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Research Article

p-Trigonometric and p-Hyperbolic Functions in Complex Domain

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We study extension of p-trigonometric functions $\sin_p$ and $\cos_p$ and of p-hyperbolic functions $\sinh_p$ and $\cosh_p$ to complex domain. Our aim is to answer the question under what conditions on p these functions satisfy well-known relations for usual trigonometric and hyperbolic functions, such as, for example, $\sin(z) = -i\cdot\sinh(i\cdot z)$. In particular, we prove in the paper that for $p = 6, 10, 14, \ldots$ the p-trigonometric and p-hyperbolic functions satisfy very analogous relations as their classical counterparts. Our methods are based on the theory of differential equations in the complex domain using the Maclaurin series for p-trigonometric and p-hyperbolic functions.

1. Introduction

The p-trigonometric functions are generalizations of regular trigonometric functions sine and cosine and arise from the study of the eigenvalue problem for the one-dimensional p-Laplacian.

In recent years, the p-trigonometric functions were intensively studied from various points of views by many researchers; see, for example, monograph [1] for systematic survey and further references. The purpose of this paper is twofold. We begin with a short survey of results from [2, 3]. Then, we extend the ideas from [3] to define corresponding generalization of hyperbolic functions and study relations of p-trigonometric and p-hyperbolic functions on a disc in the complex domain.

More precisely, our goal is to generalize the hyperbolic functions such that the relations

$$\begin{aligned} \sin z &= -i \cdot \sinh(i \cdot z), \\ \cos z &= \cosh(i \cdot z), \end{aligned} \tag{1}$$

$$\begin{aligned} \cos z &= \sin' z, \\ \cosh z &= \sinh' z, \end{aligned} \tag{2}$$

$$\begin{aligned} \cos^2 z + \sin^2 z &= 1, \\ \cosh^2 z - \sinh^2 z &= 1, \end{aligned} \tag{3}$$

where $z \in \mathbb{C}$, have their counterparts for generalized p-trigonometric and p-hyperbolic functions. It turns out that this goal can be achieved only for even integer $p > 2$.

The p-trigonometric functions in the real domain $\mathbb{R}$ originate naturally from the study of the nonlinear eigenvalue problem

$$\begin{aligned} -\left(\left|u'\right|^{p-2} u'\right)' - \lambda |u|^{p-2} u = 0 \quad \text{in } \left(0, \pi_p\right), \\ u(0) = u\left(\pi_p\right) = 0, \end{aligned} \tag{4}$$

where $p > 1$, $\lambda \in \mathbb{R}$ is a parameter, and

$$\pi_p = 2\int_0^1 \frac{1}{(1-s^p)^{1/p}} \mathrm{d}s = \frac{2\pi}{p \sin(\pi/p)}. \tag{5}$$

It was shown in Elbert [4] that all eigenfunctions of (4) can be expressed in terms of solutions of the initial-value problem

$$\begin{aligned} -\left(\left|u'\right|^{p-2} u'\right)' - (p-1)|u|^{p-2} u = 0, \quad x \in \mathbb{R}, \\ u(0) = 0, \\ u'(0) = 1. \end{aligned} \tag{6}$$

Indeed, (6) has the unique solution in $\mathbb{R}$; see, for example, [5, Lemma A.1], [6, Section 3], and [4]. Denoting the solution of (6) by $\sin_p x$, the set of all eigenvalues $\lambda_k \in \mathbb{R}$ and eigenfunctions $u_k \in W_0^{1,p}(0,\pi_p)$ of (4) can be written as

$$\lambda_k = (p-1)\,k^p,$$
$$u_k(x) = \sin_p(k\cdot x), \tag{7}$$
$$\text{where } k \in \mathbb{N}.$$

A piecewise construction of the solution of (6) was provided in [4]. At first, one sets

$$\arcsin_p x \overset{\text{def}}{=} \int_0^x \frac{1}{(1-s^p)^{1/p}}\mathrm{d}s, \quad x \in [0,1]. \tag{8}$$

Then, the restriction of $\sin_p x$ on $[0,\pi_p/2]$ is the inverse function to $\arcsin_p x$. For $x \in (\pi_p/2,\pi_p]$, $\sin_p x$ satisfies $\sin_p x = \sin_p(\pi_p - x)$, where clearly $\pi_p - x \in [0,\pi_p/2)$, and $\sin_p x = -\sin_p(-x)$ for $x \in [-\pi_p, 0)$. Finally, $\sin_p x$ is a $2\pi_p$-periodic function on $\mathbb{R}$.

We also extend $\arcsin_p x$ from (8) to $[-1,1]$ as an odd function. Then, it is the inverse function to the restriction of $\sin_p x$ to $[-\pi_p/2,\pi_p/2]$, and we have

$$\sin_p\left(\arcsin_p x\right) = x, \quad \forall x \in [-1,1]. \tag{9}$$

Finally, let us define $\cos_p x \overset{\text{def}}{=} \sin_p' x$ for all $x \in \mathbb{R}$. Then, the functions $\sin_p x$ and $\cos_p x$ satisfy the so-called p-trigonometric identity

$$\left|\cos_p x\right|^p + \left|\sin_p x\right|^p = 1 \tag{10}$$

for all $x \in \mathbb{R}$; see, for example, [4–6].

Note that there is an alternative definition of "$\cos_p x$" in [7] and/or [8] which leads to different "p-trigonometric" identity. Yet another alternative generalization of trigonometric and hyperbolic functions motivated by geometrical point of view was introduced in [9]. Studies of relations between their respective generalizations of p-trigonometric and p-hyperbolic functions were suggested in [7] and in [9], respectively.

Remark 1. In the paper, we use Gauss' hypergeometric function ${}_2F_1(a,b,c,z)$, where $a,b,c \in \mathbb{C}$ are parameters and $z \in \mathbb{C}$ is variable (for definition see [10, 15.1.1. p. 556]), to express integrals of the type

$$\int_0^x \frac{1}{(1\pm s^p)^{1/p}}\mathrm{d}s,$$
$$\int_0^z \frac{1}{(1\pm s^p)^{1/p}}\mathrm{d}s, \tag{11}$$

for $p > 1$, $x \in \mathbb{R}$, and $z \in \mathbb{C}$ (by $z^{1/p}$ we understand the principal branch thereof). Indeed,

$$z\,{}_2F_1\left(\frac{1}{p},\frac{1}{p},1+\frac{1}{p},\mp z^p\right) = \int_0^z \frac{1}{(1\pm s^p)^{1/p}}\mathrm{d}s \tag{12}$$

for $|z| < 1$ (by comparing respective series expansions). By the uniqueness of analytic extension, the equation is valid for $z \in \mathbb{C}\setminus\{x+iy : x > 1 \text{ and } y = 0\}$ (for analytic continuation of ${}_2F_1$ see, e.g., [10, 15.3.1, p. 558] and [11, Theorem 2.2.1, p. 65]).

In the definition of π_p (i.e., (5)) and in (8), we need to evaluate integral

$$\int_0^1 \frac{1}{(1-s^p)^{1/p}}\mathrm{d}s. \tag{13}$$

By [11, Theorem 2.2.2, p. 66], this is possible, since $\Re[c-a-b] = 1+1/p-1/p-1/p = 1-1/p > 0$ for $p > 1$.

Notation 1. This paper combines real variable and complex variable approach to the p-trigonometric and p-hyperbolic functions. Each of these approaches has its own natural way of how to define the functions $\sin_p$ and $\sinh_p$. Thus, we need to distinguish between real and complex definitions. By $\sin_p x$ and $\sinh_p x$, we mean functions defined by the real variable approach and by $\sin_p z$ and $\sinh_p z$ we mean functions defined by the complex variable approach, throughout the paper.

2. Real Analyticity Results for $\sin_p x$ and $\cos_p x$

It is well known that the p-trigonometric functions are not real analytic functions in general; see, for example, [12, 13]. Very detailed study of the degree of smoothness of the restriction of $\sin_p x$ to $(-\pi_p/2,\pi_p/2)$ was performed in [2] including the following two results. The first one concerns "generic" $p > 1$.

Proposition 2 (see [2], Theorem 3.2 on p. 105). *Let $p \in \mathbb{R}\setminus\{2m\}$, $m \in \mathbb{N}$, $p > 1$. Then,*

$$\sin_p x \in C^{\lceil p\rceil}\left(-\frac{\pi_p}{2},\frac{\pi_p}{2}\right), \tag{14}$$

but

$$\sin_p x \notin C^{\lceil p\rceil+1}\left(-\frac{\pi_p}{2},\frac{\pi_p}{2}\right). \tag{15}$$

Here, $\lceil p\rceil \overset{\text{def}}{=} \min\{k \in \mathbb{N}: k \ge p\}$.

The second result treats only the even integers $p > 2$ and differs significantly from the previous case in an unexpected way.

Proposition 3 (see [2], Theorem 3.1 on p. 105). *Let $p = 2(m+1)$, $m \in \mathbb{N}$. Then,*

$$\sin_p x \in C^{\infty}\left(-\frac{\pi_p}{2},\frac{\pi_p}{2}\right). \tag{16}$$

Thus, the Maclaurin series of $\sin_p x$

$$M_p(x) \overset{\text{def}}{=} \sum_{n=1}^{\infty}\frac{1}{n!}\sin_p^{(n)}(0)\cdot x^n \tag{17}$$

is well defined for $p = 2(m+1)$, $m \in \mathbb{N}$. Moreover, the following result establishes an explicit expression for the radius of convergence of M_p.

Proposition 4 (see [2], Theorem 3.3 on p. 106). *Let $p = 2(m+1)$ for $m \in \mathbb{N}$. Then, the Maclaurin series $M_p(x)$ of $\sin_p x$ converges on $(-\pi_p/2, \pi_p/2)$.*

Thus, for $p = 2(m+1)$, $m \in \mathbb{N}$, we can compute approximate values of $\sin_p x$ using Maclaurin series. It turns out that the most effective method of computing coefficients in (17) is to use formal inversion of the Maclaurin series of

$$\begin{aligned}\operatorname{arcsin}_p x &= \int_0^x \frac{1}{(1-s^p)^{1/p}} \mathrm{d}s \\ &= x \cdot {}_2F_1\left(\frac{1}{p}, \frac{1}{p}, 1+\frac{1}{p}, x^p\right) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(k+1/p)}{(kp+1)\, k!\,\Gamma(1/p)} x^{kp+1},\end{aligned} \tag{18}$$

where $p = 2(m+1)$ for some $m \in \mathbb{N}$. The procedure of inverting power series is well known; see, for example, [14]. This task can be easily performed using computer algebra systems. In Pseudocode 1, we provide an example of computing the partial sum of M_4 up to terms of order 32 in *Mathematica*® v. 9.0. In this way, we can easily get partial sums of M_p and get approximations of $\sin_p$ for any $p = 2(m+1)$, $m \in \mathbb{N}$, up to terms of orders of hundreds. Note that this formal inverse can be applied also for $p = 2m+1$, $m \in \mathbb{N}$. The question is what is the mathematical sense of the resulting formal series. Let

$$T_p(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} a_n \cdot x^n \tag{19}$$

denote the series that is the formal inverse of (18) for $p = 2m+1$, $m \in \mathbb{N}$. For $p = 2m+1$, $m \in \mathbb{N}$, let us also define

$$M_p(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{n!} \left(\lim_{x\to 0+} \sin_p^{(n)} x\right) \cdot x^n, \tag{20}$$

which is a formal Maclaurin series of some unknown function. It turns out that this unknow function is not $\sin_p x$ as the following result holds.

