TIME-DEPENDENT STOKES EQUATIONS WITH MEASURE DATA

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We establish the existence of a unique solution of an initial boundary value problem for the nonstationary Stokes equations in a bounded fixed cylindrical domain with measure data. Feedback laws yield the source and its intensity from the partial measurements of the solution in a subdomain.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^3$ with a smooth boundary and consider the initial boundary value problem

$$\begin{gathered} \mathbf{u}' - \Delta\mathbf{u} + (\mathbf{w}\cdot\nabla)\mathbf{u} = \nabla p + g(t)\boldsymbol{\mu} \quad \text{in } \Omega\times(0,T), \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in } Q = \Omega\times(0,T), \\ \mathbf{u} = 0 \quad \text{on } \partial\Omega\times(0,T), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where $\mathbf{w}$ is a given solenoidal smooth vector function, $\boldsymbol{\mu} = (\mu_1,\mu_2,\mu_3)$ is a Radon measure of bounded variation on Ω, and $\mathbf{u}_0$ is in $L^1(\Omega)$.

Let $\boldsymbol{\chi}$ be a given function in $L^1(0,T;L^1(G))$ where G is an interior subset of Ω. We associate with (1.1) the cost function

$$J(g;\boldsymbol{\mu};\mathbf{u}_0;\tau) = \int_\tau^T\int_G |\mathbf{u}-\boldsymbol{\chi}|\,dx\,dt. \tag{1.2}$$

The purpose of this paper is twofold:

(1) to establish the existence of a unique weak solution of (1.1),

(2) to determine the source $\boldsymbol{\mu}$ and the intensity of the source $g(t)$ from the partial measurements $\boldsymbol{\chi}$ of the solution $\mathbf{u}$ in a fixed subdomain $G\times(0,T)$.

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Nonlinear elliptic boundary value problems with Radon measure data have been the subject of extensive investigations by Betta et al. [1], Boccardo and Gallouët [2], Boccardo et al. [3], and others. Parabolic initial boundary value problems in cylindrical domains with Radon measure data were studied by Boccardo and Gallouët [2], Brézis and Friedman [4]. In all the cited works, scalar-valued functions are involved. For vector-valued functions with a constraint imposed on the solution, the generic truncated function used in Boccardo-Gallouët treatment in [2] does not seem applicable. As is well known, for partial differential equations with measure data, the lower-order terms give rise to difficulties. In this paper, we consider the nonstationary Stokes equations with measure data, the case of the full-time-dependent Navier-Stokes equations is open. Feedback laws for distributed systems of parabolic initial boundary value problems, with data in $L^2(Q)$, were obtained by Popa in [5, 6] and for interacting controls by the author in [7].

The existence of a unique weak solution of (1.1) is established in Section 2. The value function associated with (1.1) and (1.2) is studied in Section 3. Feedback laws are established in Section 4. The results of this paper seem new.

2. Existence theorem

We denote by $M_b(\Omega)$ the set of all Radon measures of bounded variation on Ω, by $W^{1,p}(\Omega)$ the usual Sobolev spaces, and by $J_0^{1,p}(\Omega)$ the Banach space

$$J_0^{1,p}(\Omega) = \left\{\mathbf{u} : \mathbf{u} = (u_1, u_2, u_3) \in W_0^{1,p}(\Omega),\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega\right\}, \quad 1 < p < \infty, \tag{2.1}$$

with the obvious norm.

Throughout the paper, we assume $\mathbf{w}$ to be in $C(0,T;C(\Omega)) \cap L^2(0,T;J_0^{1,2}(\Omega))$. Let $\mathcal{G}$ be a compact convex subset of $L^2(0,T)$ and let $\mathcal{U}$ be a convex subset of $M_b(\Omega)$, compact in the vague topology, that is, $\|\boldsymbol{\mu}_n\|_{M_b(\Omega)} \leq C$, then there exists a subsequence such that $\boldsymbol{\mu}_{n_k} \to \boldsymbol{\mu}$ in the distribution sense in Ω and $\|\boldsymbol{\mu}_n\|_{M_b(\Omega)} \leq C$.

Set

$$\mathcal{E}(g;\boldsymbol{\mu};\mathbf{u}_0) = \left|\left|\mathbf{u}_0\right|\right|_{L^1(\Omega)} + \|g\|_{L^2(0,T)}\|\boldsymbol{\mu}\|_{M_b(\Omega)}. \tag{2.2}$$

Let

$$\{g, \mathbf{u}_0, \boldsymbol{\mu}, \mathbf{w}\} \in L^2(0,T) \times L^1(\Omega) \times M_b(\Omega) \times C(0,T;C(\Omega)) \cap L^2(0,T;J_0^{1,2}(\Omega)), \tag{2.3}$$

then there exists $\{g_n, \mathbf{u}_0^n, \mathbf{f}^n, \mathbf{w}^n\} \in C_0^\infty(0,T) \times C_0^\infty(\Omega) \times C^1(0,T;C_0^1(\Omega))$ with $\nabla\cdot\mathbf{w}^n = 0$ such that

$$\{g_n, \mathbf{u}_0^n, \mathbf{f}^n, \mathbf{w}^n\} \longrightarrow \{g, \mathbf{u}_0, \boldsymbol{\mu}, \mathbf{w}\} \tag{2.4}$$

in $L^2(0,T)\times L^1(\Omega)\times \mathcal{D}'(\Omega)\times C(0,T;C(\Omega))\cap L^2(0,T;J_0^{1,2}(\Omega))$ and

$$||\mathbf{u}_0^n||_{L^1(\Omega)} \le ||\mathbf{u}_0||_{L^1(\Omega)}, \qquad ||\mathbf{f}^n||_{L^1(\Omega)} \le \|\boldsymbol{\mu}\|_{M_b(\Omega)}, \qquad ||g_n||_{L^2(0,T)} \le \|g\|_{L^2(0,T)}. \tag{2.5}$$

Consider the initial boundary value problem

$$\begin{gathered} \mathbf{u}_n' - \Delta \mathbf{u}_n + (\mathbf{w}^n \cdot \nabla)\mathbf{u}_n = \nabla p + g_n(t)\mathbf{f}^n \quad \text{in } Q, \\ \nabla \cdot \mathbf{u}_n = 0 \quad \text{in } Q = \Omega \times (0,T), \\ \mathbf{u}_n(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ \mathbf{u}_n(x,0) = \mathbf{u}_0^n(x) \quad \text{in } \Omega. \end{gathered} \tag{2.6}$$

With the approximating system (2.6), we have the following known result.

Lemma 2.1. *Let*

$$\{g_n, \mathbf{f}_n, \mathbf{u}_0^n, \mathbf{w}^n\} \in C_0^\infty(0,T) \times \left(C_0^\infty(\Omega)\right)^2 \times C^1\left(0,T;C_0^1(\Omega)\right), \qquad \nabla \cdot \mathbf{w}^n = 0. \tag{2.7}$$

Then there exists a unique solution $\mathbf{u}_n$ *of (2.2) in* $C^1(0,T;C(\Omega))$ *with* $\mathbf{u}_n \in L^2(0,T;$ $J_0^{1,2}(\Omega)\cap W^{3,2}(\Omega))$ *and* $\mathbf{u}_n' \in L^2(0,T;L^2(\Omega))$.

Since $\mathbf{u}_n \in C(Q)$, $\sup_{(x,t)\in Q}|u_{j,n}(x,t)| = \alpha_{j,n}$ exists. Denote

$$\alpha_n = \inf_{1\le j\le 3} \alpha_{j,n}. \tag{2.8}$$

The following decompositions of $L^2(\Omega)$ and $W_0^{1,2}(\Omega)$ are known:

$$L^2(\Omega) = J_0(\Omega) \oplus G(\Omega), \qquad W_0^{1,2}(\Omega) = J_0^{1,2}(\Omega) \oplus \left(J_0^{1,2}(\Omega)\right)^{\perp}. \tag{2.9}$$

Thus, for $\mathbf{v} \in W_0^{1,2}(\Omega)$, we get

$$\mathbf{v} = \tilde{\mathbf{v}} + \hat{\mathbf{v}}, \quad \tilde{\mathbf{v}} \in J_0^{1,2}(\Omega),\ \hat{\mathbf{v}} \in \left(J_0^{1,2}(\Omega)\right)^{\perp}. \tag{2.10}$$

We deduce from the unique decomposition of $L^2(\Omega)$ into $J_0(\Omega)m$ and $G(\Omega)$ that

$$\mathbf{v} = \tilde{\mathbf{v}} + \nabla q, \quad \tilde{\mathbf{v}} \in J_0^{1,2}(\Omega), \qquad \hat{\mathbf{v}} = \nabla q \in W_0^{1,2}(\Omega). \tag{2.11}$$

Lemma 2.2. *Let* $\{g_n, \mathbf{f}_n, \mathbf{u}_0^n, \mathbf{u}_n\}$ *be as in Lemma 2.1. Then*

$$||\mathbf{u}_n||_{L^\infty(0,T;L^1(\Omega))} + ||\nabla \mathbf{u}_n||^2_{L^2(D_n^1)} \le C\big(1 + |\Omega| + \mathcal{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\big) \tag{2.12}$$

with

$$D_n^1 = \bigcap_{j=1}^{3} \{(x,t) : (x,t) \in Q;\ |u_{n,j}(x,t)| \leq \inf\{1, \alpha_n\}\}. \tag{2.13}$$

The constant C is independent of n.

