EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR BOUNDARY VALUE PROBLEMS IN BOUNDED DOMAINS OF $\mathbb{R}^n$

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Let D be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$. We consider the following nonlinear elliptic problem: $\Delta u = f(\cdot, u)$ in D (in the sense of distributions), $u_{|\partial D} = \varphi$, where φ is a nonnegative continuous function on ∂D and f is a nonnegative function satisfying some appropriate conditions related to some Kato class of functions $K(D)$. Our aim is to prove that the above problem has a continuous positive solution bounded below by a fixed harmonic function, which is continuous on $\overline{D}$. Next, we will be interested in the Dirichlet problem $\Delta u = -\rho(\cdot, u)$ in D (in the sense of distributions), $u_{|\partial D} = 0$, where ρ is a nonnegative function satisfying some assumptions detailed below. Our approach is based on the Schauder fixed-point theorem.

1. Introduction

Let D be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ $(n \geq 2)$, and let G be the Green function for the Laplace operator with zero Dirichlet boundary condition on ∂D. In [4], Chung and Zhao have established interesting inequalities for the Green function G. In particular, they showed that there exists a constant $C > 0$ such that for each x, y in D,

$$\frac{1}{C} H(x,y) \leq G(x,y) \leq CH(x,y), \tag{1.1}$$

where

$$H(x,y) := \begin{cases} \dfrac{1}{|x-y|^{n-2}} \min\left(1, \dfrac{\delta(x)\delta(y)}{|x-y|^2}\right), & \text{if } n \geq 3, \\ \operatorname{Log}\left(1 + \dfrac{\delta(x)\delta(y)}{|x-y|^2}\right), & \text{if } n = 2, \end{cases} \tag{1.2}$$

and $\delta(x)$ denotes the Euclidean distance between x and ∂D.

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Another crucial inequality for the Green function G called $3G$-theorem is given by Kalton and Verbitsky [7] for $n \geq 3$ and by Selmi [12] for $n = 2$, namely, there exists a constant $C_0 > 0$ depending only on D such that for all x, y, and z in D,

$$\frac{G(x,z)G(z,y)}{G(x,y)} \leq C_0\left(\frac{\delta(z)}{\delta(x)}G(x,z) + \frac{\delta(z)}{\delta(y)}G(y,z)\right). \tag{1.3}$$

This $3G$-theorem was investigated by Mâagli and Zribi [10], Zeddini [13], and Mâagli and Mâatoug [9] to introduce a new class of functions denoted by $K(D)$, (see Definition 1.1 below), which contains properly the classical Kato class introduced by Aizenman and Simon [1]. Moreover, they used the properties of functions belonging to this class $K(D)$ to study some nonlinear differential equations.

Definition 1.1. A Borel measurable function q in D belongs to the class $K(D)$ if q satisfies

$$\lim_{\alpha\to 0}\left(\sup_{x\in D}\int_{D\cap B(x,\alpha)}\frac{\delta(y)}{\delta(x)}G(x,y)\,|q(y)|\,dy\right) = 0. \tag{1.4}$$

In this paper, we will exploit the properties pertaining to $K(D)$ to give some results about the existence of positive solutions of nonlinear elliptic problems. Our plan is as follows.

In Section 2, we establish some estimates on the Green function G and some properties of functions belonging to the Kato class $K(D)$.

In Section 3, we are concerned with the existence of positive continuous solutions of the nonlinear elliptic problem

$$\begin{gathered} \Delta u = f(\cdot,u) \quad \text{(in the sense of distributions)}, \\ u > 0 \quad \text{in } D, \qquad u_{|\partial D} = \varphi, \end{gathered} \tag{1.5}$$

where φ is a nontrivial nonnegative continuous function on ∂D. Then, we fix a nontrivial nonnegative harmonic function h_0 in D, which is continuous in $\overline{D}$, and we suppose that f satisfies the following hypotheses.

(H_1) $f : D\times(0,+\infty) \to [0,+\infty)$ is measurable, continuous with respect to the second variable and satisfies

$$f(x,t) \leq \theta(x,t), \quad \text{for } (x,t) \in D\times(0,+\infty), \tag{1.6}$$

where θ is a nonnegative measurable function on $D\times(0,+\infty)$ such that the function $t \to \theta(x,t)$ is nonincreasing on $(0,+\infty)$.

(H_2) The function ψ defined on D by $\psi(x) = \theta(x,h_0(x))/h_0(x)$ belongs to the class $K(D)$.

Remark 1.2. Note that the condition "$\forall c > 0$, $\theta(\cdot,c\delta(\cdot))/\delta(\cdot) \in K(D)$" implies the hypothesis (H_2). Indeed, from [14], there exists $c > 0$ such that for each $x \in D$, $h_0(x) \geq c\delta(x)$. So, using the fact that $t \to \theta(x,t)/t$ is nonincreasing function on $(0,+\infty)$, we obtain (H_2).

Under the assumptions (H_1)-(H_2), we aim at proving the following result: there exists a constant $c > 1$ such that if $\varphi \geq ch_0$ on ∂D, then problem (1.5) has a positive continuous solution u satisfying for each $x \in D$,

$$h_0(x) \leq u(x) \leq H_D\varphi(x), \tag{1.7}$$

where $H_D\varphi$ is the harmonic continuous function having boundary value φ on ∂D.

This result improves the one of Atherya [2], who considered the following problem:

$$\Delta u = g(u) \quad \text{in } \Omega, \qquad u_{|\partial\Omega} = \varphi, \tag{P}$$

where Ω is a simply connected bounded C^2-domain in $\mathbb{R}^n (n \geq 3)$ and $g(u) \leq \max(1, u^{-\alpha})$, for $0 < \alpha < 1$. He proved the existence of a positive continuous solution bounded below by a fixed positive harmonic function h_0 provided that there exists a positive constant $c > 1$ such that $\varphi \geq ch_0$ on ∂D.

