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Research Article

Numerical Solution of Riccati Equations by the Adomian and Asymptotic Decomposition Methods over Extended Domains

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We combine the Adomian decomposition method (ADM) and Adomian's asymptotic decomposition method (AADM) for solving Riccati equations. We investigate the approximate global solution by matching the near-field approximation derived from the Adomian decomposition method with the far-field approximation derived from Adomian's asymptotic decomposition method for Riccati equations and in such cases when we do not find any region of overlap between the obtained approximate solutions by the two proposed methods, we connect the two approximations by the Padé approximant of the near-field approximation. We illustrate the efficiency of the technique for several specific examples of the Riccati equation for which the exact solution is known in advance.

1. Introduction

It is well known that the Riccati equation as

$$u'(x) = p(x) + q(x)u(x) + r(x)u^2(x) \tag{1}$$

finds surprisingly many applications in physics and mathematics such as random processes, optimal control, and diffusion problems [1]. In fact, the Riccati equation naturally arises in many fields of quantum mechanics, such as in quantum chemistry [2], the Wentzel-Kramers-Brillouin approximation [3], and super symmetry theories [4]. In addition, the Riccati equation plays a prominent role in variational calculus [5], nonlinear physics [6], renormalization group equations for coupling constants in quantum field theories [7, 8], and thermodynamics [9]. It is well known that one-dimensional static Schrödinger equation is closely related to the Riccati equation. Solitary wave solutions of a nonlinear partial differential equation can be expressed as a polynomial in two elementary functions satisfying a projective Riccati equation [10]. Beside important engineering and scientific applications that are well known, the newer applications include areas such as mathematical finance [11, 12].

Adomian and his coauthors have presented a systematic methodology for practical solution of linear or nonlinear and deterministic or stochastic operator equations, including algebraic equations, ordinary differential equations, partial differential equations, and integral and integrodifferential equations [13–18]. Adomian decomposition method is a powerful technique, which provides efficient algorithms for analytic approximate solutions and numerical simulations for real-world applications in the applied sciences and engineering. Using the ADM, we calculate a series solution, but, in practice, we approximate the solution by a truncated series. The series sometimes coincides with the Taylor expansion of the exact solution in the neighborhood of the point $x = 0$. Although the series can be rapidly convergent in a small region, it has a slower convergence rate in the wider region.

Several investigators have proposed a variety of approaches to solve the Riccati equation, approximately [19–24]. In order to obtain the global approximate solution of the Riccati equation, we combine the Padé approximant of the near-field approximation as derived from the ADM with the far-field approximation as derived from the AADM [25–28] to overcome the difficulty of a finite domain of convergence. Adomian introduced a variation of his decomposition method in [25] that can be used to obtain the asymptotic value of solutions. In this method, the recursion is the same as that in the ADM, but it uses a different canonical form of the differential equation such that it yields to a steady state solution of the equation. In fact, rather than nested integrations as in decomposition, we have nested differentiations, which will be expounded later. Haldar and Datta [29] applied the AADM to calculate integrals neither expressible in terms of elementary functions nor adequately tabulated.

This paper is arranged as follows. In the next section, we present a brief review of the ADM for nonlinear IVPs. In Section 3, we present a description of the AADM for solving the Riccati equation. In Section 4, we investigate several numerical examples. In Section 5, we present our conclusions and summarize our findings.

2. Review of the Adomian Decomposition Method

We review the salient features of the Adomian decomposition method in solving IVPs for first-order nonlinear ordinary differential equations as

$$\begin{gathered} \frac{d}{dx}u(x) + \alpha(x)u(x) + f(u(x)) = g(x), \\ u(x_0) = c_0, \end{gathered} \tag{2}$$

where the functions α, g, and f are analytic.

We rewrite (2) in Adomian's usual operator-theoretic form

$$Lu + Ru + Nu = g, \tag{3}$$

where $L = (d/dx)(\cdot)$ and then $L^{-1} = \int_{x_0}^{x}(\cdot)dx$, $Ru = \alpha(x)u(x)$, and $Nu = f(u(x))$. Next we rewrite (3) as

$$Lu = g - Ru - Nu, \tag{4}$$

and we apply the integral operator L^{-1} to both sides of (4):

