COMMON FIXED POINTS OF ONE-PARAMETER NONEXPANSIVE SEMIGROUPS IN STRICTLY CONVEX BANACH SPACES

TOMONARI SUZUKI

Received 20 December 2003; Accepted 10 July 2005

One of our main results is the following convergence theorem for one-parameter nonexpansive semigroups: let C be a bounded closed convex subset of a Hilbert space E, and let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on C. Fix $u \in C$ and $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$. Define a sequence $\{x_n\}$ in C by $x_n = (1-\alpha_n)/(t_2 - t_1)\int_{t_1}^{t_2} T(s)x_n ds + \alpha_n u$ for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ converging to 0. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \in \mathbb{R}_+\}$.

1. Introduction

Let C be a closed convex subset of a Banach space E, and let T be a *nonexpansive mapping* on C, that is, $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We know that T has a fixed point in the case that E is uniformly convex and C is bounded; see Browder [4], Göhde [10], and Kirk [15]. We denote by $F(T)$ the set of fixed points of T.

Let $\{T(t) : t \in \mathbb{R}_+\}$ be a *strongly continuous semigroup of nonexpansive mappings* (*nonexpansive semigroup*, in short) on a closed convex subset C of a Banach space E, that is,

(i) for each $t \in \mathbb{R}_+$, $T(t)$ is a nonexpansive mapping on C;
(ii) $T(s+t) = T(s) \circ T(t)$ for all $s, t \in \mathbb{R}_+$;
(iii) for each $x \in C$, the mapping $t \mapsto T(t)x$ from $\mathbb{R}_+$ into C is strongly continuous.

We also know that $\{T(t) : t \in \mathbb{R}_+\}$ has a common fixed point in the case that E is uniformly convex and C is bounded; see Browder [4]. Bruck [7] prove the following theorem.

Theorem 1.1 (Bruck [7]). *Suppose a closed convex subset C of a Banach space has the fixed point property for nonexpansive mappings, and C is either weakly compact, or bounded and separable. Then for any commuting family S of nonexpansive mappings on C, the set of common fixed points of S is a nonempty nonexpansive retract of C.*

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2006, Article ID 58684, Pages 1–10
DOI 10.1155/AAA/2006/58684

This theorem yields that $\{T(t) : t \in \mathbb{R}_+\}$ has a common fixed point in the case that C has the fixed point property, and that C is weakly compact, or bounded and separable.

Several authors have studied about convergence theorems for nonexpansive semigroups; see [1, 2, 13, 16, 19, 21, 22] and others. For example, the following theorem is a corollary of Theorem 8 in [19].

Theorem 1.2 (Shioji and Takahashi [19]). *Let C be a bounded closed convex subset of a Hilbert space E. Let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on C. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $\lim_n \alpha_n = 0$, $t_n > 0$ and $\lim_n t_n = \infty$. Fix $u \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_n = \frac{1-\alpha_n}{t_n} \int_0^{t_n} T(s)x_n \, ds + \alpha_n u \tag{1.1}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \in \mathbb{R}_+\}$.

Also, Suzuki[21] proved the following theorem.

Theorem 1.3 (Suzuki [21]). *Let E, C, $\{T(t) : t \in \mathbb{R}_+\}$ be as in Theorem 1.2. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_n t_n = \lim_n \alpha_n / t_n = 0$. Fix $u \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_n = (1-\alpha_n)T(t_n)x_n + \alpha_n u \tag{1.2}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \in \mathbb{R}_+\}$.

We note that in these theorems, real sequences $\{t_n\}$ converge to 0 and ∞. So, it is natural to study convergence theorems under the assumption that $\{t_n\}$ is a constant sequence. In this paper, motivated by Theorems 1.2 and 1.3, we consider such type of convergence theorems to a common fixed point of $\{T(t) : t \in \mathbb{R}_+\}$.

2. Preliminaries

Throughout this paper we denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}_+$ the set of nonnegative real numbers, and by $\mathbb{N}$ the set of positive integers. For a Banach space E, we also denote by E^* the dual space of E.

We recall that a Banach space E is called strictly convex if $\|x+y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. We know the following lemma.

