1) \neq 0$, and $C \in M(1,2) \neq 0$. In these two sections, we will use the following notation: $A_\alpha = A + \alpha BC$, $\alpha \in \mathbb{R}$, $\omega = \operatorname{tr} A - CAB/CB$ if $CB \neq 0$ (we assume also that ω is not defined if $CB = 0$).

Lemma 6.1. *Let $a, b, c, d \in \mathbb{R}$. The system of inequalities*

$$a + b \cdot \alpha < 0, \qquad c + d \cdot \alpha > 0 \tag{6.1}$$

is solvable with respect to $\alpha \in \mathbb{R}$ if and only if one of the following conditions holds:

(1) $b = d = 0$, $a < 0$, $c > 0$;
(2) $b = 0$, $d \neq 0$, $a < 0$;
(3) $d/b < 0$;
(4) $d/b \geq 0$, $c > ad/b$.

Theorem 6.2. *The triple (A, B, C) is $\mathcal{LH}_1$-stab. if and only if one of the following conditions holds:*

(1) $CB = CAB = 0$, $\operatorname{tr} A < 0$, $\det A > 0$;
(2) $CB = 0$, $CAB \neq 0$, $\operatorname{tr} A < 0$;
(3) $\omega < 0$;
(4) $\omega \geq 0$, $\det A > \operatorname{tr} A \cdot \omega$.

Proof. Corollary 4.3 and Lemma 2.2 imply that

$$[(A, B, C) \text{ is } \mathcal{LH}_1\text{-stab.}] \Longleftrightarrow [\exists \alpha \in \mathbb{R} : \operatorname{tr} A_\alpha < 0 \text{ and } \det A_\alpha > 0]. \tag{6.2}$$

Simple direct calculations yield

$$\operatorname{tr} A_\alpha = \operatorname{tr} A + \alpha CB, \qquad \det A_\alpha = \det A + \alpha(\operatorname{tr} A \cdot CB - CAB). \tag{6.3}$$

Thus, the conditions in the right-hand side of (6.2) are equivalent to the solvability of (6.1) with respect to $\alpha \in \mathbb{R}$, where $a = \operatorname{tr} A$, $b = CB$, $c = \det A$, and $d = \operatorname{tr} A \cdot CB - CAB$. Referring to Lemma 6.1 completes the proof. $\square$

Corollary 6.3. *Assume that the matrix A is not stable. Then the triple (A,B,C) is $\mathcal{L}\mathcal{H}_1$-stab. if and only if one of conditions (2), (3), and (4) of Theorem 6.2 is fulfilled.*

7. Efficient criteria for $\mathcal{L}\mathcal{H}$-stabilizability in terms of the triple (A,B,C)

The equality $\pi_A(\lambda) = \lambda^2 - \operatorname{tr} A \cdot \lambda + \det \lambda$ implies the following lemma.

Lemma 7.1. *For a 2×2 matrix A, $\sigma(A) \cap [0,\infty) = \varnothing$ if and only if one of the following conditions holds:*

(a) $\operatorname{tr} A < 0$, $\det A > 0$,
(b) $\operatorname{tr}^2 A - 4 \det A < 0$.

Lemma 7.1, Theorem 4.6, and relation (6.2) yield the following corollary.

Corollary 7.2. *For the matrices A,B,C from (1.1), $[(A,B,C)$ is $\mathcal{L}\mathcal{H}_1$-stab.$] \Leftrightarrow [$there exists $\alpha \in \mathbb{R} : \operatorname{tr} A_\alpha < 0,\ \det A_\alpha > 0]$; $[(A,B,C)$ is $\mathcal{L}\mathcal{H}$-stab.$] \Leftrightarrow [$there exists $\alpha \in \mathbb{R} : (\operatorname{tr} A_\alpha < 0$, $\det A_\alpha > 0) \vee (\operatorname{tr}^2 A_\alpha - 4 \det A_\alpha < 0)]$.*

The main result of the paper is the following criterion.

Theorem 7.3. *The triple (A,B,C) is $\mathcal{L}\mathcal{H}$-stab. if and only if one of the following conditions is fulfilled:*

(1) $CB = 0$, $CAB = 0$, $\operatorname{tr} A < 0$, $\det A > 0$;
(2) $CB = 0$, $CAB \neq 0$;
(3) $\omega < 0$;
(4) $\omega \geq 0$, $\det A > (CAB/CB) \cdot \omega$.

