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Research Article

Direction and Stability of Hopf Bifurcation in a Delayed Solow Model with Labor Demand

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This paper is concerned with a delayed model of mutual interactions between the economically active population and the economic growth. The main purpose is to investigate the direction and stability of the bifurcating branch resulting from the increase of delay. By using a second order approximation of the center manifold, we compute the first Lyapunov coefficient for Hopf bifurcation points and we show that the system under consideration can undergo a supercritical or subcritical Hopf bifurcation and the bifurcating periodic solution is stable or unstable in a neighborhood of some bifurcation points, depending on the choice of parameters.

1. Introduction

The term "economic growth" is, generally, employed to describe the increase in the potential level of real output produced by an economy over a period of time. It is conventionally measured as the percent rate of increase in real gross domestic product (GDP). Several studies have analyzed national income and capital stock to explain an economy's growth rate in terms of the level of labor force, saving, and productivity of capital. After the pioneering works of Harrod (1939) [1], Domar (1946) [2], Cassel (1924) (see Kelley (1967, [3]), and Solow (1956) [4], the literature on the modeling of mutual interactions between the economically active population and economic growth has increased considerably in recent years (see, for example, [5–9] and references cited in these publications). The original model of economic growth put forward by Solow (1956), based on an ordinary differential equation, was employed to describe the evolution of capital stock and, consequently, to explain an economy's growth rate in terms of the labor force, the level of saving, and productivity of capital [4]. The aforementioned models play an important role in explaining phenomena related to economic growth, namely, development, population growth, unemployment, and savings and they have been developed and investigated in-depth in terms of the stability, bifurcations, oscillations, and chaotic behavior of solutions [5, 10–13].

Recently, researchers in applied mathematics have proposed systems of differential equations to analytically study the relationship between economic growth and the population concerned [14–18]. They have proved the birth of branches of bifurcated periodic solutions from a positive equilibrium when the delay, namely, the time needed to build, plan, and install new equipment, increases and crosses some critical values [9, 15, 16].

In [9], 2016, we divided the labor force population (the economically active population) into two subgroups: the unemployed and the employed persons. The national economy creates jobs to deal with persons who enter the job market every year (increasing of the number of unemployed persons). This creation reduces the unemployment rate that is calculated as follows:

$$p = \frac{U(t)}{U(t) + L(t)} 100, \tag{1}$$

where $U(t)$ is the number of unemployed persons and $L(t)$ is the number of employed persons.

Empirical studies highlight that economic growth has particularly positive impact on job creation; see, for example, the literature review by Basnett and Sen [19]. To study this impact, we proposed the following model of the mutual interactions between the economically active population and the economic growth:

$$\begin{aligned} \frac{dK}{dt} &= sf(K,L) - \delta K, \\ \frac{dL}{dt} &= \gamma\left[1 - \frac{L_\tau}{g(K_\tau)}\right]L, \end{aligned} \tag{2}$$

where K is the capital stock, s denotes the constant saving rate, δ is the depreciation rate of capital stock, f is the production function, g is the effective labor demand, τ is the time delay of the recruitment process that is the average time needed for expressing each identifying the jobs vacancy, analyzing the job requirements, reviewing applications, and screening and selecting the right candidate, and $\gamma = (r.p)/(1-p)$ with r being the employment rate and p being the unemployed rate. A significant part of our contribution focused on the existence of the Hopf bifurcation phenomenon around the positive equilibrium of the system (2), when the delay τ passes through a critical value. Yet we have not discussed the direction of Hopf bifurcation and the stability of the resulting periodic solutions of this system.

In this work, we investigate the direction and stability of the resulting bifurcating branch of the system (2) by using a second order approximation of the center manifold and computing the first Lyapunov coefficient for Hopf bifurcation points [20]. The results show that the system under investigation can undergo a supercritical Hopf bifurcation and the bifurcating periodic solution is stable in a neighborhood of some bifurcation points.

The remainder of this paper is structured as follows. In Section 2 we recall previous results. In Section 3 we prove our new result concerning the direction of Hopf bifurcation, that is, to ensure whether the bifurcating branch of periodic solution exists locally and to determine the properties of these bifurcating periodic solutions. Finally, in Section 4 we present our conclusions, some open problems, and future work.

2. Previous Results

In this section, we recall the basic results on the local asymptotic stability and the local existence of periodic solutions (Hopf bifurcation) of the positive steady state (K_*, L_*) of the system (2).

As in [9], we assume that the function g is continuously differentiable, satisfying the following hypotheses:

(H_1): $g(0) > 0$

(H_2): g is a strictly monotone increasing and concave function

(H_3): $\lim_{K\to+\infty} g(K) = L_e$

where L_e is the maximal effective labor demand [5]. Moreover, as in the Solow model [4], we consider a Cobb-Douglas function [21]:

$$f(K,L) = AK^\alpha L^{1-\alpha}, \tag{3}$$

where A is a positive constant that reflects the level of the technology and $\alpha \in (0,1)$ and $1-\alpha$ are the output elasticities of capital and labor, respectively.