In the last section, we will study the following nonlinear problem:

$$\Delta u = -\rho(\cdot, u) \quad \text{in } D \text{ (in the sense of distributions)}, \qquad u_{|\partial D} = 0, \tag{1.8}$$

where ρ is required to verify the following hypotheses.

(H_3) ρ is nonnegative Borel measurable function on $D \times (0,\infty)$, continuous with respect to the second variable.

(H_4) There exist $p, q : D \to (0,\infty)$ nontrivial Borel measurable functions and $h, k : (0,\infty) \to [0,\infty)$ nontrivial and nondecreasing Borel measurable functions satisfying

$$p(x)h(t) \leq \rho(x,t) \leq q(x)k(t), \quad \text{for } (x,t) \in D \times (0,\infty), \tag{1.9}$$

such that

(A_1) $p \in L^1_{\mathrm{loc}}(D)$,

(A_2) $q \in K(D)$,

(A_3) $\lim_{t\to 0^+}(h(t)/t) = +\infty$,

(A_4) $\lim_{t\to +\infty}(k(t)/t) = 0$.

Under these hypotheses, we will prove that (1.8) has a positive continuous solution u satisfying on D,

$$a\delta(x) \leq u(x) \leq b, \tag{1.10}$$

where a, b are positive constants.

Problem (1.8) has been studied by Dalmasso [5] on the unit ball with more restrictive conditions on ρ. Indeed, Dalmasso proved the existence of positive solutions provided that ρ is nondecreasing with respect to the second variable and satisfies

$$\lim_{t\to 0^+}\left(\min_{x\in \overline{B}} \frac{\rho(x,t)}{t}\right) = +\infty, \qquad \lim_{t\to +\infty}\left(\max_{x\in \overline{B}} \frac{\rho(x,t)}{t}\right) = 0. \tag{1.11}$$

When $\rho(x,t) = \rho(|x|,t)$, he showed the uniqueness of positive radial solution of (1.8).

On the other hand, problem (1.8) has been studied on the entire space $\mathbb{R}^n$ by Brezis and Kamin [3] for the special nonlinearity $\rho(x,t) = \nu(x)t^\alpha$, $0 < \alpha < 1$. More precisely they proved the existence and the uniqueness of positive solution for the problem below:

$$\Delta u = -\nu(x)u^\alpha \quad \text{in } \mathbb{R}^n \qquad \liminf_{|x|\to\infty} u = 0. \tag{1.12}$$

Notations and preliminaries. In order to simplify our statement, we adopt the following notations.

(i) $C_0(D) := \{f \in C(D) : \lim_{x\to\partial D} f(x) = 0\}$.

We note that $C_0(D)$ is a Banach space endowed with the uniform norm

$$\|f\|_\infty = \sup_{x\in D} |f(x)|. \tag{1.13}$$

(ii) Let f and g be two nonnegative functions on a set S.

We call $f \sim g$, if there exists a constant $c > 0$ such that

$$\frac{1}{c}g(x) \le f(x) \le cg(x) \quad \forall x \in S. \tag{1.14}$$

We call $f \preceq g$, if there exists a constant $c > 0$ such that

$$f(x) \le cg(x) \quad \forall x \in S. \tag{1.15}$$

(iii) Let f be a nonnegative function in D, then we denote by Vf the potential of f defined on D by

$$Vf(x) = \int_D G(x,y)f(y)dy. \tag{1.16}$$

We recall that if $f \in L^1_{\text{loc}}(D)$ and $Vf \in L^1_{\text{loc}}(D)$, then we have $\Delta(Vf) = -f$ in D (in the sense of distributions) (see [4, page 52]).

(iv) We denote by d the diameter of D.

(v) For $x, y \in D$, we denote $[x,y]^2 = |x-y|^2 + \delta(x)\delta(y)$.

2. Properties of the Green function and the class $K(D)$

In this section, we establish some results concerning the Green function $G(x,y)$ and the Kato class $K(D)$.

Proposition 2.1 (see [9, 10]). *Let q be a nonnegative function in $K(D)$. Then*

(i) *the potential $Vq \in C_0(D)$,*

(ii) *the function $x \to \delta(x)q(x)$ is in $L^1(D)$.*

In the sequel, we put

$$\|q\|_D = \sup_{x\in D}\int_D \frac{\delta(y)}{\delta(x)} G(x,y)\,|q(y)|\,dy, \tag{2.1}$$

$$\alpha_q = \sup_{x,y\in D}\int_D \frac{G(x,z)G(z,y)}{G(x,y)}\,|q(z)|\,dz \tag{2.2}$$

We recall that if $q \in K(D)$, then $\|q\|_D < \infty$.

Now, it is obvious to see that by (1.3), we have

$$\alpha_q \le 2C_0\|q\|_D, \tag{2.3}$$

where C_0 is the constant given by (1.3).

Next, we will prove that $\alpha_q \sim \|q\|_D$.

Proposition 2.2. *Let q be a function in $K(D)$. Then*

(i) *for any nonnegative superharmonic function h in D, we have*

$$\int_D G(x,y)\,|q(y)|\,h(y)dy \le \alpha_q h(x), \quad \forall x \in D, \tag{2.4}$$

(ii) *there exists a constant $C > 0$ such that*

$$C\|q\|_D \le \alpha_q. \tag{2.5}$$

Proof. (i) Let h be a nonnegative superharmonic function in D, then from [11, Theorem 2.1, page 164], there exists a sequence (f_k) of nonnegative measurable functions on D such that for all $y \in D$,

$$h_k(y) = \int_D G(x,z) f_k(z)dz \tag{2.6}$$

increases to $h(y)$.