Proposition 5 (see [2], Theorem 3.4 on p. 106). *Let $p = 2m+1$, $m \in \mathbb{N}$. Then, the formal Maclaurin series M_p converges on $(-\pi_p/2, \pi_p/2)$. Moreover, the formal Maclaurin series M_p converges towards $\sin_p x$ on $[0, \pi_p/2)$ but does not converge towards $\sin_p x$ on $(-\pi_p/2, 0)$.*

In Appendix A, we prove that T_p and M_p are identical.

Theorem 6. *Let $p = 2m+1$, $m \in \mathbb{N}$. Then,*

$$a_n = \frac{1}{n!}\left(\lim_{x\to 0+} \sin_p^{(n)} x\right), \quad \forall n \in \mathbb{N}, \tag{21}$$

and $T_p(x) = M_p(x)$ for all $x \in (-\pi_p/2, \pi_p/2)$.

It turns out that the pattern of zero coefficients of M_p is the same as in the Maclaurin series of $\operatorname{arcsin}_p x$; compare (18).

Theorem 7. *Let $p > 2$ be an integer. Then, $a_i = 0$ for all $i \in \mathbb{N}$ such that $i-1$ is not divisible by p.*

The proof is technical and thus postponed to Appendix B. It is based on the formal inversion of (18). Note that the structure of powers in (18) does not allow any substitution that will transform it into a power series of new variable without zero coefficients. This makes the proof technically complicated.

Using Theorem 7, we can omit zero entries and rewrite the series M_p:

$$M_p(x) = \sum_{l=0}^{\infty} \alpha_l \cdot x^{lp+1}, \tag{22}$$

where α_l can be obtained by formal inversion of the Maclaurin series of $\operatorname{arcsin}_p x$ in (18). In particular,

$$\begin{aligned}\alpha_0 &= 1, \\ \alpha_1 &= -\frac{1}{p(p+1)}, \\ \alpha_2 &= -\frac{p^2-2p-1}{2p^2(p+1)(2p+1)}, \ldots.\end{aligned} \tag{23}$$

3. Extension of $\sin_p z$ and $\cos_p z$ to the Complex Domain for Integer $p > 1$

The conclusion of this theorem follows from the discussion in [2].

Theorem 8. *Let $p = 2(m+1)$, $m \in \mathbb{N}$. Then, the Maclaurin series of $\sin_p x$ converges on the open disc*

$$B_p \stackrel{\text{def}}{=} \left\{z \in \mathbb{C} : |z| < \frac{\pi_p}{2}\right\}. \tag{24}$$

Proof. In fact, the Maclaurin series $\sum_{l=0}^{\infty} \alpha_l \cdot x^{l\cdot p+1}$ converges towards the values of $\sin_p x$ on $(-\pi_p/2, \pi_p/2)$ absolutely for $p = 2(m+1)$, $m \in \mathbb{N}$. □

For $p = 2(m+1)$, $m \in \mathbb{N}$, the expressions with powers in the initial-value problem (6) can be written without the absolute values. Thus, the resulting initial-value problem

$$\begin{aligned}(u')^{p-2} u'' + u^{p-1} &= 0, \\ u(0) &= 0, \\ u'(0) &= 1\end{aligned} \tag{25}$$

makes sense also in the complex domain (the derivatives are understood in the sense of the derivative with respect to

```
In[1] := Series[s*Hypergeometric2F1[1/4, 1/4, 5/4, s^4],
            {s, 0, 32}]
(* computes the Maclaurin series of arcsin_4 *)
```
$$\texttt{Out[1] = } s+\frac{s^5}{20}+\frac{5s^9}{288}+\frac{15s^{13}}{1664}+\frac{195s^{17}}{34816}+\frac{221s^{21}}{57344}+\frac{4641s^{25}}{1638400}+\frac{16575s^{29}}{7602176}+O(s^{33})$$
```
In[2] := InverseSeries[%]
(* computes the inverse series *)
```
$$\texttt{Out[2] = } s-\frac{s^5}{20}-\frac{7s^9}{1440}-\frac{463s^{13}}{374400}-\frac{211741s^{17}}{509184000}-\frac{104361161s^{21}}{641571840000}-\frac{8978996213s^{25}}{128314368000000}-\frac{7995735867463s^{29}}{248783228928000000}+O(s^{33})$$
```
(*which is M_4 up to terms of order 32*)
```

Pseudocode 1: *Mathematica* v. 9.0 code.

complex variable). Let us observe that, using the substitution $u' = v$, we get the following first-order system:

$$\begin{aligned} u' &= v, \\ v' &= -\frac{u^{p-1}}{v^{p-2}}, \\ u(0) &= 0, \\ v(0) &= 1. \end{aligned} \tag{26}$$

By [15, Theorem 9.1, p. 76], there exists $\delta_p > 0$ such that problems (26) and hence (25) have the unique solution on the open disc $|z| < \delta_p$.

Now we will consider initial-value problems (25) and (26) also for $p = 2m+1$, $m \in \mathbb{N}$.

Theorem 9. *Let $p = 2m+1$, $m \in \mathbb{N}$. The unique solution $u(z)$ of (25) restricted to open disc B_p is the Maclaurin series M_p.*

Proof. Let $u(z) = \sum_{k=1}^{\infty} b_k z^k$ be the unique solution of (25) in any point of the open disc $|z| < \delta_p$. Observe that the solution $u(z) = \sum_{k=1}^{\infty} b_k z^k$ solves also the real-valued Cauchy problem (6) for $x > 0$ (where there is no need for $|\cdot|$). On the other hand, $\sin_p x$ is the unique solution of the real-valued Cauchy problem (6). Since M_p given by (20) converges towards $\sin_p x$ in $[0, \pi_p/2)$, we find that M_p satisfies (6) in $[0, \pi_p/2)$. Thus, $M_p(x) = u(x + i\cdot y)$ for $x \in [0, \delta_p)$ and $y = 0$. In particular, taking a sequence of points $z_n = \delta_p/(n+1) + 0\cdot i$, $n \in \mathbb{N}$, we have $M_p(z_n) = u(z_n)$; thus, we infer from Proposition A.2 that

$$b_k = \frac{1}{k!}\left(\lim_{x\to 0+} \sin_p^{(k)} x\right). \tag{27}$$

M_p has radius of convergence $\pi_p/2$ by Proposition 5, and so does $u(z)$. Thus, $\delta_p = \pi_p/2$. □

Theorems 8 and 9 enable us to extend the range of definition of the function $\sin_p x$ to the complex open disc B_p by M_p for $p = 2(m+1)$ and for $p = 2m+1$, $m \in \mathbb{N}$, respectively. Thus, we can consider $p = m+2$ in the following definition. Note that all the powers of z are of positive-integer order $l\cdot p+1$ and the function $\sin_p z$ is an analytic complex function on B_p and thus is single-valued.

Definition 10. Let $p = m+2$, $m \in \mathbb{N}$, and $z \in B_p$. Then,

$$\begin{aligned} \sin_p z &\overset{\text{def}}{=} \sum_{l=0}^{\infty} \alpha_l \cdot z^{l\cdot p+1}, \\ \cos_p z &\overset{\text{def}}{=} \sin_p' z = \frac{\mathrm{d}}{\mathrm{d}z}\sin_p z, \end{aligned} \tag{28}$$

where the derivative $\mathrm{d}/\mathrm{d}z$ is considered in the sense of complex variables.

The following fundamental results were proved in [3] (providing explicit value for δ_p).

Proposition 11 (see [3], Theorem 2.1 on p. 226). *Let $p = 2(m+1)$, $m \in \mathbb{N}$; then, the unique solution of the initial-value problem (25) on B_p is the function $\sin_p z$.*

Proposition 12 (see [3], Theorem 3.1 on p. 229). *Let $p = 2m+1$, $m \in \mathbb{N}$. Then, the unique solution $u(z)$ of the complex initial-value problem (25) differs from the solution $\sin_p x$ of the Cauchy problem (6) for $z = x \in (-\pi_p/2, 0)$.*

In [3], it was shown that there is no hope for solutions of (25) to be entire functions for $p = m+2$, $m \in \mathbb{N}$. This result follows from the complex analogy of the p-trigonometric (10).

Lemma 13. *Let $p = m+1$, $m \in \mathbb{N}$, and $r > 0$ be such that the solution u of (25) is holomorphic on a disc $D_r = \{z \in \mathbb{C} : |z| < r\}$. Then, u satisfies the complex p-trigonometric identity*

$$\left(u'(z)\right)^p + (u(z))^p = 1 \tag{29}$$

on the disc D_r.

Proof. Multiplying (25) by u' and integrating from 0 to $z \in D_r$, we obtain

$$\left(u'(z)\right)^p - \left(u'(0)\right)^p + (u(z))^p - (u(0))^p = 0. \tag{30}$$

Now using the initial conditions of (25), we get (29), which is the first integral of (25) and we can think of it as complex p-trigonometric identity for holomorphic solutions of (25) for $p = m+1$, $m \in \mathbb{N}$. □

Now we state the very classical result from complex analysis.

Proposition 14 (see [16], Theorem 12.20 on p. 433). *Let f and g be entire functions and for some positive integer n satisfy identity*

$$f^n + g^n = 1. \tag{31}$$

(i) *If $n = 2$, then there is an entire function h such that $f = \cos \circ h$ and $g = \sin \circ h$.*

(ii) *If $n > 2$, then f and g are each constant.*

The following interesting connection between complex analysis (including the classical reference [16, Theorem 12.20]) and p-trigonometric functions was studied in [3]. We should point out that it was an interesting internet discussion [17] that called our attention towards this connection. It seems to us that this connection was overlooked by the "p-trigonometric community." Thus, we provide its more precise proof here.

Theorem 15. *The solution u of complex initial-value problem (25) cannot be entire function for any $p = m + 2$, $m \in \mathbb{N}$.*

Proof. Assume by contradiction that the solution u of (25) is entire function. Then, we can choose $r > 0$ arbitrarily large in Lemma 13. Thus, u and u' must satisfy (29) at any point $z \in \mathbb{C}$. Note that u' is an entire function too. Thus, by Proposition 14 u and u' are constant which contradicts $u'(0) = 1$. This concludes the proof. □

In particular, the solution of (25) is $u(z) = \sin_p z$ with $u'(z) = \cos_p z$. Thus, (29) becomes

$$\cos_p^p z + \sin_p^p z = 1 \tag{32}$$

and we see that $\sin_p$ and $\cos_p$ cannot be entire functions for $p = m + 2$, $m \in \mathbb{N}$.

4. Generalized Hyperbolometric Function $\operatorname{argsinh}_p x$ and Generalized Hyperbolic Function $\sinh_p x$ in the Real Domain for Real $p > 1$

In analogy to $p = 2$, we define $\sinh_p x$ for $p > 1$ as the solution to the initial-value problem:

$$\begin{gathered} -\left(\left|u'\right|^{p-2} u'\right)' + (p-1)\,|u|^{p-2}\,u = 0, \quad x \in \mathbb{R}, \\ u(0) = 0, \\ u'(0) = 1. \end{gathered} \tag{33}$$

The uniqueness of the solution of this problem can be proved in the same way as in the case of (6) using the first integral (see [4]). Note that the first integral of the real-valued initial-value problem (33) is the real p-hyperbolic identity

$$1 + |u|^p = \left|u'\right|^p, \tag{34}$$

for $p > 1$; compare [4]. Thus, $|u'| \geq 1$ on the domain of definition of solution to (33). Since $u'(0) = 1$ and u' must be absolutely continuous, we find that $u' > 0$ on the domain of definition of solution to (33) and the real p-hyperbolic identity can be rewritten in equivalent form

$$u' = \left(1 + |u|^p\right)^{1/p}, \tag{35}$$

which is a separable first-order ODE in $\mathbb{R}$. By the standard integration procedure, we obtain *inverse function* of the solution u (cf. [4]).

