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Research Article

Interval Oscillation Criteria for Forced Second-Order Nonlinear Delay Dynamic Equations with Damping and Oscillatory Potential on Time Scales

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We are concerned with the interval oscillation of general type of forced second-order nonlinear dynamic equation with oscillatory potential of the form $(r(t)g_1(x(t), x^\Delta(t)))^\Delta + p(t)g_2(x(t), x^\Delta(t))x^\Delta(t) + q(t)f(x(\tau(t))) = e(t)$, on a time scale $\mathbb{T}$. We will use a unified approach on time scales and employ the Riccati technique to establish some oscillation criteria for this type of equations. Our results are more general and extend the oscillation criteria of Erbe et al. (2010). Also our results unify the oscillation of the forced second-order nonlinear delay differential equation and the forced second-order nonlinear delay difference equation. Finally, we give some examples to illustrate our results.

1. Introduction

The theory of time scales, which has recently received a lot of attention, was originally introduced by Hilger in his Ph.D. thesis [1], in order to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus. Many authors have expounded on various aspects of this new theory; see [2–4]. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers. If the time scale equals the real numbers or integer numbers, the obtained results represent the classical theories of the differential and difference equations. Many other interesting time scales exist and give rise to many applications. The new theory of the so-called "dynamic equation" not only unifies the theories of differential equations and difference equations, but also extends these classical cases to the so-called q-difference equations (when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0 \text{ for } q > 1\}$ or $\mathbb{T} = q^{\overline{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\}$) which have important applications in quantum theory (see [5]). Also it can be applied on different types of time scales like $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = \mathbb{N}_0^2$ and the space of the harmonic numbers $\mathbb{T} = \mathbb{T}_n$.

In recent years, there have been many research activities concerning the oscillation of solutions of various forced second-order dynamic equations on time scales; we refer the reader to the articles [6–13] and the references cited therein.

In this paper, we are concerned with the interval oscillation of the second-order nonlinear dynamic equation:

$$\begin{aligned}&\left(r(t)\, g_1\left(x(t), x^\Delta(t)\right)\right)^\Delta \\ &\qquad + p(t)\, g_2\left(x(t), x^\Delta(t)\right) x^\Delta(t) + q(t)\, f(x(\tau(t))) \\ &\quad = e(t),\end{aligned} \tag{1}$$

on a time scale $\mathbb{T}$, subject to the following conditions:

(H_1) $\mathbb{T}$ is an unbounded above time scale, and $t_0 \in \mathbb{T}$ with $t_0 > 0$. We define the time scale interval $[t_0, \infty)_\mathbb{T}$ by $[t_0, \infty)_\mathbb{T} = [t_0, \infty) \cap \mathbb{T}$.

(H_2) $r(t) \in C^1_{\mathrm{rd}}([t_0, \infty)_\mathbb{T}, \mathbb{R}^+)$ such that

$$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty. \tag{2}$$

(H_3) $p(t)$, $q(t)$, and $e(t)$ are rd-continuous functions.

(H_4) $f \in C(\mathbb{R}, \mathbb{R})$ is assumed to satisfy $uf(u) > 0$, for $u \neq 0$ and $f(u)/u \geq L|u|^{\gamma-1}$ for some $L > 0$ and $\gamma \geq 1$.

(H_5) $g_1 \in C^1_{\mathrm{rd}}(\mathbb{R}^2, \mathbb{R})$, $g_2 \in C_{\mathrm{rd}}(\mathbb{R}^2, \mathbb{R})$, and $v g_1(u, v) > 0$, for all $v \neq 0$, and there exist positive constants $\alpha_1, \alpha_2, \alpha_3$ such that

(1) $g_1(u, v)v \geq \alpha_1 g_1^2(u, v)$, $(u, v) \in \mathbb{R}^2$,

(2) $g_2(u, v)uv \geq \alpha_2 g_1^2(u, v)$ for all $(u, v) \in \mathbb{R}^2$,

(3) $g_1(u, v) \geq \alpha_3 v$ for all $(u, v) \in \mathbb{R}^2$.

(H_6) $\tau : \mathbb{T} \to \mathbb{T}$ with $\tau(t) \leq t$ and $\lim_{t\to\infty}\tau(t) = \infty$.

By a solution of (1), we mean that a nontrivial real valued function x satisfies (1) for $t \in \mathbb{T}$. A solution x of (1) is said to be oscillatory if it has arbitrarily a large number of zeros; that is, there exists a sequence of zeros $\{t_n\}$ such that $x(t_n) = 0$ and $\lim_{n\to\infty} t_n = \infty$. Otherwise, x is said to be nonoscillatory. Equation (1) is said to be oscillatory if all of its solutions are oscillatory. In this work, we study the solutions of (1) which are not identically vanishing eventually.

In order to prove our results, we use the following Hardy et al. inequality [14].

Lemma 1 (Hardy et al. inequality [14]). *If X and Y are nonnegative, then*

$$\frac{1}{\lambda}X + \frac{1}{\delta}Y \geq X^{1/\lambda}Y^{1/\delta}, \quad \frac{1}{\lambda} + \frac{1}{\delta} = 1. \tag{3}$$

2. Main Results

In the following theorems, we apply Riccati techniques to establish some sufficient conditions for oscillation of (1) on a sequence of subintervals of the interval $[t_0, \infty)_{\mathbb{T}}$. Also, we do not require that $p(t)$, $q(t)$, and $e(t)$ be of definite sign.

Theorem 2. *Assume that (H_1)–(H_6) hold, and suppose that for any $T \in [t_0, \infty)_{\mathbb{T}}$ there exist points $T < s_i < t_i$ in $\mathbb{T}$ for $i = 1, 2$ such that $p(t) \geq 0$ and $q(t) \geq 0$ for $t \in [\tau(s_1), t_1)_{\mathbb{T}} \cup [\tau(s_2), t_2)_{\mathbb{T}}$ and*

$$e(t) \begin{cases} \leq 0, & t \in [\tau(s_1), t_1)_{\mathbb{T}}, \\ \geq 0, & t \in [\tau(s_2), t_2)_{\mathbb{T}}. \end{cases} \tag{4}$$

Further assume that there exist a C^1_{rd} function u such that for $i = 1, 2$, $u(t) \not\equiv 0$, on $[s_i, t_i]_{\mathbb{T}}$, $u(s_i) = u(t_i) = 0$ and a positive delta differentiable function $\rho(t) \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ such that

$$\begin{aligned} A_i[u] = \int_{s_i}^{t_i} (u^\sigma(t))^2 \Bigg[Q_i(t) &- \frac{\rho^2(t) r^2(t)}{4(u^\sigma(t))^4 \rho^\sigma(t) \delta_i(t) [\alpha_2 p(t) + \alpha_1 r(t)]} \Bigg| (u(t) \\ &+ u^\sigma(t)) u^\Delta(t) + (u^\sigma(t))^2 \frac{\rho^\Delta(t)}{\rho(t)} \Bigg|^2 \Bigg] \Delta t \geq 0, \end{aligned} \tag{5}$$

where for $t \in [s_i, t_i)_{\mathbb{T}}$ and $i = 1, 2$

$$\begin{gathered} Q_i(t) = \gamma(\gamma - 1)^{(1-\gamma)/\gamma} (Lq(t))^{1/\gamma} \delta_1(t) \rho^\sigma(t) |e(t)|^{(\gamma-1)/\gamma}, \\ \delta_i(t) = \begin{cases} \left(\left(\int_{\tau(s_i)}^{\tau(t)} \frac{\Delta s}{r(s)} \right) \left(\left(1 - \frac{1}{\alpha_1\alpha_3} \right) \int_{\tau(s_i)}^{\tau(t)} \frac{\Delta s}{r(s)} + \frac{1}{\alpha_1\alpha_3} \int_{\tau(s_i)}^{\sigma(t)} \frac{\Delta s}{r(s)} \right)^{-1}, & \tau(t) < \sigma(t), \\ 1, & \tau(t) = \sigma(t). \end{cases} \end{gathered} \tag{6}$$

Then every solution of (1) is oscillatory.