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu, \tag{5}$$

where $L^{-1}Lu = u - \Phi$ since $L\Phi = 0$. In the case of a first-order ordinary differential equation, we have $\Phi = u(x_0) = c_0$. Therefore

$$u = \Phi + L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{6}$$

For the sake of simplicity, we define the γ function as $\gamma = \Phi + L^{-1}g$, and then, upon substitution, we obtain

$$u = \gamma - L^{-1}Ru - L^{-1}Nu. \tag{7}$$

In the ADM, the solution $u(x)$ is represented by a series; say

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{8}$$

and the nonlinearity comprises the Adomian polynomials

$$Nu = \sum_{n=0}^{\infty} A_n, \tag{9}$$

where

$$A_n = A_n(u_0, u_1, \dots, u_n) \tag{10}$$

is called an Adomian polynomial, which were first defined by Adomian [13] as

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{\infty} u_k \lambda^k\right)\right]_{\lambda=0}, \quad n \geq 0. \tag{11}$$

For convenient reference, we list the first five Adomian polynomials

$$\begin{aligned} A_0 &= N(u_0), \\ A_1 &= N'(u_0)u_1, \\ A_2 &= N'(u_0)u_2 + N''(u_0)\frac{u_1^2}{2!}, \\ A_3 &= N'(u_0)u_3 + N''(u_0)u_1u_2 + N'''(u_0)\frac{u_1^3}{3!}, \\ A_4 &= N'(u_0)u_4 + N''(u_0)\left(\frac{u_2^2}{2!} + u_1u_3\right) \\ &\quad + N'''(u_0)\frac{u_1^2u_2}{2!} + N^{(4)}(u_0)\frac{u_1^4}{4!}. \end{aligned} \tag{12}$$

Several algorithms for the Adomian polynomials have been developed by Rach [30, 31], Adomian and Rach [32], Wazwaz [33], Biazar et al. [34], and several others. New, efficient algorithms with their subroutines written in Mathematica for rapid computer-generation of the Adomian polynomials have been provided by Duan in [35–37].

From (7)–(9) the solution components are determined by the classic Adomian recursion scheme:

$$\begin{aligned} u_0(x) &= \gamma(x), \\ u_{n+1}(x) &= -L^{-1}Ru_n(x) - L^{-1}A_n(x), \quad n \geq 0. \end{aligned} \tag{13}$$

Thus the m-term approximation $\phi_m(x) = \sum_{n=0}^{m-1} u_n(x)$ as obtained from the ADM can serve as the near-field approximation of the solution $u(x)$, where x is in the neighborhood of the initial point $x = x_0$.

We remark that the convergence of the Adomian decomposition series has been previously proven by several researchers [30, 38–41]. For example, Abdelrazec and Pelinovsky [41] have recently published a rigorous proof of convergence for the ADM in accordance with the Cauchy-Kovalevskaya theorem.

3. Description of Adomian's Asymptotic Decomposition Method

In this section, we advocate Adomian's asymptotic decomposition method for solving the Riccati equation. We remark that Adomian's asymptotic decomposition method does not need use of the initial condition to obtain the asymptotic solution or the solution in the large, which is another, convenient advantage in computations using this technique. Rather than nested integrations as required by decomposition, we now have nested differentiations. In effect our aim is to solve for the solution by not inverting the linear differential operator L, but instead by decomposing the nonlinear operator Nu and hence determining the asymptotic solution u. Toward this end, we rewrite (1) as

$$r(x)u^2(x) = -p(x) - q(x)u(x) + u'(x). \quad (14)$$

For the case when the coefficient $r(x) \neq 0$, we can divide both sides of (14) by $r(x)$, and we have

$$u^2(x) = -\frac{p(x)}{r(x)} - \frac{q(x)}{r(x)}u(x) + \frac{1}{r(x)}u'(x). \quad (15)$$

Substituting the respective decomposition series (i.e., $u(x) = \sum_{n=0}^{\infty} u_n(x)$ and $u^2(x) = \sum_{n=0}^{\infty} A_n(x)$), we obtain

$$\sum_{n=0}^{\infty} A_n(x) = -\frac{p(x)}{r(x)} - \frac{q(x)}{r(x)}\sum_{n=0}^{\infty} u_n(x) + \frac{1}{r(x)}\sum_{n=0}^{\infty} u_n'(x), \quad (16)$$

from which we design the asymptotic recursion scheme

$$A_0(x) = -\frac{p(x)}{r(x)}, \quad (17)$$

$$A_{n+1}(x) = -\frac{q(x)}{r(x)}u_n(x) + \frac{1}{r(x)}u_n'(x), \quad n \geq 0. \quad (18)$$