Proposition 1 (see [9]). *System (2) always has two equilibria $P_0 = (0,0)$ and $P_1 = (0, g(0))$ which exist for all parameter values. On the other hand, if hypotheses (H_1), (H_2), and (H_3) hold, then system (2) also admits a unique positive equilibrium (K_*, L_*), where K_* is the unique positive solution of*

$$g(K) = \left(\frac{\delta}{sA}\right)^{1/(1-\alpha)} K, \tag{4}$$

and L_* is determined by

$$L_* = \left(\frac{\delta}{sA}\right)^{1/(1-\alpha)} K_*. \tag{5}$$

Theorem 2 (see [9]). *If $M < 1$, then there exists $\tau_0 > 0$ such that,*

(i): for $\tau \in [0, \tau_0)$, the steady state (K_*, L_*) is locally asymptotically stable

(ii): for $\tau > \tau_0$, (K_*, L_*) is unstable

(iii): for $\tau = \tau_0$, a Hopf bifurcation of periodic solutions of system (2) occurs at (K_*, L_*) when $\tau = \tau_0$

with

$$\begin{aligned} \omega_0^2 = \frac{1}{2}\Big\{&\left(\gamma^2 - ((1-\alpha)\delta)^2\right) + \Big[\left(\gamma^2 - ((1-\alpha)\delta)^2\right)^2 \\ &- 4\delta^2\gamma^2(1-\alpha)^2(1-M)^2\Big]^{1/2}\Big\}, \end{aligned} \tag{6}$$

and

$$\tau_0 = \frac{1}{\omega_0} \arccos \frac{\omega_0^2 \delta(\alpha-1)M}{\gamma(\omega_0^2 + (\delta(1-\alpha)(1-M))^2}, \tag{7}$$

and

$$M = K_* L_*^{-1} g'(K_*). \tag{8}$$

3. Direction and Stability of the Hopf Bifurcation

In Theorem 2, we obtained a condition under which system (2) undergoes Hopf bifurcation at τ_0; that is, a family of periodic solutions bifurcate from the positive steady state point (K_*, L_*) at the critical value τ_0. One interesting question here is to determine the direction of Hopf bifurcation, that is, to ensure whether the bifurcating branch of periodic solution exists locally for $\tau > \tau_0$ or $\tau < \tau_0$ and to determine the properties of bifurcating periodic solutions, for example, stability on the center manifold. In this section we use a second order approximation of the center manifold and we compute the first Lyapunov coefficient for Hopf bifurcation points to formulate an explicit algorithm about the direction and the stability of the bifurcating branch of periodic solutions of (2) (see [20]).

Normalizing the delay τ by scaling $t \longrightarrow t/\tau$ and effecting the change $x(t) = K(\tau t) - K_*$, $y(t) = L(\tau t) - L_*$ and $\tau = \tau_0 + \mu$, where $\mu \in \mathbb{R}$. Then the system (2) is transformed into

$$\begin{aligned} \frac{dx}{dt} &= (\tau_0 + \mu)\left[a_1 x + a_2 y + a_{11} xy + a_{20} x^2 + a_{02} y^2 \right. \\ &\quad \left. + a_{12} xy^2 + a_{21} x^2 y + a_{30} x^3 + a_{03} y^3\right], \\ \frac{dy}{dt} &= (\tau_0 + \mu)\left[b_1 x_1 + b_2 y_1 + b_{11} x_1 \left[y_1 + y\right] + b_{20} x_1^2 \right. \\ &\quad \left. + b_{21} x_1^2 \left[y_1 + x\right] + b_{30} y_1^3\right], \end{aligned} \tag{9}$$

where

$$\begin{aligned} x_1 &= x(t-1); \\ y_1 &= y(t-1); \\ a_1 &= (\alpha - 1)\delta; \\ a_2 &= (1-\alpha)\delta K_* L_*^{-1}; \\ b_1 &= \gamma g'(K_*); \\ b_2 &= -\gamma; \\ a_{20} &= \frac{1}{2}\alpha(\alpha-1)\delta K_*^{-1}; \\ a_{02} &= \frac{1}{2}\alpha(\alpha-1)\delta K_* L_*^{-2}; \\ a_{11} &= \alpha(1-\alpha)\delta L_*^{-1}; \\ a_{12} &= \frac{1}{2}\alpha^2(\alpha-1)\delta L_*^{-1}; \\ a_{21} &= \frac{-1}{2}\alpha(\alpha-1)^2\delta K_*^{-1} L_*^{-1}; \\ b_{11} &= \gamma g'(K_*) L_*^{-1}; \\ b_{20} &= \frac{1}{2}\gamma\left[g''(K_*) - 2\left(g'(K_*)\right)^2 L_*^{-1}\right]; \end{aligned} \tag{10}$$

and

$$\begin{aligned} b_{21} &= \frac{1}{2}\gamma\left[g''(K_*) L_*^{-1} - 2g'(K_*) L_*^{-2}\right]; \\ b_{30} &= \frac{\gamma}{6}\left[g'''(K_*) L_*^{-3} \right. \\ &\quad \left. - 3g'(K_*) g''(K_*) L_*^{-1}\left(L_*^{-3} + 1\right) - 6g'(K_*)^3 L_*^{-2}\right]. \end{aligned} \tag{11}$$