Proof. (1) Let

$$\varphi = \begin{cases} \inf\{\alpha_n, 1\}, & \text{if } \inf\{\alpha_n, 1\} < s, \\ s, & \text{if } -\inf\{\alpha_n, 1\} \leq s \leq \inf\{\alpha_n, 1\}, \\ -\inf\{\alpha_n, 1\}, & \text{if } s < -\inf\{\alpha_n, 1\}. \end{cases} \tag{2.14}$$

With $\mathbf{u}_n \in C^1(0,T;C(\Omega))$ and with $\nabla \cdot \mathbf{u}_n = 0$ in Q, we now consider the testing vector function $\boldsymbol{\varphi}(\mathbf{u}_n) = (\varphi(u_{n,1}),\dots,\varphi(u_{n,3}))$. It is clear that

$$D_n^1 \subset \bigcap_{j=1}^{3} \operatorname{supp}(u_{n,j}) \tag{2.15}$$

is nonempty and $\partial\Omega \times (0,T) \subset \partial D_n^1$. For each $t \in [0,T]$, we have $\boldsymbol{\varphi}\mathbf{u}_n(\cdot,t) \in W_0^{1,2}(\Omega)$ and using the decomposition (2.11), we write

$$\boldsymbol{\varphi}(\mathbf{u}_n(\cdot,t)) = \tilde{\boldsymbol{\varphi}}(\mathbf{u}_n(\cdot,t)) + \nabla q(\cdot,t) \tag{2.16}$$

with

$$\tilde{\boldsymbol{\varphi}}(\mathbf{u}(\cdot,t)) \in J_0^{1,2}(\Omega), \quad \nabla q \in W_0^{1,2}(\Omega), \quad \forall t \in [0,T]. \tag{2.17}$$

It is clear that $\boldsymbol{\varphi}$ maps $L^2(\Omega)$ into $J_0(\Omega)$ and $W_0^{1,2}(\Omega)$ into $J_0^{1,2}(\Omega)$. Since

$$||\boldsymbol{\varphi}(\mathbf{u}_n(\cdot,t))||_{L^\infty(\Omega)} \leq 1, \tag{2.18}$$

we obtain, by using the decomposition (2.11),

$$\begin{aligned} ||\tilde{\boldsymbol{\varphi}}(\mathbf{u}_n(\cdot,t))||_{L^\infty(\Omega)} &\leq \inf_{\|\mathbf{h}\|_{L^1(\Omega)} \leq 1;\ \nabla\cdot\mathbf{h}=0} |(\tilde{\boldsymbol{\varphi}}(\mathbf{u}_n(\cdot,t)),\mathbf{h})| \\ &\leq \inf_{\|\mathbf{h}\|_{L^1(\Omega)} \leq 1;\ \nabla\cdot\mathbf{h}=0} |(\boldsymbol{\varphi}(\mathbf{u}_n(\cdot,t)),\mathbf{h})| \\ &\leq ||\boldsymbol{\varphi}(\mathbf{u}_n(\cdot,t))||_{L^\infty(\Omega)}, \quad \forall t \in [0,T]. \end{aligned} \tag{2.19}$$

Hence,

$$||\tilde{\boldsymbol{\varphi}}(\mathbf{u}_n)||_{L^\infty(0,T;L^\infty(\Omega))} \leq 1. \tag{2.20}$$

We have

$$
\begin{aligned}
(\Delta\mathbf{u}_n(\cdot,t),\tilde{\boldsymbol{\varphi}}(\mathbf{u}_n(\cdot,t))) &= (\Delta\mathbf{u}_n(\cdot,t),\boldsymbol{\varphi}(\mathbf{u}_n(\cdot,t))-\nabla q)\\
&= (\Delta\mathbf{u}_n(\cdot,t),\boldsymbol{\varphi}(\mathbf{u}_n(\cdot,t)))+\sum_{j,k=1}^{3}(D_ju_{n,k},D_jD_kq)\\
&= (\Delta\mathbf{u}_n(\cdot,t),\boldsymbol{\varphi}(\mathbf{u}_n(\cdot,t)))-\sum_{j,k=1}^{3}(D_ku_{n,k},\Delta q)\\
&= (\Delta\mathbf{u}_n(\cdot,t),\boldsymbol{\varphi}(\mathbf{u}_n(\cdot,t)))
\end{aligned}
\tag{2.21}
$$

as

$$
\begin{gathered}
\{\mathbf{u}_n(\cdot,t),\nabla q\}\in J_0^{1,2}(\Omega)\cap W^{3,2}(\Omega)\times W_0^{1,2}(\Omega),\\
\nabla\cdot(\Delta\mathbf{u}_n(\cdot,t))=0\quad \text{for } t\in[0,T].
\end{gathered}
\tag{2.22}
$$

Since

$$
\begin{aligned}
\|D_j\{\tilde{\boldsymbol{\varphi}}(\mathbf{v})\}\|_{L^2(\Omega)} &\le \|D_j\{\boldsymbol{\varphi}(\mathbf{v})\}\|_{L^2(\Omega)}\\
&\le \|\boldsymbol{\varphi}'(\mathbf{v})\|_{L^\infty(\Omega)}\|D_j\mathbf{v}\|_{L^2(\Omega)}\\
&\le \|D_j\mathbf{v}\|_{L^2(\Omega)}\quad \forall\mathbf{v}\in W_0^{1,2}(\Omega),
\end{aligned}
\tag{2.23}
$$

we deduce that

$$
\|\tilde{\boldsymbol{\varphi}}'(\mathbf{v})\|_{L^\infty(\Omega)}\le 1\quad \forall\mathbf{v}\in W_0^{1,2}(\Omega).
\tag{2.24}
$$

With $\mathbf{u}_n\in C^1(0,T;C(\Omega))$, $\nabla\cdot\mathbf{u}_n=0$, and thus $\mathbf{u}_n'\in L^2(0,T;J_0(\Omega))$, we get

$$
\begin{aligned}
(\mathbf{u}_n',\tilde{\boldsymbol{\varphi}}(\mathbf{u}_n(\cdot,t))) &= (\mathbf{u}_n',\boldsymbol{\varphi}(\mathbf{u}_n(\cdot,t)))-(\mathbf{u}_n',\nabla q)\\
&= (\mathbf{u}_n',\boldsymbol{\varphi}(\mathbf{u}_n(\cdot,t))).
\end{aligned}
\tag{2.25}
$$

(2) Set

$$
\phi(s)=\int_0^s\boldsymbol{\varphi}(r)\,dr.
\tag{2.26}
$$

Multiplying (2.6) by $\tilde{\boldsymbol{\varphi}}(\mathbf{u}_n)$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\phi(\mathbf{u}_n(x,t))\,dx+\sum_{j,m=1}^{3}\int_\Omega\varphi'(u_{n,j})\,|D_mu_{n,j}(x,t)|^2dx\\
&\qquad+\sum_{j,k=1}^{3}\int_\Omega w_j^nD_ju_{n,k}\tilde{\varphi}(u_{n,k})(x,t)\,dx\\
&=\int_\Omega g_n(t)f_{n,k}\tilde{\varphi}(u_{n,k})\,dx.
\end{aligned}
\tag{2.27}
$$

Since $w^n \in J_0^{1,2}(\Omega)$, we get by integration by parts that

$$\begin{aligned}
&\sum_{j,k=1}^{3} \int_\Omega w_j^n D_j u_{n,k} \tilde{\varphi}(u_{n,k})\,dx \\
&\quad = -\sum_{j,k=1}^{3} \int_\Omega w_j^n u_{n,k} \tilde{\varphi}'(u_{n,k}) D_j u_{n,k}(x,t)\,dx \\
&\quad = -\sum_{j,k=1}^{3} \int_\Omega w_j(x,t) D_j \left\{ \int_0^{u_{n,k}(x,t)} s\tilde{\varphi}'(s)\,ds \right\} dx \\
&\quad = \sum_{k=1}^{3} \int_\Omega \operatorname{div}(\mathbf{w}^n) \left\{ \int_0^{u_{n,k}(x,t)} s\tilde{\varphi}'(s)\,ds \right\} dx = 0.
\end{aligned} \tag{2.28}$$

