Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2015, Article ID 698121, 3 pages
http://dx.doi.org/10.1155/2015/698121

Letter to the Editor

Comment on "Existence Theorem for Integral and Functional Integral Equations with Discontinuous Kernels"

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Received 17 June 2015; Accepted 23 June 2015

Academic Editor: Dumitru Motreanu

We present a counterexample to the main result of the abovementioned paper showing that this result is false and cannot be improved in a simple way.

1. Introduction

In [1] author considers the nonlinear Volterra integral equation (VIE)

$$x(t) = u(t) + \int_0^t f(t, \tau, x(\tau))\, d\tau \tag{1}$$

and the nonlinear functional Volterra integral equation (FVIE)

$$x(t) = u(t) + \int_0^t f(t, \tau, x(\tau), x)\, d\tau. \tag{2}$$

Theorem 2.1 of [1] states the following.

Theorem 1. *Let $u : [0,1] \to \mathbb{R}$ and let $f : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$ be given. Suppose that (C1)–(C4) are fulfilled.*

(C1) *u is continuous.*

(C2) *For each $(t,x) \in [0,1]\times\mathbb{R}$, the function $\tau \to f(t,\tau,x)$ is Lebesgue measurable. For all $(t,x) \in [0,1]\times\mathbb{R}$ and for almost all $\tau \in [0,1]$,*

$$\left| f(t,\tau,x) \right| < M(\tau), \tag{3}$$

where $M : [0,1] \to [0,\infty]$ is a Lebesgue integrable function.

(C3) *For each $(t,\tau,x) \in [0,1]\times[0,1]\times\mathbb{R}$*

$$\limsup_{y\uparrow x} f(t,\tau,y) \le f(t,\tau,x) = \liminf_{y\downarrow x} f(t,\tau,y). \tag{4}$$

(C4) *Let $F = \{y \in \mathbb{R} : |y| \le \|u\| + |\int_0^1 M(\tau)d\tau|\}$, where $\|u\| = \max\{|u(t)|; t \in [0,1]\}$. For every $y \in F$ and all $n \in \mathbb{N}$ the functions*

$$t \longrightarrow \int_0^t \sup_{|x-y|\le 1/3^n} f(t,\tau,x)\, d\tau \tag{5}$$

are equicontinuous and tend to zero as $t \downarrow 0$.

Under the above assumptions VIE expressed by (1) has extremal solutions in the interval $[0,1]$.

In the following we present a counterexample showing that this result is false.

2. Comment on the Assumption (C4)

Define

$$h_{y,n}(t) = \int_0^t \sup_{|x-y|\le 1/3^n} f(t,\tau,x)\, d\tau, \qquad y \in F,\ n \in \mathbb{N},\ t \in [0,1]. \tag{6}$$

(C4) states that $h_{y,n}(t)$ is equicontinuous in $[0,1]$ and tends to zero as $t \downarrow 0$. In fact the last seems to be superfluous since it follows from equicontinuity (or from (C2)).

Assume that f does not depend on τ; that is, we set $f(t,\tau,x) = f(t,x)$ (with a small violation of notation). Now

(C2) gives $|f(t,x)| \le M$ in $[0,1]\times\mathbb{R}$ for some $M \ge 0$. We also have

$$h_{y,n}(t) = t \sup_{|x-y|\le 1/3^n} f(t,x) = \sup_{|x-y|\le 1/3^n} tf(t,x), \quad y \in F,\ n\in\mathbb{N},\ t\in[0,1]. \tag{7}$$

Proposition 2. *If f is Lipschitz continuous in t, that is, if there exists $L \ge 0$ such that*

$$|f(t,x) - f(\bar{t},x)| \le L|t-\bar{t}| \tag{8}$$

for all $(t,x), (\bar{t},x) \in [0,1]\times\mathbb{R}$ then (C4) is satisfied.

Proof. For all $t,\bar{t}\in[0,1]$ and $y\in F$, $n\in\mathbb{N}$, we have

$$\begin{aligned}
&\left|h_{y,n}(t) - h_{y,n}(\bar{t})\right| \\
&\quad = \left|\sup_{|x-y|\le 1/3^n} tf(t,x) - \sup_{|x-y|\le 1/3^n} \bar{t}f(\bar{t},x)\right| \\
&\quad \le \sup_{|x-y|\le 1/3^n} \left|tf(t,x) - \bar{t}f(\bar{t},x)\right| \\
&\quad \le \sup_{|x-y|\le 1/3^n} \left|tf(t,x) - \bar{t}f(t,x)\right| \\
&\qquad + \sup_{|x-y|\le 1/3^n} \left|\bar{t}f(t,x) - \bar{t}f(\bar{t},x)\right| \\
&\quad \le M|t-\bar{t}| + L|t-\bar{t}| = (M+L)|t-\bar{t}|.
\end{aligned} \tag{9}$$

This gives equicontinuity of $h_{y,n}(t)$. □

3. The Counterexample

Our example is a modification of this given in [2].