We note that the Adomian polynomials A_n for the quadratic nonlinearity $f(u) = u^2$ are

$$\begin{aligned} A_0 &= u_0^2, \\ A_1 &= 2u_0u_1, \\ A_2 &= 2u_0u_2 + u_1^2, \\ A_3 &= 2u_0u_3 + 2u_1u_2, \\ A_4 &= 2u_0u_4 + 2u_1u_3 + u_2^2, \\ &\vdots \\ A_n &= \sum_{i=0}^{n} u_i u_{n-i}. \end{aligned} \quad (19)$$

Using the form of the Adomian polynomials in (19), we rewrite the recursion schemes (17) and (18) as

$$u_0^2(x) = -\frac{p(x)}{r(x)}, \quad (20)$$

$$\sum_{i=0}^{n+1} u_i(x)u_{n+1-i}(x) = -\frac{q(x)}{r(x)}u_n(x) + \frac{1}{r(x)}u_n'(x), \quad n \geq 0. \quad (21)$$

In view of (21) and after appropriate manipulations, we obtain

$$\begin{aligned} \sum_{i=0}^{n+1} u_i(x)u_{n+1-i}(x) &= u_0(x)u_{n+1}(x) + \sum_{i=1}^{n} u_i(x)u_{n+1-i}(x) + u_{n+1}(x)u_0(x) \\ &= 2u_0(x)u_{n+1}(x) + \sum_{i=1}^{n} u_i(x)u_{n+1-i}(x). \end{aligned} \quad (22)$$

Consequently, with this result, the solution components are given by the following recursion scheme:

$$\begin{aligned} u_0(x) &= \sqrt{-\frac{p(x)}{r(x)}}, \\ u_{n+1} &= \frac{1}{2u_0(x)}\left(-\frac{q(x)}{r(x)}u_n(x) + \frac{1}{r(x)}u_n'(x) - \sum_{i=1}^{n} u_i(x)u_{n+1-i}(x)\right), \quad n \geq 0. \end{aligned} \quad (23)$$

Thus the obtained m-terms asymptotic approximation $\varphi_m(x) = \sum_{n=0}^{m-1} u_n(x)$ for the Riccati equation can serve as the far-field approximation, where x is far from the initial point $x = x_0$.

4. Numerical Examples

In this section, several numerical examples are given to illustrate the efficiency of our technique as presented in this paper. We remark that all calculations are performed by Mathematica package 8.

Example 1. Consider the following Riccati equation:

$$u'(x) + u^2(x) = 1 + x^2, \quad (24)$$

subject to initial condition $u(0) = 1$.

The exact solution is known in advance to be

$$u^*(x) = x + \frac{e^{-x^2}}{1 + \int_0^x e^{-t^2}dt}. \quad (25)$$

In Adomian's operator notation, we have

$$Lu + Nu = g, \quad (26)$$

where $L = d/dx$, $N = u^2$, and $g = 1 + x^2$.

To apply Adomian decomposition method, equation (26) should be written as the following,

$$Lu = g - Nu. \quad (27)$$

Applying the inverse operator $L^{-1}(\cdot) = \int_0^x (\cdot)dx$ to both sides yields

$$u = u(0) + L^{-1}g - L^{-1}Nu. \tag{28}$$

Next we consider the solution as a series $u(x) = \sum_{n=0}^{\infty} u_n(x)$ and the nonlinearity $f(u) = u^2 = \sum_{n=0}^{\infty} A_n(x)$, and upon substitution, we obtain

$$\sum_{n=0}^{\infty} u_n(x) = u(0) + L^{-1}g - L^{-1}\sum_{n=0}^{\infty} A_n(x). \tag{29}$$

The components of the series solution are given by the recursion scheme

$$\begin{aligned} u_0 &= u(0) + L^{-1}g, \\ u_{n+1} &= -L^{-1}A_n(x), \quad n \geq 0. \end{aligned} \tag{30}$$

The first few components are as follows:

$$\begin{aligned} u_0 &= 1 + x + \frac{x^3}{x}, \\ u_1 &= -x - x^2 - \frac{x^3}{3} - \frac{x^4}{6} - \frac{2x^5}{15} - \frac{x^7}{63}, \\ u_3 &= x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \frac{x^5}{3} + \frac{19x^6}{90} + \frac{22x^7}{315} + \frac{x^8}{56} \\ &\quad + \frac{38x^9}{2835} + \frac{2x^{11}}{2079}, \\ &\vdots \end{aligned} \tag{31}$$

The partial sums $\phi_m(x) = \sum_{n=0}^{m-1} u_n(x)$ of the Adomian decomposition series can serve as a near-field approximate solution.