Proof. By (6.3), we have for all $\alpha \in \mathbb{R}$ that

$$f(\alpha) := \operatorname{tr}^2 A_\alpha - 4 \det A_\alpha = (CB)^2 \alpha^2 - 2(CB \cdot \operatorname{tr} A - 2CAB)\alpha + \operatorname{tr}^2 A - 4 \det A. \tag{7.1}$$

Consider the inequality

$$f(\alpha) < 0 \tag{7.2}$$

in some special situations.

(a) If $CB = 0$ and $CAB \neq 0$, then (7.2) is equivalent to the inequality

$$4CAB \cdot \alpha + \operatorname{tr}^2 A - 4 \det A < 0 \tag{7.3}$$

which is clearly solvable with respect to $\alpha \in \mathbb{R}$.

(b) If $CB \neq 0$, then the solvability of (7.2) is equivalent to the positivity of the discriminant $4d$ of the quadratic equation $f(\alpha) = 0$, that is, to the condition

$$d = 4(CAB)^2 - 4CAB \cdot \operatorname{tr} A \cdot CB + 4 \det A \cdot (CB)^2 > 0. \tag{7.4}$$

The last inequality, in turn, is equivalent to

$$\det A > \frac{CAB}{CB} \cdot \omega. \tag{7.5}$$

Now we are able to continue the proof of the theorem.

(1) Let $CB = CAB = 0$. Due to Theorem 4.6, Lemma 7.1, and relation (6.3), $\mathcal{LH}$-stabilizability of (A,B,C) is equivalent to the condition $\operatorname{tr} A < 0$, $\det A > 0$.

(2) Let $CB = 0$ and $CAB \neq 0$. Because of (a), the triple (A,B,C) is $\mathcal{LH}$-stab.

(3) Let $\omega < 0$. By Theorem 6.2, (A,B,C) is $\mathcal{LH}$-stab.

(4) Let $\omega \geq 0$. By Corollary 7.2, Theorem 6.2, and (b), $\mathcal{LH}$-stabilizability of the triple (A,B,C) is equivalent either to (7.5) or to

$$\det A \geq \operatorname{tr} A \cdot \omega. \tag{7.6}$$

Moreover, $\omega = \operatorname{tr} A - CAB/CB \geq 0$ guarantees the implication (7.6)⇒(7.5). Thus, $\mathcal{LH}$-stabilizability of (A,B,C) is equivalent to (7.5). □

The next two theorems follow directly from Theorems 6.2 and 7.3 and Corollary 7.2.

Theorem 7.4. *The triple (A,B,C) is not $\mathcal{LH}_1$-stab. but $\mathcal{LH}$-stab. if and only if one of the following conditions is satisfied:*

(1) $CB = 0$, $CAB \neq 0$, $\operatorname{tr} A \geq 0$;
(2) $\omega > 0$, $CAB/CB < \det A/\omega \leq \operatorname{tr} A$.

Theorem 7.5. *Assume that (A,B,C) is not $\mathcal{LH}_1$-stab. Then (A,B,C) is $\mathcal{LH}$-stab. if and only if one of the following conditions is satisfied:*

(1) $CB = 0$, $CAB \neq 0$;
(2) $\det A > CAB/CB \cdot \omega$.

8. A detailed algorithm which tests $\mathcal{LH}_1$- and $\mathcal{LH}$-stabilizability of the triple (A,B,C)

First of all, we introduce some new notation:

(i) $\mathbf{1} = \{(A,B,C) \mid A \in M(2,2), B \in M(2,\ell), C \in M(m,2)$ for some $\ell, m \in \mathbb{N}\}$,
(ii) $\mathbf{LH_0} = \{\Omega \in \mathbf{1} \mid A$ is stable$\}$,
(iii) $\mathbf{LH_1} = \{\Omega \in \mathbf{1} \mid A$ is not stable, Ω is $\mathcal{LH}_1$-stab.$\}$,
(iv) $\mathbf{LH} = \{\Omega \in \mathbf{1} \mid \Omega$ is not $\mathcal{LH}_1$-stab., but is $\mathcal{LH}$-stab.$\}$,
(v) $\mathbf{LH_-} = \{\Omega \in \mathbf{1} \mid \Omega$ is not $\mathcal{LH}$-stab.$\}$.