Therefore, it is natural to define

$$\operatorname{argsinh}_p x \overset{\text{def}}{=} \int_0^x \frac{1}{\left(1 + |s|^p\right)^{1/p}} \mathrm{d}s, \quad x \in \mathbb{R}, \tag{36}$$

for any $p > 1$, in the real domain (cf., e.g., [18–21]). Note that the integral on the right-hand side can be evaluated in terms of the analytic extension of Gauss's ${}_2F_1$ hypergeometric function to $\mathbb{C} \setminus \{s + it : s > 1,\ t = 0\}$ (see, e.g., [10, § 15.3.1, p. 558] and [11, Theorem 2.2.1, p. 65]); thus, (taking into account that integrand in (36) is even)

$$\operatorname{argsinh}_p x = \begin{cases} x \cdot {}_2F_1\left(\dfrac{1}{p}, \dfrac{1}{p}, 1 + \dfrac{1}{p}, -x^p\right), & x \in [0, +\infty) \\ -\operatorname{argsinh}_p(-x), & x \in (-\infty, 0). \end{cases} \tag{37}$$

Since $\operatorname{argsinh}_p : \mathbb{R} \to \mathbb{R}$ is strictly increasing function on $\mathbb{R}$, its inverse exists and it is, in fact, $\sinh_p x$ by the same reasoning as in [4] (cf., e.g., [20]).

5. Generalized Hyperbolic Functions $\sinh_p z$ and $\cosh_p z$ in Complex Domain for Integer $p > 1$

In the previous section, we introduced real-valued generalization of $\sinh x$ called $\sinh_p x$. Our aim is to extend this function to complex domain. It is important to observe that, for $p = 2$, the following relations between complex functions $\sin z$ and $\sinh z$ are known:

$$\begin{array}{c|c} \sin z & \sinh z \\ \hline u'' + u = 0 & u'' - u = 0 \\ u(0) = 0 & u(0) = 0 \\ u'(0) = 1 & u'(0) = 1 \end{array} \tag{38}$$

$$\sin z = -i \cdot \sinh(i \cdot z),$$

where $z \in \mathbb{C}$ and the equations are understood in the sense of ordinary differential equations in the complex domain.

Since the function $|\cdot| : \mathbb{C} \to [0, +\infty)$ (complex modulus) is not analytic at $0 \in \mathbb{C}$, we cannot work with (6) and (33), but we need to consider (25) and

$$\begin{gathered} \left(u'\right)^{p-2} u'' - u^{p-1} = 0, \\ u(0) = 0, \\ u'(0) = 1 \end{gathered} \tag{39}$$

in our discussion in the complex domain. Thus, the direct analogy of the classical relations summarized in the table above for $p \neq 2$ is stated in the following table:

$$\begin{array}{c|c} \sin_p z & \sinh_p z \\ \hline \left(u'\right)^{p-2} u'' + u^{p-1} = 0 & \left(u'\right)^{p-2} u'' - u^{p-1} = 0 \\ u(0) = 0 & u(0) = 0 \\ u'(0) = 1 & u'(0) = 1 \end{array} \tag{40}$$

$$\sin_p z \overset{?}{=} -i \cdot \sinh_p (i \cdot z),$$

where z belongs to some complex disc centred at $0 \in \mathbb{C}$ with radius small enough such that both complex initial-value problems are solvable. However, it turns out (see below) that if we define $\sinh_p z$ as the solution (39), then the "p-analogies" of (2)-(3) are satisfied, but the "p-analogy" of the identity (1), that is,

$$\sin_p z = -i \cdot \sinh_p (i \cdot z), \tag{41}$$

is not satisfied in general. Our aim is to provide conditions when (41) holds as well.

Let us formalize the above-stated ideas. Denote

$$D_p \overset{\text{def}}{=} \left\{z \in \mathbb{C} : |z| < \gamma_p\right\} \tag{42}$$

an open disc in $\mathbb{C}$, where $\gamma_p > 0$ is given radius. At first we prove unique solvability of (39) in D_p.

Lemma 16. *Let $p = m+2$, $m \in \mathbb{N}$. Then, there exists a complex disc D_p such that the initial-value problem in complex domain (39) has a unique solution on D_p.*

Proof. Using the substitution $u' = v$, we get the following first-order system:

$$\begin{gathered} u' = v, \\ v' = \frac{u^{p-1}}{v^{p-2}}, \\ u(0) = 0, \\ v(0) = 1. \end{gathered} \tag{43}$$

By [15, Theorem 9.1] on page 76, the statement of the lemma follows. □

Now we can define $\sinh_p : D_p \to \mathbb{C}$ for any integer $p > 2$.

Definition 17. Let $p = m + 2$, $m \in \mathbb{N}$. The complex function $\sinh_p z$ is defined on D_p as the unique solution of the initial-value problem (39) and $\cosh_p z \overset{\text{def}}{=} \sinh'_p z$ for all $z \in D_p$.

Lemma 18. *Let $p = m + 1$, $m \in \mathbb{N}$, and $r > 0$ be such that the solution u of (25) is holomorphic on a disc $D_r = \{z \in \mathbb{C} : |z| < r\}$. Then, u satisfies the complex "p-hyperbolic" identity*

$$\left(u'(z)\right)^p - (u(z))^p = 1 \tag{44}$$

on the disc D_r.

The proof of Lemma 18 is analogous to the proof of Lemma 13 and thus it is omitted.

Remark 19. Les us note that the real-valued identity

$$\left(u(x)'\right)^p - (u(x))^p = 1 \tag{45}$$

for general $p > 0$ already appeared in [22], where the formal Maclaurin power series expansion of the solution to this identity was treated. Interesting recurrence formula for the coefficients of the Maclaurin power series can be found there. It will be very interesting to use the following relations between $\sin_p z$ and $\sinh_p z$ to find the analogous recurrence formulas for $\sin_p z$.

Now we are ready to state main results of Section 5.

Theorem 20. *Let $p = 4l + 2$, $l \in \mathbb{N}$. Then,*

$$\sin_p z = -i \cdot \sinh_p (i \cdot z), \tag{46}$$

$$\cos_p z = \cosh_p (i \cdot z) \tag{47}$$

for all $z \in B_p$. Moreover,

$$\sinh_p z = \sum_{k=0}^{\infty} (-1)^k \cdot \alpha_k \cdot z^{kp+1}. \tag{48}$$

On the other hand, we have also the following surprising result.

Theorem 21. *Let $p = 4l$, $l \in \mathbb{N}$. Then,*

$$\begin{gathered} \sin_p z = -i \cdot \sin_p (i \cdot z), \\ \cos_p z = \cos_p (i \cdot z) \end{gathered} \tag{49}$$

for all $z \in B_p$.

The statement of the previous theorem is closely related to similar result for p-hyperbolic functions.

Theorem 22. *Let $p = 4l$, $l \in \mathbb{N}$. Then,*

$$\sinh_p z = -i \cdot \sinh_p (i \cdot z), \tag{50}$$

$$\cosh_p z = \cosh_p (i \cdot z) \tag{51}$$

for all $z \in D_p$.

Proof of Theorem 20. Let $p = 4l + 2$, $l \in \mathbb{N}$, and $u(z) = \sinh_p z$ be the unique solution of the initial-value problem (39) on D_p.

We show that $w(z) = -i \cdot u(i \cdot z)$ satisfies (25) on $D_p \cap B_p$. Due to uniqueness of solution of (25), the identity (46) must hold on $D_p \cap B_p$.

Indeed, plugging into the left-hand side of (25), we get

$$\begin{aligned}
&\left(w'(z)\right)^{p-2} w''(z) + w(z)^{p-1} \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}z} w(z)\right)^{p-2} \frac{\mathrm{d}^2}{\mathrm{d}z^2} w(z) + w(z)^{p-1} \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}z}(-i \cdot u(i \cdot z))\right]^{p-2} \frac{\mathrm{d}^2}{\mathrm{d}z^2}(-i \cdot u(i \cdot z)) \\
&\quad + (-i \cdot u(i \cdot z))^{p-1} \\
&= i \cdot \left[\frac{\mathrm{d}}{\mathrm{d}(i \cdot z)} u(i \cdot z)\right]^{p-2} \frac{\mathrm{d}^2}{\mathrm{d}(i \cdot z)^2} u(i \cdot z) \\
&\quad + (-i \cdot u(i \cdot z))^{p-1} \\
&= i \cdot \left[\frac{\mathrm{d}}{\mathrm{d}s} u(s)\right]^{p-2} \frac{\mathrm{d}^2}{\mathrm{d}s^2} u(s) - i \cdot (-i)^{p-2} \cdot u(s)^{p-1} \\
&= i \cdot \left(\left[\frac{\mathrm{d}}{\mathrm{d}s} u(s)\right]^{p-2} \frac{\mathrm{d}^2}{\mathrm{d}s^2} u(s) - (-i)^{p-2} \cdot u(s)^{p-1}\right) \\
&= i \cdot \left(\left[\frac{\mathrm{d}}{\mathrm{d}s} u(s)\right]^{p-2} \frac{\mathrm{d}^2}{\mathrm{d}s^2} u(s) - u(s)^{p-1}\right) = 0.
\end{aligned} \tag{52}$$

Note that for $p = 4l + 2$, $l \in \mathbb{N}$, $(-i)^{p-2} = 1$. The last equality then follows from (39). The right-hand side of (25) is zero. So we have verified that $w(z) = -i \cdot \sinh_p(i \cdot z)$ satisfies the differential equation in (39). The initial conditions of (25) are also satisfied by $w(z) = -i \cdot \sinh_p(i \cdot z)$, since $u(z) = \sinh_p z$ satisfies the initial conditions of (39).

Now it remains to show that $D_p = B_p$ and hence $\sin_p z = -i \cdot \sinh_p(i \cdot z)$ on B_p. To this end, let us write $\sinh_p z = \sum_{n=1}^{\infty} c_n \cdot z^n$ for yet unknown $c_n \in \mathbb{C}$. Then,

$$\sum_{n=1}^{\infty} a_n \cdot z^n = \sin_p z = -i \cdot \sinh_p(i \cdot z) = \sum_{n=1}^{\infty} c_n \cdot i^{n-1} \cdot z^n \tag{53}$$

on $D_p \cap B_p$. From here,

$$c_n = \frac{a_n}{i^{n-1}} = i^{3n+1} \cdot a_n. \tag{54}$$

Since $|i^{3n+1}| = 1$, $D_p = B_p$.

Now taking into account that $a_n = 0$ for $n-1$ not divisible by p, we immediately get $c_n = 0$ for $n-1$ not divisible by p. Now using our notation $\alpha_k = a_{kp+1}$, $k \in \mathbb{N}$, we find that

$$c_{kp+1} = (i)^{3(kp+1)+1} \alpha_k = (-1)^k \alpha_k \tag{55}$$

which establishes (48).