Since for each $x, y \in D$, we have

$$\int_D G(x,y)\,|q(y)|\,h_k(y)dy \le \alpha_q h_k(x). \tag{2.7}$$

Thus, from the monotone convergence theorem, we deduce the result.

(ii) Let φ_1 be a positive eigenfunction corresponding to the first eigenvalue of the Dirichlet problem $\Delta u + \lambda u = 0$, $u_{|\partial D} = 0$. Then, from [8, Proposition 2.6] we have for $x \in D$

$$\varphi_1(x) \sim \delta(x). \tag{2.8}$$

Since, φ_1 is a superharmonic function in D, then by applying (i) to φ_1, we deduce (ii). □

Proposition 2.3. *Let $p > n/2$. Then for each $\lambda < 2 - n/p$, we have*

$$\frac{1}{(\delta(\cdot))^{\lambda}} L^p(D) \subset K(D). \tag{2.9}$$

To prove Proposition 2.3, we need the two next lemmas.

Lemma 2.4. *On D^2, we have*
(i) *for $n \geq 3$, $G(x,y) \sim \delta(x)\delta(y)/|x-y|^{n-2}[x,y]^2$,*
(ii) *for $n = 2$, $G(x,y) \sim (\delta(x)\delta(y)/[x,y]^2)\operatorname{Log}(1+[x,y]^2/|x-y|^2)$.*

Proof. (i) For each $a, b \geq 0$, we have

$$\min(a,b) \sim \frac{ab}{a+b}. \tag{2.10}$$

So, by (1.1) we deduce (i).

(ii) Using (1.1), the fact that for each $t \geq 0$, $\operatorname{Log}(1+t) \sim \min(1,t)\operatorname{Log}(2+t)$, and (2.10) we obtain that

$$G(x,y) \sim \min\left(1, \frac{\delta(x)\delta(y)}{|x-y|^2}\right)\operatorname{Log}\left(2+\frac{\delta(x)\delta(y)}{|x-y|^2}\right) \sim \frac{\delta(x)\delta(y)}{[x,y]^2}\operatorname{Log}\left(1+\frac{[x,y]^2}{|x-y|^2}\right). \tag{2.11}$$

□

Lemma 2.5. *Let $\lambda \in \mathbb{R}$. Then on D^2, we have*

$$\frac{1}{(\delta(y))^\lambda}\frac{\delta(y)}{\delta(x)}G(x,y) \preceq \begin{cases} \dfrac{1}{|x-y|^{n-2+\lambda^+}}, & \text{if } n \geq 3, \\[2ex] \dfrac{1}{|x-y|^{\lambda^+}}\operatorname{Log}\left(\dfrac{2d}{|x-y|}\right), & \text{if } n = 2, \end{cases} \tag{2.12}$$

where $\lambda^+ = \max(0,\lambda)$.

Proof. By Lemma 2.4, we have on D^2

$$\frac{1}{(\delta(y))^\lambda}\frac{\delta(y)}{\delta(x)}G(x,y) \preceq \begin{cases} \dfrac{1}{|x-y|^{n-2}}\dfrac{(\delta(y))^{2-\lambda}}{[x,y]^2}, & \text{if } n \geq 3, \\[2ex] \dfrac{(\delta(y))^{2-\lambda}}{[x,y]^2}\operatorname{Log}\left(1+\dfrac{[x,y]^2}{|x-y|^2}\right), & \text{if } n = 2. \end{cases} \tag{2.13}$$

Now, we remark that

$$[x,y]^2 \sim |x-y|^2 + 4\delta(x)\delta(y). \tag{2.14}$$

So, we have

$$\begin{aligned} [x,y]^2 &\succeq \max\left(\left|\delta(x)-\delta(y)\right|^2 + 4\delta(x)\delta(y), |x-y|^2\right) \\ &\succeq \max\left((\delta(y))^2, |x-y|^2\right). \end{aligned} \tag{2.15}$$

Therefore by (2.15) we have

$$\frac{1}{[x,y]^2} \preceq \frac{1}{|x-y|^{\lambda^+}(\delta(y))^{2-\lambda^+}}. \tag{2.16}$$

Hence, it follows that

$$\frac{(\delta(y))^{2-\lambda}}{[x,y]^2} \preceq \frac{1}{|x-y|^{\lambda^+}}. \tag{2.17}$$

Thus, for $n \geq 3$, we obtain

$$\frac{1}{(\delta(y))^{\lambda}}\frac{\delta(y)}{\delta(x)}G(x,y) \preceq \frac{1}{|x-y|^{n-2+\lambda^+}}. \tag{2.18}$$

Next, it is obvious to see that

$$\operatorname{Log}\left(1+\frac{[x,y]^2}{|x-y|^2}\right) \leq \operatorname{Log}\left(2\frac{[x,y]^2}{|x-y|^2}\right) \leq \operatorname{Log}\left(\frac{4d^2}{|x-y|^2}\right). \tag{2.19}$$

Then, for $n = 2$, we obtain by (2.17) and (2.19) that

$$\frac{1}{(\delta(y))^{\lambda}}\frac{\delta(y)}{\delta(x)}G(x,y) \preceq \frac{1}{|x-y|^{\lambda^+}}\operatorname{Log}\left(\frac{2d}{|x-y|}\right). \tag{2.20}$$