Solving (26) by Adomian's asymptotic decomposition method, we first rewrite it as

$$Nu = g - Lu. \tag{32}$$

Next we assume the series $u = \sum_{n=0}^{\infty} u_n$ and the nonlinearity $u^2 = \sum_{n=0}^{\infty} A_n(x)$.

Upon substitution and using the form of the Adomian polynomials in (19), we obtain the solution components of the far-field approximation $\varphi_n(x)$ according to the recursion scheme (23)

$$\begin{aligned} u_0 &= \sqrt{1 + x^2}, \\ u_1 &= -\frac{x}{2(1 + x^2)}, \\ u_2 &= \frac{2 - 3x^2}{8(1 + x^2)^{5/2}}, \\ &\vdots \end{aligned} \tag{33}$$

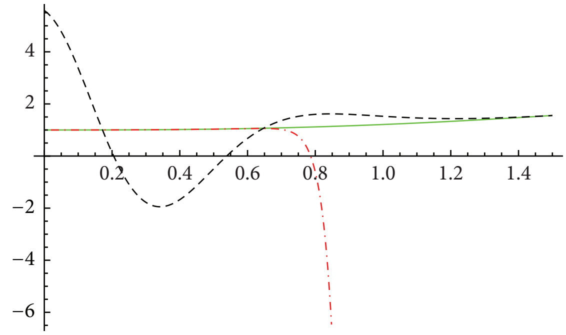

FIGURE 1: The near-field approximation $\phi_{20}(x)$ (dot-dashed line), far-field approximation $\varphi_7(x)$ (dashed line), and exact solution $u^*(x)$ (solid line).

from which we conclude that $u(x) \sim x$ is the slant asymptote of the exact solution; that is, $\lim_{x\to\infty}[u^*(x) - x] = 0$.

Computation shows that this Adomian decomposition series has a finite radius of convergence. By plotting the curves of $\phi_m(x)$ and $\varphi_n(x)$ for several values of m and n, we do not find any regions of overlap. In this case, we connect the two approximations by the Padé approximant of $\phi_m(x)$ or simply replace $\phi_m(x)$ by its Padé approximant and then match the Padé approximant with $\varphi_n(x)$. For example, we investigate ϕ_{20} and φ_7. The curves of the near-field approximation $\phi_{20}(x)$, the far-field approximation $\varphi_7(x)$, and the exact solution $u^*(x)$ are plotted in Figure 1.

We calculated the Padé approximant $[9/10]\{\phi_{20}(x)\}$ by Mathematica and found that the Padé approximant $[9/10]\{\phi_{20}(x)\}$ and the far-field approximation $\varphi_7(x)$ overlap in the approximate region $1.5 \leq x \leq 5.5$; see Figure 2. Thus we can match them as

$$\tilde{u}(x) = \left[\frac{9}{10}\right]\{\phi_{20}(x)\}\, h(\xi - x) + \varphi_7(x)\, h(x - \xi), \tag{34}$$

which is a global approximation, where ξ belongs to the region of overlap and $h(x)$ is the unit step function; that is,

$$h(x) = \begin{cases} 1, & x < 0, \\ \frac{1}{2}, & x = 0, \\ 1, & x > 0. \end{cases} \tag{35}$$

Example 2. Consider the following Riccati equation:

$$u'(x) + u^2(x) = 1, \tag{36}$$

with the initial value $u(0) = 0$.

The exact solution is known in advance to be

$$u^*(x) = \tanh x. \tag{37}$$

By the Adomian decomposition method and applying the integral operator $L^{-1}(\cdot) = \int_0^x (\cdot)dx$, we have

$$u = x - L^{-1}u^2. \tag{38}$$

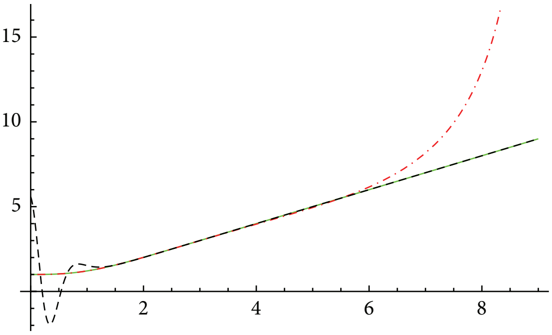

Figure 2: The Padé approximant $[9/10]\{\phi_{20}(x)\}$ (dot-dashed line), far-field approximation $\varphi_7(x)$ (dashed line), and exact solution $u^*(x)$ (solid line).

