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*Research Article*

# Bounds for Products of Zeros of Solutions to Nonhomogeneous ODE with Polynomial Coefficients

**Michael Gil'**

*Department of Mathematics, Ben Gurion University of the Negev, P.O. Box 653, 84105 Be'er-Sheva, Israel*

Correspondence should be addressed to Michael Gil'; gilmi@bezeqint.net

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We consider the equation $u'' = P(z)u + F(z)$ $(z \in \mathbb{C})$, where $P(z)$ is a polynomial and $F(z)$ is an entire function. Let $z_k(u)$ $(k = 1, 2, \ldots)$ be the zeros of a solution $u(z)$ to that equation. Lower estimates for the products $\prod_{k=1}^{j} |z_k(u)|$ $(j = 1, 2, \ldots)$ are derived. In particular, they give us a bound for the zero free domain. Applications of the obtained estimates to the counting function of the zeros of solutions are also discussed.

## 1. Introduction and Statement of the Main Result

In the present paper, we investigate the zeros of solutions to the initial problem

$$u'' = P(z)u + F(z) \quad \text{with the initial conditions } u(0) = 1,\ u'(0) = u_1 \in \mathbb{C} \ (z \in \mathbb{C}), \tag{1}$$

where

$$P(z) = \sum_{k=0}^{n} c_k z^k \quad (c_n \neq 0) \tag{2}$$

is a polynomial with complex in general coefficients and $F(z)$ is an entire function. It is assumed that there are nonnegative constants $A_F$, $B_F$ and an integer $\rho_F$, such that

$$|F(z)| \leq A_F \exp\left[B_F r^{\rho_F}\right] \quad \left(z = e^{it};\ r > 0;\ t \in [0, 2\pi)\right). \tag{3}$$

The literature devoted to the zeros of solutions of homogeneous equations is very rich. Here the main tool is the Nevanlinna theory. An excellent exposition of the Nevanlinna theory and its applications to differential equations is given in book [1]. In connection with the recent results see interesting papers [2–13] (see also [14, 15]). At the same time the zeros of solutions to nonhomogeneous ODE were not enough investigated in the available literature. Here we can point out [16], only, in which the estimates for the sums of the zeros of solutions to (1) have been derived. In the present paper, lower estimates for the products of the zeros are obtained. In addition, we refine the main result from [16].

Enumerate the zeros $z_k(u)$ of $u$ with their multiplicities in order of increasing absolute values: $|z_k(u)| \leq |z_{k+1}(u)|$ $(k = 1, 2, \ldots)$. Denote

$$s_P = \sqrt{\sum_{k=0}^{n} |c_k|}, \qquad \rho_0 = \max\left\{\rho_F, \frac{n}{2} + 1\right\}. \tag{4}$$

Now we are in a position to formulate the main result of the paper.

**Theorem 1.** *Let condition (3) hold. Then the zeros of the solution u to problem (1) satisfy the inequality*

$$\prod_{k=1}^{j} |z_k(u)| > \frac{(j!)^{1/\rho_0}}{C(u_1)\zeta^j} \quad (j = 1, 2, \ldots), \tag{5}$$

*where*

$$\zeta = 2\left(e\rho_0\left(B_F + s_P\right)\right)^{1/\rho_0},$$
$$C\left(u_1\right) = e^{s_P}\left(\frac{3}{2} + \frac{12\left|u_1\right|}{\zeta} + \frac{16a_F e^{B_F}}{\zeta^2}\right). \quad (6)$$

The proof of this theorem is presented in the next two sections. Below we also suggest the sharper but more complicated bound for the products of the zeros.

## 2. Solution Estimates

Consider the equation

$$\frac{d^2u}{dz^2} = Q(z)u + F_0(z) \quad \left(u(0) = u_0,\ u'(0) = u_1\right), \quad (7)$$

where $Q(z)$ and $F_0(z)$ are entire functions. Put $M_f(r) = \sup_{|z|\le r}|f(z)|$.