This completes the proof. □

Proof of Proposition 2.3. Let $\alpha > 0$, $p > n/2$ and $q \geq 1$ such that $(1/p)+(1/q) = 1$. To show the claim, we use Lemma 2.5 and the Hölder inequality. We distinguish two cases.
Case 1 ($n \geq 3$). Let $f \in L^p(D)$ and $\lambda < 2 - n/p$. Then, for $x \in D$, we have

$$\begin{aligned}
&\int_{D\cap B(x,\alpha)} \frac{\delta(y)}{\delta(x)}G(x,y)\frac{|f(y)|}{(\delta(y))^{\lambda}}dy \\
&\quad \preceq \int_{D\cap B(x,\alpha)} \frac{|f(y)|}{|x-y|^{n-2+\lambda^+}}dy \preceq \|f\|_p\left(\int_0^{\alpha} r^{n(1-q)+(2-\lambda^+)q-1}dr\right)^{1/q} \preceq \|f\|_p\alpha^{2-n/p-\lambda^+},
\end{aligned} \tag{2.21}$$

which tends to zero as $\alpha \to 0$.
Case 2 ($n = 2$). Let $f \in L^p(D)$ and $\lambda < 2/q$. Then, for $x \in D$, we have

$$\begin{aligned}
&\int_{D\cap B(x,\alpha)} \frac{\delta(y)}{\delta(x)}G(x,y)\frac{|f(y)|}{(\delta(y))^{\lambda}}dy \\
&\quad \preceq \int_{D\cap B(x,\alpha)} \frac{|f(y)|}{|x-y|^{\lambda^+}}\operatorname{Log}\left(\frac{2d}{|x-y|}\right)dy \preceq \|f\|_p\left(\int_0^{\alpha} r^{n-1-\lambda^+q}\left(\operatorname{Log}\frac{2d}{r}\right)^q dr\right)^{1/q},
\end{aligned} \tag{2.22}$$

which tends to zero as $\alpha \to 0$. This completes the proof. □

In the sequel, we put for $f \in \mathcal{B}(D)$ and $x \in D$,

$$v(x) = \int_D G(x,y)\frac{|f(y)|}{(\delta(y))^{\lambda}}dy. \tag{2.23}$$

Remark 2.6. From (1.1), we remark that for $x, y \in D$, we have $\delta(x)\delta(y) \preceq G(x,y)$. This implies that there exists a constant $C > 0$ such that for each $f \in \mathcal{B}(D)$ and $x \in D$,

$$C\delta(x)\int_D (\delta(y))^{1-\lambda}|f(y)|\,dy \leq v(x). \tag{2.24}$$

In the next proposition, we will give upper estimates on the function v.

PROPOSITION 2.7. *Let $p > n/2$ and $\lambda < 2 - n/p$. Then there exists a constant $c > 0$, such that for each $f \in L^p(D)$ and $x \in D$,*

$$v(x) \leq \begin{cases} c\|f\|_p(\delta(x))^{2-n/p-\lambda}, & \text{if } 1-\dfrac{n}{p} < \lambda < 2-\dfrac{n}{p}, \\ c\|f\|_p\delta(x)\left(\operatorname{Log}\dfrac{2d}{\delta(x)}\right)^{1/q}, & \text{if } \lambda = 1-\dfrac{n}{p}, \\ c\|f\|_p\delta(x), & \text{if } \lambda < 1-\dfrac{n}{p}. \end{cases} \tag{2.25}$$

To prove Proposition 2.7, we need the following lemma.

LEMMA 2.8 (see [8]). *Let $x, y \in D$. Then we have the following properties:*
(i) *if* $\delta(x)\delta(y) \leq |x-y|^2$ *then* $\min(\delta(x),\delta(y)) \leq ((\sqrt{5}+1)/2)|x-y|$,
(ii) *if* $|x-y|^2 \leq \delta(x)\delta(y)$ *then* $((3-\sqrt{5})/2)\delta(x) \leq \delta(y) \leq ((3+\sqrt{5})/2)\delta(x)$.

Proof of Proposition 2.7. Let $p > n/2$, $q \geq 1$ such that $(1/p)+(1/q) = 1$ and $\lambda < 2-n/p$. Let $f \in L^p(D)$, then for each $x \in D$, we have

$$v(x) = \int_{D_1} G(x,y)\frac{|f(y)|}{(\delta(y))^{\lambda}}dy + \int_{D_2} G(x,y)\frac{|f(y)|}{(\delta(y))^{\lambda}}dy = I_1 + I_2, \tag{2.26}$$

where

$$\begin{aligned} D_1 &= \{y \in D : \delta(x)\delta(y) \geq |x-y|^2\}, \\ D_2 &= \{y \in D : \delta(x)\delta(y) \leq |x-y|^2\}. \end{aligned} \tag{2.27}$$

Now, we remark that for each $x \in D$ and $y \in D_1$, we have by (1.1) and Lemma 2.8

$$\frac{1}{(\delta(y))^{\lambda}}G(x,y) \preceq \begin{cases} \dfrac{1}{(\delta(x))^{\lambda}}\dfrac{1}{|x-y|^{n-2}}, & \text{for } n \geq 3, \\ \dfrac{1}{(\delta(x))^{\lambda}}\operatorname{Log}\left(1+\left(\dfrac{2\delta(x)}{|x-y|}\right)^2\right), & \text{for } n = 2. \end{cases} \tag{2.28}$$

Then, by the Hölder inequality and Lemma 2.8, we obtain for $n \geq 3$

$$\begin{aligned} I_1 &\preceq \|f\|_p (\delta(x))^{-\lambda} \left(\int_{D_1} \frac{1}{|x-y|^{(n-2)q}} dy \right)^{1/q} \\ &\preceq \|f\|_p (\delta(x))^{-\lambda} \left(\int_0^{((\sqrt{5}+1)/2)\delta(x)} r^{n-1-(n-2)q} dr \right)^{1/q} \\ &\preceq \|f\|_p (\delta(x))^{2-\lambda-n/p}. \end{aligned} \tag{2.29}$$

Now, assume that $n = 2$, then since $q > 1$ and $\operatorname{Log}(1+t) \preceq t^{1/2q}$, for each $t \geq 1$, we obtain

$$\frac{1}{(\delta(y))^{\lambda}} G(x,y) \preceq \frac{(\delta(x))^{1/q-\lambda}}{|x-y|^{1/q}}. \tag{2.30}$$