Hence, system (9) becomes a functional differential equation in $C := C([-1,0], \mathbb{R}^2)$ as

$$\dot{u}(t) = L_\mu(u_t) + f(\mu, u_t), \tag{12}$$

where $u = (x, y)^T \in \mathbb{R}^2$, $L_\mu : C \longrightarrow \mathbb{R}^2$ is the linear operator and $f : \mathbb{R} \times C \longrightarrow \mathbb{R}^2$ is the nonlinear part which are given, respectively, by

$$\begin{aligned} L_\mu(\varphi) &= (\tau_0 + \mu) \\ &\quad \cdot \left[\begin{pmatrix} (\alpha-1)\delta & (1-\alpha)\delta L_*^{-1} K_* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} 0 & 0 \\ \gamma g'(K_*) & -\gamma \end{pmatrix}\begin{pmatrix} \varphi_1(-1) \\ \varphi_2(-1) \end{pmatrix}\right], \end{aligned} \tag{13}$$

and

$$f(\mu, \varphi) = (\tau_0 + \mu)\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \tag{14}$$

where

$$\begin{aligned} Q_1 &= a_{11}\varphi_1(0)\varphi_2(0) + a_{20}\varphi_1^2(0) + a_{02}\varphi_2^2(0) \\ &\quad + a_{12}\varphi_1(0)\varphi_2^2(0) + a_{21}\varphi_1^2(0)\varphi_2(0) + a_{30}\varphi_1^3(0) \\ &\quad + a_{03}\varphi_2^3(0), \end{aligned} \tag{15}$$

and

$$\begin{aligned} Q_2 &= b_{11}\varphi_1(-1)\left[\varphi_2(-1) + \varphi_2(0)\right] + b_{20}\varphi_1^2(-1) \\ &\quad + b_{21}\varphi_1^2(-1)\left[\varphi_2(-1) + \varphi_1(0)\right] + b_{30}\varphi_1(-1). \end{aligned} \tag{16}$$

Let

$$L := L_\mu : C\left([-1,0], \mathbb{R}^2\right) \longrightarrow \mathbb{R}^2. \tag{17}$$

Using the Riesz representation theorem (see [22]), we obtain

$$L(\varphi) = \int_{-1}^{0} d\eta(\theta)\varphi(\theta), \tag{18}$$

where

$$\begin{aligned} \eta(\theta, \mu) &= (\tau_0 + \mu) \\ &\quad \cdot \begin{pmatrix} (\alpha-1)\delta\delta(\theta) & (1-\alpha)\delta L_*^{-1} K_* \delta(\theta) \\ \gamma g'(K_*)\delta(\theta+1) & -\gamma\delta(\theta+1) \end{pmatrix}, \end{aligned} \tag{19}$$

with $\delta(\cdot)$ being the Dirac function. For $\varphi = (\varphi_1, \varphi_2) \in C$, define

$$A(\mu)\varphi(\theta) = \begin{cases} \dfrac{d\varphi}{d\theta}(\theta) & \text{for } \theta \in [-1, 0), \\ \displaystyle\int_{-1}^{0} d\eta(\theta, \mu)\varphi(\theta) & \text{for } \theta = 0, \end{cases} \tag{20}$$

and

$$R(\mu)\varphi = \begin{cases} 0 & \text{for } \theta \in [-1,0), \\ f(\mu,\varphi) & \text{for } \theta = 0. \end{cases} \tag{21}$$

Thus, the system (9) can be transformed into the following functional differential equation:

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{22}$$

where $u_t(\theta) = u(t+\theta), \theta \in [-1,0]$.

Now, for $\psi = (\psi_1,\psi_2) \in C([0,1],\mathbb{R}^{2*})$, let us consider the operator

$$A^* : C\left([0,1],\mathbb{R}^2\right) \longrightarrow \mathbb{R}^2, \tag{23}$$

defined by

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi}{ds}(s), & \text{for } s \in (0,1] \\ -\displaystyle\int_{-1}^{0}\psi(-s)\,d\eta(s), & \text{for } s = 0, \end{cases} \tag{24}$$

and for $\varphi \in C([-1,0],\mathbb{R}^2)$ and $\psi \in C([0,1],\mathbb{R}^{2*})$ we define a bilinear inner product:

$$\begin{aligned}\langle\psi,\varphi\rangle &= \overline{\psi}(0)\varphi(0) \\ &\quad - \int_{-1}^{0}\int_{0}^{\theta}\overline{\psi}(\xi-\theta)\,d\eta(\theta)\,\varphi(\xi)\,d\xi,\end{aligned} \tag{25}$$

where $\eta(\theta) = \eta(\theta,0)$, and then $A(0)$ and A^* are adjoint operators.

Suppose that $q(\theta)$ and $q^*(s)$ are eigenvectors of $A(0)$ and A^* corresponding to $i\omega_0\tau_0$ and $-i\omega_0\tau_0$, respectively. By a simple computation, we have

$$q(\theta) = (1,q_1)^T e^{i\theta\omega_0\tau_0}, \tag{26}$$

and

$$q^*(s) = \kappa(q_2,1)^T e^{i\theta\omega_0\tau_0}, \tag{27}$$

where $q_1 = (i\omega_0 - (\alpha-1)\delta)/(1-\alpha)\delta L_*^{-1}K_*$, and $q_2 = (i\omega_0 + \gamma)/(1-\alpha)\delta L_*^{-1}K_*$.