**Lemma 2.** *A solution $u(z)$ of (7) satisfies the inequality*

$$M_u(r) \le \left(\left|u_0\right| + r\left|u_1\right| + \int_0^r (r-s) M_{F_0}(s)\,ds\right) \cdot \cosh\left(r\sqrt{M_Q(r)}\right) \quad (r \ge 0). \quad (8)$$

*Proof.* For a fixed $t \in [0, 2\pi)$ and $z = re^{it}$ we have

$$\frac{1}{e^{2it}}\frac{d^2u\left(re^{it}\right)}{dr^2} = Q\left(re^{it}\right)u\left(re^{it}\right) + F_0\left(re^{it}\right). \quad (9)$$

Integrating twice this equation in $r$, we obtain

$$e^{-2it}u\left(re^{it}\right) = e^{-2it}\left(u_0 + ru_1\right) + \int_0^r (r-s)\left[Q\left(se^{it}\right)u\left(se^{it}\right) + F_0\left(se^{it}\right)\right]ds. \quad (10)$$

Hence,

$$M_u(r) \le \left|u_0\right| + r\left|u_1\right| + \int_0^r (r-s)\left(M_Q(s)M_u(s) + M_F(s)\right)ds \le M_Q(r)\int_0^r (r-s)M_u(s)\,ds + H(r), \quad (11)$$

where

$$H(r) = \left|u_0\right| + r\left|u_1\right| + \int_0^r (r-s)M_{F_0}(s)\,ds. \quad (12)$$

Due to the comparison lemma [17, Lemma III.2.1], we have $M_u(s) \le \hat{v}(r)$, where $\hat{v}(r)$ is a solution of the equation

$$\hat{v}(r) = H(r) + \int_0^r (r-s)M_Q(s)\hat{v}(s)\,ds = H(r) + V\hat{v}(r). \quad (13)$$

Here $V$ is the Volterra operator defined by

$$(Vv)(r) = \int_0^r (r-s)M_Q(s)v(s)\,ds, \quad (14)$$

and, therefore,

$$\hat{v} = \sum_{k=0}^{\infty} V^k H. \quad (15)$$

But for any positive nondecreasing $h(r)$ we have

$$Vh(r) = \int_0^r (r-s)M_Q(s)h(s)\,ds \le h(r)M_Q(r)\int_0^r (r-s)\,ds. \quad (16)$$

Similarly,

$$V^m H(r) \le H(r)M_Q^m(r) \cdot \int_0^r \int_0^{r_1} \cdots \int_0^{r_{m-1}} (r-r_1)\cdots(r_{m-1}-r_m)\,dr_1\cdots dr_m = H(r)M_Q^m(r)\frac{r^{2m}}{(2m)!}. \quad (17)$$

Thus from (15) it follows

$$M_u(r) \le \hat{v}(r) \le H(r)\sum_{k=0}^{\infty}\frac{M_Q^k(r)r^{2k}}{(2k)!}. \quad (18)$$

But

$$\sum_{k=0}^{\infty}\frac{M_Q^k(r)r^{2k}}{(2k)!} = \cosh\left(r\sqrt{M_Q(r)}\right). \quad (19)$$

This implies the required result. □

Note that in our reasoning $Q$ and $F_0$ can be arbitrary piecewise continuous functions.

Consider now (1). In this case

$$M_Q(r) = M_P(r) \le \hat{p}(r) := \sum_{k=0}^{n}\left|c_k\right|r^k. \quad (20)$$

In addition, since $r^{1+k/2} \le 1 + r^{1+d_0/2}$ for any $d_0 \ge k$, we have

$$r\sqrt{\hat{p}(r)} \le \sqrt{\hat{p}(1)}\max_k r^{k/2+1} \le \sqrt{\hat{p}(1)}\left(1 + r^{\rho_0}\right) = s_P\left(1 + r^{\rho_0}\right), \quad (21)$$

$$\cosh\left(r\sqrt{\hat{p}(r)}\right) \le \exp\left[s_P\left(1 + r^{\rho_0}\right)\right].$$

Now Lemma 2 yields the following.

