

Research Article

Dynamical Behavior of a System of Second-Order Nonlinear Difference Equations

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This paper is concerned with local stability, oscillatory character of positive solutions to the system of the two nonlinear difference equations $x_{n+1} = A + x_{n-1}^p/y_n^p$ and $y_{n+1} = A + y_{n-1}^p/x_n^p$, $n = 0, 1, \dots$, where $A \in (0, \infty)$, $p \in [1, \infty)$, $x_i \in (0, \infty)$, and $y_i \in (0, \infty)$, $i = -1, 0$.

1. Introduction

Difference equation or discrete dynamical system is a diverse field which impacts almost every branch of pure and applied mathematics. Every dynamical system $x_{n+1} = f(x_n)$ determines a difference equation and vice versa. Recently, there has been great interest in studying difference equation systems. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economics, probability theory, genetics, psychology, and so forth.

The theory of difference equations occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering, and economics. It is very interesting to investigate the behavior of solutions of a system of nonlinear difference equations and to discuss the local asymptotic stability of their equilibrium points.

Amleha et al. [1] investigated global stability, boundedness character, and periodic nature of the difference equation:

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \dots, \quad (1)$$

where $x_{-1}, x_0 \in \mathbb{R}$ and $\alpha > 0$.

El-Owaidey et al. [2] investigated local stability, oscillation, and boundedness character of the difference equation:

$$x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^p}, \quad n = 0, 1, \dots \quad (2)$$

And also Stević [3] studied dynamical behavior of this difference equation. Other related difference equation readers can refer to [4–18].

Papaschinopoulos and Schinas [6] studied the system of two nonlinear difference equation:

$$\begin{aligned} x_{n+1} &= A + \frac{y_n}{x_{n-p}}, \\ y_{n+1} &= A + \frac{x_n}{y_{n-q}}, \end{aligned} \quad (3)$$

$$n = 0, 1, \dots,$$

where p, q are positive integers.

Clark and Kulenović [9, 10] investigated the system of rational difference equations:

$$\begin{aligned}x_{n+1} &= \frac{x_n}{a + cy_n}, \\y_{n+1} &= \frac{y_n}{b + dx_n},\end{aligned}\quad (4)$$

$$n = 0, 1, \dots,$$

where $a, b, c, d \in (0, \infty)$ and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers.

Our aim in this paper is to investigate local stability, oscillation, and boundedness character of positive solutions of the system of difference equations:

$$\begin{aligned}x_{n+1} &= A + \frac{x_{n-1}^p}{y_n^p}, \\y_{n+1} &= A + \frac{y_{n-1}^p}{x_n^p},\end{aligned}\quad (5)$$

$$n = 0, 1, \dots,$$

where $A \in (0, \infty)$, $p \in [1, \infty)$, and initial conditions $x_i, y_i \in (0, \infty)$, $i = -1, 0$.

Firstly we recall some basic definitions that we need in the sequel.

(x_n, y_n) is bounded and persists if there exist positive constants M, N such that

$$\begin{aligned}M &\leq x_n \leq N, \\M &\leq y_n \leq N,\end{aligned}\quad (6)$$

$$n = -2, -1, 0, \dots$$

A solution (x_n, y_n) of (5) is said to be nonoscillatory about $(0, 0)$ if both x_n and y_n are either eventually positive or eventually negative. Otherwise, it is said to be oscillatory about $(0, 0)$.

A solution (x_n, y_n) of (5) is said to be nonoscillatory about equilibrium (\bar{x}, \bar{y}) if both $x_n - \bar{x}$ and $y_n - \bar{y}$ are either eventually positive or eventually negative. Otherwise, it is said to be oscillatory about equilibrium (\bar{x}, \bar{y}) .

2. Main Results

In this section, we will prove the following results concerning system (5).

Theorem 1. *The following statements are true:*

- (i) *The system (5) has a positive equilibrium point $(\bar{x}, \bar{y}) = (A + 1, A + 1)$.*
- (ii) *The equilibrium point $(A + 1, A + 1)$ of system (5) is locally asymptotically stable if $A > 2p - 1$.*
- (iii) *The equilibrium point $(A + 1, A + 1)$ of system (5) is unstable if $0 < A < 2p - 1$.*

- (iv) *The equilibrium point $(A + 1, A + 1)$ of system (5) is a sink or an attracting equilibrium if $p/(A + 1) < \sqrt{2} - 1$.*

Proof. (i) Let x, y be positive numbers such that

$$\begin{aligned}x &= A + \frac{x^p}{y^p}, \\y &= A + \frac{y^p}{x^p}.\end{aligned}\quad (7)$$

Then from (7) we have that the positive equilibrium point $(\bar{x}, \bar{y}) = (A + 1, A + 1)$.