So, by the Hölder inequality and Lemma 2.8, it follows that

$$\begin{aligned} I_1 &\preceq \|f\|_p (\delta(x))^{1/q-\lambda} \left(\int_{D_1} \frac{1}{|x-y|} dy \right)^{1/q} \\ &\preceq \|f\|_p (\delta(x))^{1/q-\lambda} \left(\int_0^{((\sqrt{5}+1)/2)\delta(x)} dr \right)^{1/q} \\ &\preceq \|f\|_p (\delta(x))^{2/q-\lambda} = \|f\|_p (\delta(x))^{2-\lambda-2/p}. \end{aligned} \tag{2.31}$$

Next, by (1.1), we have for each $x \in D$ and $y \in D_2$

$$\frac{1}{(\delta(y))^{\lambda}} G(x,y) \sim \frac{\delta(x)(\delta(y))^{1-\lambda}}{|x-y|^n}. \tag{2.32}$$

Then, using the Hölder inequality and Lemma 2.8, we obtain

$$I_2 \preceq \|f\|_p \left(\int_{D_2} \left(\frac{\delta(x)(\delta(y))^{1-\lambda}}{|x-y|^n} \right)^q dy \right)^{1/q}. \tag{2.33}$$

For each $y \in D_2$, it follows from Lemma 2.8 that $\delta(y) \preceq |x-y|$. So, we will discuss two cases.
Case 3. If $\lambda \leq 1$, it follows that

$$I_2 \preceq \|f\|_p \delta(x) \left(\int_{D_2} \frac{1}{|x-y|^{(n-1+\lambda)q}} dy \right)^{1/q} \tag{2.34}$$

$$\preceq \|f\|_p \delta(x) \left(\int_{((\sqrt{5}-1)/2)\delta(x)}^{d} r^{n-1-(n-1+\lambda)q} dr \right)^{1/q}. \tag{2.35}$$

Thus, we distinguish the following two subcases.

(a) If $\lambda \leq 1 - n/p$, then from (2.35) it follows that

$$
\begin{aligned}
I_2 &\preceq \|f\|_p \delta(x) \left(\int_{((\sqrt{5}-1)/2)\delta(x)}^{d} r^{(1-n-\lambda p)/(p-1)} dr \right)^{1/q} \\
&\preceq \|f\|_p \delta(x) \begin{cases} \left(\operatorname{Log} \dfrac{2d}{\delta(x)} \right)^{1/q} & \text{if } \lambda = 1 - \dfrac{n}{p}; \\ 1 & \text{if } \lambda < 1 - \dfrac{n}{p}. \end{cases}
\end{aligned}
\tag{2.36}
$$

(b) If $1 - n/p < \lambda \leq 1$, then by (2.34) we obtain

$$
\begin{aligned}
I_2 &\preceq \|f\|_p (\delta(x))^{2-\lambda-n/p} \left(\int_{D_2} \frac{(\delta(x))^{(\lambda+n/p-1)q}}{|x-y|^{(n-1+\lambda)q}} dy \right)^{1/q} \\
&= \|f\|_p (\delta(x))^{2-\lambda-n/p} \left(\int_{(((\sqrt{5}-1)/2)\delta(x) \leq |x-y| \leq d)} \frac{1}{|x-y|^n} dy \right)^{1/q} \\
&\preceq \|f\|_p (\delta(x))^{2-\lambda-n/p}.
\end{aligned}
\tag{2.37}
$$

Case 4. If $\lambda > 1$, then from (2.33) it follows that

$$
\begin{aligned}
I_2 &\preceq \|f\|_p (\delta(x))^{2-\lambda-n/p} \left(\int_{D_2} \left(\frac{\delta(x)}{\delta(y)} \right)^{(\lambda-1)q} \frac{(\delta(x))^{n/(p-1)}}{|x-y|^{n+n/p-1}} dy \right)^{1/q} \\
&\preceq \|f\|_p (\delta(x))^{2-\lambda-n/p} \left(\int_{D_2} \left(\frac{\delta(x)}{\delta(y)} \right)^{(\lambda-1)q} \frac{1}{|x-y|^n} dy \right)^{1/q}.
\end{aligned}
\tag{2.38}
$$

Since $(\lambda - 1)q \in]0,1[$, it follows from [8, Corollary 2.8] that

$$
I_2 \preceq \|f\|_p (\delta(x))^{2-\lambda-n/p}. \tag{2.39}
$$

This completes the proof. □

Remark 2.9. By taking $p = +\infty$ (i.e., $q = 1$), in Propositions 2.3 and 2.7, we find again the results of Mâagli in [8].

3. First existence result

In this section, we are interested in the existence of positive solutions for problem (1.5). We recall that h_0 is a fixed nontrivial nonnegative harmonic function in D, which is continuous in $\overline{D}$. Let φ be a nontrivial nonnegative continuous function on ∂D.

We denote by $H_D\varphi$ the solution of the Dirichlet problem

$$
\Delta w = 0 \quad \text{in } D, \qquad w_{|\partial D} = \varphi. \tag{3.1}
$$

The main result of this section is the following.

THEOREM 3.1. *Assume (H_1)-(H_2). Then there exists a constant $c > 1$ such that if $\varphi \geq ch_0$ on ∂D, then problem (1.5) has a positive continuous solution satisfying for each $x \in D$*

$$h_0(x) \leq u(x) \leq H_D\varphi(x). \tag{3.2}$$

To prove Theorem 3.1, we need the following lemma.