Equation (47) follows directly from $\cosh_p z = \sinh'_p z$ and (46). □

Proof of Theorem 21. Let $p = 4l$, $l \in \mathbb{N}$, and $u(z) = \sin_p z$. Now, plugging $w(z) = -i \cdot u(i \cdot z)$ into the left-hand side of (25), we proceed in the same way as in the proof of Theorem 20. The most important difference is that for $p = 4l$, $l \in \mathbb{N}$, $(-i)^{p-2} = -1$. Then, all following steps are analogous to those in proof of Theorem 20 with several obvious changes. □

Proof of Theorem 22. The proof is almost identical to the proof of Theorem 20 with obvious changes (cf. the proof of Theorem 21). □

6. Real Restrictions of the Complex Valued Solutions of (25) and (39) and Their Maximal Domains of Extension as Real Initial-Value Problems

Let us denote the restriction of the complex valued solution of (25) and (39) to the real axis by $\widehat{\mathrm{s}}_p(x)$ and by $\widehat{\mathrm{sh}}_p(x)$, respectively. Since the equation in (25) and (39) contains only integer powers of the solution and its derivatives, all coefficients in the equation are real, and the initial conditions in (25) and (39) are real, the value of $\sin_p z$ and $\sinh_p z$ must be a real number for $z = x + iy$ with $-\pi_p/2 < x < \pi_p/2$ and $-\gamma_p < x < \gamma_p$ and $y = 0$, respectively. Hence, $\widehat{\mathrm{s}}_p(x)$ and $\widehat{\mathrm{sh}}_p(x)$ attain only real values.

We start with the slightly more complicated case, which is $\widehat{\mathrm{sh}}_p(x)$. Moreover, since the solution of (39) is an analytic function, it has the series representation

$$\sinh_p z = \sum_{k=1}^{\infty} c_k z^k, \quad z \in D_p, \tag{56}$$

where $c_k \in \mathbb{R}$, $k \in \mathbb{N}$ (note that $\sinh_p z$ must be a real number for any $z = x + iy$ with $-\gamma_p < x < \gamma_p$ and $y = 0$).

Now we show that $\widehat{\mathrm{sh}}_p(x)$ solves (33) (in the sense of differential equations in real domain) for $p = 2(m+1)$, $m \in \mathbb{N}$, and does not solve (33) (in the sense of differential equations in real domain) for $p = 2m + 1$, $m \in \mathbb{N}$, and $x < 0$. For this purpose, we use an interesting consequence of omitting of the modulus function.

Theorem 23. *Let $p = 2m+1$, $m \in \mathbb{N}$, and $x \in (-\pi_p/2, \pi_p/2)$. Then,*

$$\widehat{\mathrm{sh}}_p(x) = \begin{cases} \sinh_p x, & x \in \left[0, \dfrac{\pi_p}{2}\right), \\ \sin_p x & x \in \left(-\dfrac{\pi_p}{2}, 0\right). \end{cases} \tag{57}$$

Proof. For $x \in [0, \gamma_p)$, the statement of Theorem follows directly from the definition of real function $\sinh_p x$ and the facts that $\sinh_p x \geq 0$ and $\sinh'_p x \geq 0$ on $[0, \gamma_p)$.

By the definition, the function $\sin_p x$ is the unique solution of (6); that is,

$$-\left(\left|u'\right|^{p-2}u'\right)' - (p-1)\,|u|^{p-2}\,u = 0,\quad u(0) = 0,\quad u'(0) = 1. \tag{58}$$

Assume that $x \in (-\pi_p/2, 0)$. Then, $\sin_p x < 0$ and $\sin'_p x > 0$. Hence, we can rewrite (6) as

$$\left(u'\right)^{p-2}u'' - u^{p-1} = 0,\quad u(0) = 0,\quad u'(0) = 1, \tag{59}$$

which is formally (39) but here considered in real domain.

By Lemma 16, (39) has the unique solution on D_p. Its restriction to $(-\gamma_p, 0)\cap(-\pi_p/2, 0)$ clearly satisfies (59). Hence,

$$\sin_p x = \sum_{k=1}^{\infty} c_k x^k = \widehat{\operatorname{sh}}_p(x),\quad x \in \left(-\gamma_p, 0\right) \cap \left(-\frac{\pi_p}{2}, 0\right). \tag{60}$$

Moreover, $\sin_p x = \sum_{k=0}^{\infty} \alpha_k \cdot x \cdot |x|^{kp}$, which is generalized Maclaurin series of $\sin_p x$ (see [2, Remark 6.6, p. 125]) convergent on $(-\pi_p/2, \pi_p/2)$. For $(-\pi_p/2, 0)$, we obtain

$$\sum_{k=0}^{\infty} \alpha_k \cdot x \cdot |x|^{kp} = \sum_{k=0}^{\infty} \alpha_k \cdot (-1)^{kp} \cdot x^{kp+1} =: G(x). \tag{61}$$

Hence, the Maclaurin series $G(x)$ converges on $(-\pi_p/2, \pi_p/2)$ (but not towards $\sin_p x$ for $x > 0$). From (60) we get

$$\sum_{k=1}^{\infty} c_k x^k = G(x) \quad \text{on } \left(-\gamma_p, 0\right) \tag{62}$$

and using Proposition A.2 we obtain $\gamma_p = \pi_p/2$. □

Corollary 24. *Let $p = 2m+1$, $m \in \mathbb{N}$. Then, $\widehat{\operatorname{sh}}_p(x)$ does not solve (33) for $x \in (-\pi_p/2, 0)$.*

Proof. Since $\sin_p x \neq \sinh_p x$ for $x \neq 0$, the statement of Corollary follows directly from Theorem 23 and the uniqueness of solution of (33). □

Theorem 25. *Let $p = 2(m+1)$, $m \in \mathbb{N}$. Then, $\widehat{\operatorname{sh}}_p(x)$ solves (33) for $x \in (-\gamma_p, \gamma_p)$. In particular, $\gamma_p = \pi_p/2$ for $p = 4m+2$, $m \in \mathbb{N}$.*

Proof. Since p is even, we can drop the absolute values in (33) obtaining (59), which is formally (39) but here considered in real domain. Since $\sinh_p(z)$ solves (39) on D_p, its restriction $\widehat{\operatorname{sh}}_p(x)$ to $(-\gamma_p, \gamma_p)$ must solve (59) on $(-\gamma_p, \gamma_p)$.

For $p = 4m+2$, $m \in \mathbb{N}$, we get $\gamma_p = \pi_p/2$ by (46) in Theorem 20. □

Theorem 26. *Let $p = 2m+1$, $m \in \mathbb{N}$, and $x \in (-\pi_p/2, \pi_p/2)$. Then,*

$$\widehat{\operatorname{s}}_p(x) = \begin{cases} \sin_p x, & x \in \left[0, \dfrac{\pi_p}{2}\right), \\ \sinh_p x, & x \in \left(-\dfrac{\pi_p}{2}, 0\right). \end{cases} \tag{63}$$

Proof. The proof follows the same steps as the proof of Theorem 23 with obvious modifications. □

Now we will consider (25) and (39) as real-valued problems and find their maximal domains of extension. Let $\overset{\leftrightarrow}{\operatorname{s}}_p(x)$ and $\overset{\leftrightarrow}{\operatorname{sh}}_p(x)$ denote solutions with maximal domains of (25) and (39), respectively. We also define $\overset{\leftrightarrow}{\operatorname{c}}_p(x) \overset{\text{def}}{=} (d/dx)\overset{\leftrightarrow}{\operatorname{s}}_p(x)$ and $\overset{\leftrightarrow}{\operatorname{ch}}_p(x) \overset{\text{def}}{=} (d/dx)\overset{\leftrightarrow}{\operatorname{sh}}_p(x)$.

Theorem 27. *Let $p = 2m+1$, $m \in \mathbb{N}$. Then,*

$$\overset{\leftrightarrow}{\operatorname{s}}_p x = \begin{cases} \sinh_p x, & x \in (-\infty, 0), \\ \sin_p x, & x \in \left[0, \dfrac{\pi_p}{2}\right), \end{cases}$$
$$\overset{\leftrightarrow}{\operatorname{sh}}_p x = \begin{cases} \sin_p x, & x \in \left(-\dfrac{\pi_p}{2}, 0\right), \\ \sinh_p x, & x \in [0, +\infty). \end{cases} \tag{64}$$

Theorem 28. *Let $p = 2(m+1)$, $m \in \mathbb{N}$. Then,*

$$\overset{\leftrightarrow}{\operatorname{s}}_p x = \sin_p x, \quad x \in \mathbb{R},\qquad \overset{\leftrightarrow}{\operatorname{sh}}_p x = \sinh_p x, \quad x \in \mathbb{R}. \tag{65}$$

Proof of Theorems 27 and 28. The solutions with maximal domain of extension are known for (6) and (33). The proof uses uniqueness of the solutions of real initial-value problems (6) and (33) and initial-value problems (25) and (39) considered in real domain and the fact that (6) and (33) can be rewritten as (25) and (39) depending on $u(x) \lesseqgtr 0$, $u'(x) \lesseqgtr 0$, and the parity of the positive integer p. The main ideas of how to combine these ingredients are contained in the proof of Theorem 23. □

It easily follows from Theorems 27 and 28 that $\overset{\leftrightarrow}{\operatorname{c}}_p(x)$ is defined on $(-\infty, \pi_p/2)$ for $p = 2m+1$ and on $\mathbb{R}$ for $p = 2(m+1)$, $m \in \mathbb{N}$. Similarly, $\overset{\leftrightarrow}{\operatorname{ch}}_p(x)$ is defined on $(-\pi_p/2, +\infty)$ for $p = 2m+1$ and on $\mathbb{R}$ for $p = 2(m+1)$, $m \in \mathbb{N}$. Moreover, functions $\overset{\leftrightarrow}{\operatorname{s}}_p(x)$ and $\overset{\leftrightarrow}{\operatorname{c}}_p(x)$ satisfy the complex p-trigonometric identity (29); that is,

$$\left(\overset{\leftrightarrow}{\operatorname{s}}_p(x)\right)^p + \left(\overset{\leftrightarrow}{\operatorname{c}}_p(x)\right)^p = 1. \tag{66}$$

```
In[1] := pip[p_] =
      Integrate[1/(1 - s^p)^(1/p), {s, 0, 1},
          Assumptions -> p > 1]
(* assigns definition to function pip which returns pi_p/2 *)
Out[1] = πcsc(π/p)
         ---------
             p
In[2] := s3[x_] = (u[x]/.
          NDSolve[
          {u'[x] == (v[x])^(1/2), v'[x] == - 2 u[x]^2,
            u[0] == 0, v[0] == 1},
          {u, v}, {x, -5, pip[3]}][[1]])
(* assigns definition to auxiliar function s3 *)
In[3] := sh3[x_] = (u[x]/.
          NDSolve[
          {u'[x] == (v[x])^(1/2), v'[x]== 2 u[x]^2,
            u[0]== 0, v[0] == 1},
          {u, v}, {x, -pip[3], 5}][[1]])
(* assigns definition to auxiliar function sh3 *)
In[4] := sin3[x_] = (u[x]/.
          NDSolve[{u'[x] == (Abs[v[x]])^(1/2) Sign[v[x]],
                    v'[x] == -2 Abs[u[x]]*u[x],
                    u[0] == 0, v[0] == 1},
                    {u,v}, {x, -4 pip[3], 4 pip[3]}][[1]])
(* assigns definition to auxiliar function sin3 *)
```

PSEUDOCODE 2: *Mathematica*, v. 9.0 code. Code for `pip[p_]` computes $\pi_p/2$ for given argument p. Code for `s3[x_]` computes $\overset{\leftrightarrow}{s}_3(x)$ for $x \in (-5, \pi_3/2) \subsetneq (-\infty, \pi_3/2)$, code for `sh3[x_]` computes $\overset{\leftrightarrow}{\mathrm{sh}}_3(x)$ for $x \in (\pi_3/2, 5) \subsetneq (\pi_3/2, +\infty)$, and code for `sin3[x_]` computes real-valued function $\sin_3(x)$ for $x \in (-2\pi_3, 2\pi_3) \subsetneq \mathbb{R}$.