It follows that

$$\frac{d}{dt} \int_\Omega \phi(\mathbf{u}_n(x,t))\,dx \le \int_\Omega g_n \mathbf{f}_n \tilde{\boldsymbol{\varphi}}(\mathbf{u}_n(x,t))\,dx. \tag{2.29}$$

Hence,

$$\begin{aligned}
\int_\Omega |\mathbf{u}_n(x,t)|\,dx &\le C\left\{1 + |\Omega| + \int_\Omega \phi(\mathbf{u}_n(x,t))\,dx\right\} \\
&\le C\left\{|\Omega| + \int_0^t \int_\Omega \phi(\mathbf{u}_0(x))\,dx\right\} + \mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0) \\
&\le C\{1 + |\Omega| + \mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\}.
\end{aligned} \tag{2.30}$$

The constant C is independent of n and of the data $(g;\boldsymbol{\mu};\mathbf{u}_0)$. The second assertion of the lemma is now obvious. □

Lemma 2.3. *Suppose all the hypotheses of Lemma 2.2 are satisfied. Then*

$$\|\mathbf{u}_n\|_{L^p(0,T;W_0^{1,r}(\Omega))} \le C\{1 + |\Omega| + \mathscr{E}(g;\mathbf{f};\mathbf{u}_0)\} \tag{2.31}$$

with $r < 1 < 5/4$. The constant C is independent of n and of the data $(g;\boldsymbol{\mu};\mathbf{u}_0)$.

Proof. (1) Let ψ_k be the function

$$\psi_k(s) = \begin{cases} 1, & \text{if } 1+k < s, \\ s-k, & \text{if } k \le s \le 1+k, \\ 0, & \text{if } -k \le s \le k, \\ s+k, & \text{if } -1-k \le s \le -k, \\ -1, & \text{if } s \le -1-k. \end{cases} \tag{2.32}$$

(1) Let k be a positive integer with

$$\begin{aligned} \alpha_n &= \inf_j \left\{ \sup_Q |u_{n,j}(x,t)| \right\} > k, \\ k \le K_n &= \sup_j \left\{ \sup_Q |u_{n,j}| \right\}. \end{aligned} \tag{2.33}$$

Let

$$B_k = \bigcap_{j=1}^{3} \{(x,t) : (x,t) \in Q;\ k \le |u_{n,j}(x,t)| \le 1+k\}, \tag{2.34}$$

then

$$B_k \cap \left\{ \bigcap_j \operatorname{supp}(u_{n,j}) \right\} \ne \emptyset. \tag{2.35}$$

With $\boldsymbol{\psi}_k(\mathbf{u}_n(\cdot,t)) \in W_0^{1,2}(\Omega)$ for $t \in [0,T]$, we use the decomposition (2.11), and as in the first part of the proof of Lemma 2.2, we have

$$\boldsymbol{\psi}_k(\mathbf{u}_n(\cdot,t)) = \tilde{\boldsymbol{\psi}}_k(\mathbf{u}_n(\cdot,t)) + \nabla q \tag{2.36}$$

with

$$\nabla q \in \left(J_0^{1,2}\right)^{\perp}, \qquad \tilde{\boldsymbol{\psi}}_k(\mathbf{u}_n(\cdot,t)) \in J_0^{1,2}(\Omega) \quad \forall t \in [0,T]. \tag{2.37}$$

Moreover,

$$\|\tilde{\boldsymbol{\psi}}_k(\mathbf{u}_n)\|_{L^\infty(0,T;L^\infty(\Omega))} \le k, \qquad \|\tilde{\boldsymbol{\psi}}'_k(\mathbf{u}_n)\|_{L^\infty(0,T;L^\infty(\Omega))} \le 1. \tag{2.38}$$

As in Lemma 2.2, we have

$$(\mathbf{u}'_n, \boldsymbol{\psi}_k(\mathbf{u}_n)) = (\mathbf{u}'_n, \tilde{\boldsymbol{\psi}}_k(\mathbf{u}_n)), \qquad (\Delta\mathbf{u}_n, \boldsymbol{\psi}_k(\mathbf{u}_n)) = (\Delta\mathbf{u}_n, \tilde{\boldsymbol{\psi}}_k(\mathbf{u}_n)). \tag{2.39}$$

Taking the pairing of (2.6) with $\tilde{\boldsymbol{\psi}}_k(\mathbf{u}_n(x,t))$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \boldsymbol{\Phi}_k(\mathbf{u}_n(x,t))\,dx + \sum_{j,m=1}^{3} \int_\Omega \psi'_k(u_{n,j}(x,t))\,|D_m u_{n,j}(x,t)|^2\,dx \\ &\qquad + \sum_{j,m=1}^{3} \int_\Omega w_j^n \cdot D_j u_{n,m} \tilde{\psi}_k(u_{n,m}(x,t))\,dx \\ &\quad \le C\{1 + \mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\}, \end{aligned} \tag{2.40}$$

where

$$\Phi_k(s) = \int_0^s \psi_k(\sigma)d\sigma. \tag{2.41}$$

Since $\mathbf{w}^n \in J_0^{1,2}(\Omega)$, we get by integration by parts that

$$\begin{aligned}
&\sum_{j,m=1}^{3} \int_\Omega w_j^n D_j u_{n,m} \tilde{\psi}_k(u_{n,m}(x,t))dx \\
&\quad = -\sum_{j,m=1}^{3} \int_\Omega w_j^n u_{n,m}(\tilde{\psi}_k)'(u_{n,m}) D_j u_{n,m} dx \\
&\quad = -\sum_{j,m=1}^{3} \int_\Omega w_j^n D_j \left\{ \int_0^{u_{n,m}(x,t)} s(\tilde{\psi}_k)'(s)ds \right\} dx \\
&\quad = \sum_{m=1}^{3} \int_\Omega \operatorname{div}(\mathbf{w}^n) \left\{ \int_0^{u_{n,m}(x,t)} s(\tilde{\psi}_k)'(s)ds \right\} dx = 0.
\end{aligned} \tag{2.42}$$

It follows that

$$\begin{aligned}
&\int_\Omega \mathbf{\Phi}_k(\mathbf{u}_n(x,t))dx \\
&\qquad + \sum_{j,m=1}^{3} \int_\Omega (\psi_k)'(u_{n,m}(x,t)) \left| D_j u_{n,m}(x,t) \right|^2 dx \le C\{1 + |\Omega| + \mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\}.
\end{aligned} \tag{2.43}$$

Therefore,

$$\int_{B_k} \left| \nabla \mathbf{u}_n(x,t) \right|^2 dx\,dt \le C\{1 + |\Omega| + \mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\}. \tag{2.44}$$

Let $r \in (1, 5/4)$, then an application of the Hölder inequality gives

$$\begin{aligned}
k\left| B_k \right| &\le \int_{B_k} \left| u_{n,j}(x,t) \right| dx\,dt \\
&\le \|\mathbf{u}_n\|_{L^{4r/3}(B_k)} \left| B_k \right|^{(4r-3)/4r}.
\end{aligned} \tag{2.45}$$

Therefore,

$$\left| B_k \right| \le k^{-4r/3} \|\mathbf{u}_n\|_{L^{4r/3}(B_k)}^{4r/3}. \tag{2.46}$$

An application of the Hölder inequality, together with (2.45) and (2.46), yields

$$\begin{aligned}
\|\nabla \mathbf{u}_n\|_{L^r(B_k)}^r &\le \left| B_k \right|^{(2-r)/2} \|\nabla \mathbf{u}_n\|_{L^2(B_k)}^r \\
&\le C k^{-2r(2-r)/3} \|\mathbf{u}_n\|_{L^{4r/3}(B_k)}^{2r(2-r)/3} \{1 + |\Omega| + \mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\}.
\end{aligned} \tag{2.47}$$

(2) We now consider the case

$$\alpha_n = \inf_j \left\{ \sup_Q |u_{n,j}(x,t)| \right\} < k. \tag{2.48}$$

Since $\alpha_n = \sup_Q |u_{n,s}| < k$,

$$\operatorname{supp}(u_{n,s}) \cap \{(x,t) : k \leq |u_{n,s}(x,t)| \leq k+1\} = \emptyset. \tag{2.49}$$

We have two subcases:

(i) $\inf_{j\neq s}\{\sup_Q |u_{n,j}(x,t)|\} > k$ and we treat exactly as before,
(ii) $\inf_{j\neq s}\{\sup_Q |u_{n,j}|\} = \sup_Q |u_{n,m}| < k$.

As above, we only have to consider the case

$$\sup_Q |u_{n,r}| > k, \quad r \neq m, s, \tag{2.50}$$

and again, as in the first part, the same argument carries over.