Using the normalization condition, i.e., $\langle q^*(s), q(\theta)\rangle = 1$, we get

$$\kappa = \frac{-(1-\alpha)\delta L_*^{-1}K_*}{2i\omega_0 - \gamma + (\alpha-1)\delta - \gamma\tau_0 e^{i\omega_0\tau_0}\left[(1-\alpha)\delta L_*^{-1}K_* g'(K_*) + (\alpha-1)\delta + i\omega_0\right]}. \tag{28}$$

It is easy to check that $\langle q^*, \overline{q}\rangle = 0$.

Let

$$\begin{aligned} z(t) &= \langle q^*, u_t\rangle, \\ W(z,\overline{z},\theta) &= u_t(\theta) - \operatorname{Re} z(t)q(\theta), \\ W(z,\overline{z}) &= W(z,\overline{z},\theta), \end{aligned} \tag{29}$$

and

$$f_0(z,\overline{z}) = f(0, W(z,\overline{z}) + \operatorname{Re}(z(t)q(\theta))), \tag{30}$$

where u_t is the solution of Eq. (22).

Next, we compute the coordinates describing center manifold C_0 at $\mu = 0$.

On the center manifold C_0, we have

$$\begin{aligned}\dot{z}(t) &= \langle q^*, \dot{u}_t\rangle = \langle q^*, A(0)u_t + R(0)u_t\rangle \\ &= \langle A^*(0)q^*, u_t\rangle + \langle q^*, R(0)u_t\rangle \\ &= i\omega_0\tau_0 z(t) + q^*(0)f_0(z,\overline{z}),\end{aligned} \tag{31}$$

and this last equation is written as follows:

$$\dot{z}(t) = i\omega_0\tau_0 z(t) + g(z,\overline{z}), \tag{32}$$

where

$$\begin{aligned} g(z,\overline{z}) &= q^*(0)f_0(z,\overline{z}) \\ &= g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + \cdots, \end{aligned} \tag{33}$$

and

$$\begin{aligned} W(z,\overline{z}) &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} \\ &\quad + \cdots, \end{aligned} \tag{34}$$

where z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q and $\overline{q}^*$.

Note that W is real if u_t is real.

Thus, from (22) we have

$$\dot{W} = \dot{u}_t - q\dot{z} - \overline{q}\dot{\overline{z}}, \tag{35}$$

which leads to

$$\dot{W} = A(0)W + H(z,\overline{z},\theta), \tag{36}$$

with

$$\begin{aligned} H(z,\overline{z},\theta) &= H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)\frac{z\overline{z}}{2} + H_{02}(\theta)\frac{\overline{z}^2}{2} \\ &\quad + \cdots, \end{aligned} \tag{37}$$

where

$$\begin{aligned} H_{20} &= -\left(A(0) - 2i\omega_0\tau_0\right)W_{20}(\theta); \\ H_{11} &= -A(0)W_{11}(\theta); \\ H_{02} &= -\left(A(0) + 2i\omega_0\tau_0\right)W_{02}(\theta). \end{aligned} \tag{38}$$

By $u_t = (x, y)^T = W(z, \overline{z}) + zq(\theta) + \overline{z}\,\overline{q}(\theta)$ and $q(\theta) = (1, q_1)^T e^{i\theta\omega_0\tau_0}$, we get

$$\begin{aligned} x(t) &= z + \overline{z} + \frac{1}{2}W_{20}(0)z^2 + W_{11}(0)z\overline{z} \\ &\quad + \frac{1}{2}W_{02}(0)\overline{z}^2 + \cdots, \\ x(t-1) &= ze^{-i\theta\omega_0\tau_0} + \overline{z}e^{i\theta\omega_0\tau_0} + \frac{1}{2}W_{20}(-1)z^2 \\ &\quad + W_{11}(-1)z\overline{z} + \frac{1}{2}W_{02}(0)\overline{z}^2 + \cdots, \\ y(t) &= q_1 z + \overline{q_1 z} + \frac{1}{2}W_{20}(0)z^2 + W_{11}(0)z\overline{z} \\ &\quad + \frac{1}{2}W_{02}(0)\overline{z}^2 + \cdots, \end{aligned} \tag{39}$$

and

$$\begin{aligned} y(t-1) &= q_1 ze^{-i\theta\omega_0\tau_0} + \overline{q_1 z}e^{i\theta\omega_0\tau_0} + \frac{1}{2}W_{20}(-1)z^2 \\ &\quad + \frac{1}{2}W_{11}(-1)z\overline{z} + \frac{1}{2}W_{02}(-1)\overline{z}^2 + \cdots \end{aligned} \tag{40}$$

It follows from (19) and (30) that

$$g(z, \overline{z}) = \overline{q}^* f_0(z, \overline{z}) = \overline{q}^* f_0(0, u_t). \tag{41}$$