Lemma 2.1. *Let E be a Banach space. Then the following are equivalent:*
(i) *E is strictly convex;*
(ii) *$\|\lambda x + (1-\lambda)y\| < 1$ for all $\lambda \in (0,1)$ and $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$;*
(iii) *if $\|x\| = \|y\| = \|\lambda x + (1-\lambda)y\|$ for some $\lambda \in (0,1)$, then $x = y$.*

A Banach space E is called uniformly convex if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x+y\|/2 < 1-\delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \varepsilon$. It is clear that a uniformly convex Banach space is strictly convex. The norm of E is called Fréchet differentiable if for each $x \in E$ with $\|x\| = 1$, $\lim_{t \to 0}(\|x+ty\| - \|x\|)/t$ exists and is attained uniformly in $y \in E$ with $\|y\| = 1$. A Banach space E is said to have the Opial property [17] if for each weakly convergent sequence $\{x_n\}$ in E with weak limit z, $\liminf_n \|x_n - z\| < \liminf_n \|x_n - y\|$ for all $y \in E$ with $y \neq z$. All Hilbert spaces, all finite dimensional Banach spaces and $\ell^p (1 \le p < \infty)$ have the Opial property. Gossez and Lami Dozo[11] prove that every weakly compact convex subset of a Banach space with the Opial property has normal structure. We also know that every separable Banach space can be equivalently renormed so that it has the Opial property; see [23].

3. Common fixed points

In this section, we give our main results. The following proposition plays an important role in this paper.

Proposition 3.1. *Let C be a closed convex subset of a strictly convex Banach space E. Let $\tau_\infty > 0$ and let $\{T(t) : t \in [0,\tau_\infty)\}$ be a family of mappings on C satisfying the following:*

(i) *for each $t \in [0,\tau_\infty)$, $T(t)$ is nonexpansive;*

(ii) *there exists a strictly increasing sequence $\{\tau_n\}$ in $[0,\tau_\infty)$ such that $\tau_1 = 0$, $\{\tau_n\}$ converges to τ_∞, and mappings $t \mapsto T(t)x$ are weakly continuous on $[\tau_n, \tau_{n+1})$ for all $x \in C$ and $n \in \mathbb{N}$.*

Suppose that

$$\bigcap_{t\in[0,\tau_\infty)} F(T(t)) \neq \varnothing. \tag{3.1}$$

Then

$$\bigcap_{t\in[0,\tau_\infty)} F(T(t)) = F(S), \tag{3.2}$$

where S is a nonexpansive mapping on C defined by

$$Sx = \frac{1}{\tau_\infty}\int_0^{\tau_\infty} T(s)x\,ds \tag{3.3}$$

for all $x \in C$.

Remark 3.2. We do not assume $\{T(\cdot)\}$ is a nonexpansive semigroup.

Proof. Fix $f \in E^*$. Then the functions $t \mapsto f(T(t)x)$ from $[\tau_n, \tau_{n+1})$ into $\mathbb{R}$ are continuous on $[\tau_n, \tau_{n+1})$ for $x \in C$ and $n \in \mathbb{N}$. So, the functions $t \mapsto f(T(t)x)$ from $[0,\tau_\infty)$ into $\mathbb{R}$ are measurable for $x \in C$. We also have $\{T(t)x : t \in [0,\tau_\infty)\}$ is separable for each $x \in C$. Fix $w \in \bigcap_{t\in[0,\tau_\infty)} F(T(t))$. Since

$$\begin{aligned} ||T(t)x|| &= ||T(t)x|| - ||T(t)w|| + \|w\| \le |T(t)x - T(t)w| + \|w\| \\ &\le \|x - w\| + \|w\|, \end{aligned} \tag{3.4}$$

for $x \in C$ and $t \in [0,\tau_\infty)$, we have that the mappings $t \mapsto T(t)x$ are Bochner integrable for all $x \in C$ and hence S is well-defined. Using the separation theorem, we can easily prove that S is a mapping on C. Since

$$\begin{aligned}\|Sx-Sy\| &= \left|\frac{1}{\tau_\infty}\int_0^{\tau_\infty}(T(s)x-T(s)y)\,ds\right\| \\ &\leq \frac{1}{\tau_\infty}\int_0^{\tau_\infty}\|T(s)x-T(s)y\|\,ds \\ &\leq \frac{1}{\tau_\infty}\int_0^{\tau_\infty}\|x-y\|\,ds=\|x-y\|\end{aligned} \tag{3.5}$$

for $x,y\in C$, S is nonexpansive. Therefore S is a nonexpansive mapping on C. It is obvious that $\bigcap_{t\in[0,\tau_\infty)}F(T(t))\subset F(S)$. We assume that $z\in F(S)\setminus\bigcap_{t\in[0,\tau_\infty)}F(T(t))$. Then there exists $t_1\in[0,\tau_\infty)$ such that $T(t_1)z\neq z$. Fix $g\in E^*$ with