Proof. Assume that (1) is nonoscillatory on $[t_0, \infty)_{\mathbb{T}}$. Then there is a solution x of (1) and a point $T \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)$ and $x(\tau(t))$ are of the same sign on $[T, \infty)_{\mathbb{T}}$. Consider the cases $x(t)$ and $x(\tau(t))$ that are positive on $[T, \infty)_{\mathbb{T}}$. We use Riccati substitution:

$$w(t) = \rho(t) \frac{r(t) g_1(x(t), x^\Delta(t))}{x(t)}. \tag{7}$$

Then, from (1) and using (H_5), we have

$$\begin{aligned} w^\Delta(t) &= \rho^\Delta(t) \frac{r(t) g_1(x(t), x^\Delta(t))}{x(t)} + \rho^\sigma(t) \left[\frac{x(t)\left(r(t) g_1(x(t), x^\Delta(t))\right)^\Delta - r(t) g_1(x(t), x^\Delta(t)) x^\Delta(t)}{x(t) x^\sigma(t)} \right] \\ &= \frac{\rho^\Delta(t)}{\rho(t)} w(t) \\ &\quad + \rho^\sigma(t) \left[-\frac{p(t) g_2(x(t), x^\Delta(t)) x^\Delta(t)}{x^\sigma(t)} - \frac{q(t) f(x(\tau(t)))}{x^\sigma(t)} + \frac{e(t)}{x^\sigma(t)} - \frac{r(t) g_1(x(t), x^\Delta(t)) x^\Delta(t)}{x(t) x^\sigma(t)} \right] \end{aligned}$$

$$\leq \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) + \rho^{\sigma}(t) \left[-\frac{\alpha_2 p(t) g_1^2 \left(x(t), x^{\Delta}(t)\right)}{x(t) x^{\sigma}(t)} - \frac{q(t) f(x(\tau(t)))}{x^{\sigma}(t)} + \frac{e(t)}{x^{\sigma}(t)} - \frac{\alpha_1 r(t) g_1^2 \left(x(t), x^{\Delta}(t)\right)}{x(t) x^{\sigma}(t)} \right]$$
$$\leq \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) - \rho^{\sigma}(t) \left[\alpha_2 p(t) + \alpha_1 r(t)\right] \frac{g_1^2 \left(x(t), x^{\Delta}(t)\right)}{x(t) x^{\sigma}(t)} - \rho^{\sigma}(t) q(t) \frac{f(x(\tau(t)))}{x^{\sigma}(t)} + \frac{\rho^{\sigma}(t) e(t)}{x^{\sigma}(t)}$$
$$\leq \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) - \frac{\rho^{\sigma}(t)}{\rho^2(t)} \left[\alpha_2 p(t) + \alpha_1 r(t)\right] \frac{x(t)}{r^2(t) x^{\sigma}(t)} w^2(t) - \rho^{\sigma}(t) q(t) \frac{f(x(\tau(t)))}{x^{\sigma}(t)} + \frac{\rho^{\sigma}(t) e(t)}{x^{\sigma}(t)}. \tag{8}$$

Since $t_1 > s_1 > T$ such that $p(t) \geq 0$, $q(t) \geq 0$ for $t \in [\tau(s_1), t_1)_{\mathbb{T}}$ and $e(t) \leq 0$ for $t \in [\tau(s_1), t_1)_{\mathbb{T}}$, then, from (1), we have

$$\left(r(t) g_1 \left(x(t), x^{\Delta}(t)\right)\right)^{\Delta} = -p(t) g_2 \left(x(t), x^{\Delta}(t)\right) x^{\Delta}(t) - q(t) f(x(\tau(t))) + e(t) \leq 0, \tag{9}$$

for $t \in [\tau(s_1), t_1)_{\mathbb{T}}$. Hence $(r(t)g_1(x(t), x^{\Delta}(t)))$ is nonincreasing for $t \in [\tau(s_1), t_1)_{\mathbb{T}}$. We claim that $x^{\Delta}(t) > 0$ on $[s_1, t_1)_{\mathbb{T}}$. If not, then there is $t \geq s_1$ such that

$$r(t) g_1 \left(x(t), x^{\Delta}(t)\right) \leq r(s_1) g_1 \left(x(s_1), x^{\Delta}(s_1)\right) = c < 0; \tag{10}$$

using (H_5), we have

$$x^{\Delta}(t) \leq \frac{c\alpha_1}{r(t)}, \tag{11}$$

integrating from s_1 to t, we get

$$x(t) \leq x(s_1) + c\alpha_1 \int_{s_1}^{t} \frac{\Delta s}{r(s)} \longrightarrow -\infty \quad \text{as } t \longrightarrow \infty, \quad \text{for } t \in [s_1, t_1)_{\mathbb{T}}, \tag{12}$$

which implies that $x(t)$ is eventually negative. This is a contradiction. Hence $x^{\Delta}(t) > 0$ on $[s_1, t_1)_{\mathbb{T}}$.

Therefore, for $\tau(t) < t < \sigma(t)$, we have

$$x(\tau(t)) < x(t). \tag{13}$$

Now, we use the fact that $r(t)g_1(x(t), x^{\Delta}(t))$ is nonincreasing for $t \in [\tau(s_1), t_1)_{\mathbb{T}}$ and (H_5). Then, for $t \in [s_1, t_1)_{\mathbb{T}}$, we have

$$\begin{aligned} x^{\sigma}(t) - x(\tau(t)) &= \int_{\tau(t)}^{\sigma(t)} x^{\Delta}(s) \Delta s \\ &= \int_{\tau(t)}^{\sigma(t)} \frac{r(s) x^{\Delta}(s)}{r(s)} \Delta s \\ &\leq \frac{1}{\alpha_3} \int_{\tau(t)}^{\sigma(t)} \frac{r(s) g_1 \left(x(s), x^{\Delta}(s)\right)}{r(s)} \Delta s \\ &\leq \frac{1}{\alpha_3} \left(r g_1 \left(x, x^{\Delta}\right)\right)(\tau(t)) \int_{\tau(t)}^{\sigma(t)} \frac{\Delta s}{r(s)}, \end{aligned} \tag{14}$$

which implies that

$$\frac{x^{\sigma}(t)}{x(\tau(t))} \leq 1 + \frac{\left(r g_1 \left(x, x^{\Delta}\right)\right)(\tau(t))}{\alpha_3 x(\tau(t))} \int_{\tau(t)}^{\sigma(t)} \frac{\Delta s}{r(s)}. \tag{15}$$

On the other hand, for $t \in [s_1, t_1)_{\mathbb{T}}$, we have

$$\begin{aligned} x(\tau(t)) > x(\tau(t)) - x(\tau(s_1)) &= \int_{\tau(s_1)}^{\tau(t)} x^{\Delta}(s) \Delta s \\ &= \int_{\tau(s_1)}^{\tau(t)} \frac{r(s) x^{\Delta}(s)}{r(s)} \Delta s \\ &\geq \alpha_1 \int_{\tau(s_1)}^{\tau(t)} \frac{r(s) g_1 \left(x(s), x^{\Delta}(s)\right)}{r(s)} \Delta s \\ &\geq \alpha_1 \left(r g_1 \left(x, x^{\Delta}\right)\right)(\tau(t)) \int_{\tau(s_1)}^{\tau(t)} \frac{\Delta s}{r(s)} \end{aligned} \tag{16}$$

which implies that

$$\frac{\left(r g_1 \left(x, x^{\Delta}\right)\right)(\tau(t))}{x(\tau(t))} \leq \frac{1}{\alpha_1} \left(\int_{\tau(s_1)}^{\tau(t)} \frac{\Delta s}{r(s)} \right)^{-1}, \quad \text{for } t \in (s_1, t_1)_{\mathbb{T}}; \tag{17}$$

using (15) and (17), we get

$$\begin{aligned} \frac{x^{\sigma}(t)}{x(\tau(t))} &\leq 1 + \frac{1}{\alpha_1 \alpha_3} \frac{\int_{\tau(t)}^{\sigma(t)} (\Delta s / r(s))}{\int_{\tau(s_1)}^{\tau(t)} (\Delta s / r(s))} \\ &\leq 1 - \frac{1}{\alpha_1 \alpha_3} + \frac{1}{\alpha_1 \alpha_3} \frac{\int_{\tau(s_1)}^{\sigma(t)} (\Delta s / r(s))}{\int_{\tau(s_1)}^{\tau(t)} (\Delta s / r(s))} = \frac{1}{\delta_1(t)}, \quad \text{for } t \in (s_1, t_1)_{\mathbb{T}}. \end{aligned} \tag{18}$$

Therefore,

$$x^{\sigma}(t) \leq \frac{x(\tau(t))}{\delta_1(t)}, \quad \text{for } t \in (s_1, t_1)_{\mathbb{T}}. \tag{19}$$

To see that (19) holds for $t = s_1$, we note that if $\tau(s_1) = \sigma(s_1)$, then $\delta_1(s_1) = 1$ and clearly (19) holds. If $\tau(s_1) < \sigma(s_1)$, then $\delta_1(s_1) = 0$ and thus (19) holds for $t = s_1$. Hence (19) holds for $t \in [s_1, t_1)_{\mathbb{T}}$.