Consider VIE

$$z(s) = -s\int_{-1}^{s} z(\tau)^{1/2}\,d\tau \quad s\in[-1,1], \tag{10}$$

where $z^{1/2} = |z|^{1/2}\operatorname{sgn} z$ for any $z\in\mathbb{R}$ (see [2]).

Consider VIE

$$x(t) = (2-4t)\int_0^t x(\tau)^{1/2}\,d\tau \quad t\in[0,1]. \tag{11}$$

Proposition 3. *Set $s = 2t-1$, $t\in[0,1]$, and $z(s) = x(t)$, $s\in[-1,1]$. A function $z(s)$ is a solution of (10) if and only if $x(t)$ is a solution of (11).*

Proof. Suppose that $z(s)$ is a solution of (10). Set $s = 2t-1$, $t\in[0,1]$, in (10). We have

$$x(t) = z(2t-1) = (1-2t)\int_{-1}^{2t-1} z(\tau)^{1/2}\,d\tau. \tag{12}$$

By making a substitution $\tau = 2r-1$ in the integral we get

$$\begin{aligned}
x(t) &= (2-4t)\int_0^t z(2r-1)^{1/2}\,dr \\
&= (2-4t)\int_0^t x(r)^{1/2}\,dr.
\end{aligned} \tag{13}$$

Hence, $x(t)$ satisfies (11). Similarly setting $t = (s+1)/2$, $s\in[-1,1]$, in (11) we obtain that $z(s)$ satisfies (10) if $x(t)$ satisfies (11). □

Corollary 4. *VIE (10) has a maximal (minimal) solution if and only if (11) has a maximal (minimal) solution.*

Consider VIE

$$x(t) = (2-4t)\int_0^t [I(x(\tau))]^{1/2}\,d\tau \quad t\in[0,1], \tag{14}$$

where $I(x) = (\operatorname{sgn} x)\min\{|x|,4\}$ for $x\in\mathbb{R}$.

Proposition 5. *VIE's (11) and (14) have the same (nonempty) sets of solutions.*

Proof. The statement follows from the fact that every solution of (11) and (14) takes its values in the interval $[-4,4]$ where $I(x) = x$. Indeed, if x satisfies (11) and $\|x\| = \max\{|x(t)|; t\in[0,1]\}$ then we have

$$\begin{aligned}
|x(t)| &\le |2-4t|\int_0^t |x(\tau)|^{1/2}\,d\tau \le 2\int_0^1 \|x\|^{1/2}\,d\tau \\
&\le 2\|x\|^{1/2}, \quad t\in[0,1]
\end{aligned} \tag{15}$$

which implies $\|x\| \le 2\|x\|^{1/2}$ and $\|x\|\le 4$. Since $|I(x)|\le|x|$ a similar estimation holds for (14). Of course a zero function is a solution of both equations. □

Set $f(t,\tau,x) = f(t,x) = (2-4t)[I(x)]^{1/2}$. Of course, f is Lipschitz continuous in t. It is not difficult to verify (see Proposition 2) the following.

Remark 6. VIE (14) satisfies all the assumptions of Theorem 1.

Proposition 7. *VIE (14) has no extremal solution in $[-1,1]$.*

Proof. In view of Corollary 4 and Proposition 5 we only need to show that (10) has no extremal solutions. This was in fact done in [2] where the proof is rather long and complicated. For the reader's convenience, we present an original and short explanation.

Suppose that v is not a trivial solution of the problem

$$\begin{aligned}
&v'(s) = -s^{1/2}v(s)^{1/2}, \\
&v(-1) = 0 \\
&\qquad\qquad \text{in } [-1,1].
\end{aligned} \tag{16}$$

Such solution exists since this problem, in view of the classical theory, has many solutions. Suppose that $z_M(s)$ is a maximal solution of (10). Since 0, $sv(s)$, $-sv(s)$ are all solutions of (10) we have $z_M(s) \ge \max\{0, sv(s), -sv(s)\}$; hence $z_M(s)\ge 0$ and it is not identically zero. This gives $z_M(s) = -s\int_{-1}^{s} z_M(\tau)^{1/2}d\tau < 0$ for some $s\in(0,1]$. This leads to a contradiction. We finish the proof by observing that the negative of a minimal solution of (10) must be its maximal solution. □

Remark 8. Of course, we can improve Theorem 1 by assuming that f is nondecreasing in x. In this case however, (C3) is not necessary and (C4) can be reduced to a simpler one and the result is well-known.

Remark 9. Theorem 3.1 [1] (FVIE (2)) and Theorem 4.1 [1] (system of Volterra integral equation) are false since they generalize Theorem 1 (Theorem 2.1 [1]).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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