$$\|g\|=1,\qquad g(T(t_1)z-z)=\|T(t_1)z-z\|. \tag{3.6}$$

For some $m\in\mathbb{N}$, t_1 belongs to $[\tau_m,\tau_{m+1})$. From the assumption (ii), there exists $t_2\in(t_1,\tau_{m+1})$ such that

$$g(T(t)z-z)>\frac{1}{2}\|T(t_1)z-z\| \tag{3.7}$$

for all $t\in[t_1,t_2)$. Define nonexpansive mappings S_1 and S_2 on C by

$$\begin{aligned}S_1x &= \frac{1}{t_2-t_1}\int_{t_1}^{t_2}T(s)x\,ds, \\ S_2x &= \frac{1}{\tau_\infty-t_2+t_1}\left(\int_0^{t_1}T(s)x\,ds+\int_{t_2}^{\tau_\infty}T(s)x\,ds\right)\end{aligned} \tag{3.8}$$

for all $x\in C$. We note that

$$Sx=\frac{t_2-t_1}{\tau_\infty}S_1x+\frac{\tau_\infty-t_2+t_1}{\tau_\infty}S_2x \tag{3.9}$$

for all $x\in C$. We have

$$\begin{aligned}g(S_1z-Sz) &= g\left(\frac{1}{t_2-t_1}\int_{t_1}^{t_2}T(s)z\,ds-z\right) \\ &= g\left(\frac{1}{t_2-t_1}\int_{t_1}^{t_2}(T(s)z-z)\,ds\right) \\ &= \frac{1}{t_2-t_1}\int_{t_1}^{t_2}g(T(s)z-z)\,ds \\ &\geq \frac{1}{t_2-t_1}\int_{t_1}^{t_2}\frac{1}{2}\|T(t_1)z-z\|\,ds \\ &= \frac{1}{2}\|T(t_1)z-z\|>0.\end{aligned} \tag{3.10}$$

Hence

$$g(S_2z-Sz)=\frac{t_2-t_1}{\tau_\infty-t_2+t_1}g(Sz-S_1z)<0. \tag{3.11}$$

Therefore $S_1 z \neq S_2 z$. Fix $w \in \bigcap_{t \in [0,\tau_\infty)} F(T(t))$. Then we note that $S_1 w = S_2 w = w$. We have

$$\begin{aligned} \|z-w\| = \|Sz-w\| &= \left\| \frac{t_2-t_1}{\tau_\infty} S_1 z + \frac{\tau_\infty - t_2 + t_1}{\tau_\infty} S_2 z - w \right\| \\ &\le \frac{t_2-t_1}{\tau_\infty} \|S_1 z - w\| + \frac{\tau_\infty - t_2 + t_1}{\tau_\infty} \|S_2 z - w\| \\ &= \frac{t_2-t_1}{\tau_\infty} \|S_1 z - S_1 w\| + \frac{\tau_\infty - t_2 + t_1}{\tau_\infty} \|S_2 z - S_2 w\| \\ &\le \frac{t_2-t_1}{\tau_\infty} \|z-w\| + \frac{\tau_\infty - t_2 + t_1}{\tau_\infty} \|z-w\| = \|z-w\| \end{aligned} \tag{3.12}$$

and hence

$$\|Sz-w\| = \|S_1 z - w\| = \|S_2 z - w\|. \tag{3.13}$$

This contradicts the strict convexity of E. Therefore, $F(S) \subset \bigcap_{t \in [0,\tau_\infty)} F(T(t))$. This completes the proof. □

As a direct consequence of Proposition 3.1, we can prove the following, which was proved by Bruck [6]; see also [20].

Corollary 3.3 (Bruck [6]). *Let C be a closed convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\alpha_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \alpha_n = 1$. Define a nonexpansive mapping S on C by*

$$Sx = \sum_{n=1}^{\infty} \alpha_n T_n x \tag{3.14}$$

for $x \in C$. Then $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ holds.