(ii) The linearized equation of system (5) about the equilibrium point $(\bar{x}, \bar{y}) = (A + 1, A + 1)$ is

$$\Psi_{n+1} = B\Psi_n, \quad (8)$$

where

$$\Psi_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}, \quad (9)$$

$$B = \begin{pmatrix} 0 & \frac{p}{A+1} & -\frac{p}{A+1} & 0 \\ 1 & 0 & 0 & 0 \\ -\frac{p}{A+1} & 0 & 0 & \frac{p}{A+1} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of (8) is

$$\lambda^4 - \frac{p}{A+1} \left(\frac{p}{A+1} + 2 \right) \lambda^2 - \left(\frac{p}{A+1} \right)^2 = 0. \quad (10)$$

By using the linearized stability theorem [11], the equilibrium point $(A + 1, A + 1)$ is locally asymptotically stable iff all the roots of characteristic equation (10) lie inside unit disk, in other words, iff

$$\begin{aligned}\left(\frac{p}{A+1} \right)^2 &< 1, \\1 + \left(\frac{p}{A+1} \right)^2 + \frac{p}{A+1} \left(\frac{p}{A+1} + 2 \right) &> 0, \\1 + \left(\frac{p}{A+1} \right)^2 - \frac{p}{A+1} \left(\frac{p}{A+1} + 2 \right) &> 0.\end{aligned}\quad (11)$$

That is, $A > 2p - 1$.

(iii) From the proof of (ii), it is true.

(iv) By using the linearized stability theorem [11], the equilibrium point $(A + 1, A + 1)$ is a sink or attracting equilibrium iff

$$\begin{aligned}\left| \frac{p}{A+1} \left(\frac{p}{A+1} + 2 \right) \right| &< 1, \\ \left(\frac{p}{A+1} \right)^2 &< 2.\end{aligned}\quad (12)$$

It only needs $p/(A + 1) < \sqrt{2} - 1$. This completes the proof of the theorem. \square

Theorem 2. Let $0 \leq A < 1$, and let $\{(x_n, y_n)\}$ be a solution of system (3) such that

$$\begin{aligned} 0 &< x_{-1} < 1, \\ 0 &< y_{-1} < 1, \\ x_0 &\geq \frac{1}{(1-A)^{1/p}}, \\ y_0 &\geq \frac{1}{(1-A)^{1/p}}. \end{aligned} \quad (13)$$

Then the following statements are true:

- (i) $\lim_{n \rightarrow \infty} x_{2n} = \infty$, $\lim_{n \rightarrow \infty} y_{2n} = \infty$.
- (ii) $\lim_{n \rightarrow \infty} x_{2n+1} = A$, $\lim_{n \rightarrow \infty} y_{2n+1} = A$.

Proof. Since $0 \leq A < 1$, so $(1-A)^2 < 1$ and so $1/(1-A) > A+1$; then

$$\begin{aligned} x_0^p &> A+1, \\ y_0^p &> A+1. \end{aligned} \quad (14)$$

We have

$$\begin{aligned} x_1 &= A + \frac{x_{-1}^p}{y_0^p} \leq A + \frac{1}{y_0^p} \leq 1, \\ y_1 &= A + \frac{y_{-1}^p}{x_0^p} \leq A + \frac{1}{x_0^p} \leq 1, \\ x_1 &= A + \frac{x_{-1}^p}{y_0^p} > A, \\ y_1 &= A + \frac{y_{-1}^p}{x_0^p} > A. \end{aligned} \quad (15)$$

Thus

$$\begin{aligned} x_1 &\in (A, 1], \\ y_1 &\in (A, 1]. \end{aligned} \quad (16)$$