Repeating the argument done before, we obtain

$$\int_{B_k} |\nabla \mathbf{u}_n(x,t)|^2 dx\,dt \leq C\{1 + |\Omega| + \mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\}. \tag{2.51}$$

As before, let $1 < r < 5/4$, and an identical argument as in the first part yields

$$\begin{aligned} \|\nabla \mathbf{u}_n\|^r_{L^r(B_k)} &\leq \|\nabla \mathbf{u}_n\|^r_{L^2(B_k)} |B_k|^{(2-r)/2} \\ &\leq Ck^{-2r(2-r)/3}\|\mathbf{u}_n\|^{2r(2-r)/3}_{L^{4r/3}(B_k)}\{1 + |\Omega| + \mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\}. \end{aligned} \tag{2.52}$$

Combining the two cases, we get from (2.47) and (2.52),

$$\|\nabla \mathbf{u}_n\|^r_{L^r(B_k)} \leq Ck^{-2r(2-r)/3}\|\mathbf{u}_n\|^{2r(2-r)/3}_{L^{4r/3}(B_k)}(|\Omega| + \mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)). \tag{2.53}$$

The constant C is independent of n, k and of g, $\boldsymbol{\mu}$, $\mathbf{u}_0$.

(2) Let k_0 be a fixed positive number and let φ_n be the truncated function

$$\varphi_n(s) = \begin{cases} \inf\{k_0, \alpha_n\}, & \text{if } s > \inf\{k_0, \alpha_n\}, \\ s, & \text{if } -\inf\{k_0, \alpha_n\} \leq s \leq \inf\{k_0, \alpha_n\}, \\ -\inf\{k_0, \alpha_n\}, & \text{if } s < -\inf\{k_0, \alpha_n\}. \end{cases} \tag{2.54}$$

Then as in Lemma 2.2, we have

$$\int_{D_n^{k_0}} |\nabla \mathbf{u}_n|^2 dx\,dt \leq C\{k_0 + |\Omega| + \mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\}. \tag{2.55}$$

Now a proof as in that of Lemma 2.2 gives

$$\begin{aligned}\int_{D_n^{k_0}} |\nabla \mathbf{u}_n|^r\,dx\,dt &\le |\Omega|^{(2-r)/2}\|\nabla \mathbf{u}_n\|_{L^2(D_n^{k_0})}^{r/2}|\Omega|^{(2-r)/2}\\ &\le C\{k_0+|\Omega|+\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\},\end{aligned}\tag{2.56}$$

where $D_n^{k_0}$ is as in Lemma 2.2.

(3) Combining (2.53) and (2.56), we obtain

$$\begin{aligned}\|\nabla \mathbf{u}_n\|_{L^r(0,T;L^r(\Omega))}^r &\le C(k_0+1+|\Omega|+\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0))\\ &\quad + C(1+|\Omega|+\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0))\\ &\quad \times \sum_{k=k_0}^{K_n}\|\mathbf{u}_n\|_{L^{4r/3}(B_k)}^{2r(2-r)/3}k^{-(2-r)2r/3}.\end{aligned}\tag{2.57}$$

Applying the Hölder inequality to (2.57), we get

$$\begin{aligned}\|\nabla \mathbf{u}_n\|_{L^r(0,T;L^r(\Omega))}^r &\le C\{1+|\Omega|+\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\}+C\{1+|\Omega|+\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\}\\ &\quad\times\|\mathbf{u}_n\|_{L^{4r/3}(0,T;L^{4r/3}(\Omega))}^{4r/3}\left\{\sum_{k=k_0}^{K_n}k^{-4(2-r)/3}\right\}^{r/2}\\ &\le C\{1+|\Omega|+\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\}\|\mathbf{u}_n\|_{L^{4r/3}(0,T;L^{4r/3}(\Omega))}^{4r/3}\\ &\quad\times\left\{\sum_{k=k_0}^{\infty}k^{-4(2-r)/3}\right\}^{r/2}.\end{aligned}\tag{2.58}$$

The series converges since $1<r<5/4$.

(4) Again, the Hölder inequality gives

$$\|\mathbf{u}_n\|_{L^{4r/3}(\Omega)} \le \|\mathbf{u}_n\|_{L^1(\Omega)}^{1/4}\|\mathbf{u}_n\|_{L^{3r/(3-r)}(\Omega)}^{3/4}.\tag{2.59}$$

With the estimate of Lemma 2.2, we obtain

$$\begin{aligned}\|\mathbf{u}_n\|_{L^{4r/3}(0,T;L^{4r/3}(\Omega))}^{2r(2-r)/3} &\le C\{1+|\Omega|+\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\}\\ &\quad + C\{1+|\Omega|+\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\}\\ &\quad \times\|\mathbf{u}_n\|_{L^r(0,T;L^{3r/(3-r)}(\Omega))}^{r(2-r)/2}.\end{aligned}\tag{2.60}$$

(5) The Sobolev imbedding theorem yields

$$\|\mathbf{u}_n\|_{L^r(0,T;L^{3r/(3-r)}(\Omega))}^r \le C\|\mathbf{u}_n\|_{L^r(0,T;W_0^{1,r}(\Omega))}^r.\tag{2.61}$$

It follows from (2.57), (2.59), and (2.60) that

$$\begin{aligned}
\|\mathbf{u}_n\|^r_{L^r(0,T;L^{3r/(3-r)}(\Omega))} &\le C\left(1+|\Omega|+\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\right) \\
&\quad + C\left\{1+|\Omega|+\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\right\} \\
&\quad \times \|\mathbf{u}_n\|^{4r/3}_{L^{4r/3}(0,T;L^{4r/3}(\Omega))}\left\{\sum_{k=k_0} k^{-4(2-r)/3}\right\}^{r/2} \\
&\le C\left\{1+|\Omega|+\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\right\} \\
&\quad + C\|\mathbf{u}_n\|^r_{L^r(0,T;L^{3r/(3-r)}(\Omega))}\left\{\sum_{k=k_0} k^{-4(2-r)/3}\right\}^{r/2}.
\end{aligned} \tag{2.62}$$

Since $1<r<5/4$, the series converges and there exists k_0 such that

$$\|\mathbf{u}_n\|^r_{L^r(0,T;L^{3r/(3-r)}(\Omega))} \le C\left(1+|\Omega|+\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\right). \tag{2.63}$$

Using the estimate (2.63) in (2.60), we obtain

$$\|\mathbf{u}_n\|^{2r(2-r)/3}_{L^{4r/3}(0,T;L^{4r/3}(\Omega))} \le C\left(1+|\Omega|+\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\right)^{r(4-r)/2}. \tag{2.64}$$

Taking the estimate (2.64) into (2.57), we have

$$\|\mathbf{u}_n\|^r_{L^r(0,T;J_0^{1,r}(\Omega))} \le C\left(1+|\Omega|+\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\right). \tag{2.65}$$

The lemma is proved. □

Lemma 2.4. *Suppose all the hypotheses of Lemmas 2.1 and 2.2 are satisfied. Then*

$$\|\mathbf{u}_n'\|_{L^r(0,T;(J_0^{1,r}(\Omega))^*)} \le C\left(1+|\Omega|+\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\right). \tag{2.66}$$

The constant C is independent of n, g, $\mathbf{u}_0$, and $\boldsymbol{\mu}$.

Proof. Since $r\in(1,5/4)$, we have $W^{1,r/(r-1)}(\Omega)\subset L^\infty(\Omega)$ as Ω is a bounded open subset of $\mathbb{R}^3$. Let $\mathbf{v}$ be in $L^{r/(r-1)}(0,T;J_0^{1,r}(\Omega))$, then we get from (2.2)

$$(\mathbf{u}_n',\mathbf{v})+\nu(\nabla\mathbf{u}_n,\nabla\mathbf{v})+((\mathbf{w}^n\cdot\nabla)\mathbf{u}_n,\mathbf{v}) = (g\mathbf{f}_n,\mathbf{v}). \tag{2.67}$$

Thus,

$$\begin{aligned}
\int_0^T |(\mathbf{u}_n',\mathbf{v})|\,dt &\le \|\mathbf{u}_n\|_{L^r(0,T;J_0^{1,r}(\Omega))}\|\mathbf{v}\|_{L^{r/(r-1)}(0,T;J^{1,r/(r-1)}(\Omega))} \\
&\quad + \|\mathbf{w}\|_{L^\infty(0,T;L^\infty(\Omega))}\|\mathbf{u}_n\|_{L^r(0,T;J_0^{1,r}(\Omega))}\|\mathbf{v}\|_{L^{r/(r-1)}(0,T;J_0^{1,r/(r-1)}(\Omega))} \\
&\quad + C\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)\|\mathbf{v}\|_{L^{r/(r-1)}(0,T;L^\infty(\Omega))}.
\end{aligned} \tag{2.68}$$

It follows from the estimate of Lemma 2.2 that

$$\int_0^T |(\mathbf{u}_n', \mathbf{v})|\, dt \le C(1 + |\Omega| + \mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)) \|\mathbf{v}\|_{L^{r/(r-1)}(0,T;J_0^{1,r/(r-1)}(\Omega))}. \tag{2.69}$$

Hence,

$$\|\mathbf{u}_n'\|_{L^r(0,T;(J_0^{1,r/(r-1)}(\Omega))^*)} \le C(1 + |\Omega| + \mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0)). \tag{2.70}$$

The lemma is proved. □

The main result of this section is the following theorem.