For a fixed $q \in K^+(D)$, put

$$\Gamma_q = \{v \in K(D) : |v| \leq q\}, \tag{3.3}$$

then, we have

LEMMA 3.2. *Let q be a nonnegative function belonging to $K(D)$, the family of functions*

$$\mathfrak{F}_q = \left\{ \int_D G(\cdot, y)v(y)dy : v \in \Gamma_q \right\} \tag{3.4}$$

is uniformly bounded and equicontinuous in $\overline{D}$, and consequently, it is relatively compact in $C_0(D)$.

Proof. Let $q \in K(D)$ and T be the operator defined on $\mathfrak{F}_q$ by

$$Tv(x) = \int_D G(x, y)v(y)dy. \tag{3.5}$$

By Proposition 2.1(i), we obtain

$$\sup_{x\in D} \left| Tv(x) \right| \leq \sup_{x\in D} \int_D G(x, y)q(y)dy < \infty. \tag{3.6}$$

Then the family $T(\mathfrak{F}_q)$ is uniformly bounded.

Next, we propose to prove the equicontinuity of $T(\mathfrak{F}_q)$ in $\overline{D}$.

Let $v \in \mathfrak{F}_q$, $x_0 \in D$, and $\alpha > 0$. Let $x, x' \in B(x_0, \alpha) \cap D$.

Then

$$\left| Tv(x) - Tv(x') \right| \leq \left| Vq(x) - Vq(x') \right|. \tag{3.7}$$

Since, by Proposition 2.1(i), $Vq \in C_0(D)$, it follows that

$$\left| Tv(x) - Tv(x') \right| \longrightarrow 0 \quad \text{as } |x - x'| \longrightarrow 0. \tag{3.8}$$

Similarly, we have $\lim_{x\to\partial D} Tv(x) = 0$. Which implies that the family $T(\mathfrak{F}_q)$ is equicontinuous in $\overline{D}$.

Finally, by Ascoli's theorem, the family $T(\mathfrak{F}_q)$ is relatively compact in $C_0(D)$. Which completes the proof. □

Proof of Theorem 3.1. We will use a fixed-point argument.

Let $c = 1 + \alpha_\psi$, where α_ψ is the constant defined by (2.2) associated to the function ψ given in (H_2). Let $\varphi \in C^+(\partial D)$ such that $\varphi \geq ch_0$ on ∂D.

We consider the set Λ given by

$$\Lambda = \{u \in C(\overline{D}) : h_0 \leq u \leq H_D\varphi\}. \tag{3.9}$$

Since $\varphi \geq ch_0$ on ∂D, we obtain

$$H_D\varphi \geq ch_0 \quad \text{on } D. \tag{3.10}$$

So Λ is a nonempty closed bounded and convex set in $C(\overline{D})$.

For each $u \in \Lambda$, define

$$Tu(x) = H_D\varphi(x) - \int_D G(x,y)f(y,u(y))\,dy, \quad \forall x \in D. \tag{3.11}$$

Now, we will prove that the family $T\Lambda$ is relatively compact in $C(\overline{D})$.

For each $y \in D$ and $u \in \Lambda$, we have by (H_2)

$$0 \leq f(y,u(y)) \leq \frac{\theta(y,h_0(y))}{h_0(y)}h_0(y) \preceq c\psi(y). \tag{3.12}$$

with $c = \sup_{y\in D} h_0(y)$. Then, the function $y \to f(y,u(y)) \in \Gamma_{c\psi}$.

Hence the family

$$\left\{\int_D G(\cdot,y)f(y,u(y))\,dy : u \in \Lambda\right\} \subseteq \mathfrak{F}_{c\psi}. \tag{3.13}$$

So, using Lemma 3.2 and the fact that $H_D\varphi$ is continuous in $\overline{D}$, we conclude that $T\Lambda$ is a relatively compact set in $C(\overline{D})$.

Next, we intend to show that T maps Λ to itself.

It's obvious to see that

$$Tu(x) \leq H_D\varphi(x), \quad \forall x \in D. \tag{3.14}$$

Moreover, from (H_1), and by using (3.11), (2.4), and (3.10), we obtain that for each $x \in D$

$$Tu(x) \geq H_D\varphi(x) - \alpha_\psi h_0(x) \geq h_0(x), \tag{3.15}$$

which proves that $T\Lambda \subset \Lambda$.

Now, let us prove the continuity of the operator T in Λ in the supremum norm. Let $(u_k)_k$ be a sequence in Λ which converges uniformly to a function u in Λ. Then, for each $x \in D$, we have

$$|Tu_k(x) - Tu(x)| \leq \int_D G(x,y)\,|f(y,u_k(y)) - f(y,u(y))|\,dy. \tag{3.16}$$

On the other hand, by hypothesis (H_1), we have

$$|f(y,u_k(y)) - f(y,u(y))| \leq 2h_0(y)\psi(y) \preceq \psi(y). \tag{3.17}$$

Since $V\psi \in C_0(D)$, we conclude by the continuity of f with respect to the second variable and the dominated convergence theorem that

$$\forall x \in \overline{D}, \quad Tu_k(x) \longrightarrow Tu(x) \quad \text{as } k \longrightarrow +\infty. \tag{3.18}$$

Since $T\Lambda$ is a relatively compact family in $C(\overline{D})$, therefore the pointwise convergence implies the uniform convergence, namely,

$$||Tu_k - Tu||_\infty \longrightarrow 0 \quad \text{as } k \longrightarrow +\infty. \tag{3.19}$$

Thus, T is a compact mapping on Λ.

Finally the Schauder fixed-point theorem implies the existence of $u \in \Lambda$ such that $Tu = u$, that is, for each $x \in D$

$$u(x) = H_D\varphi(x) - \int_D G(x,y)f(y,u(y))\,dy. \tag{3.20}$$

Now, let us verify that u is a solution of problem (1.5).