From (8), (13), and (19), we get

$$\begin{aligned}
w^{\Delta}(t) &\le \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) \\
&\quad - \frac{\rho^{\sigma}(t)}{\rho^2(t)} \left[\alpha_2 p(t) + \alpha_1 r(t)\right] \frac{x(t)}{r^2(t)\, x^{\sigma}(t)} w^2(t) \\
&\quad - \rho^{\sigma}(t) q(t) \frac{f(x(\tau(t)))}{x^{\sigma}(t)} + \frac{\rho^{\sigma}(t) e(t)}{x^{\sigma}(t)} \\
&\le \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) \\
&\quad - \frac{\rho^{\sigma}(t)}{\rho^2(t)} \left[\alpha_2 p(t) + \alpha_1 r(t)\right] \frac{x(\tau(t))}{r^2(t)\, x^{\sigma}(t)} w^2(t) \\
&\quad - \rho^{\sigma}(t) q(t) \frac{f(x(\tau(t)))}{x^{\sigma}(t)} + \frac{\rho^{\sigma}(t) e(t)}{x^{\sigma}(t)} \\
&\le \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) \\
&\quad - \frac{\rho^{\sigma}(t)}{\rho^2(t)} \left[\alpha_2 p(t) + \alpha_1 r(t)\right] \frac{\delta_1(t)}{r^2(t)} w^2(t) \\
&\quad - \rho^{\sigma}(t) \delta_1(t) q(t) \frac{f(x(\tau(t)))}{x(\tau(t))} + \frac{\rho^{\sigma}(t) e(t)}{x^{\sigma}(t)}.
\end{aligned} \tag{20}$$

Using (H_4), we get

$$\begin{aligned}
w^{\Delta}(t) \le \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) &- \frac{\rho^{\sigma}(t)}{\rho^2(t)} \left[\alpha_2 p(t) + \alpha_1 r(t)\right] \frac{\delta_1(t)}{r^2(t)} w^2(t) \\
&- L\rho^{\sigma}(t) \delta_1(t) q(t) \left|x(\tau(t))\right|^{\gamma-1} \\
&+ \frac{\rho^{\sigma}(t) e(t)}{x^{\sigma}(t)}.
\end{aligned} \tag{21}$$

Or

$$\begin{aligned}
&L\rho^{\sigma}(t) \delta_1(t) q(t) \left|x(\tau(t))\right|^{\gamma-1} - \frac{\rho^{\sigma}(t) e(t)}{x^{\sigma}(t)} \\
&\quad \le -w^{\Delta}(t) + \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) \\
&\quad\quad - \frac{\rho^{\sigma}(t)}{\rho^2(t)} \left[\alpha_2 p(t) + \alpha_1 r(t)\right] \frac{\delta_1(t)}{r^2(t)} w^2(t).
\end{aligned} \tag{22}$$

Since $x(t) > 0$ for all $t \ge T$ and $e(t) \le 0$ on the interval $[\tau(s_1), t_1)_{\mathbb{T}}$, then from (22) we have for $t \in [\tau(s_1), t_1)_{\mathbb{T}}$

$$\begin{aligned}
&L\rho^{\sigma}(t) \delta_1(t) q(t) \left|x(\tau(t))\right|^{\gamma-1} + \frac{\rho^{\sigma}(t) \left|e(t)\right|}{x^{\sigma}(t)} \\
&\quad \le -w^{\Delta}(t) + \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) \\
&\quad\quad - \frac{\rho^{\sigma}(t)}{\rho^2(t)} \left[\alpha_2 p(t) + \alpha_1 r(t)\right] \frac{\delta_1(t)}{r^2(t)} w^2(t).
\end{aligned} \tag{23}$$

Now, we consider the following two cases: Case 1: $\gamma > 1$; Case 2: $\gamma = 1$.

Case 1 ($\gamma > 1$). Set $\lambda = \gamma$, $\delta = \gamma/(\gamma - 1)$, and

$$\begin{aligned}
X &= \gamma L\rho^{\sigma}(t) \delta_1(t) q(t) \left|x(\tau(t))\right|^{\gamma-1}, \\
Y &= \frac{\gamma}{\gamma - 1} \frac{\rho^{\sigma}(t) \left|e(t)\right|}{x^{\sigma}(t)}.
\end{aligned} \tag{24}$$

By Lemma 1, we get that

$$\begin{aligned}
&L\rho^{\sigma}(t) \delta_1(t) q(t) \left|x(\tau(t))\right|^{\gamma-1} + \frac{\rho^{\sigma}(t) \left|e(t)\right|}{x^{\sigma}(t)} \\
&\quad = \frac{1}{\lambda} X + \frac{1}{\delta} Y \ge X^{1/\lambda} Y^{1/\delta} \\
&\quad = \gamma (\gamma - 1)^{(1-\gamma)/\gamma} \left(Lq(t)\right)^{1/\gamma} \delta_1(t) \rho^{\sigma}(t) \left|e(t)\right|^{(\gamma-1)/\gamma} \\
&\quad = Q_1(t),
\end{aligned} \tag{25}$$

for $t \in [\tau(s_1), t_1)_{\mathbb{T}}$. From (23) and (25), we have

$$\begin{aligned}
Q_1(t) \le -w^{\Delta}(t) &+ \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) \\
&- \frac{\rho^{\sigma}(t)}{\rho^2(t)} \left[\alpha_2 p(t) + \alpha_1 r(t)\right] \frac{\delta_1(t)}{r^2(t)} w^2(t).
\end{aligned} \tag{26}$$

Case 2 ($\gamma = 1$). When $\gamma = 1$,

$$\begin{aligned}
&L\rho^{\sigma}(t) \delta_1(t) q(t) \left|x(\tau(t))\right|^{\gamma-1} + \frac{\rho^{\sigma}(t) \left|e(t)\right|}{x^{\sigma}(t)} \\
&\quad \ge Lq(t) \delta_1(t) \rho^{\sigma}(t) = Q_1(t),
\end{aligned} \tag{27}$$

this implies that (25) and (26) hold for $\gamma = 1$.

Multiplying (26) by $(u^{\sigma}(t))^2$ and integrating from s_1 to t_1, we have

$$\begin{aligned}
&\int_{s_1}^{t_1} \left(u^{\sigma}(t)\right)^2 Q_1(t) \Delta t \le -\int_{s_1}^{t_1} \left(u^{\sigma}(t)\right)^2 w^{\Delta}(t) \Delta t \\
&\quad + \int_{s_1}^{t_1} \left(u^{\sigma}(t)\right)^2 \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) \Delta t - \int_{s_1}^{t_1} \left(u^{\sigma}(t)\right)^2 \\
&\quad \cdot \frac{\rho^{\sigma}(t) \delta_1(t) \left[\alpha_2 p(t) + \alpha_1 r(t)\right]}{\rho^2(t) r^2(t)} w^2(t) \Delta t.
\end{aligned} \tag{28}$$

Using integration by parts, we get

$$\begin{aligned}
\int_{s_1}^{t_1} (u^\sigma(t))^2 Q_1(t)\,\Delta t &\le -u^2(t)\,w(t)\Big|_{s_1}^{t_1} \\
&+ \int_{s_1}^{t_1} [u(t)+u^\sigma(t)]\,u^\Delta(t)\,w(t)\,\Delta t \\
&+ \int_{s_1}^{t_1} (u^\sigma(t))^2 \frac{\rho^\Delta(t)}{\rho(t)} w(t)\,\Delta t - \int_{s_1}^{t_1} (u^\sigma(t))^2 \\
&\cdot \frac{\rho^\sigma(t)\,\delta_1(t)\,[\alpha_2 p(t)+\alpha_1 r(t)]}{\rho^2(t)\,r^2(t)} w^2(t)\,\Delta t.
\end{aligned} \tag{29}$$

Using the fact that $u(s_1) = u(t_1) = 0$, we obtain

$$\begin{aligned}
\int_{s_1}^{t_1} (u^\sigma(t))^2 Q_1(t)\,\Delta t &\le \int_{s_1}^{t_1} [u(t)+u^\sigma(t)]\,u^\Delta(t)\,w(t)\,\Delta t \\
&+ \int_{s_1}^{t_1} (u^\sigma(t))^2 \frac{\rho^\Delta(t)}{\rho(t)} w(t)\,\Delta t - \int_{s_1}^{t_1} (u^\sigma(t))^2 \\
&\cdot \frac{\rho^\sigma(t)\,\delta_1(t)\,[\alpha_2 p(t)+\alpha_1 r(t)]}{\rho^2(t)\,r^2(t)} w^2(t)\,\Delta t \le \int_{s_1}^{t_1} \Big|[u(t) \\
&+ u^\sigma(t)]\,u^\Delta(t) + (u^\sigma(t))^2 \frac{\rho^\Delta(t)}{\rho(t)}\Big|\,|w(t)|\,\Delta t - \int_{s_1}^{t_1} (u^\sigma(t))^2 \\
&\cdot \frac{\rho^\sigma(t)\,\delta_1(t)\,[\alpha_2 p(t)+\alpha_1 r(t)]}{\rho^2(t)\,r^2(t)} w^2(t)\,\Delta t \\
&\le -\int_{s_1}^{t_1} \Bigg\{ \sqrt{\frac{(u^\sigma(t))^2 \rho^\sigma(t)\,\delta_1(t)\,[\alpha_2 p(t)+\alpha_1 r(t)]}{\rho^2(t)\,r^2(t)}}\,|w(t)| \\
&- \frac{1}{2}\frac{1}{u^\sigma(t)} \sqrt{\frac{\rho^2(t)\,r^2(t)}{\rho^\sigma(t)\,\delta_1(t)\,[\alpha_2 p(t)+\alpha_1 r(t)]}}\,\Big|[u(t)+u^\sigma(t)] \\
&\cdot u^\Delta(t) + (u^\sigma(t))^2 \frac{\rho^\Delta(t)}{\rho(t)}\Big| \Bigg\}^2 \Delta t \\
&+ \int_{s_1}^{t_1} \frac{\rho^2(t)\,r^2(t)}{4(u^\sigma(t))^2 \rho^\sigma(t)\,\delta_1(t)\,[\alpha_2 p(t)+\alpha_1 r(t)]} \Big|[u(t) \\
&+ u^\sigma(t)]\,u^\Delta(t) + (u^\sigma(t))^2 \frac{\rho^\Delta(t)}{\rho(t)}\Big|^2 \Delta t \\
&\le \int_{s_1}^{t_1} \frac{\rho^2(t)\,r^2(t)}{4(u^\sigma(t))^2 \rho^\sigma(t)\,\delta_1(t)\,[\alpha_2 p(t)+\alpha_1 r(t)]} \Big|[u(t) \\
&+ u^\sigma(t)]\,u^\Delta(t) + (u^\sigma(t))^2 \frac{\rho^\Delta(t)}{\rho(t)}\Big|^2 \Delta t.
\end{aligned} \tag{30}$$