THEOREM 2.5. *Let* $\{g, \boldsymbol{\mu}, \mathbf{u}_0\}$ *be in* $L^2(0,T) \times M_b(\Omega) \times L^1(\Omega)$ *and let* $\mathbf{w}$ *be in*

$$L^\infty(0,T;L^\infty(\Omega)) \cap L^2(0,T;J_0^{1,2}(\Omega)). \tag{2.71}$$

Then there exists a unique solution $\mathbf{u}$ *of (1.1). Moreover,*

$$\|\mathbf{u}\|_{L^\infty(0,T;L^1(\Omega))} + \|\mathbf{u}\|_{L^r(0,T;J_0^{1,r}(\Omega))} + \|\mathbf{u}'\|_{L^r(0,T;(J_0^{1,r(r-1)}(\Omega))^*)} \le C\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0) \tag{2.72}$$

for $r \in (1, 5/4)$. *The constant* C *is independent of* g, $\boldsymbol{\mu}$, *and* $\mathbf{u}_0$.

Proof. Let $\mathbf{u}_n$ be as in Lemmas 2.1, 2.2, and 2.3. Then from the estimates of those lemmas, we get, by taking subsequences,

$$\{\mathbf{u}_n, \mathbf{u}_n'\} \longrightarrow \{\mathbf{u}, \mathbf{u}'\} \tag{2.73}$$

in

$$\begin{aligned}(L^\infty(0,T;L^1(\Omega)))_{\text{weak}^*} \cap L^r(0,T;L^r(\Omega)) \cap (L^r(0,T;J_0^{1,r}(\Omega)))_{\text{weak}} \\ \times (L^r(0,T;(J_0^{1,r/(r-1)}(\Omega))^*))_{\text{weak}}.\end{aligned} \tag{2.74}$$

It is now clear that $\mathbf{u}$ is a solution of (1.1) with

$$\{\mathbf{u}, \mathbf{u}'\} \in L^\infty(0,T;L^1(\Omega)) \cap L^r(0,T;J_0^{1,r}(\Omega)) \times L^r\left(0,T;(J_0^{1,r/(r-1)}(\Omega))^*\right). \tag{2.75}$$

It is trivial to check that the solution is unique. Since the solution is unique, an application of the closed-graph theorem yields the stated estimate. The constant C depends on Ω. □

3. Value function

Let $\boldsymbol{\chi}$ be an element of $L^1(0,T;L^1(\Omega))$ representing the observed values of the solution $\mathbf{u}$ of (1.1) in an interior subdomain G. Let

$$\mathscr{G} = \{g : \|g\|_{W^{1,2}(0,T)} \le 1\}, \qquad \mathscr{U} = \{\boldsymbol{\mu} : \|\boldsymbol{\mu}\|_{M_b(\Omega)} \le 1\}. \tag{3.1}$$

It is clear that $\{\mathcal{G}, \mathcal{U}\}$ are closed convex subsets of $L^2(0,T) \times M_b(\Omega)$. Let

$$V(\mathbf{u}_0;\tau) = \inf\{J(g;\boldsymbol{\mu};\mathbf{u}_0;\tau), \forall \boldsymbol{\mu} \in \mathcal{U}, \ \forall g \in \mathcal{G}\}. \tag{3.2}$$

The value function exists. The main result of this section is the following theorem which shows that one can determine the source and its intensity from the observed values of the solution in an interior subdomain.

Theorem 3.1. *Suppose all the hypotheses of Theorem 2.5 are satisfied. Then there exists* $\{\tilde{g}, \tilde{\boldsymbol{\mu}}\} \in \mathcal{G} \times \mathcal{U}$ *such that*

$$V(\mathbf{u}_0;\tau) = J(\tilde{g};\tilde{\boldsymbol{\mu}};\mathbf{u}_0;\tau). \tag{3.3}$$

Moreover,

$$|V(\mathbf{u}_0;\tau) - V(\mathbf{v}_0;\tau)| \le C||\mathbf{u}_0 - \mathbf{v}_0||_{L^1(\Omega)} \quad \forall \mathbf{u}_0, \mathbf{v}_0 \in L^1(\Omega). \tag{3.4}$$

The constant C is independent of $\mathbf{u}_0$, $\mathbf{v}_0$, *and* τ. *The generalized Clarke subgradients* $\partial V(\mathbf{u}_0;\tau)$ *exist and are set-valued mappings of* $L^1(\Omega)$ *into the closed convex subsets of* $(L^1(\Omega))^* = L^\infty(\Omega)$ *with*

$$||f(\mathbf{u}_0;\tau)||_{L^\infty(\Omega)} \le C \quad \forall f \in \partial V(\mathbf{u}_0;\tau). \tag{3.5}$$

Proof. Let $\{g_n, \boldsymbol{\mu}_n\}$ be a minimizing sequence of (3.2) with

$$V(\mathbf{u}_0;\tau) \le J(g_n;\boldsymbol{\mu}_n;\mathbf{u}_0;\tau) \le V(\mathbf{u}_0;\tau) + \frac{1}{n}. \tag{3.6}$$

Let $\mathbf{u}_n$ be the unique solution of (1.1) with the controls $\{g_n, \boldsymbol{\mu}_n\}$, given by Theorem 2.5. From the estimates of the theorem, we have

$$||\mathbf{u}_n||_{L^\infty(0,T;L^1(\Omega))} + ||\mathbf{u}_n||_{L^r(0,T;J_0^{1,r}(\Omega))} + ||\mathbf{u}_n'||_{L^r(0,T;(J_0^{1,r/(r-1)}(\Omega))^*)} \le C\mathscr{E}(g;\boldsymbol{\mu};\mathbf{u}_0). \tag{3.7}$$

Thus, there exists a subsequence such that

$$\{\mathbf{u}_n, \mathbf{u}_n'; g_n, \boldsymbol{\mu}_n\} \longrightarrow \{\tilde{\mathbf{u}}, \tilde{\mathbf{u}}', \tilde{g}, \tilde{\boldsymbol{\mu}}\} \tag{3.8}$$

in

$$\begin{aligned}
&(L^r(0,T;J_0^{1,r}(\Omega)))_{\text{weak}} \cap L^r(0,T;L^r(\Omega)) \cap (L^\infty(0,T;L^1(\Omega)))_{\text{weak}^*} \\
&\quad \times \left(L^r\left(0,T;\left(J_0^{1,r/(r-1)}(\Omega)\right)^*\right)\right)_{\text{weak}} \\
&\quad \times (W^{1,2}(0,T))_{\text{weak}} \times \mathscr{D}'(\Omega).
\end{aligned} \tag{3.9}$$

It is now trivial to check that $\tilde{\mathbf{u}}$ is the unique solution of (1.1) with the controls $\{\tilde{g}, \tilde{\boldsymbol{\mu}}\}$, and moreover

$$V(\mathbf{u}_0;\tau) = J(\tilde{g};\tilde{\boldsymbol{\mu}};\mathbf{u}_0;\tau). \tag{3.10}$$

(2) Let $\mathbf{v}_0$ be in $L^1(\Omega)$, then we have

$$V(\mathbf{v}_0;\tau) - V(\mathbf{u}_0;\tau) \le J(\tilde{g};\tilde{\boldsymbol{\mu}};\mathbf{v}_0;\tau) - J(\tilde{g};\tilde{\boldsymbol{\mu}};\mathbf{u}_0;\tau) \le \int_\tau^T \int_G |\tilde{\mathbf{u}} - \hat{\mathbf{v}}|\,dx\,dt, \tag{3.11}$$

where $\hat{\mathbf{v}}$ is the unique solution of (1.1) with the controls $\{\tilde{g},\tilde{\boldsymbol{\mu}}\}$ and initial value $\mathbf{v}_0$. It follows from Lemma 2.1 that

$$\|\tilde{\mathbf{u}} - \hat{\mathbf{v}}\|_{L^\infty(0,T;L^1(\Omega))} \le C\|\mathbf{u}_0 - \mathbf{v}_0\|_{L^1(\Omega)}. \tag{3.12}$$