The coefficients in (33) are

$$\begin{aligned} g_{20} &= 2\tau_0\overline{\kappa}\left(\overline{q}_2\left(a_{02}q_1^2 + a_{11}q_1 + a_{20}\right)\right. \\ &\quad \left.+ b_{11}e^{i\tau_0\omega_0}q_1\left(e^{i\tau_0\omega_0} + 1\right) + b_{20}e^{i2\tau_0\omega_0}\right), \\ g_{02} &= 2\tau_0\overline{\kappa}\left(\overline{q}_2\left(a_{02}\overline{q}_1^2 + a_{11}\overline{q}_1 + a_{20}\right)\right. \\ &\quad \left.+ b_{11}e^{-i\tau_0\omega_0}\overline{q}_1\left(e^{-i\tau_0\omega_0} + 1\right) + b_{20}e^{-i2\tau_0\omega_0}\right), \\ g_{11} &= \tau_0\overline{\kappa}(\overline{q}_2\left(2a_{02}q_1\overline{q}_1 + a_{11}\overline{q}_1 + a_{11}q_1 + 2a_{20}\right) \\ &\quad + b_{11}\left(\overline{q}_1 + q_1 + \overline{q}_1 e^{i\tau_0\omega_0} + q_1 e^{-i\tau_0\omega_0} + 2b_{20}\right), \end{aligned} \tag{42}$$

and

$$\begin{aligned} g_{21} &= 2\tau_0\overline{\kappa}(\overline{q}_2\left(a_{02}\left(\overline{q}_1 W_{2_{20}}(0) + 2q_1 W_{2_{11}}(0)\right)\right. \\ &\quad + a_{11}\left(\frac{1}{2}W_{2_{20}}(0) + W_{1_{11}}(0)q_1 + W_{2_{11}}(0)\right. \\ &\quad + \frac{1}{2}W_{1_{20}}(0)\overline{q}_1 + a_{12}\left(2\overline{q}_1 q_1 + q_1^2\right) \\ &\quad + a_{20}\left(W_{1_{20}} + 2W_{1_{11}}\right) + a_{02}\left(\overline{q}_1 W_{2_{20}} + 2q_1 W_{2_{11}}\right) \\ &\quad \left.\left.+ a_{21}\left(\overline{q}_1 + 2q_1\right)\right) + b_{11}\left(\frac{1}{2}e^{-i\tau_0\omega_0}W_{2_{20}}(-1)\right.\right. \\ &\quad + W_{1_{11}}(-1)q_1 e^{i\tau_0\omega_0} + e^{i\tau_0\omega_0}W_{2_{11}}(-1) \\ &\quad + \frac{1}{2}W_{1_{20}}(-1)\overline{q}_1 e^{-i\tau_0\omega_0} + \frac{1}{2}e^{-i\tau_0\omega_0}W_{2_{20}}(0) \\ &\quad \left.+ e^{i\tau_0\omega_0}W_{2_{11}}(0) + W_{1_{11}}(-1)q_1 + \frac{1}{2}W_{1_{20}}(-1)\overline{q}_1\right) \\ &\quad + b_{20}\left(2e^{i\tau_0\omega_0}W_{1_{11}}(-1) + e^{-i\tau_0\omega_0}W_{1_{20}}(-1)\right. \\ &\quad + b_{21}\left(2q_1 + e^{2i\tau_0\omega_0}\overline{q}_1 + 2e^{i\tau_0\omega_0}q_1 + e^{i\tau_0\omega_0}\overline{q}_1\right) \\ &\quad \left.\left.+ 3b_{30}e^{i\tau_0\omega_0}\right)\right). \end{aligned} \tag{43}$$

In order to compute g_{21}, we need to compute $W_{11}(\theta)$ and $W_{20}(\theta)$.

For $\theta \in [-1, 0)$, we have

$$\begin{aligned} H(z, \overline{z}, \theta) &= \overline{q}^*(0) f_0 q(0) - q^*(0)\overline{f}_0\overline{q}(0) \\ &= -g(z, \overline{z})q(\theta) - \overline{g}(z, \overline{z})\overline{q}(\theta) \\ &= -\left(g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta)\right)\frac{z^2}{2} \\ &\quad - \left(g_{11}q(\theta) + \overline{g}_{11}\overline{q}(\theta)\right)z\overline{z} + \cdots, \end{aligned} \tag{44}$$

which on comparing the coefficients with (37) gives

$$\begin{aligned} H_{20}(\theta) &= -\left(g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta)\right), \\ H_{11}(\theta) &= -\left(g_{11}q(\theta) + \overline{g}_{11}\overline{q}(\theta)\right). \end{aligned} \tag{45}$$

From (38) and (45) and the definition of A, we get

$$\dot{W}_{20} = 2i\tau_0\omega_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta). \tag{46}$$