Since $\psi \in K(D)$, it follows from Proposition 2.1(ii), that $\psi \in L^1_{\text{loc}}(D)$.

Furthermore, we have $f(\cdot,u) \leq c\psi$, then $f(\cdot,u) \in L^1_{\text{loc}}(D)$ and $V(f(\cdot,u)) \in \mathfrak{F}_{c\psi}$. So by Lemma 3.2, we have

$$V(f(\cdot,u)) \in C_0(D) \subset L^1_{\text{loc}}(D). \tag{3.21}$$

Thus, applying Δ to both sides of (3.20) and using the fact that $\Delta(Vf) = -f$, we obtain, that u satisfies the elliptic differential equation

$$\Delta u = f(\cdot,u) \quad \text{in } D \text{ (in the sense of distributions)}. \tag{3.22}$$

Moreover, since $H_D\varphi = \varphi$ in ∂D and $V(f(\cdot,u)) \in C_0(D)$, we conclude that $u_{|\partial D} = \varphi$. So u is a positive continuous solution of problem (1.5). $\square$

Now, let us state another comparison result for the solution u of problem (1.5), in the case of the special nonlinearity $f(x,t) = q(x)\Phi(t)$.

For this aim, suppose that the following hypotheses on q and Φ are adopted.

(i) $\Phi : (0,\infty) \to (0,\infty)$ is continuously differentiable nonincreasing function.

(ii) q is a nontrivial nonnegative function on D such that

$$q \in C^\alpha_{\text{loc}}(D),\ 0 < \alpha < 1,\ \forall c > 0, \quad x \longrightarrow \frac{q(x)}{\delta(x)}\Phi(c\delta(x)) \in K(D). \tag{3.23}$$

Moreover, let F be the function defined on $[0,\infty)$ by

$$F(t) = \int_0^t \frac{1}{\Phi(s)}\,ds. \tag{3.24}$$

It is obviously seen, from hypotheses adopted on Φ, that the function F is a bijection from $[0,\infty)$ to itself. Then, we have the following.

Theorem 3.3. *Let u be the solution given by (3.20) of the following problem:*

$$\Delta u + q\Phi(u) = 0, \quad \text{in } D, \qquad u_{|\partial D} = \varphi. \tag{3.25}$$

Then, we have $u \in C^{2+\alpha}(D) \cap C(\overline{D})$. Further, u satisfies on D

$$u(x) \le \min\left(H_D\varphi(x), F^{-1}\left(H_D(F \circ \varphi)(x) - Vq(x)\right)\right). \tag{3.26}$$

Proof. Let v be the function defined on D by

$$v = F(u) - H_D(F \circ \varphi) + Vq. \tag{3.27}$$

Then $v \in C^2(D)$ and we have

$$\Delta v = \frac{1}{\Phi(u)}\Delta u - \frac{\Phi'(u)}{(\Phi(u))^2}|\nabla u|^2 - q = -\frac{\Phi'(u)}{(\Phi(u))^2}|\nabla u|^2. \tag{3.28}$$

Thus, $\Delta v \ge 0$. In addition, since $Vq \in C_0(D)$, it follows that $v \in C_0(D)$. Then, the maximum principle (see [6, pages 465-466]) implies that $v \le 0$, in D. This completes the proof. □

Remark 3.4. (1) Let $\lambda > 0$ and $\varphi(x) = \lambda$, $\forall x \in \partial D$. Then, we have for each $x \in D$,

$$H_D(F \circ \varphi)(x) - Vq(x) = F(\lambda)(x) - Vq(x) \le F(\lambda). \tag{3.29}$$

Thus for each $x \in D$,

$$F^{-1}\left(H_D(F \circ \varphi)(x) - Vq(x)\right) \le \lambda = H_D\varphi(x). \tag{3.30}$$

Therefore, from (3.26) we have for each $x \in D$,

$$h_0(x) \le u(x) \le F^{-1}\left(H_D(F \circ \varphi)(x) - Vq(x)\right). \tag{3.31}$$

(2) By hypothesis (i), we have

$$\Phi\left(\|u\|_\infty\right) \ge \Phi\left(\|\varphi\|_\infty\right). \tag{3.32}$$

Therefore,

$$h_0(x) \le u(x) \le H_D\varphi(x) - \Phi\left(\|\varphi\|_\infty\right)Vq(x). \tag{3.33}$$

Then we have

$$h_0 \le u \le \min\left(H_D\varphi - \Phi\left(\|\varphi\|_\infty\right), F^{-1}\left(H_D(F \circ \varphi) - Vq\right)\right). \tag{3.34}$$

Example 3.5. Let h_0 be a nontrivial nonnegative harmonic function, which is continuous on $\overline{D}$. Then, from [14], there exists c_1 such that for each $x \in D$

$$h_0(x) \ge c_1\delta(x). \tag{3.35}$$

Let $\alpha > 0$, and f be a nonnegative measurable function on $D \times (0,\infty)$, continuous with respect to the second variable satisfying

$$f(x,t) \preceq t^{-\alpha}(\delta(x))^{\alpha+1}q(x), \tag{3.36}$$

where the function q belongs to $K^+(D)$.

Then, there exists $c > 0$ such that if $\varphi \geq (1+c)h_0$ on ∂D, the problem

$$\begin{gathered} \Delta u = f(\cdot,u) \quad \text{(in the sense of distributions)} \\ u > 0 \quad \text{in } D, \qquad u_{|\partial D} = \varphi, \end{gathered} \tag{3.37}$$

has a positive continuous solution in $\overline{D}$ satisfying

$$h_0(x) \leq u(x) \leq H_D\varphi(x). \tag{3.38}$$

4. Second existence result

In this section, we prove the following result for problem (1.8).