Similarly we have

$$\begin{aligned} x_2 &= A + \frac{x_0^p}{y_1^p} \geq A + x_0^p, \\ y_2 &= A + \frac{y_0^p}{x_1^p} \geq A + y_0^p, \\ x_3 &= A + \frac{x_1^p}{y_2^p} \leq A + \frac{1}{(A+y_0^p)^p} \leq A + \frac{1}{A+x_0^p} \\ &\leq A+1-A=1, \\ y_3 &= A + \frac{y_1^p}{x_2^p} \leq A + \frac{1}{(A+x_0^p)^p} \leq A + \frac{1}{A+x_0^p} \\ &\leq A+1-A=1. \end{aligned} \quad (17)$$

Therefore

$$\begin{aligned} x_3 &\in (A, 1], \\ y_3 &\in (A, 1]. \end{aligned} \quad (18)$$

Also

$$\begin{aligned} x_4 &= A + \frac{x_2^p}{y_3^p} > A + x_2^p \geq A + (A+x_0^p)^p \\ &\geq A + (A+x_0^p) = 2A+x_0^p, \\ y_4 &= A + \frac{y_2^p}{x_3^p} > A + y_2^p \geq A + (A+y_0^p)^p \\ &\geq A + (A+y_0^p) = 2A+y_0^p. \end{aligned} \quad (19)$$

Thus

$$\begin{aligned} x_4 &\geq 2A+x_0^p, \\ y_4 &\geq 2A+y_0^p. \end{aligned} \quad (20)$$

By induction, we have

$$\begin{aligned} x_{2n} &\geq nA+x_0^p, \\ y_{2n} &\geq nA+y_0^p, \\ A &< x_{2n+1} < 1, \\ A &< y_{2n+1} < 1. \end{aligned} \quad (21)$$

Thus

$$\begin{aligned} \lim_{n \rightarrow \infty} x_{2n} &= \infty, \\ \lim_{n \rightarrow \infty} y_{2n} &= \infty, \\ \lim_{n \rightarrow \infty} \frac{x_{2n-1}^p}{y_{2n}^p} &= 0, \\ \lim_{n \rightarrow \infty} \frac{y_{2n-1}^p}{x_{2n}^p} &= 0. \end{aligned} \quad (22)$$

So

$$\begin{aligned} \lim_{n \rightarrow \infty} x_{2n+1} &= A + \lim_{n \rightarrow \infty} \frac{x_{2n-1}^p}{y_{2n}^p} = A, \\ \lim_{n \rightarrow \infty} y_{2n+1} &= A + \lim_{n \rightarrow \infty} \frac{y_{2n-1}^p}{x_{2n}^p} = A. \end{aligned} \quad (23)$$

This completes the proof. \square

Theorem 3. Let $\{(x_n, y_n)\}$ be a positive solution of system (5) which consists of at least two semicycles. Then $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is oscillatory.

Proof. Consider the following two cases.

Case 1. Let

$$\begin{aligned} x_{N-1} &< A+1 \leq x_N, \\ y_{N-1} &< A+1 \leq y_N, \end{aligned} \quad (24)$$

for some $N \geq 0$.

Then

$$\begin{aligned} x_{N+1} &= A + \frac{x_{N-1}^p}{y_N^p} < A+1, \\ y_{N+1} &= A + \frac{y_{N-1}^p}{x_N^p} < A+1, \\ x_{N+2} &= A + \frac{x_N^p}{y_{N+1}^p} > A+1, \\ y_{N+2} &= A + \frac{y_N^p}{x_{N+1}^p} > A+1. \end{aligned} \quad (25)$$

Therefore

$$\begin{aligned} x_{N+1} &< A+1 < x_{N+2}, \\ y_{N+1} &< A+1 < y_{N+2}. \end{aligned} \quad (26)$$

Case 2. Let

$$\begin{aligned} x_N &< A+1 \leq x_{N-1}, \\ y_N &< A+1 \leq y_{N-1}, \end{aligned} \quad (27)$$

for some $N \geq 0$.

Then

$$\begin{aligned} x_{N+1} &= A + \frac{x_{N-1}^p}{y_N^p} > A+1, \\ y_{N+1} &= A + \frac{y_{N-1}^p}{x_N^p} > A+1, \\ x_{N+2} &= A + \frac{x_N^p}{y_{N+1}^p} < A+1, \\ y_{N+2} &= A + \frac{y_N^p}{x_{N+1}^p} < A+1. \end{aligned} \quad (28)$$

Therefore

$$\begin{aligned} x_{N+2} &< A+1 < x_{N+1}, \\ y_{N+2} &< A+1 < y_{N+1}. \end{aligned} \quad (29)$$

This completes the proof. \square

Theorem 4. Let $\{(x_n, y_n)\}$ be a positive solution of system (5). Then the following statements are true:

(i) $x_{2n} > A+1, y_{2n+1} < A+1$, for $n \geq 1$, if $x_{-1} < y_0 < A+1 < y_{-1} < x_0$.