Proof. Define a strictly increasing sequence $\{\tau_n\}$ in $[0,1)$ by $\tau_1 = 0$ and

$$\tau_n = \sum_{k=1}^{n-1} \alpha_k \tag{3.15}$$

for $n \in \mathbb{N}$ with $n \ge 2$. We note that $\lim_n \tau_n = 1$. Define a family $\{T(t) : t \in [0,1)\}$ of nonexpansive mappings as follows: If $\tau_n \le t < \tau_{n+1}$, then

$$T(t)x = T_n x \tag{3.16}$$

for all $x \in C$. Then we note that

$$Sx = \sum_{n=1}^{\infty} \alpha_n T_n x = \sum_{n=1}^{\infty} \int_{\tau_n}^{\tau_{n+1}} T(s)x\, ds = \int_0^1 T(s)x\, ds = \frac{1}{1} \int_0^1 T(s)x\, ds \tag{3.17}$$

for $x \in C$ and

$$\bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{t \in [0,1)} F(T(t)). \tag{3.18}$$

So, by Proposition 3.1, we obtain the desired result. □

As another direct consequence of Proposition 3.1, we obtain the following proposition.

Proposition 3.4. *Let C be a closed convex subset of a strictly convex Banach space E. Let $\tau > 0$ and let $\{T(t) : t \in [0,\tau)\}$ be a family of mappings on C satisfying the following:*
(i) *for each $t \in [0,\tau)$, $T(t)$ is nonexpansive;*
(ii) *mappings $t \mapsto T(t)x$ are weakly continuous on $[0,\tau)$ for all $x \in C$.*

Suppose that

$$\bigcap_{t \in [0,\tau)} F(T(t)) \neq \varnothing. \tag{3.19}$$

Then

$$\bigcap_{t \in [0,\tau)} F(T(t)) = F(S), \tag{3.20}$$

where S is a nonexpansive mapping on C defined by

$$Sx = \frac{1}{\tau} \int_0^{\tau} T(s)x\, ds \tag{3.21}$$

for all $x \in C$.

Now, we prove one of our main results.

Theorem 3.5. *Let C be a closed convex subset of a strictly convex Banach space E and let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on C. Suppose that*

$$\bigcap_{t \in \mathbb{R}_+} F(T(t)) \neq \varnothing. \tag{3.22}$$

Fix $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$, and define a nonexpansive mapping S on C by

$$Sx = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} T(s)x\, ds \tag{3.23}$$

for all $x \in C$. Then

$$\bigcap_{t \in \mathbb{R}_+} F(T(t)) = F(S) \tag{3.24}$$

holds.

Proof. It is clear that $\bigcap_{t\in\mathbb{R}_+} F(T(t)) \subset F(S)$. Fix $w \in F(S)$. By Proposition 3.4, we have

$$\bigcap_{t\in[t_1,t_2)} F(T(t)) = F(S). \tag{3.25}$$

So, $T(t)w = w$ for $t \in [t_1,t_2)$. Hence, for every $t \in [0,(t_2-t_1)/2]$, we have

$$T(t)w = T(t)\circ T(t_1)w = T(t+t_1)w = w. \tag{3.26}$$

Let $t \in \mathbb{R}_+$ be fixed. Then there exist $m \in \mathbb{N}\cup\{0\}$ and $u \in [0,(t_2-t_1)/2)$ such that $t = u+m(t_2-t_1)/2$. We have

$$T(t)w = T\left(u+m\frac{t_2-t_1}{2}\right)w = T(u)\circ T\left(\frac{t_2-t_1}{2}\right)^m w = T(u)w = w, \tag{3.27}$$

where $T((t_2-t_1)/2)^0$ is the identity mapping on C. Therefore w is a common fixed point of $\{T(t) : t\in\mathbb{R}_+\}$. This completes the proof. $\square$

Similarly we can prove the following theorem.

Theorem 3.6. *Let C be a closed convex subset of a strictly convex Banach space E and let $\{\{T_n(t) : t\in\mathbb{R}_+\} : n\in\mathbb{N}\}$ be a sequence of strongly continuous semigroups of nonexpansive mappings on C. Let $\{U_n : n\in\mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose that*

$$\bigcap_{n=1}^{\infty}\bigcap_{t\in\mathbb{R}_+} F(T_n(t)) \cap \bigcap_{n=1}^{\infty} F(U_n) \neq \varnothing. \tag{3.28}$$

Let $\{t_n\}$, $\{u_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences such that $0\le t_n<u_n$, $\alpha_n>0$ and $\beta_n>0$ for all $n\in\mathbb{N}$, and $\sum_{n=1}^{\infty}\alpha_n+\sum_{n=1}^{\infty}\beta_n = 1$. Define a nonexpansive mapping S on C by

$$Sx = \sum_{n=1}^{\infty}\frac{\alpha_n}{u_n-t_n}\int_{t_n}^{u_n} T_n(s)x\,ds + \sum_{n=1}^{\infty}\beta_n U_n x \tag{3.29}$$

for all $x\in C$. Then

$$\bigcap_{n=1}^{\infty}\bigcap_{t\in\mathbb{R}_+} F(T_n(t)) \cap \bigcap_{n=1}^{\infty} F(U_n) = F(S). \tag{3.30}$$

holds.