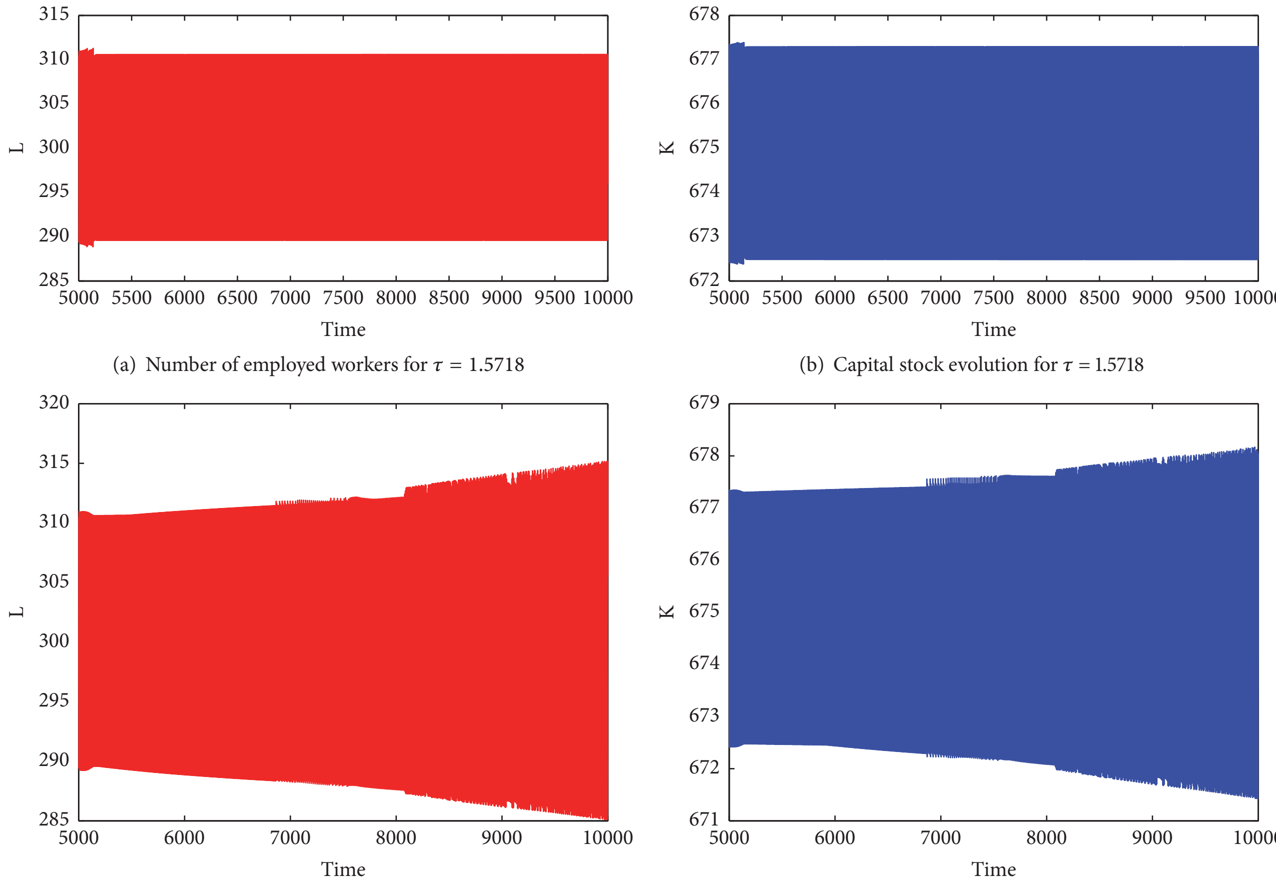
Since $q(\theta) = q(0)e^{i\theta\tau_0\omega_0}$, we have

$$\begin{aligned} W_{20} &= \frac{ig_{20}e^{i\tau_0\omega_0\theta}}{\tau_0\omega_0}q(0) + \frac{i\overline{g}_{02}e^{-i\tau_0\omega_0\theta}}{3\tau_0\omega_0}\overline{q}(0) \\ &\quad + E_1 e^{2i\tau_0\omega_0\theta}, \end{aligned} \tag{47}$$

and

$$W_{11} = \frac{-ig_{11}e^{i\tau_0\omega_0\theta}}{\tau_0\omega_0}q(0) + \frac{i\overline{g}_{11}e^{-i\tau_0\omega_0\theta}}{\tau_0\omega_0}\overline{q}(0) + E_2, \tag{48}$$

where $E_k = (E_{k_1}, E_{k_2}))^T \in \mathbb{R}^2$, for $k = 1$ and $k = 2$, are two vectors given by



(a) Number of employed workers for $\tau = 1.5718$

(b) Capital stock evolution for $\tau = 1.5718$

(c) Number of employed workers for $\tau = 1.572$

(d) Capital stock evolution for $\tau = 1.572$

Figure 1: For $\alpha = 0.5$, an unstable periodic solution bifurcates from the positive equilibrium: (a and b) for $1.5718 < \tau_0$, existence of bifurcating periodic solution and (c and d), for $1.572 > \tau_0$, absence of bifurcating periodic solution (see Theorem 3).