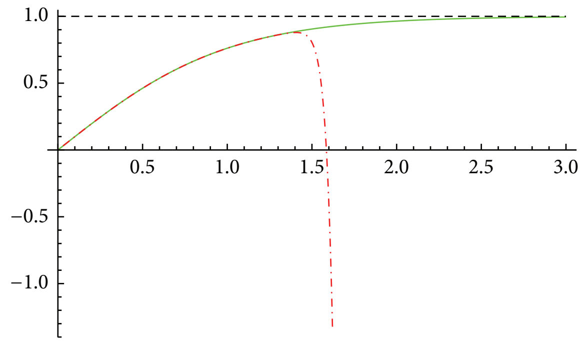

Figure 3: The near-field approximation $\phi_{20}(x)$ (dot-dashed line), far-field approximation $\varphi_3(x)$ (dashed line), and exact solution $u^*(x)$ (solid line).

As before, we decompose u and u^2 as

$$u = \sum_{n=0}^{\infty} u_n,$$
$$u^2 = \sum_{n=0}^{\infty} A_n. \tag{39}$$

Thus the solution components of the near-field approximation $\phi_m(x)$ are determined recursively as

$$\begin{aligned} u_0 &= x, \\ u_1 &= -\frac{x^3}{3}, \\ &\vdots \\ u_n &= -\int_0^x A_{n-1}\,dx, \quad n \geq 1. \end{aligned} \tag{40}$$

By Adomian's asymptotic decomposition method according to the recursion scheme (23), the solution components of the far-field approximation $\varphi_n(x)$ are computed as

$$\begin{aligned} u_0 &= 1, \\ u_1 &= 0, \\ u_2 &= 0, \\ &\vdots \end{aligned} \tag{41}$$

from which we conclude that $u(x) = 1$ is the horizontal asymptote of the exact solution when the independent variable x approaches infinity.

The curves of the near-field approximation $\phi_{20}(x)$, the far-field approximation $\varphi_3(x)$, and the exact solution $u^*(x)$ are plotted in Figure 3.

We calculated the Padé approximant $[13/12]\{\phi_{20}(x)\}$ by Mathematica and found that the Padé approximant

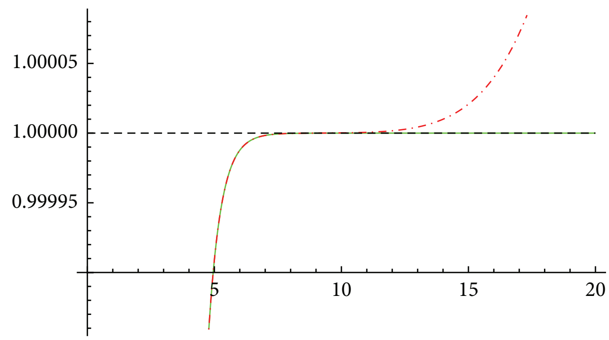

Figure 4: The Padé approximant $[13/12]\{\phi_{20}(x)\}$ (dot-dashed line), far-field approximation $\varphi_3(x)$ (dashed line), and exact solution $u^*(x)$ (solid line).

$[13/12]\{\phi_{20}(x)\}$ and the far-field approximation $\varphi_3(x)$ overlap almost in the approximate region $7 < x < 11$; see Figure 4. Thus we can match them as

$$\tilde{u}(x) = \left[\frac{13}{12}\right]\{\phi_{20}(x)\}\,h(\xi - x) + \varphi_3(x)\,h(x - \xi), \tag{42}$$

which is a global approximation, where ξ belongs to the region of overlap.

5. Conclusion

In this work, we combined the ADM and the AADM to approximate the global solution of the Riccati equation. We evaluated the approximate solution by matching the Padé approximant of the near-field approximation derived from the ADM with the far-field approximation derived from the AADM. Furthermore we have shown that the AADM can be an important complement in analysis of the solution's asymptote.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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