**Corollary 3.** *A solution $u(z)$ of problem (1) satisfies the inequality*

$$M_u(r) \le \left(1 + r\left|u_1\right| + \int_0^r (r-s)M_F(s)\,ds\right) \cdot \exp\left[s_P\left(1 + r^{\rho_0}\right)\right] \quad (r > 0). \quad (22)$$

This corollary is sharp: as it is well known a solution of the homogeneous equation

$$u'' = P(z)u \tag{23}$$

is an entire function of order no more than $(n+2)/2$; see, for example, [1, Proposition 5.1]. Besides, our proof is absolutely different.

**Corollary 4.** *Let condition (3) hold. Then a solution of problem (1) satisfies the inequality*

$$M_u(r) \le e^{s_P(1+r^{\rho_0})}\left(1 + r|u_1| + A_F r^2 \exp\left[B_F r^{\rho_F}\right]\right) \quad (r > 0). \tag{24}$$

## 3. Proof of Theorem 1

**Lemma 5.** *Let an entire function $f(z)$ satisfy the inequality*

$$M_f(r) \le r^m \exp\left[Br^\rho\right] \quad (B = const > 0;\ \rho \ge 1;\ r > 0) \tag{25}$$

*with an integer $m \ge 0$. Then its Taylor coefficients $f_j$ are subject to the inequalities*

$$\left|f_j\right| \le \frac{(eB\rho)^{(j-m)/\rho}}{\left[(j-m)!\right]^{1/\rho}} \quad (j \ge m), \quad f_0 = \cdots = f_{m-1} = 0. \tag{26}$$

*Proof.* Let $\hat{f}(z) = (1/z^m)f(z)$. Then the Taylor coefficients $\hat{f}_j$ of $\hat{f}$ satisfy the relation $f_j = \hat{f}_{j-m}$ and $M_{\hat{f}}(r) \le \exp[Br^\rho]$. By the well-known inequality for the coefficients of a power series,

$$\left|\hat{f}_j\right| \le \frac{M_{\hat{f}}(r)}{r^j} \le \frac{e^{Br^\rho}}{r^j}. \tag{27}$$

Employing the usual method for finding extrema it is easy to see that the function $r^{-j}e^{Br^\rho}$ $(j \ge 1)$ takes its smallest value in the range $r > 0$ for $r_0 = r_0(j)$ defined by

$$r_0 = \left(\frac{j}{B\rho}\right)^{1/\rho}$$
$$\text{and therefore } \left|\hat{f}_j\right| \le \frac{M_{\hat{f}}(r_0)}{r_0^j} \le \left(\frac{eB\rho}{j}\right)^{j/\rho} \le \frac{(eB\rho)^{j/\rho}}{(j!)^{1/\rho}}. \tag{28}$$

Since $f_j = \hat{f}_{j-m}$, the lemma is proved. □

The solution $u(z)$ to (1) can be represented as $u(z) = v(z) + w(z) + y(z)$, where $v(z)$ is the solution to (1) with $v(0) = 1$, $v'(0) = 0$, and $F(z) \equiv 0$; $w(z)$ is the solution to (1) with $w(0) = 0$, $w'(0) = u_1$, and $F(z) \equiv 0$; $y(z)$ is the solution to (1) with $y(0) = y'(0) = 0$.