$$E_{1_1} = \frac{\left((1-\alpha)\,\delta L_*^{-1} K_* \left(b_{110} e^{i\tau_0\omega_0} q_1 \left(e^{i\tau_0\omega_0}+1\right) + b_{200} e^{2i\tau_0\omega_0}\right) + \left(\gamma e^{-2i\tau_0\omega_0} + 2i\tau_0\omega_0\right)\left(a_{02} q_1^2 + a_{11} q_1 + a_{20}\right)\right)}{-\left(\gamma e^{-2i\tau_0\omega_0} + 2i\tau_0\omega_0\right)\left((\alpha-1)\,\delta - 2i\tau_0\omega_0\right) - (1-\alpha)\,\delta\gamma L_*^{-1} K_* g'\left(K_*\right) e^{-2i\tau_0\omega_0}},$$

$$E_{1_2} = \frac{\left((\alpha-1)\,\delta - 2i\tau_0\omega_0\right)\left(b_{110} e^{i\tau_0\omega_0} q_1 \left(e^{i\tau_0\omega_0}+1\right) + b_{200} e^{2i\tau_0\omega_0}\right) - \gamma g'\left(K_*\right) e^{-2i\tau_0\omega_0}\left(a_{02} q_1^2 + a_{11} q_1 + a_{20}\right)}{(1-\alpha)\,\delta\gamma L_*^{-1} K_* g'\left(K_*\right) e^{-2i\tau_0\omega_0} + (\alpha-1)\,\delta - 2i\tau_0\omega_0)\left(\gamma e^{-2i\tau_0\omega_0} + 2i\tau_0\omega_0\right)}, \tag{49}$$

$$E_{2_1} = \frac{\left(2a_{02} q_1 \overline{q}_1 + a_{11}\left(\overline{q}_1 + q_1\right) + 2a_{20} + L_*^{-1} K_* (b_{11}\left(\overline{q}_1 + q_1 + e^{i\tau_0\omega_0}\overline{q}_1 + e^{-i\tau_0\omega_0}\overline{q}_1 + 2b_{20}\right)\right.}{\gamma\left(1 - L_*^{-1} K_* g'\left(K_*\right)\right)},$$

and

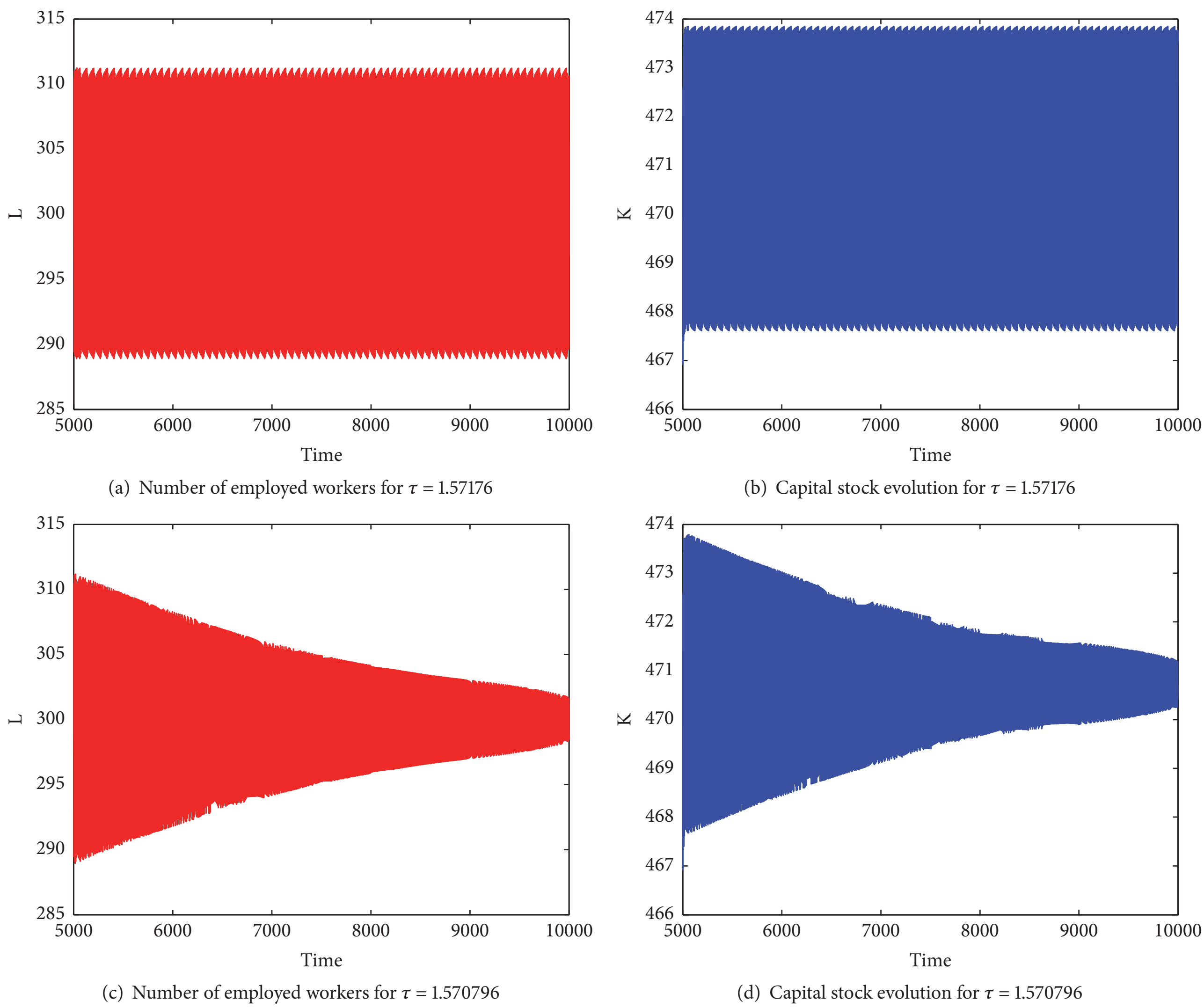
$$E_{2_2} = \frac{\gamma\left(2a_{02} q_0 q_1 + a_{11} q_0 + a_{11} q_1 + 22a_{20}\right) + (\alpha-1)\,\delta (b_{11}\left(\overline{q}_1 + q_1 + e^{i\tau_0\omega_0}\overline{q}_1 + e^{-i\tau_0\omega_0} q_1\right) + 2a_{20}}{\gamma\delta\,(1-\alpha)\left(1 - L_*^{-1} K_*\right)}. \tag{50}$$