(ii) $x_{2n} < A+1, y_{2n+1} > A+1$, for $n \geq 1$, if $x_0 < y_{-1} < A+1 < y_0 < x_{-1}$.

Proof. (i) Since

$$\begin{aligned} x_1 &= A + \frac{x_{-1}^p}{y_0^p} < A+1, \\ y_1 &= A + \frac{y_{-1}^p}{x_0^p} < A+1, \end{aligned} \quad (30)$$

thus

$$\begin{aligned} x_2 &= A + \frac{x_0^p}{y_1^p} > A+1, \\ y_3 &= A + \frac{y_1^p}{x_2^p} < A+1. \end{aligned} \quad (31)$$

By induction, assuming, for $n = k$, we have

$$\begin{aligned} x_{2k} &> A+1, \\ y_{2k+1} &< A+1, \end{aligned} \quad (32)$$

then, for $n = k+1$, we have

$$\begin{aligned} x_{2(k+1)} &= A + \frac{x_{2k}^p}{y_{2k+1}^p} > A+1, \\ y_{2(k+1)+1} &= A + \frac{y_{2k+1}^p}{x_{2(k+1)}^p} < A+1. \end{aligned} \quad (33)$$

Thus

$$\begin{aligned} x_{2n} &> A+1, \\ y_{2n+1} &< A+1, \end{aligned} \quad (34)$$

for $n \geq 1$.

(ii) The proof of (ii) is similar to (i), so we omit it. \square

Remark 5. If $0 \leq A \leq 2p-1$, then system (5) has a unique positive equilibrium $(A+1, A+1)$. If $A > 2p-1$, then system (5) has multipositive equilibrium; however system (5) always has an equilibrium $(A+1, A+1)$. In this paper, we only investigate the dynamical behavior of the solution to this system associated with the equilibrium $(A+1, A+1)$. It is of further interest to study the behavior of system (5) about other equilibriums in the future.

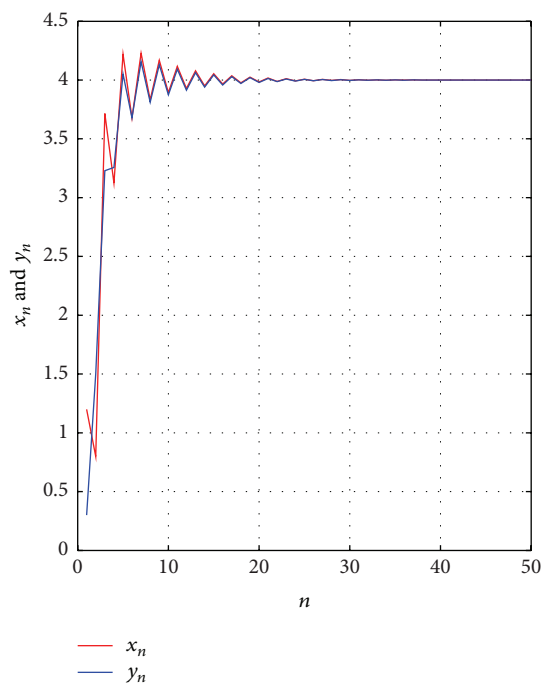


FIGURE 1

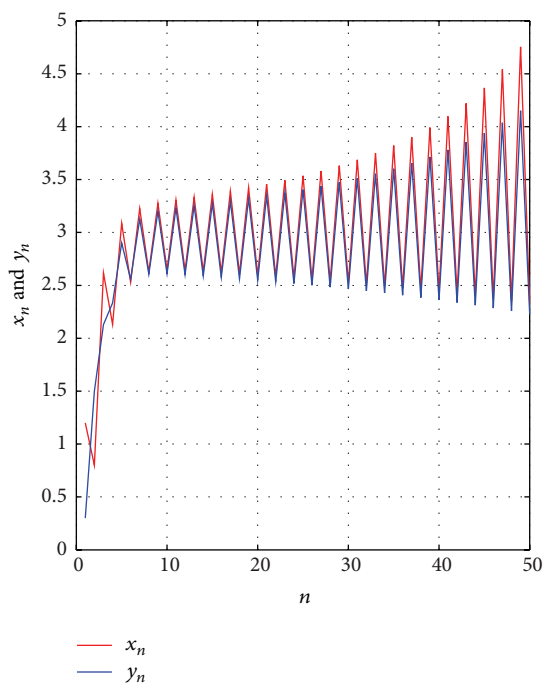


FIGURE 2

3. Numerical Results

For confirming the results of this section, we consider numerical examples, which represent different types of solutions to (5).