Evidently, $\mathbf{1} = \mathbf{LH_0} \sqcup \mathbf{LH_1} \sqcup \mathbf{LH} \sqcup \mathbf{LH_-}$.

Algorithm 8.1 tests if a given triple $\Omega = (A,B,C) \in \mathbf{1}$ belongs to one of the classes $\mathbf{LH_0}$, $\mathbf{LH_1}$, $\mathbf{LH}$, and $\mathbf{LH_-}$.

Remark 8.1. The items **1**(YES) and **3**(YES) of the algorithm follow from Lemma 2.2 and Corollary 4.5, respectively. The items **5**(YES) and **7**(YES) are implied by Theorems 5.5

1 Condition: $\operatorname{tr} A < 0$, $\det A > 0$.
(YES) $\longrightarrow \Omega \in \mathbf{LH_0} \longrightarrow$ END
(NO) $\longrightarrow$ CONTINUE
2 Condition: B or C is the zero matrix.
(YES) $\longrightarrow \Omega \in \mathbf{LH}_- \longrightarrow$ END
(NO) $\longrightarrow$ CONTINUE
3 Condition: $\operatorname{rank} B \geq 2$, $\operatorname{rank} C \geq 2$.
(YES) $\longrightarrow \Omega \in \mathbf{LH_1} \longrightarrow$ END
(NO) $\longrightarrow$ find $\sigma(A) = \{\lambda_1, \lambda_2\} \longrightarrow$ CONTINUE
4 Condition: $\operatorname{rank} B = 1$.
(YES) $\longrightarrow$ find a nonzero column b
of the matrix $B \longrightarrow$ CONTINUE
(NO) $\longrightarrow$ find a nonzero row c
of the matrix $C \longrightarrow$ go to **7**
5 Condition: $[\lambda_1 \geq 0, \lambda_2 \geq 0, \det(b\ Ab) = 0] \vee [\lambda_1 < 0 \leq \lambda_2, Ab = \lambda_1 b]$
$\vee [\lambda_2 < 0 \leq \lambda_1, Ab = \lambda_2 b]$.
(YES) $\longrightarrow \Omega \in \mathbf{LH}_- \longrightarrow$ END
(NO) $\longrightarrow$ CONTINUE
6 Condition: $\operatorname{rank} C \geq 2$.
(YES) $\longrightarrow \Omega \in \mathbf{LH_1} \longrightarrow$ END
(NO) $\longrightarrow$ find a nonzero row c
of the matrix $C \longrightarrow$ go to **8**
7 Condition: $\left[\lambda_1 \geq 0, \lambda_2 \geq 0, \det\begin{pmatrix} c \\ cA \end{pmatrix} = 0\right] \vee [\lambda_1 < 0 \leq \lambda_2, cA = \lambda_1 c]$
$\vee [\lambda_2 < 0 \leq \lambda_1, cA = \lambda_2 c]$.
(YES) $\longrightarrow \Omega \in \mathbf{LH}_- \longrightarrow$ END
(NO) $\longrightarrow \Omega \in \mathbf{LH_1} \longrightarrow$ END
8 Condition: $[cb = 0, cAb \neq 0, \operatorname{tr} A < 0] \vee [\operatorname{tr} A - cAb/cb < 0]$
$\vee [\operatorname{tr} A - cAb/cb \geq 0, \det A > \operatorname{tr} A(\operatorname{tr} A - cAb/cb)]$.
(YES) $\longrightarrow \Omega \in \mathbf{LH_1} \longrightarrow$ END
(NO) $\longrightarrow$ CONTINUE
9 Condition: $[cb = 0, cAb \neq 0] \vee [\det A > (cAb/cb)(\operatorname{tr} A - cAb/cb)]$.
(YES) $\longrightarrow \Omega \in \mathbf{LH} \longrightarrow$ END
(NO) $\longrightarrow \Omega \in \mathbf{LH}_- \longrightarrow$ END.

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and 5.8, and the items **6**(YES) and **7**(NO) can be obtained from Corollary 4.4. In the items **4**(YES) and **6**(NO), we use Lemma 4.1 to reduce the stabilization problem for the triple (A, B, C) to that for either (A, b, C) or (A, b, c), where b is a nonzero column of B and c is a nonzero row of C ($B = b$ and $C = c$ if $\ell = m = 1$). For **8**(YES), the statement follows from Corollary 6.3. Finally, **9** is implied by Theorem 7.5.

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