Hence, the first Lyapunov coefficient is given by

$$l_1(\tau) = \frac{\operatorname{Re}(c_1)}{\omega\tau} + \sigma\frac{\operatorname{Im}(c_1)}{\omega^2\tau^2}, \tag{51}$$

where $c_1 = g_{21}/2 + |g_{11}|^2/\lambda + |g_{02}|^2/2(2\lambda - \overline{\lambda}) + g_{20} g_{11}(2\lambda + \overline{\lambda})/2|\lambda|^2$.

For $\tau = \tau_0$, we have $\omega = \omega_0$ and $\sigma = 0$. Thus, $l_1(\tau_0) = (1/2\omega_0\tau_0)[\operatorname{Re}(g_{21}) - (1/\omega_0\tau_0)\operatorname{Im}(g_{20} g_{11})]$, and consequently,



(a) Number of employed workers for $\tau = 1.57176$

(b) Capital stock evolution for $\tau = 1.57176$

(c) Number of employed workers for $\tau = 1.570796$

(d) Capital stock evolution for $\tau = 1.570796$

Figure 2: For $\alpha = 0.5$, an unstable periodic solution bifurcates from the positive equilibrium: (a and b) for $1.57176 > \tau_0$, existence of bifurcating periodic solution and (c and d), for $1.570796 < \tau_0$, absence of bifurcating periodic solution (see Theorem 3).

we obtain the following result on the stability of limit cycle bifurcating from the positive equilibrium.

Theorem 3. *Assume that conditions of Theorem 2 hold. Then, the following is considered.*

The direction of the Hopf bifurcation is determined by the sign of $l_1(\tau_0)$:

(1) *if $l_1(\tau_0) < 0$, then it is a supercritical bifurcation and the bifurcating periodic solutions existing for $\tau > \tau_0$ are stable*

(2) *if $l_1(\tau_0) > 0$, then it is a subcritical bifurcation and the bifurcating periodic solutions existing for $\tau < \tau_0$ are unstable.*

4. Numerical Simulations

In this section, we study how the dynamics of the model (2) change when the time delay and the output elasticities of capital vary. Let us consider the following examples where we suppose the parameters of the model take the following values:

$$\begin{gathered} s = 0.3; \\ A = 1; \\ \delta = 0.2; \\ \gamma = 1; \\ \text{and } g(K) = \frac{300e^K}{1+e^K}. \end{gathered} \tag{52}$$

Example 4. If $\alpha = 0.5$, then system (2) has a unique positive equilibrium point $E^* = (675; 300)$, the critical value of time delay $\tau_0 = 1.5719$, and the first Lyapunov coefficient $l_1|_{\tau=\tau_0} = 0.0056151 > 0$, and consequently an unstable periodic solution bifurcates from the positive equilibrium (see Figure 1).

Example 5. If $\alpha = 0.1$, then system (2) has a unique positive equilibrium point $E^* = (470, 73; 300)$. The critical value of time delay is $\tau_0 = 1.57079$ and the first Lyapunov coefficient is $l_1|_{\tau=\tau_0} = -0.026613 < 0$. Thus a stable periodic solution bifurcates from the positive equilibrium (see Figure 2).

5. Conclusion

In this paper, we considered the dynamic behavior of a delayed model of mutual interactions between the economically active population and the economic growth. The direction of the Hopf bifurcation and the stability of the bifurcated periodic solution of this model are informed by a second order approximation of the center manifold [20]. From some numerical simulations we conclude that, for some parameters, the Hopf bifurcation can appear and our proposed model can undergo a supercritical or subcritical Hopf bifurcation and the bifurcating periodic solution is stable or unstable in a neighborhood of some bifurcation points, depending on the choice of parameters. Our theoretical and experimental results would highlight that the choice of time delay influences the dynamic behavior of the economic system, which provides an effective way to control the evolution of the active population and economic growth of the system. These results could also help decision makers to better understand the fluctuations in economic growth.

For further research, we suggest a study of the Bautin bifurcation for the case when the first Lyapunov coefficient equals zero.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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