This implies that

$$\begin{aligned}
A_1[u] &= \int_{s_1}^{t_1} (u^\sigma(t))^2 \left[Q_1(t) - \frac{\rho^2(t)\,r^2(t)}{4(u^\sigma(t))^4 \rho^\sigma(t)\,\delta_1(t)\,[\alpha_2 p(t)+\alpha_1 r(t)]} \left| (u(t)+u^\sigma(t))\,u^\Delta(t) + (u^\sigma(t))^2 \frac{\rho^\Delta(t)}{\rho(t)} \right|^2 \right] \Delta t \\
&\le 0,
\end{aligned} \tag{31}$$

which is a contradiction of (5). The case $x(t) < 0$, $x(\tau(t)) < 0$ on $T \in [t_0,\infty)_{\mathbb{T}}$ is similar (in this case, we use $e(t) \ge 0$ on $[\tau(s_2), t_2)_{\mathbb{T}}$ to get a similar contradiction). Therefore, any solution of (1) is oscillatory. This completes the proof. □

If $p(t) = 0$ and $g_1(u,v) = v$ ($\alpha_1 = \alpha_3 = 1$), then (1) reduces to equation

$$\left(r(t)\,x^\Delta(t)\right)^\Delta + q(t)\,f(x(\tau(t))) = e(t). \tag{32}$$

Using Theorem 2 and choosing $\rho(t) = 1$, we have the following corollary.

Corollary 3. *Assume that* (H_1)–(H_4) *and* (H_6) *hold, and suppose that for any* $T \in [t_0,\infty)_{\mathbb{T}}$ *there exist points* $T < s_i < t_i$ *in* $\mathbb{T}$ *for* $i = 1, 2$ *such that* $q(t) \ge 0$ *for* $t \in [\tau(s_1), t_1)_{\mathbb{T}} \cup [\tau(s_2), t_2)_{\mathbb{T}}$ *and*

$$e(t) \begin{cases} \le 0, & t \in [\tau(s_1), t_1)_{\mathbb{T}}, \\ \ge 0, & t \in [\tau(s_2), t_2)_{\mathbb{T}}. \end{cases} \tag{33}$$

Further assume that there exist a C^1_{rd} *function* u *such that for* $i = 1, 2$, $u(t) \not\equiv 0$, *on* $[s_i, t_i]_{\mathbb{T}}$, $u(s_i) = u(t_i) = 0$ *and a positive delta differentiable function* $\rho(t) \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ *such that*

$$\begin{aligned}
B_i[u] &= \int_{s_i}^{t_i} (u^\sigma(t))^2 \Big[Q_i(t) \\
&- \frac{r(t)}{4(u^\sigma(t))^4 \delta_i(t)} \left[(u(t)+u^\sigma(t))\,u^\Delta(t) \right]^2 \Big] \Delta t \\
&\ge 0,
\end{aligned} \tag{34}$$

where for $t \in [s_i, t_i)_{\mathbb{T}}$ *and* $i = 1, 2$,

$$\begin{aligned}
Q_i(t) &= \gamma(\gamma-1)^{(1-\gamma)/\gamma} (Lq(t))^{1/\gamma} \delta_i(t)\,\rho^\sigma(t)\,|e(t)|^{(\gamma-1)/\gamma},
\end{aligned}$$

$$\delta_i(t) = \begin{cases} \left(\int_{\tau(s_i)}^{\tau(t)} \frac{\Delta s}{r(s)}\right)\left(\int_{\tau(s_i)}^{\sigma(t)} \frac{\Delta s}{r(s)}\right)^{-1}, & \tau(t) < \sigma(t), \\ 1, & \tau(t) = \sigma(t). \end{cases} \tag{35}$$

Then every solution of (32) is oscillatory.

Proof. Assume that (32) is nonoscillatory on $[t_0, \infty)_{\mathbb{T}}$. Then there is a solution x of (32) and a $T \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)$ and $x(\tau(t))$ have the same sign on $[T, \infty)_{\mathbb{T}}$. Consider that the cases $x(t)$ and $x(\tau(t))$ are positive on $[T, \infty)_{\mathbb{T}}$. As in the proof of Theorem 2, for $\gamma \geq 1$ and $t \in [\tau(s_1), t_1)_{\mathbb{T}}$, inequality (26) holds for all eventually positive solutions of (32), where $p(t) = 0$, $g_1(u, v) = v$ $(\alpha_1 = \alpha_3 = 1)$, and $\rho(t) = 1$. Thus, we get

$$Q_1(t) \leq -w^{\Delta}(t) - \frac{\delta_1(t)}{r(t)} w^2(t). \tag{36}$$

Multiplying (36) by $(u^{\sigma}(t))^2$ and integrating from s_1 to t_1, we have

$$\begin{aligned} \int_{s_1}^{t_1} (u^{\sigma}(t))^2 Q_1(t) \Delta t &\leq -\int_{s_1}^{t_1} (u^{\sigma}(t))^2 w^{\Delta}(t) \Delta t \\ &\quad - \int_{s_1}^{t_1} \frac{(u^{\sigma}(t))^2 \delta_1(t)}{r(t)} w^2(t) \Delta t. \end{aligned} \tag{37}$$

Using integration by parts, we get

$$\begin{aligned} \int_{s_1}^{t_1} (u^{\sigma}(t))^2 Q_1(t) \Delta t &\leq -u^2(t) w(t)\Big|_{s_1}^{t_1} \\ &\quad + \int_{s_1}^{t_1} [u(t) + u^{\sigma}(t)] u^{\Delta}(t) w(t) \Delta t \\ &\quad - \int_{s_1}^{t_1} \frac{(u^{\sigma}(t))^2 \delta_1(t)}{r(t)} w^2(t) \Delta t. \end{aligned} \tag{38}$$

Using the fact that $u(s_1) = u(t_1) = 0$, we obtain

$$\begin{aligned} \int_{s_1}^{t_1} (u^{\sigma}(t))^2 Q_1(t) \Delta t &\leq \int_{s_1}^{t_1} [u(t) + u^{\sigma}(t)] u^{\Delta}(t) \\ &\quad \cdot w(t) \Delta t - \int_{s_1}^{t_1} \frac{(u^{\sigma}(t))^2 \delta_1(t)}{r(t)} w^2(t) \Delta t \\ &\leq -\int_{s_1}^{t_1} \left(\frac{u^{\sigma}(t)\sqrt{\delta_1(t)}}{\sqrt{r(t)}} w(t) \right. \\ &\quad \left. - \frac{\sqrt{r(t)}}{2u^{\sigma}(t)\sqrt{\delta_1(t)}} [u(t) + u^{\sigma}(t)] u^{\Delta}(t) \right)^2 \Delta t \\ &\quad + \int_{s_1}^{t_1} \frac{r(t)}{4(u^{\sigma}(t))^2 \delta_1(t)} \left[(u(t) + u^{\sigma}(t)) u^{\Delta}(t)\right]^2 \Delta t \\ &\leq \int_{s_1}^{t_1} \frac{r(t)}{4(u^{\sigma}(t))^2 \delta_1(t)} \left[(u(t) + u^{\sigma}(t)) u^{\Delta}(t)\right]^2 \Delta t. \end{aligned} \tag{39}$$

This implies that

$$\begin{aligned} B_1[u] &= \int_{s_1}^{t_1} (u^{\sigma}(t))^2 \left[Q_1(t) \right. \\ &\quad \left. - \frac{r(t)}{4(u^{\sigma}(t))^4 \delta_1(t)} \left[(u(t) + u^{\sigma}(t)) u^{\Delta}(t)\right]^2 \right] \Delta t \\ &\leq 0 \end{aligned} \tag{40}$$

which is a contradiction of (34). The case $x(t) < 0$, $x(\tau(t)) < 0$ on $T \in [t_0, \infty)_{\mathbb{T}}$ is similar (in this case, we use $e(t) \geq 0$ on $[\tau(s_2), t_2)_{\mathbb{T}}$ to reach a similar contradiction). Therefore, any solution of (32) is oscillatory. This completes the proof. □

Now, we assume that

$$g_2(u, v) v = g_1(u, v), \quad \forall (u, v) \in \mathbb{R}^2. \tag{41}$$

Then, (1) reduces to

$$\begin{aligned} &\left(r(t) g_1\left(x(t), x^{\Delta}(t)\right)\right)^{\Delta} + p(t) g_1\left(x(t), x^{\Delta}(t)\right) \\ &\quad + q(t) f(x(\tau(t))) = e(t). \end{aligned} \tag{42}$$

Therefore, we have the following theorem.