Theorem 4.1. *Assume (H_3)-(H_4). Then problem (1.8) has a positive solution $u \in C_0(D)$. Moreover there exist positive constants a and b, such that*

$$a\delta(x) \leq u(x) \leq b. \tag{4.1}$$

Proof. By (A_2) and (H_4), the function $q \in K^+(D)$. Then, from Proposition 2.1(i), we have $Vq \in C_0(D)$. So, $M := \sup_{x\in D}(Vq(x)) < \infty$.

From (A_4), there exists $b > 0$ such that $Mk(b) \leq b$.

On the other hand, by (A_1), there exists a compact $K \subset D$ such that

$$0 < \int_K \delta(y)p(y)dy < \infty. \tag{4.2}$$

Furthermore, by (1.1), there exists $\alpha > 0$ such that for each x, y in D

$$G(x,y) \geq \alpha\delta(x)\delta(y). \tag{4.3}$$

Next, let r be the constant given by

$$r := \inf_{y\in K} \delta(y). \tag{4.4}$$

Then, from (H_4), there exists $a > 0$ such that

$$\alpha h(ar)\int_K \delta(y)p(y)dy \geq a. \tag{4.5}$$

Now, let Ω be the convex set

$$\Omega := \{u \in C_0(D) : a\delta(x) \leq u(x) \leq b\} \tag{4.6}$$

and S be the operator defined on Ω by

$$Su(x) = \int_D G(x,y)\rho(y,u(y))\,dy. \tag{4.7}$$

We will prove that S is a compact mapping on Ω.

By (H_4), we have for each $u \in \Omega$

$$\rho(\cdot,u) \le k(b)q = \tilde{q}. \tag{4.8}$$

Since $q \in K^+(D)$, it follows that the function $y \to \rho(y,u(y)) \in \Gamma_{\tilde{q}}$.

Hence, the family

$$\left\{ \int_\Omega G(\cdot,y)\rho(y,u(y))\,dy : u \in \Omega \right\} \subseteq \mathfrak{F}_{\tilde{q}}. \tag{4.9}$$

Consequently, by Lemma 3.2, the family $S(\Omega)$ is relatively compact in $C_0(D)$. Next, we need to verify that for $u \in \Omega$ and $x \in D$, we have

$$a\delta(x) \le Su(x) \le b. \tag{4.10}$$

Let $u \in \Omega$ and $x \in D$, then by (H_4), we have

$$\begin{aligned} Su(x) &\le \int_D G(x,y)q(y)k(u(y))\,dy \\ &\le k(b)\int_D G(x,y)q(y)\,dy \\ &\le Mk(b) \le b. \end{aligned} \tag{4.11}$$

On the other hand, from (H_4) and using (1.1) and (4.5), we have

$$\begin{aligned} Su(x) &\ge \alpha\delta(x)\int_D \delta(y)p(y)h(u(y))\,dy \\ &\ge \alpha\delta(x)\int_K \delta(y)p(y)h(a\delta(y))\,dy \\ &\ge \delta(x)\left[\alpha h(ar)\int_K \delta(y)p(y)\,dy\right] \ge a\delta(x). \end{aligned} \tag{4.12}$$

Thus, it follows that $S(\Omega) \subset \Omega$.

Now, we consider a sequence $(u_k)_k$ in Ω which converges uniformly to u in Ω. Since ρ is continuous with respect to the second variable, we deduce by the dominated convergence theorem that for all $x \in D$,

$$Su_k(x) \longrightarrow Su(x) \quad \text{as } k \longrightarrow +\infty. \tag{4.13}$$

Therefore, using the fact that $S(\Omega)$ is relatively compact in $C_0(D)$, we conclude that $\|Su_k - Su\|_\infty$ as $k \to +\infty$. Hence S is a compact mapping from Ω to itself. Then by the

Schauder fixed-point theorem, there exists a function $u \in \Omega$ such that

$$u(x) = \int_D G(x,y)\rho(y,u(y))\,dy. \tag{4.14}$$

Now, since $q \in K^+(D)$ then by Proposition 2.1(ii), we have $\rho(\cdot,u) \in L^1_{\mathrm{loc}}(D)$ and $V(\rho(\cdot, u)) \in C_0(D) \subset L^1_{\mathrm{loc}}(D)$.

So, u satisfies (in the sense of distributions) $\Delta u = -\rho(\cdot,u)$ in D. Moreover, $\lim_{x\to\partial D} u(x) = \lim_{x\to\partial D} V(\rho(\cdot,u(\cdot)))(x) = 0$. So u is a solution of problem (1.8). □

Example 4.2. Let $p > n/2$ and $f \in L^p_+(D)$. Assume that the function $g:(0,\infty) \to [0,\infty)$ is a nontrivial continuous and nondecreasing function satisfying

$$\lim_{t\to 0^+} \frac{g(t)}{t} = +\infty, \qquad \lim_{t\to+\infty} \frac{g(t)}{t} = 0. \tag{4.15}$$

Then for each $\lambda < 2 - n/p$ the problem

$$\Delta u = -(\delta(x))^{-\lambda} f(x) g(u) \quad \text{in } D, \qquad u_{|\partial D} = 0, \tag{4.16}$$

has a positive solution $u \in C_0(D)$. Moreover, from Proposition 2.7, we have for each $x \in D$,

$$u(x) \le \begin{cases} c\|f\|_p (\delta(x))^{2-n/p-\lambda}, & \text{if } 1 - \dfrac{n}{p} < \lambda < 2 - \dfrac{n}{p}, \\ c\|f\|_p \delta(x) \left(\operatorname{Log} \dfrac{2d}{\delta(x)}\right)^{(p-1)/p}, & \text{if } \lambda = 1 - \dfrac{n}{p}, \\ c\|f\|_p \delta(x), & \text{if } \lambda < 1 - \dfrac{n}{p}. \end{cases} \tag{4.17}$$

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