Therefore,

$$V(\mathbf{v}_0;\tau) - V(\mathbf{u}_0;\tau) \le C\|\mathbf{v}_0 - \mathbf{u}_0\|_{L^1(\Omega)}. \tag{3.13}$$

Reversing the role of $\mathbf{u}_0$ and $\mathbf{v}_0$, we have

$$V(\mathbf{u}_0;\tau) - V(\mathbf{v}_0;\tau) \le C\|\mathbf{u}_0 - \mathbf{v}_0\|_{L^1(\Omega)}. \tag{3.14}$$

Thus,

$$|V(\mathbf{u}_0;\tau) - V(\mathbf{v}_0;\tau)| \le C\|\mathbf{u}_0 - \mathbf{v}_0\|_{L^1(\Omega)} \quad \forall \mathbf{u}_0,\mathbf{v}_0 \in L^1(\Omega). \tag{3.15}$$

It follows that the Clarke subgradients $\partial V(\mathbf{u}_0;\tau)$ exist and are set-valued mappings of $L^1(\Omega)$ into the closed convex subsets of $L^\infty(\Omega)$. Furthermore,

$$\|f\|_{L^\infty(\Omega)} \le C \quad \forall f \in \partial V(\mathbf{u}_0;\tau). \tag{3.16}$$

□

4. Feedback laws

To establish the feedback laws of problems (1.1) and (1.2), we first establish the existence of a solution of a nonlinear parabolic initial boundary value problem.

Let $\mathcal{P}$ be the projection of $L^2(0,T)$ into $\mathcal{G}$ and let $\mathcal{L}$ be the linear compact mapping of $L^\infty(\Omega)$ into $C_0(\Omega)$ given by $\mathcal{L}h = v$ with

$$v - \Delta v = h \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega. \tag{4.1}$$

It is known that for $h \in L^\infty(\Omega)$, there exists a unique $v \in W_0^{1,r}(\Omega) \cap W^{2,r}(\Omega)$ for any $2 \le r < \infty$, and that

$$\|v\|_{C_0(\Omega)} \le C\|v\|_{W_0^{1,r}(\Omega)\cap W^{2,r}(\Omega)} \le C_1\|h\|_{L^\infty(\Omega)}. \tag{4.2}$$

Let k^* be the lower semicontinuous (l.s.c.) convex mapping of $C_0(\Omega)$ into $\mathbb{R}$ given by

$$k^*(\mathbf{f}) = \sup\{\boldsymbol{\mu}(\mathbf{f}), \forall \boldsymbol{\mu} \in \mathcal{U}\}. \tag{4.3}$$

Since k^* is a l.s.c. convex mapping of $C_0(\Omega)$ into $\mathbb{R}$, its subgradient exists and maps $C_0(\Omega)$ into $(C_0(\Omega))^* = M_b(\Omega)$. A standard argument shows that there exists $\tilde{\boldsymbol{\mu}} \in \mathcal{U}$ such that

$$k^*(\mathbf{f}) = \tilde{\boldsymbol{\mu}}(\mathbf{f}) = \sup\{\boldsymbol{\mu}(\mathbf{f}), \forall \boldsymbol{\mu} \in \mathcal{U}\}. \tag{4.4}$$

Since $k^*(0) = 0$ and

$$k^*(\mathbf{g}) - k^*(\mathbf{f}) \geq \partial k^*_{\mathbf{f}}(\mathbf{g} - \mathbf{f}) \quad \forall \mathbf{g} \in C_0(\Omega), \tag{4.5}$$

we deduce that $k^*(\mathbf{f}) = \tilde{\boldsymbol{\mu}}(\mathbf{f})$ and $\partial k^*_{\mathbf{f}} = \{\tilde{\boldsymbol{\mu}}\}$.

We consider the initial boundary value problem

$$\begin{gathered} \mathbf{u}' - \nu\Delta\mathbf{u} + (\mathbf{w}\cdot\nabla)\mathbf{u} = \nabla p + g_u\boldsymbol{\mu}_{\mathbf{u}} \quad \text{in } Q, \\ \mathbf{u}(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \text{in } \Omega, \end{gathered} \tag{4.6}$$

with

$$\begin{gathered} g_u = \mathcal{P}\left\{\int_\Omega \mathbf{u}(x,t)\,dx\right\} \\ \boldsymbol{\mu}_u = k^*\left(\int_0^T \mathcal{L}(\mathbf{f}_u)\,dt\right), \quad \mathbf{f}_u \in \partial V(\mathbf{u}(\cdot,t);t). \end{gathered} \tag{4.7}$$

The main result of this section is the following theorem.

Theorem 4.1. *Let $\{\mathbf{w},\mathbf{u}_0\}$ be in $C(0,T;C(\Omega)) \cap L^2(0,T;J_0^{1,2}(\Omega)) \times L^1(\Omega)$. Let $\boldsymbol{\chi}$ be in $L^1(0,T;L^1(G))$, then there exists a solution $\tilde{\mathbf{u}}$ of (4.6) and (4.7) with*

$$\|\tilde{\mathbf{u}}\|_{L^\infty(0,T;L^1(\Omega))} + \|\tilde{\mathbf{u}}\|_{L^r(0,T;J_0^{1,r}(\Omega))} + \|\tilde{\mathbf{u}}'\|_{L^r(0,T;(J_0^{1,r/(r-1)}(\Omega))^*)} \leq C \tag{4.8}$$

and $1 < r < 5/4$. Moreover,

$$V(\mathbf{u}_0;0) = \int_0^T\int_G |\tilde{\mathbf{u}}(x,t) - \boldsymbol{\chi}(x,t)|\,dx\,dt. \tag{4.9}$$

The value function $V(\mathbf{u}_0;0)$ is given by Theorem 3.1.

First, we establish the existence of a solution of (4.6) and (4.7) by applying Kakutani's fixed-point theorem. Let

$$\mathcal{B}_C = \Big\{\mathbf{u} : \|\mathbf{u}\|_{L^\infty(0,T;L^1(\Omega))} + \|\mathbf{u}\|_{L^r(0,T;J_0^{1,r}(\Omega))} + \|\mathbf{u}'\|_{L^r(0,T;(J_0^{1,r/(r-1)}(\Omega))^*)} \le C1 + \|\mathbf{u}_0\|_{L^1(\Omega)}\Big\} \tag{4.10}$$

with C as in Theorem 2.5. Let $\mathbf{v} \in \mathcal{B}_C$, then $g(\mathbf{v})$ is in $\mathcal{G}$. Since $\mathbf{v}(\cdot,t)$ is in $L^1(\Omega)$ for almost all t, it follows from Theorem 3.1 that $\partial V(\mathbf{v}(\cdot,t);t)$ is a set-valued mapping of $L^1(\Omega)$ into the closed convex subsets of $L^\infty(\Omega)$. Moreover,

$$\|\mathbf{f}_\mathbf{v}(\cdot,t)\|_{L^\infty(\Omega)} \le C_1 \quad \forall \mathbf{f}_\mathbf{v} \in \partial V(\mathbf{v}(\cdot,t);t) \tag{4.11}$$

and for almost all t. The constant C_1 is independent of t. Thus, $\partial V(\mathbf{v}(\cdot,t);t)$ is a closed bounded convex subset of $L^\infty(\Omega)$ and hence is also a closed convex subset of $L^2(\Omega)$. It follows that there exists a unique element $\tilde{\mathbf{f}}_\mathbf{v}$ of $\partial V(\mathbf{v}(\cdot,t);t)$ of minimum $L^2(\Omega)$-norm. Set

$$S = \{\boldsymbol{\mu}_\mathbf{v} : \boldsymbol{\mu}_\mathbf{v} \in \mathcal{U},\ \boldsymbol{\mu}_\mathbf{v}(\tilde{\mathbf{f}}_\mathbf{v}) = k^*(\tilde{\mathbf{f}}_\mathbf{v})\} \tag{4.12}$$

with k^* as in (4.3). Then S is a convex subset of $M_b(\Omega)$ and if $\boldsymbol{\mu}_n \in S$ with $\boldsymbol{\mu}_n \to \boldsymbol{\mu}$ in $\mathcal{D}'(\Omega)$, then $\boldsymbol{\mu} \in M_b(\Omega)$.