Theorem 4. *Assume that (H_1)–(H_6) hold, and suppose that for any $T \in [t_0, \infty)_{\mathbb{T}}$ there exist points $T < s_i < t_i$ in $\mathbb{T}$ for $i = 1, 2$ such that $p(t) \geq 0$ and $q(t) \geq 0$ for $t \in [\tau(s_1), t_1)_{\mathbb{T}} \cup [\tau(s_2), t_2)_{\mathbb{T}}$ and*

$$e(t) \begin{cases} \leq 0, & t \in [\tau(s_1), t_1)_{\mathbb{T}}, \\ \geq 0, & t \in [\tau(s_2), t_2)_{\mathbb{T}}. \end{cases} \tag{43}$$

Further assume that there exist a C_{rd}^1 function u such that for $i = 1, 2$, $u(t) \not\equiv 0$, on $[s_i, t_i]_{\mathbb{T}}$, $u(s_i) = u(t_i) = 0$ and a positive delta differentiable function $\rho(t) \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ such that

$$\begin{aligned} D_i[u] &= \int_{s_i}^{t_i} (u^{\sigma})^2 \left[Q_i \right. \\ &\quad \left. - \frac{\rho^2 r}{4\alpha_1 \rho^{\sigma} \delta_i} \left[\frac{\rho^{\Delta}}{\rho} - \frac{\rho^{\sigma} \delta_i p}{\rho r} + \frac{[u + u^{\sigma}] u^{\Delta}}{(u^{\sigma})^2} \right]^2 \right] \Delta t \\ &> 0, \end{aligned} \tag{44}$$

where for $t \in [s_i, t_i)_{\mathbb{T}}$ *and* $i = 1, 2$:

$$Q_i(t) = \gamma(\gamma-1)^{(1-\gamma)/\gamma}(Lq(t))^{1/\gamma}\delta_i(t)\rho^\sigma(t)|e(t)|^{(\gamma-1)/\gamma},$$
$$\delta_i(t) = \begin{cases} \left(\left(\int_{\tau(s_i)}^{\tau(t)}\frac{\Delta s}{r(s)}\right)\left(\left(1-\frac{1}{\alpha_1\alpha_3}\right)\int_{\tau(s_i)}^{\tau(t)}\frac{\Delta s}{r(s)}+\frac{1}{\alpha_1\alpha_3}\int_{\tau(s_i)}^{\sigma(t)}\frac{\Delta s}{r(s)}\right)^{-1}, & \tau(t)<\sigma(t), \\ 1, & \tau(t)=\sigma(t). \end{cases} \quad (45)$$

Then every solution of (42) is oscillatory.

Proof. Assume that (42) is nonoscillatory on $[t_0,\infty)_{\mathbb{T}}$. Then there is a solution x of (42) and a $T \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$ and $x(\tau(t))$ have the same sign on $[T,\infty)_{\mathbb{T}}$. Consider that the cases $x(t)$ and $x(\tau(t))$ are positive on $[T,\infty)_{\mathbb{T}}$. Let $w(t)$ be defined by (7). Then

$$w^\Delta(t) = \frac{\rho^\Delta(t)}{\rho(t)}w(t) + \rho^\sigma(t) \cdot \left[-\frac{p(t)g_1(x(t),x^\Delta(t))}{x^\sigma(t)} - \frac{q(t)f(x(\tau(t)))}{x^\sigma(t)} + \frac{e(t)}{x^\sigma(t)} - \frac{r(t)g_1(x(t),x^\Delta(t))x^\Delta(t)}{x(t)x^\sigma(t)}\right]. \quad (46)$$

Using (13) and (19) and applying (H_4) and (H_5), we get

$$w^\Delta(t) = \left[\frac{\rho^\Delta(t)}{\rho(t)} - \frac{\rho^\sigma(t)\delta_1(t)p(t)}{\rho(t)r(t)}\right]w(t) - L\rho^\sigma(t)\delta_1(t)q(t)|x(\tau(t))|^{\gamma-1} + \frac{\rho^\sigma(t)e(t)}{x^\sigma(t)} - \frac{\alpha_1\rho^\sigma(t)\delta_1(t)}{\rho^2(t)r(t)}w^2(t). \quad (47)$$

Or

$$L\rho^\sigma(t)\delta_1(t)q(t)|x(\tau(t))|^{\gamma-1} - \frac{\rho^\sigma(t)e(t)}{x^\sigma(t)} \le -w^\Delta(t) + \left[\frac{\rho^\Delta(t)}{\rho(t)} - \frac{\rho^\sigma(t)\delta_1(t)p(t)}{\rho(t)r(t)}\right]w(t) - \frac{\alpha_1\rho^\sigma(t)\delta_1(t)}{\rho^2(t)r(t)}w^2(t). \quad (48)$$

Since $x(t) > 0$ for all $t \ge T$ and $e(t) \le 0$ on the interval $[\tau(s_1),t_1)_{\mathbb{T}}$, then from (48) we have for $t \in [\tau(s_1),t_1)_{\mathbb{T}}$

$$\rho^\sigma(t)\delta_1(t)q(t)|x(\tau(t))|^{\gamma-1} + \frac{\rho^\sigma(t)|e(t)|}{x^\sigma(t)} \le -w^\Delta(t) + \left[\frac{\rho^\Delta(t)}{\rho(t)} - \frac{\rho^\sigma(t)\delta_1(t)p(t)}{\rho(t)r(t)}\right]w(t) - \frac{\alpha_1\rho^\sigma(t)\delta_1(t)}{\rho^2(t)r(t)}w^2(t). \quad (49)$$

From (25) and (49), we have for $t \in [\tau(s_1),t_1)_{\mathbb{T}}$

$$Q_1(t) \le -w^\Delta(t) + \left[\frac{\rho^\Delta(t)}{\rho(t)} - \frac{\rho^\sigma(t)\delta_1(t)p(t)}{\rho(t)r(t)}\right]w(t) - \frac{\alpha_1\rho^\sigma(t)\delta_1(t)}{\rho^2(t)r(t)}w^2(t). \quad (50)$$

Multiplying (50) by $(u^\sigma(t))^2$ and integrating from s_1 to t_1, we have

$$\int_{s_1}^{t_1}(u^\sigma(t))^2Q_1(t)\Delta t \le \int_{s_1}^{t_1}(u^\sigma(t))^2w^\Delta(t)\Delta t + \int_{s_1}^{t_1}(u^\sigma(t))^2\left[\frac{\rho^\Delta(t)}{\rho(t)} - \frac{\rho^\sigma(t)\delta_1(t)p(t)}{\rho(t)r(t)}\right] \cdot w(t)\Delta t - \int_{s_1}^{t_1}(u^\sigma(t))^2\frac{\alpha_1\rho^\sigma(t)\delta_1(t)}{\rho^2(t)r(t)}w^2(t)\Delta t. \quad (51)$$

Using integration by parts, we get

$$\int_{s_1}^{t_1}(u^\sigma(t))^2Q_1(t)\Delta t \le -u^2(t)w(t)\Big|_{s_1}^{t_1} + \int_{s_1}^{t_1}[u(t)+u^\sigma(t)]u^\Delta(t)w(t)\Delta t + \int_{s_1}^{t_1}(u^\sigma(t))^2\left[\frac{\rho^\Delta(t)}{\rho(t)} - \frac{\rho^\sigma(t)\delta_1(t)p(t)}{\rho(t)r(t)}\right] \cdot w(t)\Delta t - \int_{s_1}^{t_1}(u^\sigma(t))^2 \cdot \frac{\alpha_1\rho^\sigma(t)\delta_1(t)}{\rho^2(t)r(t)}w^2(t)\Delta t. \quad (52)$$

From the fact that $u(s_1) = u(t_1) = 0$, we obtain

$$\int_{s_1}^{t_1}(u^\sigma(t))^2Q_1(t)\Delta t \le \int_{s_1}^{t_1}[u(t)+u^\sigma(t)]u^\Delta(t) \cdot w(t)\Delta t + \int_{s_1}^{t_1}(u^\sigma(t))^2\left[\frac{\rho^\Delta(t)}{\rho(t)} - \frac{\rho^\sigma(t)\delta_1(t)p(t)}{\rho(t)r(t)}\right]w(t)\Delta t - \int_{s_1}^{t_1}(u^\sigma(t))^2 \cdot \frac{\alpha_1\rho^\sigma(t)\delta_1(t)}{\rho^2(t)r(t)}w^2(t)\Delta t \le \int_{s_1}^{t_1}(u^\sigma(t))^2$$

$$\cdot \left[\left(\frac{\rho^{\Delta}(t)}{\rho(t)} - \frac{\rho^{\sigma}(t)\,\delta_1(t)\,p(t)}{\rho(t)\,r(t)} + \frac{[u(t) + u^{\sigma}(t)]\,u^{\Delta}(t)}{(u^{\sigma}(t))^2} \right) w(t) - \frac{\alpha_1 \rho^{\sigma}(t)\,\delta_1(t)}{\rho^2(t)\,r(t)} \cdot w^2(t) \right] \Delta t$$
$$\leq \int_{s_1}^{t_1} \frac{\rho^2(t)\,r(t)\,(u^{\sigma}(t))^2}{4\alpha_1 \rho^{\sigma}(t)\,\delta_1(t)} \left[\frac{\rho^{\Delta}(t)}{\rho(t)} - \frac{\rho^{\sigma}(t)\,\delta_1(t)\,p(t)}{\rho(t)\,r(t)} + \frac{[u(t) + u^{\sigma}(t)]\,u^{\Delta}(t)}{(u^{\sigma}(t))^2} \right]^2 \Delta t. \tag{53}$$