Example 1. When the initial conditions $x_{-1} = 1.2$, $x_0 = 0.8$, $y_{-1} = 0.3$, $y_0 = 1.5$, and $p = 1.5$, $A = 3$, then the solution of

system (5) converges to $(A + 1, A + 1) = (4, 4)$ (see Figure 1, Theorem 1).

Example 2. When the initial conditions $x_{-1} = 1.2$, $x_0 = 0.8$, $y_{-1} = 0.3$, $y_0 = 1.5$, and $p = 1.5$, $A = 1.9 < 2p - 1$, then the solution of system (5) is locally unstable (see Figure 2, Theorem 1).

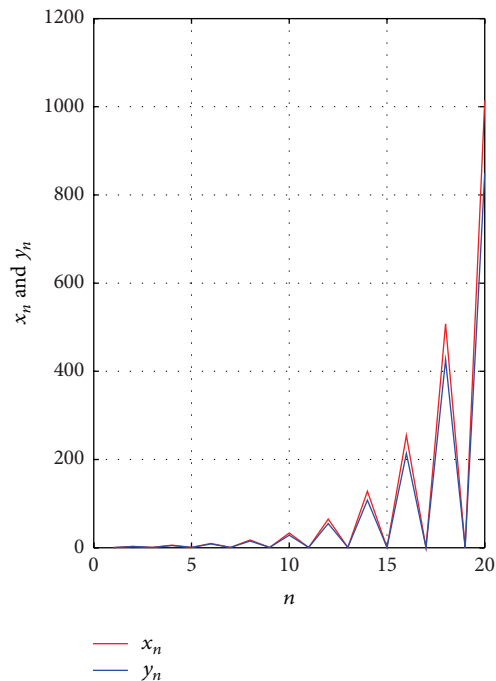


FIGURE 3

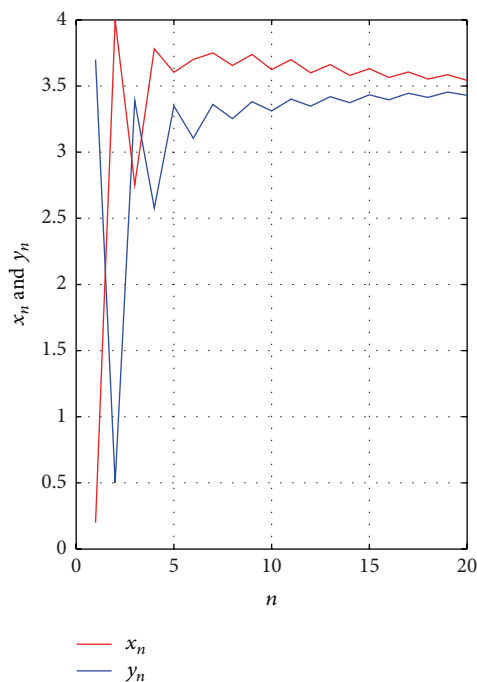


FIGURE 4

Example 3. When the initial conditions $x_{-1} = 0.2$, $x_0 = 3$, $y_{-1} = 0.3$, $y_0 = 2.5$, and $p = 1$, $A = 0.5$, then $\lim_{n \rightarrow \infty} x_{2n} = \infty$, $\lim_{n \rightarrow \infty} y_{2n} = \infty$, $\lim_{n \rightarrow \infty} x_{2n+1} = 0.5$, and $\lim_{n \rightarrow \infty} y_{2n+1} = 0.5$ (see Figure 3, Theorem 2).

Example 4. When the initial conditions $x_{-1} = 0.2$, $x_0 = 4$, $y_{-1} = 3.7$, $y_0 = 0.5$, and $p = 1.5$, $A = 2.5$, then $x_{2n} > 3.5$ and $y_{2n+1} < 3.5$ (see Figure 4, Theorem 4).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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