Corollary 4 implies

$$\begin{aligned} M_v(r) &\le e^{s_P(1+r^{\rho_0})}, \\ M_w(r) &\le r\left|u_1\right| e^{s_P(1+r^{\rho_0})}, \\ M_y(r) &\le A_F e^{s_P} r^2 \exp\left[s_P r^{\rho_0} + B_F r^{\rho_F}\right] \quad (r > 0). \end{aligned} \tag{29}$$

Since $r^{\rho_F} \le 1 + r^{\rho_0}$, we obtain

$$M_y(r) \le A_F e^{s_P + B_F} r^2 \exp\left[\left(s_P + B_F\right) r^{\rho_0}\right] \quad (r > 0). \tag{30}$$

Introduce the notations

$$\begin{aligned} b &= \left(e s_P \rho_0\right)^{1/\rho_0}, \\ c &= \left(e\left(B_F + s_P\right)\rho_0\right)^{1/\rho_0} = \frac{\zeta}{2}. \end{aligned} \tag{31}$$

Denote by $v_j$, $w_j$, and $y_j$ the Taylor coefficients of $v(z)$, $w(z)$, and $y(z)$, respectively. Then Lemma 5 yields

$$\begin{aligned} \left|v_j\right| &\le e^{s_P}\frac{b^j}{[j!]^{1/\rho_0}}, \\ \left|w_j\right| &\le \left|u_1\right| e^{s_P}\frac{b^{j-1}}{[(j-1)!]^{1/\rho_0}}, \\ \left|y_j\right| &\le A_F e^{s_P + B_F}\frac{c^{j-2}}{[(j-2)!]^{1/\rho_0}} \quad (j \ge 2). \end{aligned} \tag{32}$$

Let $u_j$ be the Taylor coefficients of $u$. Since $|u_j| \le |v_j| + |w_j| + |y_j|$, we have

$$\begin{aligned} \left|u_j\right| \le e^{s_P}\Bigg(&\frac{b^j}{[j!]^{1/\rho_0}} + \left|u_1\right|\frac{b^{j-1}}{[(j-1)!]^{1/\rho_0}} \\ &+ A_F e^{B_F}\frac{c^{j-2}}{[(j-2)!]^{1/\rho_0}}\Bigg) \quad (j \ge 2). \end{aligned} \tag{33}$$

Let us consider the entire function

$$f(z) = \sum_{k=0}^{\infty}\frac{d_k z^k}{(k!)^{1/\rho}} \quad \left(\rho \ge 1,\ z \in \mathbb{C},\ d_0 = 1,\ d_k \in \mathbb{C}\right). \tag{34}$$

Enumerate the zeros $z_k(f)$ $(k = 1, 2, \ldots)$ of $f$ with their multiplicities in order of nondecreasing absolute values and assume that

$$\theta(f) := \left[\sum_{k=1}^{\infty}\left|d_k\right|^2\right]^{1/2} < \infty. \tag{35}$$

**Lemma 6.** *Let $f$ be represented by (34) and let condition (35) hold. Then*

$$\prod_{k=1}^{j}\left|z_k(f)\right| > \frac{(j!)^{1/\rho}}{2^{-1/\rho}+\theta(f)}, \quad j=1,2,\ldots. \tag{36}$$

This lemma is a particular case of Theorem 2.1 proved in [18].

Furthermore, consider the function $u_\zeta(z) = u(z/\zeta)$, where $u(z)$ is a solution to (1). Recall that $\zeta = 2c$. Due to (33), the Taylor coefficients $a_j = u_j/\zeta^j$ of $u_\zeta(z)$ satisfy the inequalities

$$\begin{aligned}\left|a_j\right| \le \frac{e^{s_P}}{2^j}\Bigg(&\frac{b^j}{c^j\,[j!]^{1/\rho_0}} + \frac{\left|u_1\right| b^{j-1}}{c^j\,[(j-1)!]^{1/\rho_0}}\\ &+ \frac{A_F e^{B_F}}{c^2\,[(j-2)!]^{1/\rho_0}}\Bigg) \quad (j\ge 2)\end{aligned} \tag{37}$$

and $a_1 = u_1/2c$. Denote

$$\theta(u_\zeta) := \left[\sum_{k=1}^{\infty}\left|a_k\right|^2 (k!)^{2/\rho_0}\right]^{1/2}. \tag{38}$$