This implies that

$$D_1[u] = \int_{s_1}^{t_1} (u^{\sigma})^2 \left[Q_1 - \frac{\rho^2 r}{4\alpha_1 \rho^{\sigma} \delta_1} \left[\frac{\rho^{\Delta}}{\rho} - \frac{\rho^{\sigma}\delta_1 p}{\rho r} + \frac{[u + u^{\sigma}]\,u^{\Delta}}{(u^{\sigma})^2} \right]^2 \right] \Delta t \leq 0 \tag{54}$$

which is a contradiction with (44). The case $x(t) < 0$, $x(\tau(t)) < 0$ on $T \in [t_0, \infty)_{\mathbb{T}}$ is similar (in this case, we use $e(t) \geq 0$ on $[\tau(s_2), t_2)_{\mathbb{T}}$ and $D_2 \leq 0$). □

In the following, we assume that $r(t)$ is an rd-continuous function (i.e., $r(t) \in C^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$) and employ the generalized Riccati technique to establish new oscillation criteria for (1).

Theorem 5. *Assume that (H_1) and (H_3)–(H_6) hold, and suppose that for any $T \in [t_0, \infty)_{\mathbb{T}}$ there exist points $T < s_i < t_i$ in $\mathbb{T}$ for $i = 1, 2$ such that $p(t) \geq 0$ and $q(t) \geq 0$ for $t \in [\tau(s_1), t_1)_{\mathbb{T}} \cup [\tau(s_2), t_2)_{\mathbb{T}}$ and*

$$e(t) \begin{cases} \leq 0, & t \in [\tau(s_1), t_1)_{\mathbb{T}}, \\ \geq 0, & t \in [\tau(s_2), t_2)_{\mathbb{T}}. \end{cases} \tag{55}$$

Further assume that there is a function $a(t)$ such that $a(t)r(t)$ is a delta differentiable, a positive delta differentiable, function $\rho(t)$ and there exist a C^1_{rd} function u such that for $i = 1, 2$, $u(t) \not\equiv 0$, on $[s_i, t_i]_{\mathbb{T}}$, $u(s_i) = u(t_i) = 0$,

$$E_i[u] = \int_{s_i}^{t_i} (u^{\sigma})^2 \left[\psi_i - \frac{\rho^2 r^2}{4\rho^{\sigma}\beta_i [\alpha_2 p + \alpha_1 r]} \left(\left[\frac{\rho^{\Delta}}{\rho} + \frac{2\rho^{\sigma}\beta_i a [\alpha_2 p + \alpha_1 r]}{\rho r} \right] + \frac{[u + u^{\sigma}]\,u^{\Delta}}{(u^{\sigma})^2} \right)^2 \right] \Delta t \geq 0, \tag{56}$$

where for $t \in [s_i, t_i)_{\mathbb{T}}$ and $i = 1, 2$

$$\psi_i(t) = \rho^{\sigma}(t) \left[\gamma(\gamma - 1)^{(1-\gamma)/\gamma} (Lq(t))^{1/\gamma} \delta_i(t) |e(t)|^{(\gamma-1)/\gamma} - (a(t)\,r(t))^{\Delta} + \frac{\rho^{\sigma}(t)\,[\alpha_2 p(t) + \alpha_1 r(t)]}{\beta_i(t)} a^2(t) \right],$$
$$\beta_i(t) = \begin{cases} \left(\int_{s_i}^{t} \frac{\Delta s}{r(s)} \right) \left(\left(1 - \frac{1}{\alpha_1 \alpha_3} \right) \int_{s_i}^{t} \frac{\Delta s}{r(s)} + \frac{1}{\alpha_1 \alpha_3} \int_{s_i}^{\sigma(t)} \frac{\Delta s}{r(s)} \right)^{-1}, & t < \sigma(t), \\ 1, & t = \sigma(t). \end{cases} \tag{57}$$

Then every solution of (1) is oscillatory.

Proof. Assume that (1) is nonoscillatory on $[t_0, \infty)_{\mathbb{T}}$. Then there is a solution x of (1) and a $T \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)$ and $x(\tau(t))$ have the same sign on $[T, \infty)_{\mathbb{T}}$. Consider the case where $x(t)$ and $x(\tau(t))$ are positive on $[T, \infty)_{\mathbb{T}}$. Define the function $w(t)$ by the generalized Riccati substitution:

$$w(t) = \rho(t) \left[\frac{r(t)\,g_1(x(t), x^{\Delta}(t))}{x(t)} + a(t)\,r(t) \right], \quad \text{for } t \in [s_1, t_1)_{\mathbb{T}}. \tag{58}$$

Then, from (1) and using (H_5), we have

$$w^{\Delta}(t) = \rho^{\Delta}(t) \left[\frac{r(t)\,g_1(x(t), x^{\Delta}(t))}{x(t)} + a(t)\,r(t) \right] + \rho^{\sigma}(t) \left[\frac{r(t)\,g_1(x(t), x^{\Delta}(t))}{x(t)} + a(t)\,r(t) \right]^{\Delta}$$

$$
\begin{aligned}
&= \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) + \rho^{\sigma}(t)(a(t)r(t))^{\Delta} + \rho^{\sigma}(t)\left[\frac{r(t)g_1\left(x(t), x^{\Delta}(t)\right)}{x(t)}\right]^{\Delta} \\
&= \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) + \rho^{\sigma}(t)(a(t)r(t))^{\Delta} + \rho^{\sigma}(t)\left[\frac{x(t)\left[r(t)g_1\left(x(t), x^{\Delta}(t)\right)\right]^{\Delta} - r(t)g_1\left(x(t), x^{\Delta}(t)\right)x^{\Delta}(t)}{x(t)x^{\sigma}(t)}\right] \\
&= \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) + \rho^{\sigma}(t)(a(t)r(t))^{\Delta} + \rho^{\sigma}(t)\left[-\frac{p(t)g_1\left(x(t), x^{\Delta}(t)\right)x^{\Delta}(t)}{x^{\sigma}(t)} - \frac{q(t)f(x(\tau(t)))}{x^{\sigma}(t)} + \frac{e(t)}{x^{\sigma}(t)}\right] \\
&\quad - \frac{\rho^{\sigma}(t)r(t)g_1\left(x(t), x^{\Delta}(t)\right)x^{\Delta}(t)}{x(t)x^{\sigma}(t)} \\
&\le \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) + \rho^{\sigma}(t)(a(t)r(t))^{\Delta} - \rho^{\sigma}(t)\left[\alpha_2 p(t) + \alpha_1 r(t)\right]\frac{g_1^2\left(x(t), x^{\Delta}(t)\right)}{x(t)x^{\sigma}(t)} + \frac{\rho^{\sigma}(t)e(t)}{x^{\sigma}(t)} \\
&\quad - \frac{\rho^{\sigma}(t)q(t)f(x(\tau(t)))}{x^{\sigma}(t)} \\
&= \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) + \rho^{\sigma}(t)(a(t)r(t))^{\Delta} - \rho^{\sigma}(t)\left[\alpha_2 p(t) + \alpha_1 r(t)\right]\frac{x(t)}{x^{\sigma}(t)}\left[\frac{g_1\left(x(t), x^{\Delta}(t)\right)}{x(t)}\right]^2 + \frac{\rho^{\sigma}(t)e(t)}{x^{\sigma}(t)} \\
&\quad - \frac{\rho^{\sigma}(t)q(t)f(x(\tau(t)))}{x^{\sigma}(t)}.
\end{aligned} \tag{59}
$$

Using (19), we get

$$
\begin{aligned}
w^{\Delta}(t) &\le \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) + \rho^{\sigma}(t)(a(t)r(t))^{\Delta} - \rho^{\sigma}(t) \\
&\quad \cdot \left[\alpha_2 p(t) + \alpha_1 r(t)\right]\frac{x(t)}{x^{\sigma}(t)}\left[\frac{g_1\left(x(t), x^{\Delta}(t)\right)}{x(t)}\right]^2 \\
&\quad + \frac{\rho^{\sigma}(t)e(t)}{x^{\sigma}(t)} - \frac{\rho^{\sigma}(t)q(t)f(x(\tau(t)))}{\delta_1(t)x(\tau(t))} \le \frac{\rho^{\Delta}(t)}{\rho(t)} \\
&\quad \cdot w(t) + \rho^{\sigma}(t)(a(t)r(t))^{\Delta} - \rho^{\sigma}(t) \\
&\quad \cdot \left[\alpha_2 p(t) + \alpha_1 r(t)\right]\frac{x(t)}{x^{\sigma}(t)}\left[\frac{g_1\left(x(t), x^{\Delta}(t)\right)}{x(t)}\right]^2 \\
&\quad + \frac{\rho^{\sigma}(t)e(t)}{x^{\sigma}(t)} - L\rho^{\sigma}(t)\delta_1(t)q(t)|x(\tau(t))|^{\gamma-1}.
\end{aligned} \tag{60}
$$