Consider the initial boundary value problem

$$\begin{gathered} \mathbf{u}' - \nu\Delta\mathbf{u} + \mathbf{w}\cdot\nabla\mathbf{u} = \nabla p + g(\mathbf{v})\boldsymbol{\mu}_\mathbf{v} \quad \text{in } Q, \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in } Q, \qquad \mathbf{u}(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \text{in } \Omega, \end{gathered} \tag{4.13}$$

with $\boldsymbol{\mu} \in S$. Then for a given $\mathbf{v} \in \mathcal{B}_C$, it follows from Theorem 2.5 that there exists a unique solution $\mathbf{u} \in \mathcal{B}_C$. Let $\mathcal{A}$ be the set-valued mapping

$$\mathcal{A}(\mathbf{v}) = \{\mathbf{u} : \mathbf{u} \text{ is a unique solution of (4.13)};\ \boldsymbol{\mu}_\mathbf{v} \in S\}. \tag{4.14}$$

It is clear that $\mathcal{A}$ maps $\mathcal{B}_C$, considered as a subset of $L^r(0,T;L^r(\Omega))$, into the subsets of $L^r(0,T;L^r(\Omega))$. To prove the theorem, we show that $\mathcal{A}$ has a fixed point, by applying Kakutani's fixed-point theorem.

Lemma 4.2. *Suppose all the hypotheses of Theorem 4.1 are satisfied and let $\mathcal{A}$ be as in (4.7). Then the images of $\mathcal{A}$ are closed, convex subsets of $L^r(0,T;L^r(\Omega))$.*

Proof. The images of $\mathcal{A}$ are clearly convex subsets of $L^r(0,T;L^r(\Omega))$ as problem (4.13) is linear. We now show that the images are closed subsets of $L^r(0,T;L^r(\Omega))$.

Let $\mathbf{u}_n \in \mathcal{A}(\mathbf{v})$ and suppose that $\mathbf{u}_n \to \mathbf{u}$ in $L^r(0,T;L^r(\Omega))$. From the estimates of Theorem 2.5, we get

$$\|\mathbf{u}_n\|_{L^\infty(0,T;L^1(\Omega))} + \|\mathbf{u}_n\|_{L^r(0,T;J_0^{1,r}(\Omega))} + \|\mathbf{u}_n'\|_{L^r(0,T;(J_0^{1,r/(r-1)}(\Omega))^*)} \le C. \tag{4.15}$$

It follows that there exists a subsequence such that

$$\{\mathbf{u}_n, \mathbf{u}_n'\} \longrightarrow \{\mathbf{u}, \mathbf{u}'\} \tag{4.16}$$

in

$$\begin{aligned} &L^r(0,T;L^r(\Omega)) \cap (L^r(0,T;J_0^{1,r}(\Omega)))_{\text{weak}} \cap (L^\infty(0,T;L^1(\Omega)))_{\text{weak}^*} \\ &\quad \times \left(L^r\left(0,T;\left(J_0^{1,r/(r-1)}(\Omega)\right)^*\right)\right)_{\text{weak}}. \end{aligned} \tag{4.17}$$

It is now trivial to check that $\mathbf{u} \in \mathcal{A}(\mathbf{v})$. □

Lemma 4.3. *Let $\mathcal{A}$ be as in (4.7), then it is an upper semicontinuous (u.s.c.) set-valued mapping of $L^r(0,T;L^r(\Omega))$ into the closed convex subsets of $L^r(0,T;L^r(\Omega))$.*

Proof. Since $\mathcal{B}_C$ is a compact convex subset of $L^r(0,T;L^r(\Omega))$ and $\mathcal{A}$ maps $\mathcal{B}_C$ into $\mathcal{B}_C$, to show that $\mathcal{A}$ is an u.s.c. set-valued mapping, it suffices to show that its graph is closed.

Let $\{\mathbf{v}_n, \mathbf{u}_n\} \in \mathcal{B}_C \times \mathcal{A}(\mathbf{v}_n)$ and suppose that $\{\mathbf{u}_n, \mathbf{v}_n\} \to \{\mathbf{u}, \mathbf{v}\}$ in $L^r(0,T;L^r(\Omega))$. We have to show that $\mathbf{u} \in \mathcal{A}(\mathbf{v})$.

(1) From Theorem 2.5, we get

$$\{\mathbf{u}_n, \mathbf{v}_n, \mathbf{u}_n', \mathbf{v}_n'\} \longrightarrow \{\mathbf{u}, \mathbf{v}, \mathbf{u}', \mathbf{v}'\} \tag{4.18}$$

in

$$\begin{aligned} &L^r(0,T;L^r(\Omega)) \times (L^r(0,T;J_0^{1,r}(\Omega)))_{\text{weak}} \times (L^\infty(0,T;L^1(\Omega)))_{\text{weak}^*} \\ &\quad \times (L^r(0,T;(J_0^{1,r/(r-1)}(\Omega))^*))_{\text{weak}}. \end{aligned} \tag{4.19}$$

(2) Since $\mathcal{P}$ is continuous from $(L^2(0,T))_{\text{weak}}$ into $(W^{1,2}(0,T))_{\text{weak}}$, we have

$$g(\mathbf{v}_n) = \mathcal{P}\left\{\int_\Omega \mathbf{v}_n(x,t)\,dx\right\} \longrightarrow g(\mathbf{v}) \quad \text{in } (W^{1,2}(\Omega))_{\text{weak}}. \tag{4.20}$$

(3) We have

$$\|\mathbf{f}_{\mathbf{v}_n}\|_{L^\infty(0,T;L^\infty(\Omega))} \le C \quad \forall \mathbf{f}_{\mathbf{v}_n} \in \partial V(\mathbf{v}_n(t);t). \tag{4.21}$$

Thus, $\mathbf{f}_{\mathbf{v}_n} \to \mathbf{f}$ in $(L^\infty(0,T;L^\infty(\Omega)))_{\text{weak}^*}$. On the other hand,

$$\int_0^T \{V(\mathbf{w};t) - V(\mathbf{v}_n;t)\}\,dt \ge \int_0^T (\mathbf{f}_{\mathbf{v}_n}, \mathbf{w} - \mathbf{v}_n)\,dt \quad \forall \mathbf{w} \in L^\infty(0,T;L^1(\Omega)). \tag{4.22}$$

Hence,

$$\int_0^T \{V(\mathbf{w};t) - V(\mathbf{v};t)\}dt \geq \int_0^T (\mathbf{f}, \mathbf{w} - \mathbf{v})dt \quad \forall \mathbf{w} \in L^\infty(0,T;L^1(\Omega)). \tag{4.23}$$

It follows that $\mathbf{f} \in \partial V(\mathbf{v};t)$.

We now show that it is the unique element of $\partial V(\mathbf{v};t)$ with minimum $L^2(0,T;L^2(\Omega))$-norm. Let

$$\begin{aligned}\mathcal{B}_\varepsilon(\mathbf{v}) = \Big\{\mathbf{v}^\varepsilon : \mathbf{v}^\varepsilon \in \mathcal{B}_C,\ \|\mathbf{v}^\varepsilon - \mathbf{v}\|_{L^r(0,T;J_0^{1,r}(\Omega))} \\ + \|\mathbf{v}^\varepsilon - \mathbf{v}'\|_{L^\infty(0,T;L^1(\Omega))} + \|(\mathbf{v}^\varepsilon - \mathbf{v})'\|_{L^r(0,T;(J_0^{1,r/(r-1)}(\Omega))^*)} \leq C\Big\}.\end{aligned} \tag{4.24}$$

Then

$$\bigcap_\varepsilon \{\partial V(\mathbf{v}^\varepsilon;t) : \mathbf{v}^\varepsilon \in \mathcal{B}_\varepsilon(\mathbf{v})\} \subset \partial V(\mathbf{v}_n;t) \tag{4.25}$$

since $\mathbf{v}_n \in \mathcal{B}_\varepsilon(\mathbf{v})$ for $n \geq n_0$. With $\mathbf{f}_{\mathbf{v}_n}$ the unique element of $\partial V(\mathbf{v}_n;t)$ with minimum $L^2(0,T;L^2(\Omega))$-norm, we obtain

$$\|\mathbf{f}_{\mathbf{v}_n}\|_{L^2(0,T;L^2(\Omega))} \leq \|\mathbf{f}_{\mathbf{v}^\varepsilon}\|_{L^2(0,T;L^2(\Omega))} \quad \forall \mathbf{f}_{\mathbf{v}^\varepsilon} \in \partial V(\mathbf{v}^\varepsilon;t),\ \forall \varepsilon > 0. \tag{4.26}$$

In particular

$$\|\mathbf{f}_{\mathbf{v}_n}\|_{L^2(0,T;L^2(\Omega))} \leq \|\mathbf{f}_{\mathbf{v}}\|_{L^2(0,T;L^2(\Omega))} \quad \forall \mathbf{f}_{\mathbf{v}} \in \partial V(\mathbf{v};t). \tag{4.27}$$

Thus,

$$\|\mathbf{f}\|_{L^2(0,T;L^2(\Omega))} \leq \|\mathbf{f}_{\mathbf{v}}\|_{L^2(0,T;L^2(\Omega))} \quad \forall \mathbf{f}_{\mathbf{v}} \in \partial V(\mathbf{v};t). \tag{4.28}$$

Since $\partial V(\mathbf{v};t)$ is a closed convex subset of $L^2(0,T;L^2(\Omega))$, $\mathbf{f}$ is the unique element of minimum $L^2(0,T;L^2(\Omega))$-norm of the set.