Analogously, functions $\overset{\leftrightarrow}{\mathrm{sh}}_p(x)$ and $\overset{\leftrightarrow}{\mathrm{ch}}_p(x)$ satisfy the complex p-hyperbolic identity (44); that is,

$$\left(\overset{\leftrightarrow}{\mathrm{ch}}_p(x)\right)^p - \left(\overset{\leftrightarrow}{\mathrm{sh}}_p(x)\right)^p = 1. \tag{67}$$

7. Visualizations

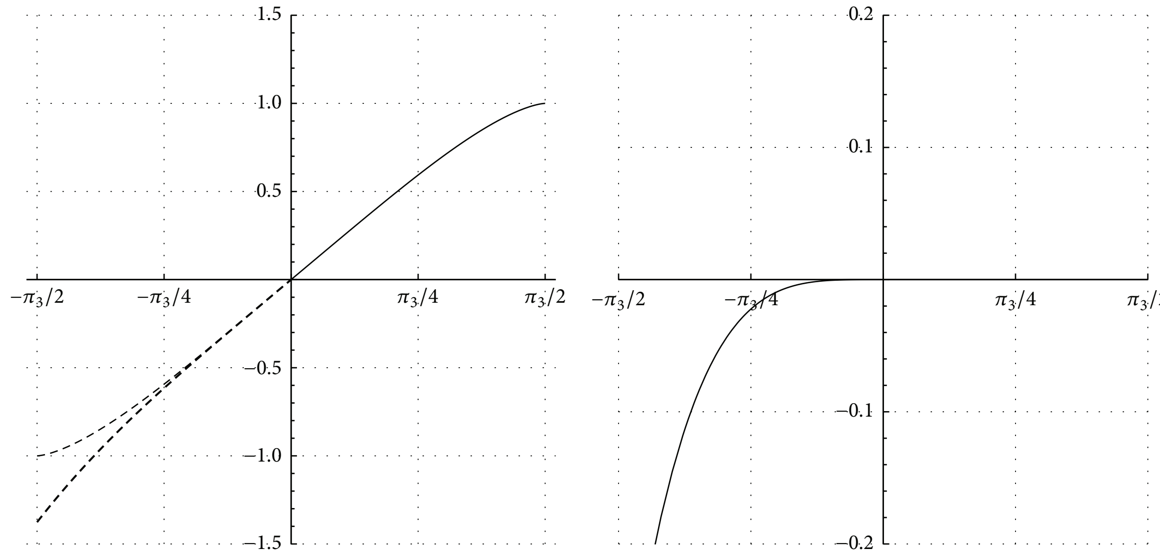
In this section, we provide visualizations of theoretical results from previous sections. To generate graphical output, we need to approximate special functions from previous sections numerically. Note that the standard numerical methods (available in *Mathematica* or *Matlab*®) can handle only initial-value problems on real intervals. Thus, in our numerical calculations we need to consider initial-value problems in real domain. This is not a problem for (6) and (33). For (25) and (39), we calculate either the partial sum of the Maclaurin series of solutions or we calculate functions $\overset{\leftrightarrow}{s}_p(x)$ and $\overset{\leftrightarrow}{\mathrm{sh}}_p(x)$ which come from real initial-value problems. In our graphical outputs, the solutions of real initial-value problems are numerically approximated by the `NDSolve` command of *Mathematica*, version 9.0. For the convenience of the reader, we provide some source code. In Pseudocode 2, we list source code for approximation of functions $\overset{\leftrightarrow}{s}_3(x)$, $\overset{\leftrightarrow}{\mathrm{sh}}_3(x)$, and $\sin_3 x$.

Figure 1 compares graphs of $\sin_3 x$ and $\overset{\leftrightarrow}{s}_3(x)$ for $x \in (-\pi_3/2, \pi_3/2)$. Figure 2 compares graphs of $\overset{\leftrightarrow}{s}_3(x)$ and $\widehat{M}_{3;28}(x)$ for $x \in (-\pi_3/2, \pi_3/2)$. Here, $\widehat{M}_{3;28}(x)$ is partial sum of $M_3(x)$ up to the order 28, which is

$$\begin{aligned}\widehat{M}_{3;28}(x) = x &- \frac{x^4}{12} - \frac{x^7}{252} - \frac{83\,x^{10}}{90\,720} - \frac{1\,817\,x^{13}}{7\,076\,160} \\ &- \frac{199\,691\,x^{16}}{2\,377\,589\,760} - \frac{12\,324\,719\,x^{19}}{406\,567\,848\,960} \\ &- \frac{22\,008\,573\,061\,x^{22}}{1\,878\,343\,462\,195\,200} \\ &- \frac{107\,355\,387\,043\,x^{25}}{22\,540\,121\,546\,342\,400} \\ &- \frac{89\,152\,153\,354\,993\,x^{28}}{44\,304\,862\,911\,490\,621\,440}.\end{aligned} \tag{68}$$

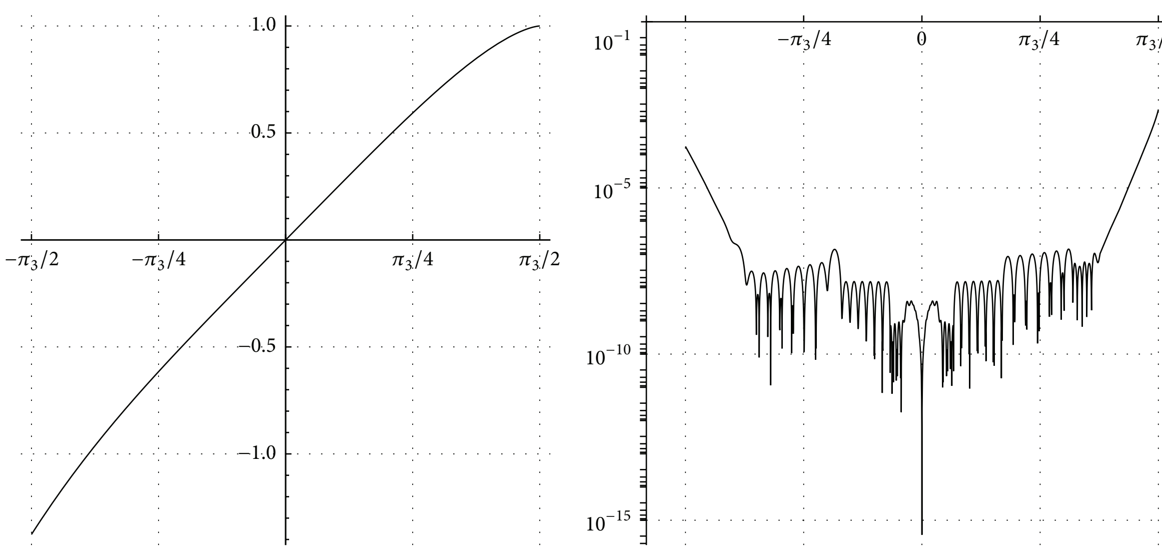
Since the difference $|\overset{\leftrightarrow}{s}_3(x) - \widehat{M}_{3;28}(x)|$ varies in order of several magnitudes throughout the radius of convergence $\pi_3/2$, we use logarithmic scale on the vertical axis.

We can also compute the functions $\overset{\leftrightarrow}{s}_p(x)$ and $\overset{\leftrightarrow}{\mathrm{sh}}_p(x)$ by inverting formulas (8) and (36) for $\arcsin_p x$ and $\operatorname{argsinh}_p x$, respectively. It turns out that this approach provides more precision than solving differential equation and enables computing values of $\overset{\leftrightarrow}{c}_p(x)$ and $\overset{\leftrightarrow}{\mathrm{ch}}_p(x)$ using identity (66) and identity (67), respectively. We provide sample code for computing $\overset{\leftrightarrow}{\mathrm{sh}}_3(x)$ for $x \in (\pi_3/2, 5) \subsetneq (\pi_3/2, +\infty)$ in



(a) (b)

Figure 1: (a) Real function $\sin_3 x$ (short-dashed line) versus $\overleftrightarrow{s}_3(x)$ (dashed line). (b) Plot of $\sin_3 x - \overleftrightarrow{s}_3(x)$.



(a) (b)

Figure 2: (a) $\overleftrightarrow{s}_3(x)$ versus $\widehat{M}_{3;28}$ (the partial sum of $M_3(x)$ up to the order 28, which is (68)). (b) The logarithmic plot of $|\overleftrightarrow{s}_3(x) - M_3(x)|$.

Pseudocode 3. Analogously, we wrote a code for computing $\overleftrightarrow{\mathrm{sh}}_4(x)$, $\overleftrightarrow{\mathrm{sh}}_{30}(x)$, $\overleftrightarrow{\mathrm{sh}}_{31}(x)$, $\overleftrightarrow{s}_3(x)$, $\overleftrightarrow{s}_4(x)$, $\overleftrightarrow{s}_{30}(x)$, and $\overleftrightarrow{s}_{31}(x)$.

In the same way, as we defined real function $\operatorname{argsinh}_p x$ by (36), we can define complex valued function $\operatorname{argsinh}_p z$ by

$$\operatorname{argsinh}_p z \overset{\text{def}}{=} \int_0^z \frac{1}{(1+s^p)^{1/p}} \mathrm{d}s = z\,{}_2F_1\left(\frac{1}{p}, \frac{1}{p}, 1+\frac{1}{p}, -z^p\right), \quad z \in \mathscr{D} \subsetneq \mathbb{C} \tag{69}$$

for any $p = m + 2$. Note that the integrand has poles at z satisfying $1 + z^p = 0$. Thus, the function $\operatorname{argsinh}_p z$ is not an entire function. In particular, for $p = 2m + 1$, there is a pole at $z = -1$. In Figure 3, we compare graphs of $\operatorname{argsinh}_p x$ for $p = 2, 3, 31$ and $x \in (-3, 3) \subsetneq \mathbb{R}$, with the restriction of the complex valued function $\operatorname{argsinh}_p z$ for $p = 2, 3, 31$, where $z = x + iy$, where $y = 0$, $x \in (-3, 3)$ for $p = 2$ and $x \in (-1, 3)$ for $p = 3, 31$, and with $\operatorname{argsinh}_p x = \operatorname{argsinh}_p z$ for $p = 2, 4, 30$, $x \in (-3, 3)$, $y = 0$, and $z = x + iy$.