Therefore,

$$
\begin{aligned}
&L\rho^{\sigma}(t)\delta_1(t)q(t)|x(\tau(t))|^{\gamma-1} - \frac{\rho^{\sigma}(t)e(t)}{x^{\sigma}(t)} \\
&\quad \le -w^{\Delta}(t) + \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) + \rho^{\sigma}(t)(a(t)r(t))^{\Delta} \\
&\quad - \rho^{\sigma}(t)\left[\alpha_2 p(t) + \alpha_1 r(t)\right] \\
&\quad \cdot \frac{x(t)}{x^{\sigma}(t)}\left[\frac{g_1\left(x(t), x^{\Delta}(t)\right)}{x(t)}\right]^2.
\end{aligned} \tag{61}
$$

Since $x(t) > 0$ for all $t \ge T$ and $e(t) \le 0$ on the interval $[\tau(s_1), t_1)_{\mathbb{T}}$, then from (61) we have for $t \in [\tau(s_1), t_1)_{\mathbb{T}}$

$$
\begin{aligned}
&L\rho^{\sigma}(t)\delta_1(t)q(t)|x(\tau(t))|^{\gamma-1} + \frac{\rho^{\sigma}(t)|e(t)|}{x^{\sigma}(t)} \\
&\quad \le -w^{\Delta}(t) + \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) + \rho^{\sigma}(t)(a(t)r(t))^{\Delta} \\
&\quad - \rho^{\sigma}(t)\left[\alpha_2 p(t) + \alpha_1 r(t)\right] \\
&\quad \cdot \frac{x(t)}{x^{\sigma}(t)}\left[\frac{g_1\left(x(t), x^{\Delta}(t)\right)}{x(t)}\right]^2.
\end{aligned} \tag{62}
$$

From (25) and (62), we have

$$
\begin{aligned}
&\gamma(\gamma-1)^{(1-\gamma)/\gamma}(Lq(t))^{1/\gamma}\rho^{\sigma}(t)\delta_1(t)|e(t)|^{(\gamma-1)/\gamma} \\
&\quad \le -w^{\Delta}(t) + \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) + \rho^{\sigma}(t)(a(t)r(t))^{\Delta} \\
&\quad - \rho^{\sigma}(t)\left[\alpha_2 p(t) + \alpha_1 r(t)\right] \\
&\quad \cdot \frac{x(t)}{x^{\sigma}(t)}\left[\frac{g_1\left(x(t), x^{\Delta}(t)\right)}{x(t)}\right]^2.
\end{aligned} \tag{63}
$$

From the definition of $w(t)$, we see that

$$\left[\frac{g_1\left(x(t), x^{\Delta}(t)\right)}{x(t)}\right]^2 = \left[\frac{w(t)}{\rho(t) r(t)} - a(t)\right]^2 = \left[\frac{w(t)}{\rho(t) r(t)}\right]^2 + a^2(t) - \frac{2w(t) a(t)}{\rho(t) r(t)}. \tag{64}$$

As in the proof of (19) in Theorem 2, we have

$$\beta_1(t) x^{\sigma}(t) \le x(t) \quad \text{for } t \in [s_1, t_1)_{\mathbb{T}}. \tag{65}$$

Using (64) and (65) in (63), we get

$$\begin{aligned}
&\gamma(\gamma-1)^{(1-\gamma)/\gamma}(Lq(t))^{1/\gamma}\rho^{\sigma}(t)\delta_1(t)|e(t)|^{(\gamma-1)/\gamma} \\
&\quad \le -w^{\Delta}(t) + \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) + \rho^{\sigma}(t)(a(t) r(t))^{\Delta} \\
&\qquad - \frac{\rho^{\sigma}(t)\beta_1(t)[\alpha_2 p(t) + \alpha_1 r(t)]}{\rho^2(t) r^2(t)} w^2(t) \\
&\qquad - \rho^{\sigma}(t)\beta_1(t)[\alpha_2 p(t) + \alpha_1 r(t)] a^2(t) \\
&\qquad + \frac{2\rho^{\sigma}(t)\beta_1(t) a(t)[\alpha_2 p(t) + \alpha_1 r(t)]}{\rho(t) r(t)} w(t).
\end{aligned} \tag{66}$$

Hence,

$$\begin{aligned}
\psi_1(t) \le& -w^{\Delta}(t) + \left[\frac{\rho^{\Delta}(t)}{\rho(t)} + \frac{2\rho^{\sigma}(t)\beta_1(t) a(t)[\alpha_2 p(t) + \alpha_1 r(t)]}{\rho(t) r(t)}\right] w(t) \\
&- \frac{\rho^{\sigma}(t)\beta_1(t)[\alpha_2 p(t) + \alpha_1 r(t)]}{\rho^2(t) r^2(t)} w^2(t).
\end{aligned} \tag{67}$$

Multiplying (67) by $(u^{\sigma}(t))^2$ and integrating from s_1 to t_1, we have

$$\begin{aligned}
\int_{s_1}^{t_1} (u^{\sigma}(t))^2 \psi_1(t)\Delta t \le& -\int_{s_1}^{t_1} (u^{\sigma}(t))^2 w^{\Delta}(t)\Delta t \\
&+ \int_{s_1}^{t_1} (u^{\sigma}(t))^2 \cdot \left[\frac{\rho^{\Delta}(t)}{\rho(t)} + \frac{2\rho^{\sigma}(t)\beta_1(t) a(t)[\alpha_2 p(t) + \alpha_1 r(t)]}{\rho(t) r(t)}\right] \\
&\cdot w(t)\Delta t - \int_{s_1}^{t_1} (u^{\sigma}(t))^2 \cdot \frac{\rho^{\sigma}(t)\beta_1(t)[\alpha_2 p(t) + \alpha_1 r(t)]}{\rho^2(t) r^2(t)} w^2(t)\Delta t.
\end{aligned} \tag{68}$$

Using integration by parts, we get

$$\begin{aligned}
\int_{s_1}^{t_1} (u^{\sigma}(t))^2 \psi_1(t)\Delta t \le& -u^2(t) w(t)\Big|_{s_1}^{t_1} + \int_{s_1}^{t_1} [u(t) + u^{\sigma}(t)] u^{\Delta}(t) w(t)\Delta t \\
&+ \int_{s_1}^{t_1} (u^{\sigma}(t))^2 \cdot \left[\frac{\rho^{\Delta}(t)}{\rho(t)} + \frac{2\rho^{\sigma}(t)\beta_1(t) a(t)[\alpha_2 p(t) + \alpha_1 r(t)]}{\rho(t) r(t)}\right] \\
&\cdot w(t)\Delta t - \int_{s_1}^{t_1} (u^{\sigma}(t))^2 \cdot \frac{\rho^{\sigma}(t)\beta_1(t)[\alpha_2 p(t) + \alpha_1 r(t)]}{\rho^2(t) r^2(t)} w^2(t)\Delta t.
\end{aligned} \tag{69}$$

From the fact that $u(s_1) = u(t_1) = 0$, we obtain

$$\begin{aligned}
\int_{s_1}^{t_1} (u^{\sigma}(t))^2 \psi_1(t)\Delta t \le& \int_{s_1}^{t_1} [u(t) + u^{\sigma}(t)] u^{\Delta}(t) \cdot w(t)\Delta t + \int_{s_1}^{t_1} (u^{\sigma}(t))^2 \left[\frac{\rho^{\Delta}(t)}{\rho(t)} + \frac{2\rho^{\sigma}(t)\beta_1(t) a(t)[\alpha_2 p(t) + \alpha_1 r(t)]}{\rho(t) r(t)}\right] w(t)\Delta t \\
&- \int_{s_1}^{t_1} (u^{\sigma}(t))^2 \cdot \frac{\rho^{\sigma}(t)\beta_1(t)[\alpha_2 p(t) + \alpha_1 r(t)]}{\rho^2(t) r^2(t)} w^2(t)\Delta t \\
\le& \int_{s_1}^{t_1} \frac{\rho^2(t) r^2(t)(u^{\sigma}(t))^2}{4\rho^{\sigma}(t)\beta_1(t)[\alpha_2 p(t) + \alpha_1 r(t)]} \left(\left[\frac{\rho^{\Delta}(t)}{\rho(t)} + \frac{2\rho^{\sigma}(t)\beta_1(t) a(t)[\alpha_2 p(t) + \alpha_1 r(t)]}{\rho(t) r(t)}\right] + \frac{[u(t) + u^{\sigma}(t)] u^{\Delta}(t)}{(u^{\sigma}(t))^2}\right)^2 \Delta t.
\end{aligned} \tag{70}$$

This implies that

$$E_1[u] = \int_{s_1}^{t_1} (u^{\sigma})^2 \left[\psi_1 - \frac{\rho^2 r^2}{4\rho^{\sigma}\beta_1[\alpha_2 p + \alpha_1 r]} \left(\left[\frac{\rho^{\Delta}}{\rho} + \frac{2\rho^{\sigma}\beta_1 a[\alpha_2 p + \alpha_1 r]}{\rho r}\right] + \frac{[u + u^{\sigma}] u^{\Delta}}{(u^{\sigma})^2}\right)^2\right]\Delta t \le 0 \tag{71}$$

which is a contradiction with (56). The case $x(t) < 0$, $x(\tau(t)) < 0$ on $T \in [t_0, \infty)_{\mathbb{T}}$ is similar (in this case, we use $e(t) \ge 0$ on $[\tau(s_2), t_2)_{\mathbb{T}}$ and $E_2 \le 0$). □

Remark 6. From Theorem 5, we can establish different sufficient conditions for the oscillation of (1) by using different choices of $\rho(t)$ and $a(t)$.