Clearly,

$$\theta(u_\zeta) \le \sum_{k=1}^{\infty}\left|a_k\right| (k!)^{1/\rho_0}. \tag{39}$$

So due to (37)

$$\begin{aligned}\theta(u_\zeta) \le \frac{\left|u_1\right|}{2c} + e^{s_P}\sum_{j=2}^{\infty}\frac{1}{2^j}\Bigg(&\left(\frac{b}{c}\right)^j + \left|u_1\right|\frac{j^{1/\rho_0}}{c}\left(\frac{b}{c}\right)^{j-1}\\ &+ A_F e^{B_F}\frac{[j(j-1)]^{1/\rho_0}}{c^2}\Bigg) < \infty.\end{aligned} \tag{40}$$

Now Lemma 6 implies

$$\prod_{k=1}^{j}\left|z_k(u_\zeta)\right| > \frac{(j!)^{1/\rho_0}}{2^{-1/\rho_0}+\theta(u_\zeta)}, \quad j=1,2,\ldots. \tag{41}$$

Since $z_k(u_\zeta)/\zeta = z_k(u)$, we have proved the following result.

**Lemma 7.** *Let condition (3) hold. Then*

$$\prod_{k=1}^{j}\left|z_k(u)\right| > \frac{(j!)^{1/\rho_0}}{\zeta^j C(P,F,u_1)}, \quad j=1,2,\ldots, \tag{42}$$

*where*

$$C(P,F,u_1) = 2^{-1/\rho_0} + \theta(u_\zeta). \tag{43}$$

Let us estimate $C(P,F,u_1)$. Recall that $b$ is defined by (31).

**Lemma 8.** *One has $\theta(u_\zeta) \le \widehat{\theta}(P,F,u_1)$, where*

$$\begin{aligned}\widehat{\theta}(P,F,u_1) := e^{s_P}\Bigg(&\frac{b^2}{\zeta^2(1-(b/\zeta))} + \frac{3\left|u_1\right|}{\zeta(1-(b/\zeta))^2}\\ &+ \frac{16A_F e^{B_F}}{\zeta^2}\Bigg),\end{aligned} \tag{44}$$

*and therefore $C(P,F,u_1) \le 2^{-1/\rho_0} + \widehat{\theta}(P,F,u_1)$.*

*Proof.* Taking into account that

$$\begin{aligned}\frac{1}{1-x} &= \sum_{k=0}^{\infty}x^k,\\ \frac{1}{(1-x)^2} &= \frac{d}{dx}\frac{1}{(1-x)} = \sum_{k=1}^{\infty}kx^{k-1},\\ \frac{2}{(1-x)^3} &= \frac{d^2}{dx^2}\frac{1}{1-x} = \sum_{k=2}^{\infty}k(k-1)x^{k-2}\\ &\qquad (0<x<1),\end{aligned} \tag{45}$$

we obtain

$$\begin{aligned}\sum_{j=2}^{\infty}\left(\frac{b}{2c}\right)^j &= \left(\frac{b}{2c}\right)^2\left(1-\left(\frac{b}{2c}\right)\right)^{-1}\\ &= \left(\frac{b}{\zeta}\right)^2\left(1-\left(\frac{b}{\zeta}\right)\right)^{-1},\\ 1+\sum_{j=2}^{\infty}j\left(\frac{b}{2c}\right)^{j-1} &= \left(1-\left(\frac{b}{2c}\right)\right)^{-2} = \left(1-\left(\frac{b}{\zeta}\right)\right)^{-2},\\ \sum_{j=2}^{\infty}j(j-1)\frac{1}{2^j} &= \frac{1}{4}\sum_{j=2}^{\infty}j(j-1)\frac{1}{2^{j-2}}\\ &= \frac{1}{2}\frac{1}{(1-1/2)^3} = 4.\end{aligned} \tag{46}$$

Since $\rho_0 \ge 1$, (40) implies $\theta(u_\zeta) \le \widehat{\theta}(P,F,u_1)$. This and (43) prove the lemma. □