(4) Since $\mathcal{L}$ is a compact linear mapping of $L^\infty(\Omega)$ into $C_0(\Omega)$, we have

$$\mathcal{L}\left(\int_0^T \mathbf{f}_{\mathbf{v}_n}(x,t)dt\right) \longrightarrow \mathcal{L}\left(\int_0^T \mathbf{f}(x,t)dt\right) \quad \text{in } C_0(\Omega). \tag{4.29}$$

On the other hand,

$$\boldsymbol{\mu}_n\left(\mathcal{L}\left(\int_0^T \mathbf{f}_{\mathbf{v}_n}dt\right)\right) = k^*\left(\mathcal{L}\left\{\int_0^T \mathbf{f}_{\mathbf{v}_n}dt\right\}\right). \tag{4.30}$$

Since $\boldsymbol{\mu}_n \in \mathcal{U}$ and $\boldsymbol{\mu}_n \to \boldsymbol{\mu}$ in $\mathcal{D}'(\Omega)$, we get

$$\left|\boldsymbol{\mu}_n\left\{\mathcal{L}\left(\int_0^T (\mathbf{f}_{\mathbf{v}_n} - \mathbf{f})dt\right)\right\}\right| \leq C\|\mathbf{f}_{\mathbf{v}_n} - \mathbf{f}\|_{L^\infty(0,T;C_0(\Omega))}. \tag{4.31}$$

Thus,

$$\boldsymbol{\mu}_n\left\{\mathcal{L}\left(\int_0^T (\mathbf{f}_{\mathbf{v}_n}-\mathbf{f})\,dt\right)\right\} \longrightarrow 0. \tag{4.32}$$

Since $\boldsymbol{\mu}_n \to \boldsymbol{\mu}$ in $\mathcal{D}'(\Omega)$, it follows that

$$\boldsymbol{\mu}_n\left\{\mathcal{L}\left(\int_0^T \mathbf{f}_{\mathbf{v}_n}\,dt\right)\right\} \longrightarrow \boldsymbol{\mu}\left\{\mathcal{L}\left(\int_0^T \mathbf{f}\,dt\right)\right\}. \tag{4.33}$$

It is now clear that

$$\boldsymbol{\mu}\left\{\mathcal{L}\left(\int_0^T \mathbf{f}\,dt\right)\right\} = k^*\left(\mathcal{L}\left\{\int_0^T \mathbf{f}\,dt\right\}\right). \tag{4.34}$$

We deduce that $\mathbf{u} \in \mathcal{A}(\mathbf{v})$. The lemma is proved. □

Proof of Theorem 4.1. (1) It follows from Lemmas 4.2 and 4.3 that the nonlinear mapping $\mathcal{A}$ satisfies all the hypotheses of the Kakutani fixed-point theorem and thus there exists $\tilde{\mathbf{u}}$ such that $\tilde{\mathbf{u}} \in \mathcal{A}(\tilde{\mathbf{u}})$. We now show that

$$V(\mathbf{u}_0;0) = \int_0^T\int_G \left|\tilde{\mathbf{u}}(x,t)-\boldsymbol{\chi}(x,t)\right|dx\,dt. \tag{4.35}$$

(2) We know that $\tilde{\mathbf{u}} \in L^\infty(0,T;L^1(\Omega))$, and thus $\tilde{\mathbf{u}}(x,t)$ is in $L^1(\Omega)$ for almost all t. Let $\{g,\boldsymbol{\mu}\}$ be in $\mathcal{G}\times\mathcal{U}$ and consider the initial boundary value problem

$$\begin{gathered}
\mathbf{u}' - \nu\Delta\mathbf{u} + \mathbf{w}\cdot\nabla\mathbf{u} = \nabla p + g\boldsymbol{\mu} \quad \text{in } \Omega\times(t,T),\\
\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times(t,T),\\
\mathbf{u}(x,t) = 0 \quad \text{on } \partial\Omega\times(t,T),\\
\mathbf{u}(x,t) = \tilde{\mathbf{u}}(x,t) \quad \text{in } \Omega.
\end{gathered} \tag{4.36}$$

From Theorem 2.5, we deduce the existence of a unique solution $\mathbf{u}$ in $L^\infty(t,T;L^1(\Omega))\cap L^r(t,T;J_0^{1,r}(\Omega))$ with $\mathbf{u}'$ in $L^{r}(t,T;(J_0^{1,r/(r-1)}(\Omega))^*)$. Let

$$V(\tilde{\mathbf{u}}(\cdot,t);t) = \inf\left\{\int_t^T\int_G \left|\mathbf{u}(x,s)-\boldsymbol{\chi}(x,s)\right|dx\,ds,\ \forall g\in\mathcal{G},\ \forall\boldsymbol{\mu}\in\mathcal{U}\right\}. \tag{4.37}$$

Using a minimizing sequence, we deduce that there exists $\{\hat{\mathbf{u}},\hat{g},\hat{\boldsymbol{\mu}}\}$ such that

$$V(\tilde{\mathbf{u}}(\cdot,t);t) = \int_t^T\int_G \left|\hat{\mathbf{u}}(x,s)-\boldsymbol{\chi}(x,s)\right|dx\,ds. \tag{4.38}$$

The solution $\hat{\mathbf{u}}$ of (4.36) with the controls $\hat{g}$, $\hat{\boldsymbol{\mu}}$ depends on t through its interval of definition.

The dynamic programming principle gives

$$V(\tilde{\mathbf{u}}(\cdot,t);t) = \inf\Big\{V(\mathbf{u}(\cdot,t+h);t+h) + \int_t^{t+h}\int_G |\mathbf{u}(x,s)-\boldsymbol{\chi}(x,s)|\,dx\,ds : \mathbf{u} \text{ is a solution of (4.36) with } \mathbf{u}(x,0)=\mathbf{u}_0,\ \forall\{g,\boldsymbol{\mu}\}\in\mathcal{G}\times\mathcal{U}\Big\}. \tag{4.39}$$

Since $\hat{\mathbf{u}}(x,t) = \tilde{\mathbf{u}}(x,t)$, it follows that

$$\begin{aligned} V(\tilde{\mathbf{u}}(\cdot,t);t) &= V(\hat{\mathbf{u}}(\cdot,t);t) \\ &\le V(\hat{\mathbf{u}}(\cdot,t+h);t+h) + \int_t^{t+h}\int_G |\hat{\mathbf{u}}(x,s)-\boldsymbol{\chi}(x,s)|\,dx\,ds. \end{aligned} \tag{4.40}$$

Thus,

$$\int_{t+h}^{T}\int_G |\hat{\mathbf{u}}(x,s)-\boldsymbol{\chi}(x,s)|\,dx\,ds \le V(\hat{\mathbf{u}}(\cdot,t+h);t+h). \tag{4.41}$$

It follows from the definition of the value function that

$$V(\hat{\mathbf{u}}(\cdot,t+h);t+h) = \int_{t+h}^{T}\int_G |\hat{\mathbf{u}}(x,s)-\boldsymbol{\chi}(x,s)|\,dx\,ds. \tag{4.42}$$

Hence,

$$V(\hat{\mathbf{u}}(\cdot,t+h);t+h) - V(\hat{\mathbf{u}}(\cdot,t);t) = -\int_t^{t+h}\int_G |\hat{\mathbf{u}}(x,s)-\boldsymbol{\chi}(x,s)|\,dx\,ds. \tag{4.43}$$

Therefore,

$$\frac{d}{dt}V(\hat{\mathbf{u}}(\cdot,t);t) = -\int_G |\hat{\mathbf{u}}(x,t)-\boldsymbol{\chi}(x,t)|\,dx. \tag{4.44}$$

But, $\tilde{\mathbf{u}}(x,t) = \hat{\mathbf{u}}(x,t)$ in Ω and therefore

$$\frac{d}{dt}V(\hat{\mathbf{u}}(\cdot,t);t) = -\int_G |\tilde{\mathbf{u}}(x,t)-\boldsymbol{\chi}(x,t)|\,dx. \tag{4.45}$$

It follows that

$$V(\hat{\mathbf{u}}(\cdot,T);T) - V(\hat{\mathbf{u}}(\cdot;0);0) = -V(\mathbf{u}_0;0) = -\int_0^T\int_G |\tilde{\mathbf{u}}(x,s)-\boldsymbol{\chi}(x,s)|\,dx\,ds. \tag{4.46}$$

The theorem is proved. $\square$

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