3. Examples

In this section, we give some examples to illustrate our results.

Example 1. Let $\mathbb{T} = \mathbb{R}$ and consider the following nonlinear forced delay differential equation:

$$\begin{aligned}&\left(\left(\sin^2 t + 1\right)(2 + \sin t)\, g_1\left(x, x'\right)\right)' \\ &\quad + \left(\sin^2 t + 1\right)(2 - \sin t)\, g_2\left(x, x'\right) x' \\ &\quad + 3\cos t f\left(x\left(\tau\left(t\right)\right)\right) = \sin t,\end{aligned} \tag{72}$$

where

$$\begin{aligned}&g_1\left(x, x'\right) = x', \\ &g_2\left(x, x'\right) = xx'\left(1 + x^2 + x'^2\right), \\ &\qquad\qquad\qquad\qquad \left(\alpha_1 = \alpha_2 = \alpha_3 = 1\right), \\ &f(x) = x(2 + \cos x), \\ &\tau(t) = t, \\ &\qquad\qquad\qquad\qquad \left(L = \gamma = 1\right).\end{aligned} \tag{73}$$

Equation (72) is of the form (1) where

$$\begin{aligned}r(t) &= \left(\sin^2 t + 1\right)(2 + \sin t), \\ p(t) &= \left(\sin^2 t + 1\right)(2 - \sin t), \\ q(t) &= 3\cos t, \\ e(t) &= \sin t.\end{aligned} \tag{74}$$

Applying Theorem 2 and for any T, we choose n sufficiently large so that $2n\pi - \pi \geq T$. Let

$$\begin{aligned}s_1 &= 2n\pi - \pi, \\ t_1 = s_2 &= 2n\pi, \\ t_2 &= 2n\pi + \pi.\end{aligned} \tag{75}$$

Note that $p(t) \geq 0$, $q(t) \geq 0$ on

$$\begin{aligned}&\left[\tau\left(s_1\right), t_1\right) \cup \left[\tau\left(s_2\right), t_2\right) \\ &\quad = [2n\pi - \pi, 2n\pi) \cup [2n\pi, 2n\pi + \pi) \\ &\quad = [2n\pi - \pi, 2n\pi + \pi), \\ &e(t) \\ &\quad = \sin t \begin{cases} \leq 0, & t \in \left[\tau\left(s_1\right), t_1\right) = [2n\pi - \pi, 2n\pi), \\ \geq 0, & t \in \left[\tau\left(s_2\right), t_2\right) = [2n\pi, 2n\pi + \pi). \end{cases}\end{aligned} \tag{76}$$

If we take $\rho(t) = 1$ and $u(t) = \sin t$, then $u(s_i) = u(t_i) = 0$, $u(t) \not\equiv 0$ on $[s_i, t_i]$, $i = 1, 2$.

It is easy to verify that

$$\begin{aligned}&\int_{s_1}^{t_1} u^2(t)\, Q_1(t)\, dt = 3\int_{2n\pi-\pi}^{2n\pi} \sin^2 t \cos t\, dt = 0, \\ &\int_{s_1}^{t_1} \frac{r^2(t)}{4u^2(t)\left[p(t) + r(t)\right]}\left(2u(t)\, u'(t)\right) dt \\ &\quad = \int_{2n\pi-\pi}^{2n\pi} \frac{\left(\sin^2 t + 1\right)^2 (2 + \sin t)^2}{4\sin^2 t\left(4\sin^2 t + 4\right)}(2\sin t\cos t)\, dt \\ &\quad = 0,\end{aligned} \tag{77}$$

where $\delta_1(t) = 1$. Hence $A_1[u] = 0$; that is, (5) holds for $i = 1$. Similarly, for s_2 and t_2, we can show that (5) holds for $i = 2$. Therefore, by Theorem 2, we get that (72) is oscillatory.

Remark 7. The results of [15] cannot be applied to (72) for $r(t) = (\sin^2 t + 1)(2 + \sin t) \neq 1$ and $p(t) = (\sin^2 t + 1)(2 - \sin t) \neq 0$. But, according to Theorem 2, when ($\mathbb{T} = \mathbb{R}$), this equation is oscillatory.

Example 2. Let $\mathbb{T} = \mathbb{R}$ and consider the following nonlinear forced delay differential equation:

$$\begin{aligned}&-2x'' + xx'^2 + \frac{m}{\sqrt{t}}x(x + 1)\operatorname{sgn}(x)(t) = \sin\sqrt{t}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad t \geq 1,\end{aligned} \tag{78}$$

where

$$\begin{aligned}&g_1\left(x, x'\right) = x', \\ &g_2\left(x, x'\right) = xx', \\ &\qquad\qquad\qquad\qquad \left(\alpha_1 = \alpha_2 = \alpha_3 = 1\right), \\ &f(x) = x(x + 1)\operatorname{sgn}(x), \\ &\tau(t) = t, \\ &\qquad\qquad\qquad\qquad \left(L = \gamma = 1\right).\end{aligned} \tag{79}$$

Equation (78) is of the form (1) where

$$\begin{aligned}r(t) &= -2, \\ p(t) &= 1, \\ q(t) &= \frac{m}{\sqrt{t}}, \\ m &\geq 0, \\ e(t) &= \sin\sqrt{t}.\end{aligned} \tag{80}$$

Applying Theorem 5 and for any $T \geq 1$, we choose n sufficiently large so that $(2n-1)^2\pi^2 \geq T$. Let

$$\begin{aligned} s_1 &= (2n-1)^2\pi^2, \\ t_1 = s_2 &= (2n\pi)^2, \\ t_2 &= (2n+1)^2\pi^2. \end{aligned} \quad (81)$$

Note that $p(t) \geq 0, q(t) \geq 0$ on

$$\begin{aligned} &[\tau(s_1), t_1) \cup [\tau(s_2), t_2) \\ &= \left[(2n-1)^2\pi^2, (2n\pi)^2\right) \cup \left[(2n\pi)^2, (2n+1)^2\pi^2\right), \\ &e(t) \\ &= \sin\sqrt{t}\begin{cases} \leq 0, & t \in [\tau(s_1), t_1) = \left[(2n-1)^2\pi^2, (2n\pi)^2\right), \\ \geq 0, & t \in [\tau(s_2), t_2) = \left[(2n\pi)^2, (2n+1)^2\pi^2\right). \end{cases} \end{aligned} \quad (82)$$

If we take $\rho(t) = 1$, $a(t) = 1$, and $u(t) = \sin\sqrt{t}$, then $u(s_i) = u(t_i) = 0$, $u(t) \not\equiv 0$ on $[s_i, t_i]$, $i = 1, 2$.

It is easy to verify that

$$\begin{aligned} E_1[u] &= \int_{s_1}^{t_1} (u^\sigma)^2 \left[\psi_1 \right. \\ &- \frac{\rho^2 r^2}{4\rho^\sigma \beta_1 [\alpha_2 p + \alpha_1 r]} \left(\left[\frac{\rho^\Delta}{\rho} + \frac{2\rho^\sigma \beta_1 a [\alpha_2 p + \alpha_1 r]}{\rho r}\right]\right. \\ &+ \left.\left.\frac{[u + u^\sigma] u^\Delta}{(u^\sigma)^2}\right)^2\right] \Delta t = \int_{(2n-1)^2\pi^2}^{(2n\pi)^2} \left(\frac{m}{\sqrt{t}} \sin^3\sqrt{t} + 4\right. \\ &\cdot \left.\cos^2\sqrt{t} + 4\cos\sqrt{t}\sin\sqrt{t}\right) dt = \int_{(2n-1)\pi}^{2n\pi} \left(2m\sin^3 s\right. \\ &+ \left. 8s\cos^2 s + 8s\cos s\sin s\right) ds = \frac{8}{3}m + 2\pi^2(4n-1) \\ &- 2\pi > 0, \end{aligned} \quad (83)$$

where $\delta_1(t) = \beta_1(t) = 1$. Hence (56) holds for $i = 1$. Similarly, for s_2 and t_2, we can show that (56) holds for $i = 2$. Therefore, by Theorem 5, we get that (78) is oscillatory.

Remark 8. The results obtained in the above examples cannot be obtained by the results in either [9] or [10].

Competing Interests

The authors declare that they have no competing interests.

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