We recall that a closed convex subset C of a Banach space E is said to have the *f*ixed point property for nonexpansive mappings (*F*PP, in short) if for every bounded closed convex subset D of C, every nonexpansive mapping on D has a fixed point. So, by the results of Browder [4] and Göhde [10], every uniformly convex Banach space has FPP. Also, by Kirk's fixed point theorem [15], every weakly compact convex subset with normal structure has FPP.

As a direct consequence of Theorem 3.6, we obtain the following corollary.

Corollary 3.7. *Let E, C, $\{\{T_n(t) : t \in \mathbb{R}_+\} : n \in \mathbb{N}\}$, $\{U_n : n \in \mathbb{N}\}$, $\{t_n\}$, $\{u_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ be as in Theorem 3.6. Assume that C is weakly compact and has FPP, and*

$$T_m(s) \circ T_n(t) = T_n(t) \circ T_m(s), \qquad U_m \circ U_n = U_n \circ U_m, \quad U_m \circ T_n(t) = T_n(t) \circ U_m \tag{3.31}$$

for all $s, t \in \mathbb{R}_+$ and $m, n \in \mathbb{N}$. Define a nonexpansive mapping S on C as in Theorem 3.6. Then

$$\bigcap_{n=1}^{\infty} \bigcap_{t \in \mathbb{R}_+} F(T_n(t)) \cap \bigcap_{n=1}^{\infty} F(U_n) = F(S) \neq \varnothing. \tag{3.32}$$

holds.

4. Convergence theorems

Using Theorem 3.5, we can prove many convergence theorems to a common fixed point of nonexpansive semigroups. In this section, we state some of them.

From the result of Ishikawa [14], we obtain the following theorem see also Edelstein [8].

Theorem 4.1. *Let C be a compact convex subset of a strictly convex Banach space E. Let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on C. Fix $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = \frac{\alpha_n}{t_2 - t_1} \int_{t_1}^{t_2} T(s)x_n \, ds + (1 - \alpha_n)x_n \tag{4.1}$$

for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_n \alpha_n < 1$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \in \mathbb{R}_+\}$.

From the results of Edelstein and O'Brien [9], and Reich [18], we obtain the following theorem.

Theorem 4.2. *Let E be a Banach space. Suppose either of the following holds:*

(i) *E is strictly convex and has the Opial property; or*

(ii) *E is uniformly convex and its norm is Fréchet differentiable.*

Let C be a weakly compact convex subset of E, and let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on C. Fix $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$x_{n+1} = \frac{\alpha}{t_2 - t_1} \int_{t_1}^{t_2} T(s)x_n \, ds + (1 - \alpha)x_n \tag{4.2}$$

for $n \in \mathbb{N}$, where α is a constant number in $(0,1)$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T(t) : t \in \mathbb{R}_+\}$.

We note that

$$x \longmapsto (1 - \alpha)Tx + \alpha u \tag{4.3}$$

is a contractive mapping if T is a nonexpansive mapping and $\alpha \in (0,1)$. By the Banach contraction principle [3], such mappings have a unique fixed point. From the results of Browder [5], and Wittmann [24], we obtain the following theorem; see also [12]. Compare Theorem 4.3 with Theorems 1.2 and 1.3.

THEOREM 4.3. *Let C be a bounded closed convex subset of a Hilbert space E, and let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on C. Fix $u \in C$ and $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$. Define a sequence $\{x_n\}$ in C by*

$$x_n = \frac{1-\alpha_n}{t_2 - t_1} \int_{t_1}^{t_2} T(s)x_n \, ds + \alpha_n u \tag{4.4}$$

for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ converging to 0. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \in \mathbb{R}_+\}$.

THEOREM 4.4. *Let E, C, $\{T(t) : t \in \mathbb{R}_+\}$, u, t_1 and t_2 be as in Theorem 4.3. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = \frac{1-\alpha_n}{t_2 - t_1} \int_{t_1}^{t_2} T(s)x_n \, ds + \alpha_n u \tag{4.5}$$

for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying the following:

$$\lim_{n\to\infty} \alpha_n = 0; \qquad \sum_{n=1}^{\infty} \alpha_n = \infty; \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \tag{4.6}$$

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \in \mathbb{R}_+\}$.