*Proof of Theorem 1.* Since $b/\zeta \le 1/2$, from the previous lemma we get

$$\begin{aligned}\theta(u_\zeta) &\le \widehat{\theta}(P,F,u_1)\\ &\le e^{s_P}\left(\frac{1}{2} + \frac{12\left|u_1\right|}{\zeta} + \frac{16A_F e^{B_F}}{\zeta^2}\right).\end{aligned} \tag{47}$$

But $2^{-1/\rho_0} \le 1 \le e^{s_P}$, and therefore

$$\begin{aligned}2^{-1/\rho_0} + \theta(u_\zeta) &\le C(P,F,u_1)\\ &\le e^{s_P}\left(\frac{3}{2} + \frac{12\left|u_1\right|}{\zeta} + \frac{16A_F e^{B_F}}{\zeta^2}\right)\\ &= C(u_1).\end{aligned} \tag{48}$$

This and Lemma 7 prove the theorem. □

## 4. Sums of Zeros and the Counting Function

In this section we derive a bound for sums of the zeros of solutions. To this end we need the following.

**Theorem 9.** *Let $f$ be defined by (34) and let condition (35) hold. Then*

$$\sum_{k=1}^{j} \frac{1}{|z_k(f)|} < \theta(f) + \sum_{k=1}^{j} \frac{1}{(k+1)^{1/\rho}} \quad (j = 1, 2, \ldots). \quad (49)$$

This theorem is proved in [19] (see also [20, Section 5.1]). It gives us the inequality

$$\sum_{k=1}^{j} \frac{1}{|z_k(u_\zeta)|} < \theta(u_\zeta) + \sum_{k=1}^{j} \frac{1}{(k+1)^{1/\rho_0}} \quad (j = 1, 2, \ldots). \quad (50)$$

Since $z_k(u_\zeta) = \zeta z_k(u)$, we get

$$\sum_{k=1}^{j} \frac{1}{|z_k(u)|} < \zeta \left( \theta(u_\zeta) + \sum_{k=1}^{j} \frac{1}{(k+1)^{1/\rho_0}} \right) \quad (j = 1, 2, \ldots). \quad (51)$$

Now (47) implies the following.

**Theorem 10.** *Let condition (3) hold. Then the zeros of the solution $u$ to problem (1) satisfy the inequality*

$$\sum_{k=1}^{j} \frac{1}{|z_k(u)|} < \zeta \left( \theta_0(u_1) + \frac{1}{(k+1)^{1/\rho_0}} \right) \quad (j = 1, 2, \ldots), \quad (52)$$

*where*

$$\theta_0(u_1) = e^{s_P} \left( \frac{1}{2} + \frac{12|u_1|}{\zeta} + \frac{16 A_F e^{B_F}}{\zeta^2} \right). \quad (53)$$

This theorem refines the main result from [16].

Furthermore, since $|z_k(u)| \le |z_{k+1}(u)|$, Theorem 1 implies that

$$|z_j(u)|^j > \frac{(j!)^{1/\rho_0}}{C(u_1)\zeta^j} \quad (j = 1, 2, \ldots). \quad (54)$$

Denote by $n(a, f)$ $(a > 0)$ the counting function of the zeros of $f$ in the circle $|z| \le a$. We thus get the following.

**Corollary 11.** *With the notation*

$$\eta_j(u) := \frac{1}{\zeta} \sqrt[j]{\frac{(j!)^{1/\rho_0}}{C(u_1)}}, \quad (55)$$

*the inequality $|z_j(u)| > \eta_j(u)$ holds and thus $u(z)$ does not have zeros in the disc*

$$\left\{ z \in \mathbb{C} : |z| \le \frac{1}{C(u_1)\zeta} \right\}. \quad (56)$$

*Moreover, $n(a, u) \le j - 1$ for any positive $a \le \eta_j(u)$ $(j = 1, 2, \ldots)$.*

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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