In Figures 4, 5, and 6, we compare graphs of real-valued functions: $\sinh x$, $\sinh_4 x$, $\sinh_{30} x$, $\cosh x$, $\cosh_4 x$, $\cosh_{30} x$

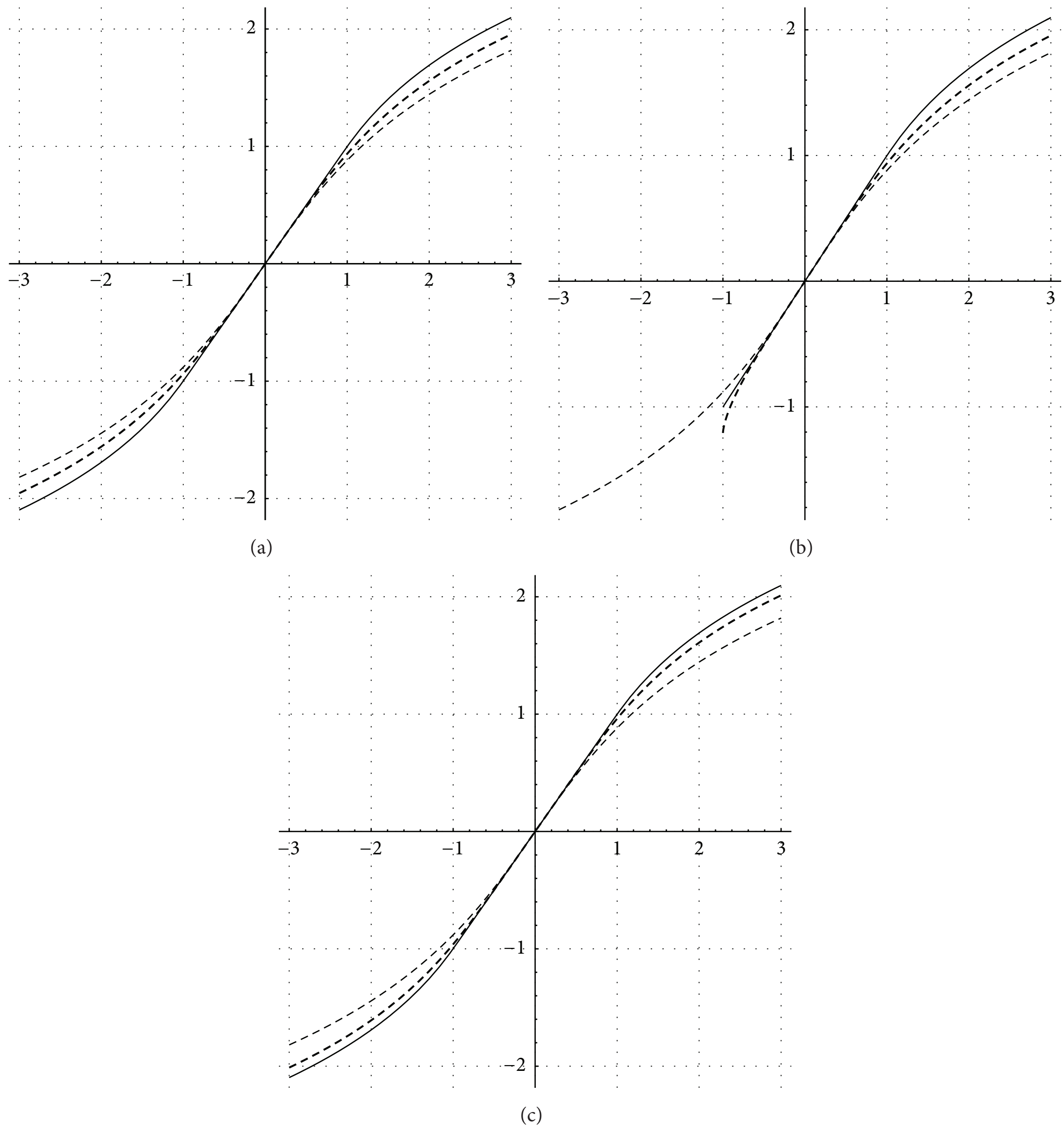
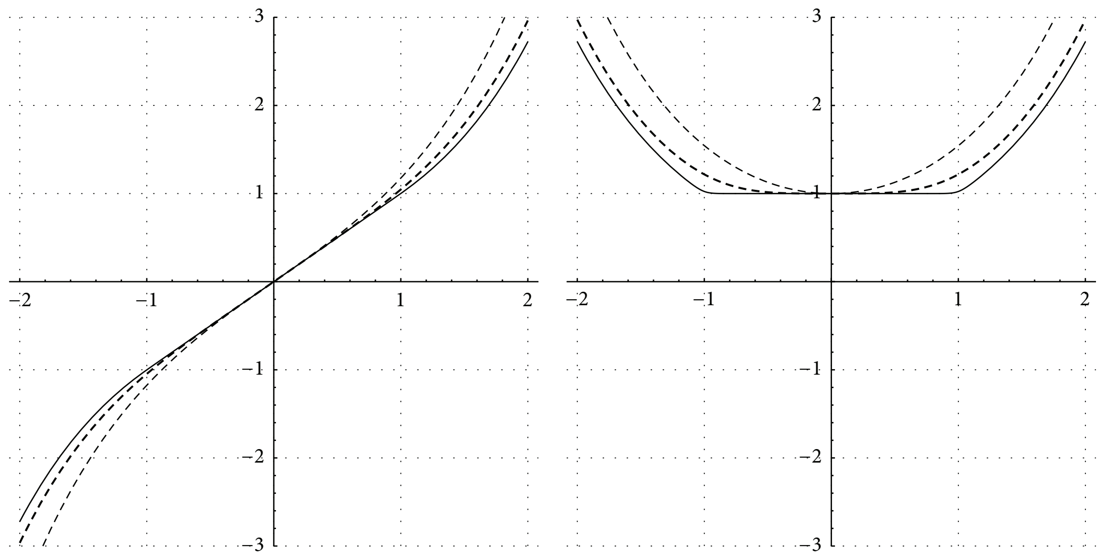


FIGURE 3: (a) $\operatorname{argsinh}_p x (= x\ {}_2F_1(1/p, 1/p, 1+1/p, -|x|^p))$ for $p = 2, 3, 31$, and $x \in (-3, 3)$. (b) $\operatorname{argsinh}_p z (= z\ {}_2F_1(1/p, 1/p, 1+1/p, -z^p))$ for $p = 2, 3, 31$. Here, $z = x + iy$, where $y = 0$, $x \in (-3, 3)$ for $p = 2$, and $x \in (-1, 3)$ for $p = 3, 31$. (c) $\operatorname{argsinh}_p x = \operatorname{argsinh}_p z$ for $p = 2, 4, 30$, $x \in (-3, 3)$, $y = 0$, and $z = x + iy$.

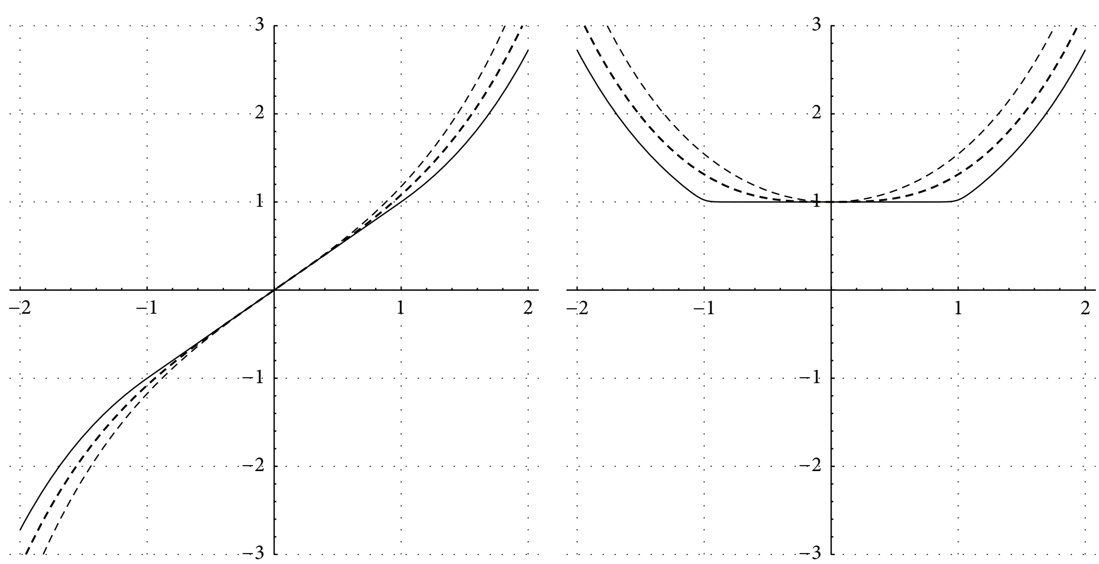
```
In[2] := auxAgSh3[s_?NumberQ] :=
      s Hypergeometric2F1[1/3, 1/3, 1 + 1/3, -s^3]
(* assigns definition to auxiliar function auxAgSh3 *)
In[3] := sh3inv[x_?NumericQ] :=
      (s/. FindRoot[auxAgSh3[s] - x, {s, 0, -1, 20}])
(* assigns definition to function sh3inv *)
In[4] := Plot[
      sh3inv[x], {x, -pip[3], 2},
      PlotStyle -> Thick, PlotRange -> {-2, 7}]
(* plots the graph *)
```

PSEUDOCODE 3: *Mathematica* v. 9.0 code. Function sh3inv returns values of $\overleftrightarrow{\mathrm{sh}}_3(x)$ for $x \in (\pi_3/2, 5) \subsetneq (\pi_3/2, +\infty)$ using inversion of the formula (36) for $\operatorname{argsinh}_3 x$. This approach provides more precision than solving differential equation.



(a) (b)

Figure 4: (a) $\sinh x$ (short-dashed line), $\sinh_4 x$ (dashed line), and $\sinh_{30} x$ (solid line). (b) $\cosh x$ (short-dashed line), $\cosh_4 x$ (dashed line), and $\cosh_{30} x$ (solid line).



(a) (b)

Figure 5: (a) $\sinh x$ (short-dashed line), $\sinh_3 x$ (dashed line), and $\sinh_{31} x$ (solid line). (b) $\cosh x$ (short-dashed line), $\cosh_3 x$ (dashed line), and $\cosh_{31} x$ (solid line).

(see Figure 4), $\sinh x$, $\sinh_3 x$, $\sinh_{31} x$, $\cosh x$, $\cosh_3 x$, $\cosh_{31} x$ (see Figure 5), $\overleftrightarrow{\mathrm{sh}}_3(x)$, $\overleftrightarrow{\mathrm{sh}}_{31}(x)$, $\overleftrightarrow{\mathrm{ch}}_3(x)$, and $\overleftrightarrow{\mathrm{ch}}_{31}(x)$ (see Figure 6).

Figure 7 illustrates the relation between $\sin_3 x$, $\sinh_3 x$, $\overleftrightarrow{\mathrm{s}}_3(x)$, and $\overleftrightarrow{\mathrm{sh}}_3(x)$, which are due to Theorem 27.