Acknowledgment

The author is supported in part by Grants-in-Aid for Scientific Research from the Japanese Ministry of Education, Culture, Sports, Science and Technology.

References

[1] S. Atsushiba and W. Takahashi, *Strong convergence theorems for one-parameter nonexpansive semigroups with compact domains*, Fixed Point Theory and Applications (Y. J. Cho, J. K. Kim, and S. M. Kang, eds.), vol. 3, Nova Science, New York, 2002, pp. 15–31.

[2] J.-B. Baillon, *Quelques propriétés de convergence asymptotique pour les semi-groupes de contractions impaires*, Comptes Rendus de l'Académie des Sciences. Série A-B **283** (1976), no. 3, A75–A78.

[3] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fundamenta Mathematicae **3** (1922), 133–181.

[4] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proceedings of the Academy of Natural Sciences of Philadelphia U.S.A **54** (1965), 1041–1044.

[5] ———, *Convergence of approximants to fixed points of nonexpansive non-linear mappings in Banach spaces*, Archive for Rational Mechanics and Analysis **24** (1967), no. 1, 82–90.

[6] R. E. Bruck Jr., *Properties of fixed-point sets of nonexpansive mappings in Banach spaces*, Transactions of the American Mathematical Society **179** (1973), 251–262.

[7] ______, *A common fixed point theorem for a commuting family of nonexpansive mappings*, Pacific Journal of Mathematics **53** (1974), no. 1, 59–71.
[8] M. Edelstein, *A remark on a theorem of M. A. Krasnoselski*, The American Mathematical Monthly **73** (1966), no. 5, 509–510.
[9] M. Edelstein and R. C. O'Brien, *Nonexpansive mappings, asymptotic regularity and successive approximations*, The Journal of the London Mathematical Society. Second Series **17** (1978), no. 3, 547–554.
[10] D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, Mathematische Nachrichten **30** (1965), 251–258.
[11] J.-P. Gossez and E. Lami Dozo, *Some geometric properties related to the fixed point theory for nonexpansive mappings*, Pacific Journal of Mathematics **40** (1972), no. 3, 565–573.
[12] B. Halpern, *Fixed points of nonexpanding maps*, American Mathematical Society. Bulletin **73** (1967), 957–961.
[13] N. Hirano, *Nonlinear ergodic theorems and weak convergence theorems*, Journal of the Mathematical Society of Japan **34** (1982), no. 1, 35–46.
[14] S. Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proceedings of the American Mathematical Society **59** (1976), no. 1, 65–71.
[15] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, The American Mathematical Monthly **72** (1965), no. 9, 1004–1006.
[16] I. Miyadera and K. Kobayasi, *On the asymptotic behaviour of almost-orbits of nonlinear contraction semigroups in Banach spaces*, Nonlinear Analysis **6** (1982), no. 4, 349–365.
[17] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, American Mathematical Society. Bulletin **73** (1967), 591–597.
[18] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, Journal of Mathematical Analysis and Applications **67** (1979), no. 2, 274–276.
[19] N. Shioji and W. Takahashi, *Strong convergence theorems for asymptotically nonexpansive semigroups in Hilbert spaces*, Nonlinear Analysis **34** (1998), no. 1, 87–99.
[20] T. Suzuki, *Convergence theorems to common fixed points for infinite families of nonexpansive mappings in strictly convex Banach spaces*, Nihonkai Mathematical Journal **14** (2003), no. 1, 43–54.
[21] ______, *On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces*, Proceedings of the American Mathematical Society **131** (2003), no. 7, 2133–2136.
[22] T. Suzuki and W. Takahashi, *Strong convergence of Mann's type sequences for one-parameter nonexpansive semigroups in general Banach spaces*, Journal of Nonlinear and Convex Analysis **5** (2004), no. 2, 209–216.
[23] D. van Dulst, *Equivalent norms and the fixed point property for nonexpansive mappings*, The Journal of the London Mathematical Society. Second Series **25** (1982), no. 1, 139–144.
[24] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Archiv der Mathematik **58** (1992), no. 5, 486–491.

Tomonari Suzuki: Department of Mathematics, Kyushu Institute of Technology, Sensuicho, Tobata, Kitakyushu 804-8550, Japan
E-mail address: suzuki-t@mns.kyutech.ac.jp