It follows from identity (66) and identity (67) that the pairs of functions $(\overleftrightarrow{\mathrm{s}}_p(x), \overleftrightarrow{\mathrm{c}}_p(x))$ and $(\overleftrightarrow{\mathrm{sh}}_p(x), \overleftrightarrow{\mathrm{ch}}_p(x))$, respectively, are parametrizations of Lamé curves restricted to the first and fourth quadrant, see [23, Book V, Chapter V, pp. 384–407] and [24]. Since the Lamé curves are frequently used in geometrical modeling, we provide graphical comparison of the Lamé curves and phase portraits of initial-value problems (25) and (39) in real domain on Figures 8 and 9, respectively.

8. Conclusion

We have discussed real and complex approaches of how to define generalized trigonometric and hyperbolic functions.

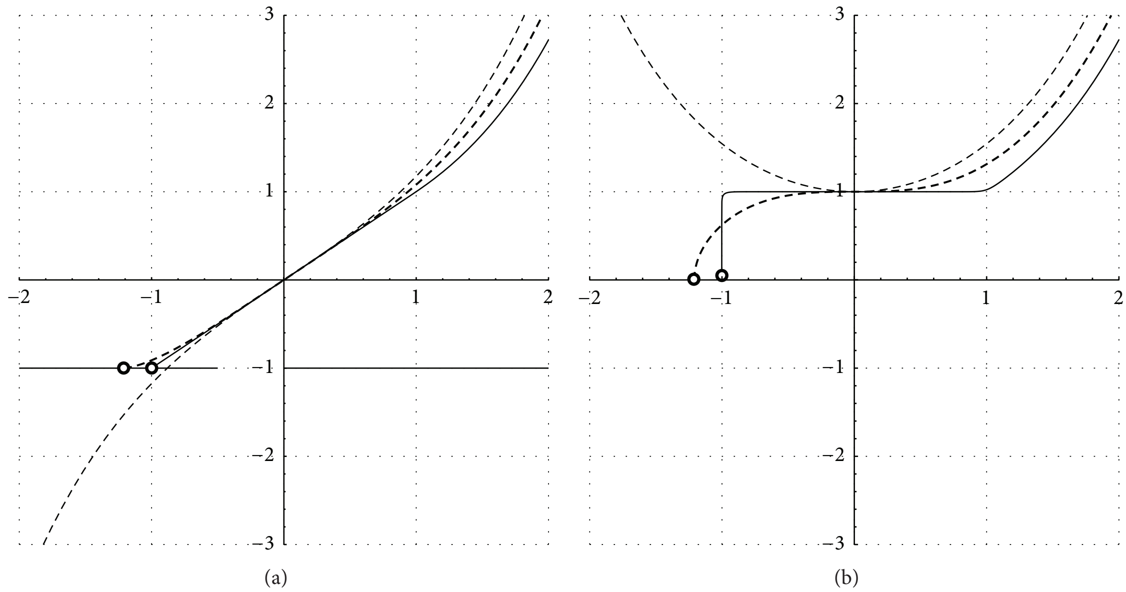


FIGURE 6: (a) $\sinh x$ (short-dashed line), $\overset{\leftrightarrow}{\mathrm{sh}}_3(x)$ (dashed line), and $\overset{\leftrightarrow}{\mathrm{sh}}_{31}(x)$ (solid line). (b) $\cosh x$ (short-dashed line), $\overset{\leftrightarrow}{\mathrm{ch}}_3(x)$ (dashed line), and $\overset{\leftrightarrow}{\mathrm{ch}}_{31}(x)$ (solid line). Note that $\overset{\leftrightarrow}{\mathrm{sh}}_p(x)$ and $\overset{\leftrightarrow}{\mathrm{ch}}_p(x)$ are defined on $(-\pi_p/2, +\infty)$ for integer $p > 1$ odd.

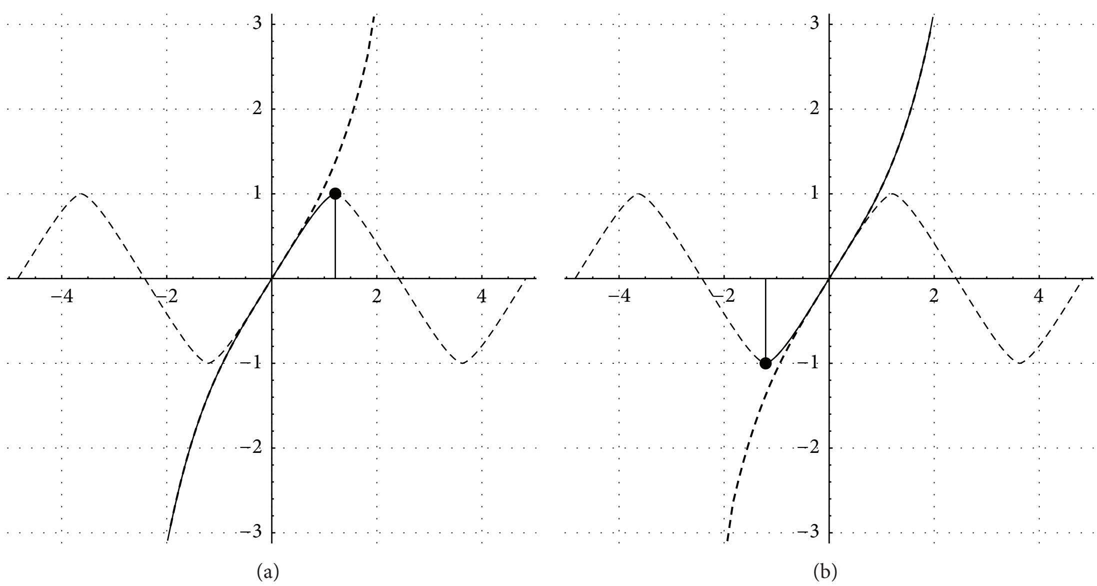


FIGURE 7: (a) $\sin_3 x$, $\sinh_3 x$, and $\overset{\leftrightarrow}{\mathrm{s}}_3(x)$. (b) $\sin_3 x$, $\sinh_3 x$, and $\overset{\leftrightarrow}{\mathrm{sh}}_3(x)$.

The real approach is motivated by minimization of Rayleigh quotient (see, e.g., [1, Equation (3.4), p. 51] and references therein):

$$\frac{\int_0^{\pi_p} \left|u'\right|^p \mathrm{d}x}{\int_0^{\pi_p} |u|^p \mathrm{d}x} \tag{70}$$

in $W_0^{1,p}(0, \pi_p)$. This leads to (4) with $\lambda = p - 1$ and to initial-value problem (6) in turn. Thus, from this point of view, the solution of (6) and its derivative can be seen as natural generalizations of the functions sine and cosine. Unfortunately, presence of absolute value in (6) does not allow for extension to complex domain for general $p > 1$.

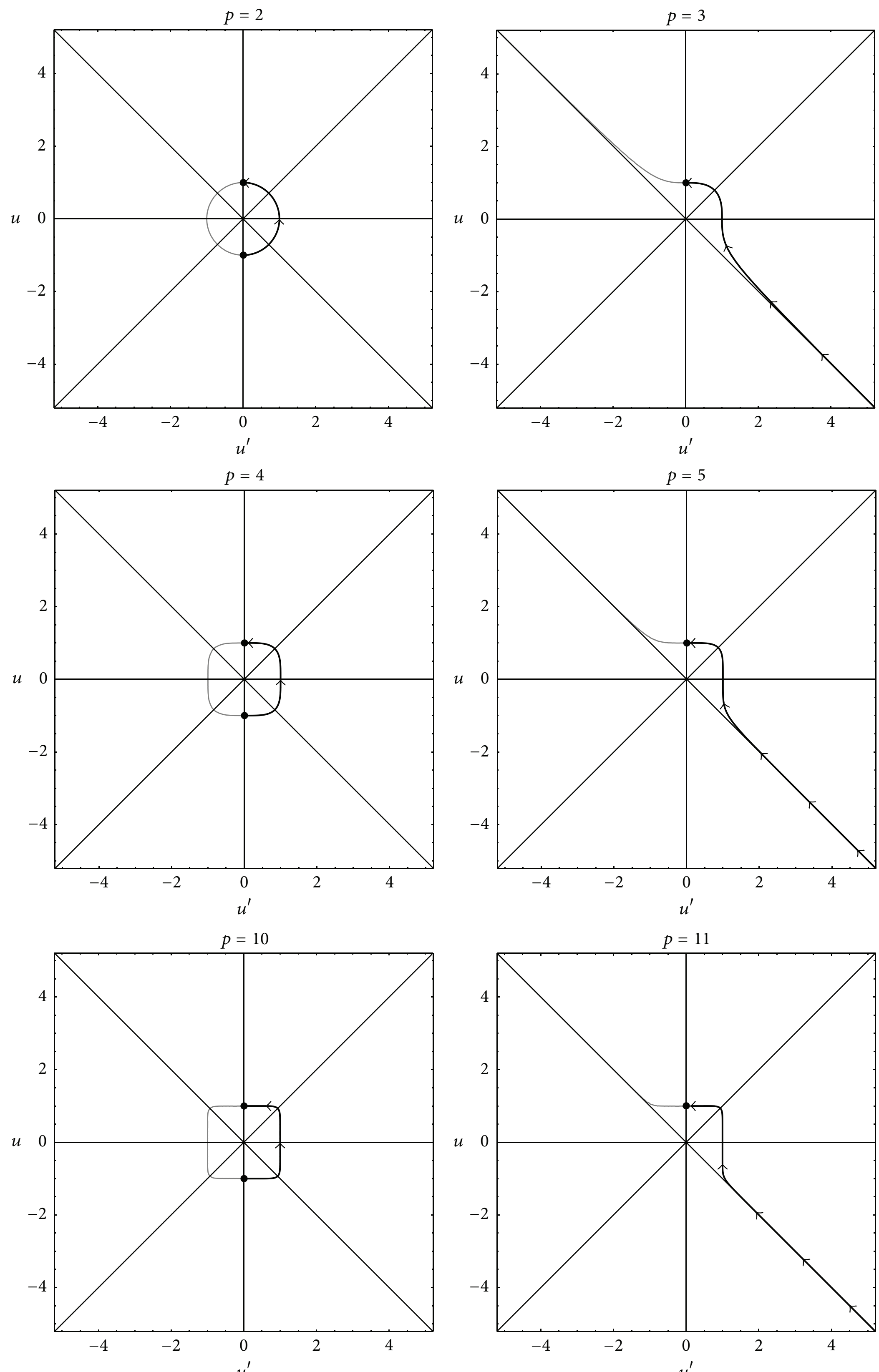


FIGURE 8: Bold lines: dependence of $u = \overset{\leftrightarrow}{s}_p(x)$ on $u' = \overset{\leftrightarrow}{c}_p(x)$. Both bold and thin lines: dependence of restriction to real axes of derivative of solution of (29) on the restriction to real axes of the solution itself (Lamé curves).

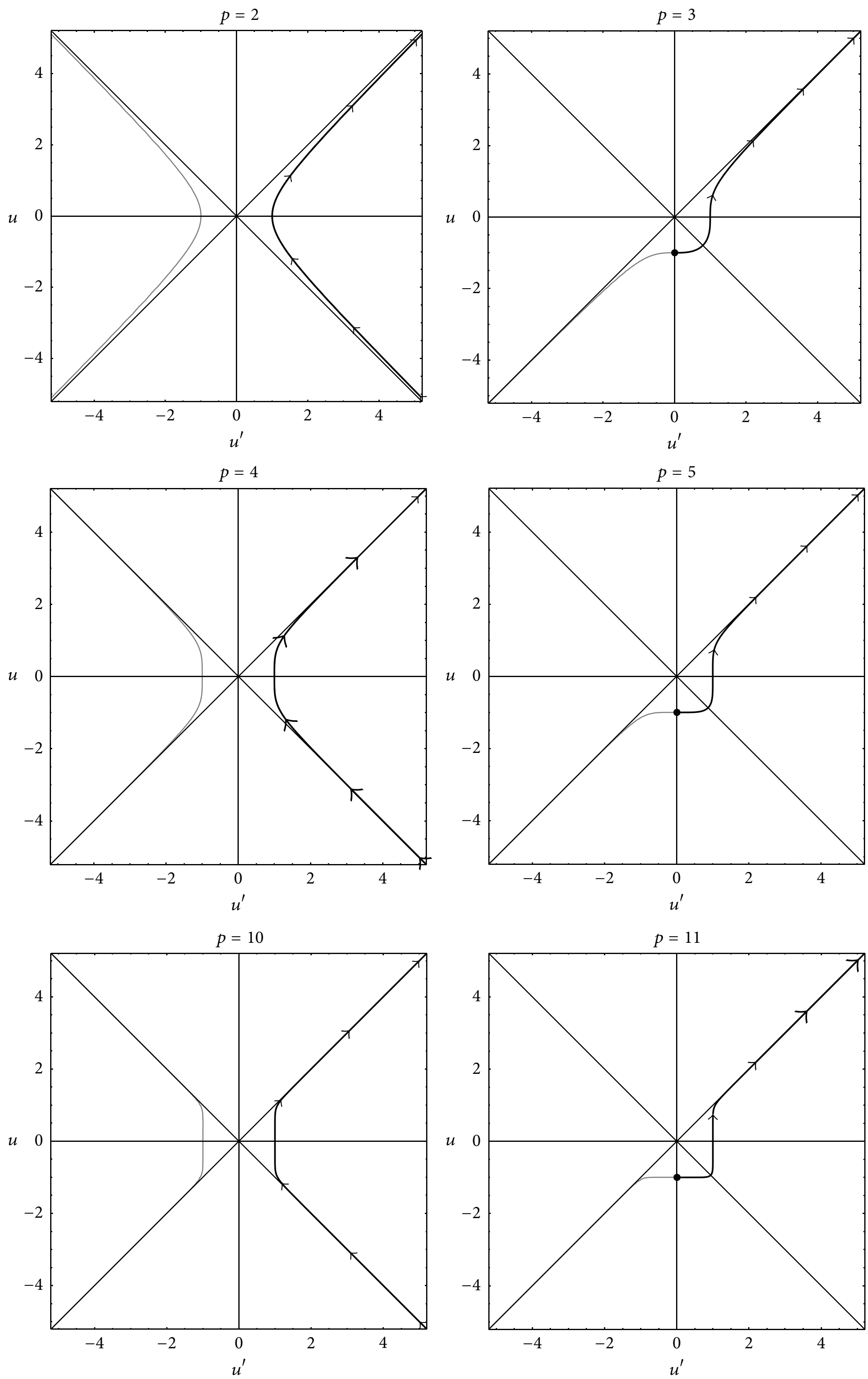


Figure 9: Bold lines: dependence of $u = \overset{\leftrightarrow}{\mathrm{sh}}_p(x)$ on $u' = \overset{\leftrightarrow}{\mathrm{ch}}_p(x)$. Both bold and thin lines: dependence of restriction to real axes of derivative of solution of (44) on the restriction to real axes of the solution itself (Lamé curves).

Table 1: Summary of results according to discussed functions and their domain.

$p \in$	Function	Domain	IVP	Results
$(1, +\infty)$	$\sin_p x$	$\mathbb{R}$	(6)	Propositions 2–5; Theorems 6, 7, 8, 23, and 26–28
$(1, +\infty)$	$\sinh_p x$	$\mathbb{R}$	(33)	Theorems 23 and 26–28
$\mathbb{N}\backslash\{1\}$	$\sin_p z$	B_p	(25)	Propositions 11 and 12; Lemma 13; Theorems 6, 7, 8, 9, 15, 20, and 21
$\mathbb{N}\backslash\{1\}$	$\sinh_p z$	D_p	(39)	Lemmas 16 and 18; Theorems 20 and 22

Table 2: Interrelations between p-trigonometric and p-hyperbolic functions. Note that the relations must be symmetric.

	$\sin_p x$	$\sinh_p x$	$\sin_p z$	$\sinh_p z$
$\sin_p x$	—	(57), (63)–(64)	—	—
$\sinh_p x$	sym.	—	—	—
$\sin_p z$	sym.	sym.	(49)	(46)
$\sinh_p z$	sym.	sym.	sym.	(50)

It was shown in [2] that functions $\sin_p x$ are real analytic functions for *any even integer* $p > 2$. Moreover, there is no need to write absolute value in (6) for $x \in [-\pi_p/2, \pi_p/2]$ provided $p > 2$ is an *even integer*.

It turns out that the relation between the real and complex approach is not as smooth as in the classical case $p = 2$. Thus, we summarize our results in Tables 1 and 2.

We also discussed the Lamé curves, which are important curves in geometrical modeling. We hope this will stimulate interest in p-trigonometric and p-hyperbolic functions among the geometric-modeling community.

Appendix

A. Proof of Theorem 6

We will use the following result to prove Theorem 6.

Proposition A.1 (see [15], Theorem 2.4b on p. 97). *Let the formal power series $F \overset{\text{def}}{=} a_1 x + a_2 x^2 + \cdots$, $a_1 \neq 0$, have a positive radius of convergence. The inversion F^{-1} of F then also has positive radius of convergence.*

Let us note that the term reversion of series is used in [15] instead of inversion of series (see [15], p. 46).

Proposition A.2 (see [25], Theorem 16.6 on p. 352). *If the sum of two power series in the variable $z - z_0$ coincides on a set of points E for which z_0 is a limit point and $z_0 \notin E$, then identical powers of $z - z_0$ have identical coefficients; that is, there is a unique power series in the variable $z - z_0$ which has given sum on the set E.*

Proof of Theorem 6. Let us remember that $T_p(x)$ is given by (19), which is the formal inverse of $\arcsin_p x$ and $M_p(x)$ is given by formula (20). The idea is to prove that there exists $\delta_p > 0$ small enough such that both series $T_p(x)$ and $M_p(x)$ have the sum equal to uniquely defined value $\sin_p x$ at any $x \in [0, \delta_p)$. Then, $T_p(x) = M_p(x)$ on $[0, \delta_p)$ and the assumptions of Proposition A.2 are satisfied on $z_n = \delta_p/(n+1)$. It follows that $T_p(x)$ has identical coefficients as $M_p(x)$ has and so $T_p(x)$ also converges on $(-\pi_p/2, \pi_p/2)$.

By Propositions 4 and 5, $M_p(x)$ converges to $\sin_p x$ for $p > 2$ even on $(-\pi_p/2, \pi_p/2)$ and for $p > 1$ odd on $[0, \pi_p/2)$, respectively. It remains to show that there exists $\sigma_p > 0$ such that

$$T_p(x) = \sin_p x \tag{A.1}$$

on $[0, \sigma_p)$. Since $T_p(x)$ is defined as formal inverse of $\arcsin_p x$, (A.1) holds on domain of convergence of $T_p(x)$. Since for $x \in [0, 1]$

$$\begin{aligned} \arcsin_p x &= x \cdot {}_2F_1\left(\frac{1}{p}, \frac{1}{p}, 1 + \frac{1}{p}, x^p\right) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(k + 1/p)}{(k \cdot p + 1) \cdot k! \cdot \Gamma(1/p)} x^{kp+1}, \end{aligned} \tag{A.2}$$

where right-hand side series has radius of convergence equal to 1, hence the existence of $\sigma_p \leq \pi_p/2$ is provided by Proposition A.1 and $\delta_p = \sigma_p$. □

B. Proof of Theorem 7

By Theorem 6, $M_p = T_p$. Hence, we can prove the statement of Theorem 7 for T_p instead of M_p.

Assume by contradiction that there exists $a_n \neq 0$ for some n such that $n - 1$ is *not* divisible by p. For this purpose, let us denote by b_j the jth coefficient of the Maclaurin series of $\arcsin_p$ corresponding to jth power. From (18), we get

$$b_j = \begin{cases} \dfrac{\Gamma(l + 1/p)}{(lp + 1) \cdot l! \cdot \Gamma(1/p)} & \text{if } j = l \cdot p + 1 \text{ for some } l \in \mathbb{N} \cup \{0\}, \\ 0 & \text{otherwise}. \end{cases} \tag{B.1}$$

Since T_p is the formal inverse series of

$$\arcsin_p x = \sum_{j=1}^{\infty} b_j \cdot x^j, \tag{B.2}$$

the coefficients a_n can be computed from the formula

$$a_n = \frac{1}{n \cdot b_1^n} \sum_{m_1, m_2, \ldots, m_{n-1}} (-1)^{m_1+m_2+\cdots+m_{n-1}} \cdot \frac{n(n+1) \cdot \ldots \cdot (n-1+m_1+m_2+m_3+\cdots+m_i+\cdots+m_{n-1})}{m_1! m_2! m_3! \ldots} \cdot \left(\frac{b_2}{b_1}\right)^{m_1} \cdot \left(\frac{b_3}{b_1}\right)^{m_2} \left(\frac{b_4}{b_1}\right)^{m_3} \cdots \left(\frac{b_{i+1}}{b_1}\right)^{m_i} \cdots \left(\frac{b_n}{b_1}\right)^{m_{n-1}}, \tag{B.3}$$

where the summation is taken over all $m_1, m_2, m_3, \ldots \in \mathbb{N} \cup \{0\}$ such that

$$m_1 + 2m_2 + 3m_3 + \cdots + im_i \cdots (n-1) m_{n-1} = n - 1, \tag{B.4}$$

and if $m_i = 0$, then the corresponding term $(b_{i+1}/b_1)^{m_i}$ is dropped from the product on the last line of (B.3).

Let us note that this procedure is fully described in [14], p. 411–413 and it requires that $b_1 \neq 0$. Note that

$$b_1 = \frac{\Gamma(1/p)}{1 \cdot 1! \cdot \Gamma(1/p)} = 1 \tag{B.5}$$

by (B.1).

Now, let us fix $m_1, m_2, m_3, \ldots, m_{n-1}$ satisfying (B.4). If $b_i = 0$ and $m_i \neq 0$ for at least one $i = 1, 2, 3, \ldots, n-1$, the summand of sum (B.3) corresponding to $m_1, m_2, m_3, \ldots$ equals 0. Taking into account (B.1), $b_i = 0$ whenever $i \neq lp+1$ for any $l \in \mathbb{N} \cup \{0\}$. This leads us to conclusion that nonzero terms in (B.3) can be formed only from m_i's where i is divisible by p. Thus, (B.4) implies that the following equation must be satisfied:

$$p \cdot m_p + 2p \cdot m_{2p} + 3p \cdot m_{3p} + \cdots + l \cdot p \cdot m_{l \cdot p} + \cdots + k \cdot p \cdot m_{k \cdot p} = n - 1, \tag{B.6}$$

where

$$k = \left\lfloor \frac{n-1}{p} \right\rfloor. \tag{B.7}$$

But here the left-hand side is a multiple of p while the right-hand side $n-1$ is not divisible by p by our assumption. This is a contradiction.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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